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LOCAL WEAK SOLVABILITY OF A MOVING BOUNDARY PROBLEM DESCRIBING SWELLING ALONG A HALFLINE

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Abstract. We obtain the local well-posedness of a moving boundary problem that describes the swelling of a pocket of water within an infinitely thin elongated pore (i.e. on $[a, +\infty)$, $a > 0$). Our result involves fine *a priori* estimates of the moving boundary evolution, Banach fixed point arguments as well as an application of the general theory of evolution equations governed by subdifferentials.

1. Introduction.

We wish to understand which effect the water-triggered micro-swelling of pores can have at observable scales of concrete-based materials. Such topic is especially relevant in cold regions, where buildings exposed to extremely low temperatures undergo freezing and build microscopic ice lenses that ultimately lead to the mechanical damage of the material; see, for instance, [19]. One way to tackle this issue from a theoretical point of view is to get a better picture of the transport of moisture. Our long-term goal is to build a macro-micro model for moisture transport suitable for cementitious mixtures, where at the macroscopic scale the transport of moisture follows a porous-media-like equation, while at the microscopic scale the moisture is involved in an adsorption-desorption process leading to a strong local swelling of the pores. Such a perspective would lead to a system of partial differential equations with distributed microstructures, see [8, 10] for related settings. In this paper, we propose a one-dimensional microscopic problem posed on a halfline with a moving boundary at one of the ends. The moving boundary conditions encode the swelling mechanism, while a diffusion equation is responsible to providing water content for the swelling to take place.

Since we are interested in how far the water content can actually push the *a priori unknown* moving boundary of swelling, we assume that pore depth is infinite although the actual physical length is finite. Our target here is to show the well-posedness of the pore-level model.

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Let us now describe briefly the setting of our equations. The timespan is $[0,T]$ while the pore is $[a,+\infty)$, with $a,T\in(0,+\infty)$. The variables are $t\in[0,T]$ and $z\in[a,+\infty)$. The boundary $z=a$ denotes the edge of the pore in contact with wetness. The interval $[a,s(t)]$ indicates the region of diffusion of the water content $u(t,z)$, where $s(t)$ is the moving interface of the water region. The function $u(t,z)$ acts in the non-cylindrical region $Q_s(T)$ defined by

$$Q_s(T):=\{(t,z)|0<t<T,\ a<z<s(t)\}.$$

Our free boundary problem, which we denote by $(\mathrm{P})_{u_0,s_0,h}$, reads:
Find the pair $(u(t,z),s(t))$ satisfying

$$u_t-ku_{zz}=0 \text{ for } (t,z)\in Q_s(T), \tag{1.1}$$

$$-ku_z(t,a)=\beta(h(t)-Hu(t,a)) \text{ for } t\in(0,T), \tag{1.2}$$

$$-ku_z(t,s(t))=u(t,s(t))s_t(t) \text{ for } t\in(0,T), \tag{1.3}$$

$$s_t(t)=a_0(u(t,s(t))-\varphi(s(t))) \text{ for } t\in(0,T), \tag{1.4}$$

$$s(0)=s_0, u(0,z)=u_0(z) \text{ for } z\in[a,s_0]. \tag{1.5}$$

Here k is a diffusion constant, β is a given adsorption function on $\mathbb{R}$ that is equal to 0 for negative input and takes a positive value for positive input, h is a given moisture threshold function on $[0,T]$, H and a_0 are further given (positive) constants, φ is our swelling function defined on $\mathbb{R}$, while s_0 and u_0 are the initial data.

From the physical perspective, (1.1) is the diffusion equation displacing u in the unknown region $[a,s]$; the boundary condition (1.2), imposed at $z=a$, implies that the moisture content h inflows if h is present at $z=a$ in a larger amount than u. The boundary condition (1.3) at $z=s(t)$ describes the mass conservation at the moving boundary. Indeed, if the flux $u_z(t,a)$ at $z=a$ is active on the time interval $[t,t+\Delta t]$ for $t>0$, namely, $s_t(t)>0$, then, it holds that

$$\int_a^{s(t)}u(t,z)dz-ku_z(t,a)\Delta t=\int_a^{s(t+\Delta t)}u(t+\Delta t,z)dz.$$

Hence, by dividing Δt in both side and letting $\Delta t\to 0$ we formally obtain that

$$-ku_z(t,a)=\int_a^{s(t)}u_t(t,z)dz+s_tu(t,s(t)).$$

By $u_t=ku_{zz}$ in (1.1), we derive that

$$\begin{aligned}-ku_z(t,a)&=\int_a^{s(t)}ku_{zz}(t,z)dz+s_tu(t,s(t))\\&=ku_z(t,s(t))-ku_z(t,a)+s_tu(t,s(t)).\end{aligned}$$

This formal argument motivates the structure of the moving boundary condition (1.3). The ordinary differential equation (1.4) describes the growth rate of the free boundary s and it is determined by the balance between the water content $u(t,s(t))$ at $z=s(t)$ and the swelling expression $\varphi(s(t))$. It is worth mentioning at this stage that the function $\varphi(s(t))$ limits the growth of the moving boundary.

From the mathematical point of view, our free boundary problem resembles remotely the classical one phase Stefan problem and its variations for handling superheating, phase transitions, evaporation; compare [9, 16, 17, 20] and references cited therein. Our work contributes to the existing mathematical modeling work of swelling by Fasano and collaborators (see [6, 7], e.g.) as well as other authors cf. e.g. [21]. The main difference between these papers and our formulation lies in the choice of the boundary conditions (1.2) and (1.3). Most of the cited settings impose an homogeneous Dirichlet boundary condition at one of the boundaries, while we impose flux boundary conditions at both boundaries. Relation (1.2) will be used in a forthcoming work to connect the microscopic moving boundary discussed here to a macroscopic transport equation.

It is worth mentioning that the literature contains already a number of free boundary problems posed for the corrosion of porous materials. We review here the closest contributions to our setting. For instance, we refer to Muntean and Böhm [14] who proposed a well-posed free boundary problem as mathematical model for the concrete carbonation process in one space dimension; Aiki and Muntean [3, 4, 5] proved the existence and uniqueness of a solution for a simplified Muntean-Böhm-model and obtained the large-time behavior of the free boundary as $t \to \infty$. Also, in [1, 18], Sato et al. proposed a free boundary problem as a mathematical model of single pore adsorption, a setting very close to ours, and showed the existence of a solution locally in time; Aiki and Murase guaranteed in [3] the existence of a solution globally in time and established the large time behaviour of this solution. Recently, based on the results of Sato et al. [18] and Aiki and Murase [2], Kumazaki et al. proposed in [12] a multiscale model of moisture transport with adsorption, coupling in a particular fashion a macroscopic diffusion equation with the microscopic picture of the model proposed by Sato et al. in [18] and ensured the local existence of a solution of this two-scale problem. We refer the reader to [8, 10, 15] and references cited therein for comprehensive descriptions of modeling, mathematical analysis and numerical approximation of reaction-diffusion systems posed on multiple space scales in the absence of free or moving boundaries.

It is worth mentioning that the main reason why we are handling the one-dimensional case only is that we do not know how the sharp interface moves in higher dimensions; hence, we are unable to write down the proper boundary conditions to close the model formulation. A similar issue is present in the case of the concrete carbonation problem mentioned above or in settings involving freely moving redox fronts in porous materials. To be more precise, it is not at all clear how the sharp interface behaves close to corners, e.g.

The paper is organized as follows: In Section 2, we state the used notation and assumptions as well as our main theorem concerning the existence and uniqueness of a solution for the moving boundary problem. In Section 3, we consider an auxiliary problem focused on finding u for given s and prove the existence of a solution of this problem by relying on the abstract theory of evolution equations governed by time-dependent subdifferentials. By using the result of Section 4, we finally prove our main theorem by suitably applying Banach's fixed point theorem and the maximum principle.

2. **Notation and assumptions.** In this framework, we use the following basic notations. We denote by $|\cdot|_X$ the norm for a Banach space X. The norm and the inner product of a Hilbert space H are denoted by $|\cdot|_H$ and $(\cdot,\cdot)_H$, respectively. Particularly, for $\Omega \subset \mathbb{R}$, we use the standard notation of the usual Hilbert spaces $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$.

Throughout this paper, we assume the following restrictions on the model parameters and functions:

(A1) a, a_0, H, k and T are positive constants.

(A2) $h \in W^{1,2}(0,T) \cap L^\infty(0,T)$ with $h \geq 0$ on $(0,T)$.

(A3) $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that $\beta = 0$ on $(-\infty, 0]$, and there exists $r_\beta > 0$ such that $\beta' > 0$ on $(0, r_\beta)$ and $\beta \equiv k_0$ on $[r_\beta, +\infty)$, where k_0 is a positive constant. Also, we put $c_\beta = k_0 + \sup_{r\in\mathbb{R}}\beta'(r)$.

(A4) $\varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that $\varphi = 0$ on $(-\infty, 0]$, and there exists $r_\varphi > 0$ such that $\varphi' > 0$ on $(0, r_\varphi)$ and $\varphi \equiv c_0$ on $[r_\varphi, +\infty)$, where $0 < c_0 \leq \min\{2\varphi(a), |h|_{L^\infty(0,T)}H^{-1}\}$. Also, we put $c_\varphi = \sup_{r\in\mathbb{R}}\varphi(r) + \sup_{r\in\mathbb{R}}\varphi'(r)$.

(A5) $s_0 > a$ and $u_0 \in H^1(a, s_0)$ such that $\varphi(a) \leq u_0(z) \leq |h|_{L^\infty(0,T)}H^{-1}$ on $[a, s_0]$.

For $T > 0$, let s be a function on $[0, T]$ and u be a function on $Q_s(T) := \{(t, z)|0 \leq t \leq T, a < s(t)\}$.

Next, we define our concept of solution to $(\mathrm{P})_{u_0,s_0,h}$ on $[0, T]$ in the following way:

Definition 2.1. We call that pair (s, u) a solution to $(\mathrm{P})_{u_0,s_0,h}$ on $[0, T]$ if the following conditions (S1)-(S6) hold:

(S1) s, $s_t \in L^\infty(0, T)$, $a < s$ on $[0, T]$, $u \in L^\infty(Q_s(T))$, u_t, $u_{zz} \in L^2(Q_s(T))$ and $t \in [0, T] \to |u_z(t, \cdot)|_{L^2(a,s(t))}$ is bounded;

(S2) $u_t - ku_{zz} = 0$ on $Q_s(T)$;

(S3) $-ku_z(t, a) = \beta(h(t) - Hu(t, a))$ for a.e. $t \in [0, T]$;

(S4) $-ku_z(t, s(t)) = u(t, s(t))s_t(t)$ for a.e. $t \in [0, T]$;

(S5) $s_t(t) = a_0(u(t, s(t)) - \varphi(s(t)))$ for a.e. $t \in [0, T]$;

(S6) $s(0) = s_0$ and $u(0, z) = u_0(z)$ for $z \in [a, s_0]$.

The main result of this paper is concerned with the existence and uniqueness of a locally in time solution in the sense of Definition 2.1 to the problem $(\mathrm{P})_{u_0,s_0,h}$. This result is stated in the next Theorem.

Theorem 2.2. *Let $T > 0$. If (A1)-(A5) hold, then there exists $T^* < T$ such that $(P)_{u_0,s_0,h}$ has a unique solution (s, u) on $[0, T^*]$ satisfying $\varphi(a) \leq u \leq |h|_{L^\infty(0,T)}H^{-1}$ on $Q_s(T^*)$.*

To be able to prove Theorem 2.2, we transform $(\mathrm{P})_{u_0,s_0,h}$, initially posed in a non-cylindrical domain, to a cylindrical domain. Let $T > 0$. For given $s \in W^{1,2}(0, T)$ with $a < s(t)$ on $[0, T]$, we introduce the following new function obtained by the indicated change of variables, "freezing" the moving domain:

$$\tilde{u}(t, y) = u(t, (1 - y)a + ys(t)) \text{ for } (t, y) \in Q(T) := (0, T) \times (0, 1).$$

Such a change of variable fixing the moving a priori unknown sharp interface is sometimes referred as Landau transformation. By using the function $\tilde{u}$, $(\mathrm{P})_{u_0,s_0h}$ becomes the following problem $(\mathrm{P})_{\tilde{u}_0,s_0,h}$:

$$\tilde{u}_t(t, y) - \frac{k}{(s(t) - a)^2}\tilde{u}_{yy}(t, y) = \frac{ys_t(t)}{s(t) - a}\tilde{u}_y(t, y) \text{ for } (t, y) \in Q(T), \tag{2.1}$$

$$-\frac{k}{s(t) - a}\tilde{u}_y(t, 0) = \beta(h(t) - H\tilde{u}(t, 0)) \text{ for } t \in (0, T), \tag{2.2}$$

$$-\frac{k}{s(t) - a}\tilde{u}_y(t, 1) = \tilde{u}(t, 1)s_t(t) \text{ for } t \in (0, T), \tag{2.3}$$

$$s_t(t) = a_0(\tilde{u}(t, 1) - \varphi(s(t))) \text{ for } t \in (0, T), \tag{2.4}$$

$$s(0) = s_0, \tag{2.5}$$

$$\tilde{u}(0, y) = u_0(1 - y)a + ys(0))(:= \tilde{u}_0(y)) \text{ for } y \in [0, 1]. \tag{2.6}$$

Definition 2.3. For $T > 0$, let s be functions on $[0, T]$ and $\tilde{u}$ be a function on $Q(T)$, respectively. We call that a pair $(s, \tilde{u})$ is a solution of $(\mathrm{P})_{\tilde{u}_0, s_0, h}$ on $[0, T]$ if the conditions (S'1)-(S'2) hold:

(S'1) $s,\ s_t \in L^\infty(0, T)$, $a < s$ on $[0, T]$, $\tilde{u} \in W^{1,2}(Q(T)) \cap L^\infty(0, T; H^1(0, 1)) \cap L^2(0, T; H^2(0, 1)) \cap L^\infty(Q(T))$.

(S'2) (2.1)–(2.6) hold.

To prove the existence of a solution of $(\mathrm{P})_{\tilde{u}_0, s_0, h}$, we consider now the following problem $(\mathrm{P})^\sigma_{\tilde{u}_0, s_0, h}$:

$$\tilde{u}_t(t, y) - \frac{k}{(s(t) - a)^2} \tilde{u}_{yy}(t, y) = \frac{y s_t(t)}{s(t) - a} \tilde{u}_y(t, y) \text{ for } (t, y) \in Q(T),$$

$$- \frac{k}{s(t) - a} \tilde{u}_y(t, 0) = \beta(h(t) - H\tilde{u}(t, 0)) \text{ for } t \in (0, T),$$

$$- \frac{k}{s(t) - a} \tilde{u}_y(t, 1) = \sigma(\tilde{u}(t, 1)) s_t(t) \text{ for } t \in (0, T),$$

$$s_t(t) = a_0(\sigma(\tilde{u}(t, 1)) - \varphi(s(t))) \text{ for } t \in (0, T),$$

$$s(0) = s_0,\ \tilde{u}(0, y) = \tilde{u}_0(y) \text{ for } y \in [0, 1],$$

where σ is a lower cut-off function on $\mathbb{R}$ given by

$$\sigma(r) = \begin{cases} r & \text{if } r > \varphi(a), \\ \varphi(a) & \text{if } r \leq \varphi(a). \end{cases}$$

The definition of a solution of $(\mathrm{P})^\sigma_{\tilde{u}_0, s_0, h}$ is Definition 2.3 replaced $\tilde{u}(t, 1)$ by $\sigma(\tilde{u}(t, 1))$. Now, we state the existence and uniqueness of a solution of $(\mathrm{P})^\sigma_{\tilde{u}_0, s_0, h}$.

Theorem 2.4. *Let $T > 0$. If (A1)-(A5) hold, then there exists $T^* < T$ such that $(P)^\sigma_{\tilde{u}_0, s_0, h}$ has a unique solution $(s, \tilde{u})$ on $[0, T^*]$.*

By Theorem 2.4, we see that for a solution $(s, \tilde{u})$ of $(\mathrm{P})^\sigma_{\tilde{u}_0, s_0, h}$ on $[0, T^*]$, a pair of the function (s, u) with the variable

$$u(t, z) := \tilde{u}\left(t, \frac{z - a}{s(t) - a}\right) \text{ for } z \in [a, s(t)] \tag{2.7}$$

is a solution of $(\mathrm{P})^\sigma_{u_0, s_0, h} := (\mathrm{P})_{u_0, s_0, h}$ replaced $u(t, s(t))$ by $\sigma(u(t, s(t)))$ on $[0, T^*]$. Finally, by proving that (s, u) satisfies $\varphi(a) \leq u \leq |h|_{L^\infty(0,T)} H^{-1}$ on $Q_s(T^*)$, the pair (s, u) is the desired solution satisfying Theorem 2.2. Therefore, in the rest of the paper, we focus on proving Theorem 2.4 and the boundedness of a solution of $(\mathrm{P})_{u_0, s_0, h}$.

Remark 1. Theorem 2.2 is proven here by Banach's fixed point theorem, and hence, the existence and uniqueness of a locally in time solution is a direct consequence. To reach a globally in time solution of $(\mathrm{P})_{u_0, s_0, h}$, we attempted to extend the existing locally in time solution to $(\mathrm{P})^\sigma_{\tilde{u}_0, s_0, h}$. However, as seen $(\mathrm{P})^\sigma_{\tilde{u}_0, s_0, h}$, if the free boundary s equals to a, then degeneracies occur (i.e. there is no domain to find a solution). Therefore, we have to ensure that s is strictly grater than a at the maximal existence time. Since the free boundary s is not always monotone with respect to time t, it is not easy to prove such a strict lower bound on the sharp interface position. In the forthcoming paper [13], we will show the existence and uniqueness of a globally in time solution of $(\mathrm{P})_{u_0, s_0, h}$.

Remark 2. It is worth noting the similarities and differences between our setting and the one in Ref. [18]. In both works the mathematical approach in handling the well-posedness of the FBP is similar in spirit, i.e. in both cases the free boundary is fixed by Landau-like transformations and weak solutions are searched by using Banach's fixed point argument. However, differences exist and are major. In our work, we require flux boundary conditions at both sides of the one-dimensional interval, thus very different ad hoc estimates have now to be built to ensure a weak maximum principle. As mentioned in the introduction, our motivation to work with our "flux" formulation of the FBP is mainly because we wish to couple our FBP to another PDE posed at a second (macro) spatial scale in an eventually fixed domain, the FBP staying then at a micro spatial level. The structure of the flux boundary conditions is motivated by what we expect from the way the mathematical theory of homogenization applies to such reaction-diffusion set-up with one slowly moving free boundary. These are prerequisites needed to build so-called distributed-microstructure (or two-scale, or micro-macro) models for swelling.

3. Auxiliary Problem.

In this section, for $T > 0$, $L > a$ and given $s \in W^{1,2}(0,T)$ with $a < s < L$ on $[0,T]$, we devote our attention to show the existence of a solution to the following auxiliary problem $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s,h}$:

$$\tilde{u}_t(t,y) - \frac{k}{(s(t)-a)^2}\tilde{u}_{yy}(t,y) = \frac{ys_t(t)}{s(t)-a}\tilde{u}_y(t,y) \text{ for } (t,y) \in Q(T), \tag{3.1}$$

$$-\frac{k}{s(t)-a}\tilde{u}_y(t,0) = \beta(h(t) - H\tilde{u}(t,0)) \text{ for } t \in (0,T), \tag{3.2}$$

$$-\frac{k}{s(t)-a}\tilde{u}_y(t,1) = a_0\sigma(\tilde{u}(t,1))(\sigma(\tilde{u}(t,1)) - \varphi(s(t))) \text{ for } t \in (0,T), \tag{3.3}$$

$$\tilde{u}(0,y) = \tilde{u}_0(y) \text{ for } y \in [0,1], \tag{3.4}$$

In the proof of the existence of solutions, we use the abstract theory of evolution equations in Hilbert spaces governed by time-dependent subdifferentials which is characterized by the following form (cf. [11] and references cited therein):

$$u_t(t) + \partial\varphi^t(u(t)) \ni l(t) \text{ in } H \text{ for } t \in [0,T],$$

where φ^t is a proper, lower semi-continuous, convex function on Hilbert spaces H for $t \in [0,T]$, and $\partial\varphi^t$ is the subdifferential of φ^t defined by

$$\partial\varphi^t(u) := \{z^* \in H \mid (z^*, v-u)_H \le \varphi^t(v) - \varphi^t(u) \text{ for } v \in H\},$$

and l is a given H-valued function on $[0,T]$. For $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s,h}$, we set φ^t on $H = L^2(0,1)$ suitably such that its subdifferential realizes the second term in the left hand side of (3.1) with the boundary conditions (3.2) and (3.3), and consider $\frac{ys_t(t)}{s(t)-a}\tilde{u}_y(t)$ as $l(t)$. To guarantee that $l \in L^2(Q(T))$, we first deal with the case that $s \in W^{1,\infty}(0,T)$ (Lemmas 3.1, 3.2 and 3.3).

For $s \in W^{1,2}(0,T)$, we take a sequence $\{s_n\} \subset W^{1,\infty}(0,T)$ such that $s_n \to s$ in $W^{1,2}(0,T)$ as $n \to \infty$, and prove that $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s,h}$ has a solution $\tilde{u}$ on $[0,T]$ by the limiting process with respect to n using some energy estimates of $\tilde{u}_n$ independent of n, where $\tilde{u}_n$ is a solution on $[0,T]$ of $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s_n,h}$ for each n (Lemma 3.4).

First of all, to solve $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s,h}$, for given $s \in W^{1,\infty}(0,T)$ with $a < s < L$ and $f \in W^{1,2}(Q(T)) \cap L^2(0,T;H^1(0,1))$, we consider the problem $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,f,s,h}$:

$$\tilde{u}_t(t,z) - \frac{k}{(s(t)-a)^2}\tilde{u}_{yy}(t,z) = \frac{ys_t(t)}{s(t)-a}f_y(t,z) \text{ for } (t,z)\in Q(T),$$
$$-\frac{k}{s(t)-a}\tilde{u}_y(t,0) = \beta(h(t)-H\tilde{u}(t,0)) \text{ for } t\in(0,T),$$
$$-\frac{k}{s(t)-a}\tilde{u}_y(t,1) = a_0\sigma(\tilde{u}(t,1))(\sigma(\tilde{u}(t,1))-\varphi(s(t))) \text{ for } t\in(0,T),$$
$$\tilde{u}(0,y) = \tilde{u}_0(y) \text{ for } y\in[0,1].$$

Now, we define a family $\{\psi^t\}_{t\in[0,T]}$ of time-dependent functionals $\psi^t : L^2(0,1)\to \mathbb{R}\cup\{+\infty\}$ for $t\in[0,T]$ as follows:

$$\psi^t(u) := \begin{cases} \dfrac{k}{2(s(t)-a)^2}\displaystyle\int_0^1 |u_y(y)|^2dy + \frac{1}{s(t)-a}\int_0^{u(1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s(t)))d\xi \\ -\dfrac{1}{s(t)-a}\displaystyle\int_0^{u(0)}\beta(h(t)-H\xi)d\xi \text{ if } u\in D(\psi^t), \\ +\infty \text{ if } otherwise, \end{cases}$$

where $D(\psi^t) = \{z\in H^1(0,1) | z\geq 0 \text{ on } [0,1]\}$ for $t\in[0,T]$. What concerns the function ψ^t, we prove a number of structural properties (as they are stated in the following Lemmas).

Lemma 3.1. *Let $s\in W^{1,2}(0,T)$ with $a<s(t)<L$ on $[0,T]$. Assuming (A1)-(A5), then the following statements hold:*

(1) There exists positive constant C_0 and C_1 such that the following inequalities hold:

$$(i)\ |u(0)|^2 \leq C_0\psi^t(u)+C_1 \text{ for } u\in D(\psi^t)$$
$$(ii)\ |u(1)|^2 \leq C_0\psi^t(u)+C_1 \text{ for } u\in D(\psi^t)$$
$$(iii)\ \frac{k}{2(s(t)-a)^2}|u_y|^2_{L^2(0,1)} \leq C_0\psi^t(u)+C_1 \text{ for } u\in D(\psi^t)$$

(2) For $t\in[0,T]$, the functional ψ^t is proper, lower semi-continuous, and convex on $L^2(0,1)$.

Proof. First, we note that for $t\in[0,T]$ if $u\in D(\psi^t)$ then, $u(0)$ and $u(1)$ are non negative. Let $t\in[0,T]$ and $u\in D(\psi^t)$. Then, if $u(1)>\varphi(a)$, then

$$\begin{aligned}
&\int_0^{u(1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s(t)))d\xi \\
=&\int_0^{\varphi(a)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s(t)))d\xi + \int_{\varphi(a)}^{u(1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s(t)))d\xi \\
=&\int_0^{\varphi(a)} a_0\varphi(a)(\varphi(a)-\varphi(s(t)))d\xi + \int_{\varphi(a)}^{u(1)} a_0\xi(\xi-\varphi(s(t)))d\xi \\
=&a_0\varphi^2(a)(\varphi(a)-\varphi(s(t))) \\
&+a_0\frac{u^3(1)}{3} - a_0\varphi(s(t))\frac{u^2(1)}{2} - \left(a_0\frac{\varphi^3(a)}{3} - a_0\varphi(s(t))\frac{\varphi^2(a)}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
=&a_0\frac{u^3(1)}{3}-a_0\varphi(s(t))\frac{u^2(1)}{2}+\left(a_0\frac{2\varphi^3(a)}{3}-a_0\varphi(s(t))\frac{\varphi^2(a)}{2}\right)\\
\geq&\frac{a_0}{3}u^3(1)(1-2\eta^{3/2})-\frac{a_0}{3}\left(\frac{c_\varphi}{2\eta}\right)^3+\left(a_0\frac{2\varphi^3(a)}{3}-a_0\frac{\varphi^2(a)c_\varphi}{2}\right), \qquad (3.5)
\end{aligned}
$$

where η is arbitrary positive constant. By taking η suitably in (3.5) and putting $\delta_s \leq s(t)-a$ for $t\in[0,T]$, we see that there exists $c_0=c_0(\eta)$, $c_1=c_1(\eta)$ such that

$$
\begin{aligned}
&\frac{1}{s(t)-a}\int_0^{u(1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s(t)))d\xi\\
\geq&\frac{1}{s(t)-a}\left(\frac{a_0}{3}u^3(1)(1-2\eta^{3/2})-\left(\frac{a_0}{3}\left(\frac{c_\varphi}{2\eta}\right)^3+a_0\frac{\varphi^2(a)c_\varphi}{2}\right)\right)\\
\geq&\frac{c_0}{L-a}u^3(1)-\frac{c_1}{\delta_s}\geq\frac{c_0\varphi(a)}{L-a}u^2(1)-\frac{c_1}{\delta_s}. \qquad (3.6)
\end{aligned}
$$

In the case $u(1)\leq\varphi(a)$, then $\sigma(u(1))=\varphi(a)$ so that we have the similarly inequality (3.6). Also, we have that

$$
\begin{aligned}
&\frac{-1}{s(t)-a}\int_0^{u(0)}\beta(h(t)-H\xi)d\xi\geq\frac{-c_\beta}{s(t)-a}u(0)=\frac{-c_\beta}{s(t)-a}\left(u(1)-\int_0^1 u_y(y)dy\right)\\
&\geq-\frac{c_0\varphi(a)}{2(L-a)}u^2(1)-\frac{L-a}{2c_0\varphi(a)}\left(\frac{c_\beta}{\delta_s}\right)^2-\frac{k}{4(s(t)-a)^2}\int_0^1|u_y(y)|^2dy-\frac{c_\beta^2}{k}\\
&\geq-\frac{c_0\varphi(a)}{2(L-a)}u^2(1)-\frac{k}{4(s(t)-a)^2}\int_0^1|u_y(y)|^2dy-\left(\frac{L-a}{2c_0\varphi(a)}\left(\frac{c_\beta}{\delta_s}\right)^2+\frac{c_\beta^2}{k}\right), \qquad (3.7)
\end{aligned}
$$

where c_β is the same constant as in (A3). By adding (3.6) and (3.7), it yields

$$
\begin{aligned}
\psi^t(u)\geq&\frac{k}{4(s(t)-a)^2}\int_0^1|u_y(y)|^2dy\\
&+\frac{c_0\varphi(a)}{2(L-a)}u^2(1)-\frac{c_1}{\delta_s}-\left(\frac{L-a}{2c_0\varphi(a)}\left(\frac{c_\beta}{\delta_s}\right)^2+\frac{c_\beta^2}{k}\right). \qquad (3.8)
\end{aligned}
$$

Also, it holds that

$$
\begin{aligned}
|u(0)|^2=&\left|\int_0^1 u_y(y)dy+u(1)\right|^2\leq2\left(\int_0^1|u_y(y)|^2dy+|u(1)|^2\right)\\
\leq&2\left(\frac{2(L-a)^2}{k}\frac{k}{2(s(t)-a)^2}\int_0^1|u_y(y)|^2dy+|u(1)|^2\right).
\end{aligned}
$$

Therefore, by (3.8) and the estimate of $u(0)$ we see that the statement (1) of Lemma 3.1 holds.

We now prove statement (2). For $r\in\mathbb{R}$, put

$$
\begin{aligned}
g_1(s(t),r)=&\frac{1}{s(t)-a}\int_0^r a_0\sigma(\xi)(\sigma(\xi)-\varphi(s(t)))d\xi,\\
g_2(s(t),h(t),r)=&-\frac{1}{s(t)-a}\int_0^r\beta(h(t)-H\xi)d\xi.
\end{aligned}
$$

Then, by $a<s(t)$, $\beta'\geq0$ in (A3) and (A4) we see that $r\mapsto a_0\sigma(r)(\sigma(r)-\varphi(s(t)))$ and $r\mapsto-\beta(h(t)-Hr)$ are also monotone increasing. This means that ψ^t is convex on $L^2(0,1)$. Also, the lower semi-continuity of ψ^t is enough to prove that

the level set of ψ^t is closed in $L^2(0,1)$. This is easy to prove by using Lemma 3.1 and the Sobolev's embedding $H^1(0,1) \hookrightarrow C([0,1])$ in one dimensional case. Thus, we see that for $t \geq 0$, ψ^t is a proper, lower semi-continuous, convex function on $L^2(0,1)$. □

By Lemma 3.1 we obtain the following existence result concerning the solutions to problem $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,f,s,h}$.

Lemma 3.2. *Let $T > 0$ and $L > a$. If (A1)-(A5) hold, then, for given $s \in W^{1,2}(0,T)$ with $a < s < L$ on $[0,T]$ and $f \in W^{1,2}(Q(T)) \cap L^\infty(0,T;H^1(0,1))$, the problem $(AP)^{\sigma}_{\tilde{u}_0,s,f,h}$ admits a unique solution $\tilde{u}$ on $[0,T]$ such that $\tilde{u} \in W^{1,2}(Q(T)) \cap L^\infty(0,T;H^1(0,1))$. Moreover, the function $t \to \psi^t(\tilde{u}(t))$ is absolutely continuous on $[0,T]$.*

Proof. By Lemma 3.1, for $t \in [0,T]$ ψ^t is a proper lower semi-continuous convex function on $L^2(0,1)$. By the definition of the subdifferential in the first of section 3, we see that for $t \in [0,T]$, $z^* \in \partial\psi^t(u)$ is characterized by

$$\begin{aligned} z^* &= -\frac{k}{(s(t)-a)^2} u_{yy} \text{ on } (0,1), \\ -\frac{k}{s(t)-a} u_z(0) &= \beta(h(t) - Hu(0)), \\ -\frac{k}{s(t)-a} u_z(1) &= a_0\sigma(u(1))(\sigma(u(1)) - \varphi(s(t))). \end{aligned}$$

Also, there exists a positive constant C such that for each t_1, $t_2 \in [0,T]$ with $t_1 \leq t_2$, and for any $u \in D(\psi^{t_1})$, there exists $\bar{u} \in D(\psi^{t_2})$ such that

$$|\bar{u} - u|_{L^2(0,1)} \leq |s(t_1) - s(t_2)|(1 + |\varphi^{t_1}(u)|^{1/2}), \tag{3.9}$$

$$|\psi^{t_2}(\bar{u}) - \psi^{t_1}(u)| \leq C(|s(t_1) - s(t_2)| + |h(t_1) - h(t_2)|)(1 + |\psi^{t_1}(u)|). \tag{3.10}$$

Indeed, by taking $\bar{u} := u$ it is easy to prove that (3.9) and (3.10) holds. Now, we consider the following Cauchy problem (CP):

$$\begin{cases} \tilde{u}_t + \partial\psi^t(\tilde{u}(t)) = \frac{y s_t(t)}{s(t)-a} f_y(t) \text{ in } L^2(0,1) \\ \tilde{u}(0,y) = \tilde{u}_0(y) \text{ for } y \in [0,1]. \end{cases}$$

Here, we notice that since $f \in L^2(0,T;H^1(0,1))$ and $s \in W^{1,2}(0,T)$ then $\frac{y f_y(t) s_t(t)}{s(t)-a} \in L^2(0,T;L^2(0,1))$. Then, by the general theory of evolution equations governed by time dependent subdifferentials (see [11] and references cited therein), we conclude that (CP) has a solution $\tilde{u}$ on $[0,T]$ such that $\tilde{u} \in W^{1,2}(Q(T))$, $\psi^t(\tilde{u}(t)) \in L^\infty(0,T)$ and $t \to \psi^t(\tilde{u}(t))$ is absolutely continuous on $[0,T]$. This implies that $\tilde{u}$ is a unique solution of $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,f,s,h}$ on $[0,T]$. □

Lemma 3.3. *Let $T > 0$, $L > a$ and $s \in W^{1,\infty}(0,T)$ with $a < s < L$ on $[0,T]$. If (A1)-(A5) hold, then, $(AP)^{\sigma}_{\tilde{u}_0,s,h}$ has a unique solution $\tilde{u}$ on $[0,T]$ such that $\tilde{u} \in W^{1,2}(Q(T)) \cap L^\infty(0,T;H^1(0,1))$.*

Proof. By Lemma 3.2, we can define the solution operator $\Gamma_T(f) = \tilde{u}$, where $\tilde{u}$ is a unique solution of $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,f,s,h}$ for given $f \in W^{1,2}(Q(T)) \cap L^\infty(0,T;H^1(0,1))$. Now, for $i = 1,2$ we put $\Gamma(f_i) = \tilde{u}_i$ and $f = f_1 - f_2$ and $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$. Then, we have

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}|^2_{L^2(0,1)} - \int_0^1 \frac{k}{(s(t)-a)^2}\tilde{u}_{yy}\tilde{u}dy = \int_0^1 \frac{ys_t}{s(t)-a}f_y\tilde{u}dy. \tag{3.11}$$

Using the structure of the boundary conditions, we obtain

$$\begin{aligned}
&-\int_0^1 \frac{k}{(s(t)-a)^2}\tilde{u}_{yy}\tilde{u}dy\\
&= \frac{k}{(s(t)-a)^2}\left(-\tilde{u}_y(t,1)\tilde{u}(t,1) + \tilde{u}_y(t,0)\tilde{u}(t,0) + \int_0^1 |\tilde{u}_y(t)|^2 dy\right)\\
&= \frac{a_0}{s(t)-a}\times\\
&\quad\Big(\sigma(\tilde{u}_1(t,1))(\sigma(\tilde{u}_1(t,1)) - \varphi(s(t))) - \sigma(\tilde{u}_2(t,1))(\sigma(\tilde{u}_2(t,1)) - \varphi(s(t)))\Big)\tilde{u}(t,1)\\
&\quad - \frac{1}{s(t)-a}\Big(\beta(h(t) - H\tilde{u}_1(t,0)) - \beta(h(t) - H\tilde{u}_2(t,0))\Big)\tilde{u}(t,0)\\
&\quad + \frac{k}{(s(t)-a)^2}\int_0^1 |\tilde{u}_y(t)|^2 dy\\
&\geq -\frac{a_0}{s(t)-a}\varphi(s(t))|\tilde{u}(t,1)|^2 - \frac{c_\beta H}{s(t)-a}|\tilde{u}(t,0)|^2 + \frac{k}{(s(t)-a)^2}\int_0^1 |\tilde{u}_y(t)|^2 dy.
\end{aligned}$$

Combining this inequality with (3.11), it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|^2_{L^2(0,1)} + \frac{k}{(s(t)-a)^2}\int_0^1 |\tilde{u}_y(t)|^2 dy\\
&\leq \int_0^1 \frac{ys_t(t)}{s(t)-a}f_y(t)\tilde{u}(t)dy + \frac{a_0}{s(t)-a}\varphi(s(t))|\tilde{u}(t,1)|^2 + \frac{c_\beta H}{s(t)-a}|\tilde{u}(t,0)|^2.
\end{aligned} \tag{3.12}$$

Here, we use the Sobolev's embedding theorem in one dimensional case:

$$|u(y)|^2 \leq C_e|u|_{H^1(0,1)}|u|_{L^2(0,1)} \text{ for } u \in H^1(0,1) \text{ and } y \in [0,1], \tag{3.13}$$

where C_e is a positive constant in Sobolev's embedding. By using (3.13), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|^2_{L^2(0,1)} + \frac{k}{(s(t)-a)^2}\int_0^1 |u_y(t)|^2 dy\\
&\leq \int_0^1 \frac{ys_t(t)}{s(t)-a}f_y(t)\tilde{u}(t)dy + C_e\left(\frac{a_0c_\varphi}{s(t)-a} + \frac{c_\beta H}{s(t)-a}\right)|\tilde{u}(t)|_{H^1(0,1)}|\tilde{u}(t)|_{L^2(0,1)}.
\end{aligned} \tag{3.14}$$

Taking $C_2 = C_e(a_0c_\varphi + c_\beta H)$ and using Young's inequality leads to

$$\begin{aligned}
&\int_0^1 \frac{ys_t(t)}{s(t)-a}f_y(t)\tilde{u}(t)dy\\
&\leq |s_t|_{L^\infty(0,T)}|\tilde{u}(t)|_{L^2(0,1)}\left(\int_0^1 \frac{1}{(s(t)-a)^2}|f_y(t)|^2 dy\right)^{1/2},\\
&\frac{C_2}{s(t)-a}|\tilde{u}|_{H^1(0,1)}|\tilde{u}|_{L^2(0,1)} \leq \frac{C_2}{s(t)-a}(|\tilde{u}_y|_{L^2(0,1)}|\tilde{u}|_{L^2(0,1)} + |\tilde{u}|^2_{L^2(0,1)})\\
&\leq \frac{k}{2(s(t)-a)^2}|\tilde{u}_y|^2_{L^2(0,1)} + \left(\frac{C_2^2}{2k} + \frac{C_2}{s(t)-a}\right)|\tilde{u}|^2_{L^2(0,1)}.
\end{aligned}$$

Now, we put δ_s such that $s(t) - a \geq \delta_s$ for $t \in [0, T]$. By (3.14), we obtain

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|^2_{L^2(0,1)} + \frac{k}{2(s(t)-a)^2}\int_0^1 |\tilde{u}_y(t)|^2 dy$$
$$\leq \frac{|f_y(t)|^2_{L^2(0,1)}}{2} + \left(\frac{|s_t|^2_{L^\infty(0,T)}}{2\delta_s^2} + \frac{C_2^2}{2k} + \frac{C_2}{\delta_s}\right)|\tilde{u}(t)|^2_{L^2(0,1)}. \tag{3.15}$$

Now, by setting

$$I(t) := \frac{1}{2}|\tilde{u}(t)|^2_{L^2(0,1)} + \frac{k}{2(L-a)^2}\int_0^t |\tilde{u}_y(\tau)|^2_{L^2(0,1)} d\tau$$

for $t \in [0, T]$, we have

$$\frac{d}{dt}I(t) \leq \frac{|f_y(t)|^2_{L^2(0,1)}}{2} + \left(\frac{|s_t|^2_{L^\infty(0,T)}}{2\delta_s^2} + \frac{C_2^2}{2k} + \frac{C_2}{\delta_s}\right)I(t). \tag{3.16}$$

Denote by C_3 the coefficient of $I(t)$ arising in the right-hand side. Using Gronwall's inequality to (3.16) gives

$$I(t) \leq \left(\frac{1}{2}\int_0^t |f_y(\tau)|^2_{L^2(0,1)} d\tau\right) e^{C_3 T} \text{ for } t \in [0, T].$$

This implies that that there exists a small $T_1 \leq T$ such that Γ_{T_1} is a contraction mapping on $W^{1,2}(Q(T)) \cap L^\infty(0,T; H^1(0,1))$. Therefore, by Banach's fixed point theorem we can find $\tilde{u} \in W^{1,2}(Q(T)) \cap L^\infty(0,T; H^1(0,1))$ such that $\Gamma_{T_1}(\tilde{u}) = \tilde{u}$. In other words, we can find a solution $\tilde{u}$ of $(\mathrm{AP})^\sigma_{\tilde{u}_0,s,h}$ on $[0, T_1]$. Since T_1 is independent of the choice of initial value, by repeating the argument of the local existence result, we can extend the solution $\tilde{u}$ beyond T_1. This argument completes the proof of the Lemma. □

As next step, for given $s \in W^{1,2}(0,T)$ with $a < s < L$ on $[0, T]$, we construct a solution to problem $(\mathrm{AP})^\sigma_{\tilde{u}_0,s,h}$.

Lemma 3.4. *Let $T > 0$ and $L > a$. If (A1)-(A5) hold, then, for given $s \in W^{1,2}(0,T)$ with $a < s < L$ on $[0,T]$, the problem $(AP)^\sigma_{\tilde{u}_0,s,h}$ has a unique solution $\tilde{u}$ on $[0,T]$.*

Proof. We choose a sequence $\{s_n\} \subset W^{1,\infty}(0,T)$ and $a < \delta < L$ satisfying $s_n(t) - a \geq \delta$ on $[0,T]$ for each $n \in \mathbb{N}$, $s_n \to s$ in $W^{1,2}(0,T)$ as $n \to \infty$. By Lemma 3.3 we can take a sequence $\{\tilde{u}_n\}$ of solutions to $(\mathrm{AP})^\sigma_{\tilde{u}_0,s_n,h}$ on $[0,T]$. Then, we see that $t \to \psi^t(\tilde{u}_n(t))$ is absolutely continuous on $[0,T]$ so that $t \to \frac{k}{(s_n(t)-a)^2}|\tilde{u}_{ny}(t)|^2_{L^2(0,1)}$ is continuous on $[0,T]$. First, we have

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}_n(t)|^2_{L^2(0,1)} - \int_0^1 \frac{k}{(s_n(t)-a)^2}\tilde{u}_{nyy}(t)\tilde{u}_n(t)dy = \int_0^1 \frac{y s_{nt}(t)}{s_n(t)-a}\tilde{u}_{ny}(t)\tilde{u}_n(t)dy.$$

For the second term in the left hand side, it holds that

$$-\int_0^1 \frac{k}{(s_n(t)-a)^2}\tilde{u}_{nyy}(t)\tilde{u}_n(t)dy$$
$$= \frac{1}{s_n(t)-a}a_0\sigma(\tilde{u}_n(t,1))(\sigma(\tilde{u}_n(t,1)) - \varphi(s_n(t)))\tilde{u}_n(t,1)$$
$$- \frac{1}{s_n(t)-a}\beta(h(t) - H\tilde{u}_n(t,0))\tilde{u}_n(t,0) + \frac{k}{(s_n(t)-a)^2}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy.$$

Accordingly, by $a_0(\sigma(\tilde{u}_n(t,1)))^2\tilde{u}_n(t,1) \geq 0$ we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}_n(t)|^2_{L^2(0,1)} + \frac{k}{(s_n(t)-a)^2}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy \\
\leq &\int_0^1 \frac{y s_{nt}(t)}{s_n(t)-a}\tilde{u}_{ny}(t)\tilde{u}_n(t)dy + \frac{1}{s_n(t)-a}a_0\varphi(s_n(t))\sigma(\tilde{u}_n(t,1))\tilde{u}_n(t,1) \\
&+ \frac{1}{s_n(t)-a}\beta(h(t)-H\tilde{u}_n(t,0))\tilde{u}_n(t,0) \text{ for} t \in [0,T]. \qquad (3.17)
\end{aligned}$$

Using (3.13) it follows that

$$\begin{aligned}
&\int_0^1 \frac{y s_{nt}(t)}{s_n(t)-a}\tilde{u}_{ny}(t)\tilde{u}_n(t)dy \\
\leq &\frac{k}{4(s_n(t)-a)^2}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy + \frac{|s_{nt}(t)|^2}{k}\int_0^1 |\tilde{u}_n(t)|^2 dy,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{s_n(t)-a}a_0\varphi(s(t))\sigma(\tilde{u}_n(t,1))\tilde{u}_n(t,1) \leq \frac{a_0 c_\varphi}{s_n(t)-a}\left(|\tilde{u}_n(t,1)|^2 + \tilde{u}_n(t,1)\varphi(a)\right) \\
\leq &\frac{a_0 c_\varphi}{s_n(t)-a}\left(\frac{3}{2}|\tilde{u}_n(t,1)|^2 + \frac{\varphi^2(a)}{2}\right) \\
\leq &\frac{3a_0 c_\varphi C_e}{2(s_n(t)-a)}\left(|\tilde{u}_{ny}(t)|_{L^2(0,1)}|\tilde{u}_n(t)|_{L^2(0,1)} + |\tilde{u}_n(t)|^2_{L^2(0,1)}\right) + \frac{a_0 c_\varphi}{s_n(t)-a}\frac{\varphi^2(a)}{2} \\
\leq &\frac{k}{4(s_n(t)-a)^2}|\tilde{u}_{ny}(t)|^2_{L^2(0,1)} \\
&+ \left(\frac{(3a_0 c_\varphi C_e)^2}{4k} + \frac{3a_0 c_\varphi C_e}{2\delta}\right)|\tilde{u}_n(t)|^2_{L^2(0,1)} + \frac{a_0 c_\varphi}{\delta}\frac{\varphi^2(a)}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{s_n(t)-a}\beta(h(t)-H\tilde{u}_n(t,0))\tilde{u}_n(t,0) \leq \frac{c_\beta}{s_n(t)-a}|\tilde{u}_n(t,0)| \\
\leq &\frac{c_\beta C_e}{2(s_n(t)-a)}\left(|\tilde{u}_{ny}(t)|_{L^2(0,1)}|\tilde{u}_n(t)|_{L^2(0,1)} + |\tilde{u}_n(t)|^2_{L^2(0,1)}\right) + \frac{c_\beta}{2(s_n(t)-a)} \\
\leq &\frac{k}{4(s_n(t)-a)^2}|\tilde{u}_{ny}(t)|^2_{L^2(0,1)} + \left(\frac{(c_\beta C_e)^2}{4k} + \frac{c_\beta C_e}{2\delta}\right)|\tilde{u}_n(t)|^2_{L^2(0,1)} + \frac{c_\beta}{2\delta}.
\end{aligned}$$

As a consequence, we see from the above two estimates and (3.17) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}_n(t)|^2_{L^2(0,1)} + \frac{k}{4(s_n(t)-a)^2}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy \\
\leq &\left(\frac{|s_{nt}(t)|^2}{k} + \frac{(3a_0 c_\varphi C_e)^2}{4k} + \frac{3a_0 c_\beta C_e}{2\delta} + \frac{(c_\beta C_e)^2}{4k} + \frac{c_\beta C_e}{2\delta}\right)|\tilde{u}_n(t)|^2_{L^2(0,1)} \\
&+ \frac{a_0 c_\varphi}{\delta}\frac{\varphi^2(a)}{2} + \frac{c_\beta}{2\delta} \text{ for } t \in [0,T].
\end{aligned}$$

We denote now the coefficient of $|\tilde{u}_n|^2_{L^2(0,1)}$ in the above inequality by $F(t)$. Then, $F \in L^1(0,T)$ and Gronwall's inequality yields that

$$
\begin{aligned}
&\frac{1}{2}|\tilde{u}_n(t)|^2_{L^2(0,1)} + \int_0^t \frac{k}{4(s_n(t)-a)^2}|\tilde{u}_{ny}(t)|^2_{L^2(0,1)} d\tau \\
&\leq \left(\frac{1}{2}|\tilde{u}(0)|^2_{L^2(0,1)} + \left(\frac{a_0 c_\varphi}{\delta}\frac{\varphi^2(a)}{2} + \frac{c_\beta}{2\delta}\right)T\right) e^{\int_0^t F(\tau)d\tau} \text{ for } t \in [0,T]. \qquad (3.18)
\end{aligned}
$$

Next, for each $n \in \mathbb{N}$ and $h > 0$, we can write

$$
\begin{aligned}
&\int_0^1 \tilde{u}_{nt}(t)\frac{\tilde{u}_n(t)-\tilde{u}_n(t-h)}{h}dy - \int_0^1 \frac{k}{(s_n(t)-a)^2}\tilde{u}_{nyy}(t)\frac{\tilde{u}_n(t)-\tilde{u}_n(t-h)}{h}dy \\
&= \int_0^1 \frac{y s_{nt}(t)}{s_n(t)-a}\tilde{u}_{ny}(t)\frac{\tilde{u}_n(t)-\tilde{u}_n(t-h)}{h}dy. \qquad (3.19)
\end{aligned}
$$

For the second term of (3.19), we obtain

$$
\begin{aligned}
&-\int_0^1 \frac{k}{(s(t)-a)^2}\tilde{u}_{nyy}(t)\frac{\tilde{u}_n(t)-\tilde{u}_n(t-h)}{h}dy \\
&= -\frac{k\tilde{u}_{ny}(t,1)}{(s_n(t)-a)^2}\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-h,1)}{h} + \frac{k\tilde{u}_{ny}(t,0)}{(s_n(t)-a)^2}\frac{\tilde{u}_n(t,0)-\tilde{u}_n(t-h,0)}{h} \\
&\quad + \int_0^1 \frac{k\tilde{u}_{ny}(t)}{(s(t)-a)^2}\frac{\tilde{u}_{ny}(t)-\tilde{u}_{ny}(t-h)}{h}dy.
\end{aligned}
$$

We name as I_1, I_2 and I_3 the three terms in the last identity. We proceed with estimating them from bellow. For the first term I_1, using the same notation g_1 and g_2 cf. the proof of Lemma 3.1, it holds that

$$
\begin{aligned}
I_1 \geq& \frac{1}{h}\frac{1}{s_n(t)-a}\times \\
&\left(\int_0^{\tilde{u}_n(t,1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s_n(t)))d\xi - \int_0^{\tilde{u}_n(t-h,1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s_n(t)))d\xi\right) \\
=& \frac{g_1(s_n(t),\tilde{u}_n(t,1)) - g_1(s_n(t-h),\tilde{u}_n(t-h,1))}{h} \\
&+ \frac{1}{h}\left(\frac{1}{s_n(t-h)-a} - \frac{1}{s_n(t)-a}\right)\int_0^{\tilde{u}_n(t-h,1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s_n(t-h)))d\xi \\
&+ \frac{1}{h}\frac{1}{s_n(t)-a}\times \\
&\int_0^{\tilde{u}_n(t-h,1)}\Big(a_0\sigma(\xi)(\sigma(\xi)-\varphi(s_n(t-h))) - a_0\sigma(\xi)(\sigma(\xi)-\varphi(s_n(t)))\Big)d\xi.
\end{aligned}
$$

Next, for the term I_2 we have

$$
\begin{aligned}
I_2 \geq& \frac{1}{h}\frac{1}{s_n(t)-a}\left(-\int_0^{\tilde{u}_n(t,0)}\beta(h(t)-H\xi)d\xi + \int_0^{\tilde{u}_n(t-h,0)}\beta(h(t)-H\xi)d\xi\right) \\
=& \frac{g_2(s_n(t),h(t),\tilde{u}_n(t,0)) - g_2(s_n(t-h),h(t-h),\tilde{u}_n(t-h,0))}{h} \\
&+ \frac{1}{h}\left(-\frac{1}{s_n(t-h)-a} + \frac{1}{s_n(t)-a}\right)\int_0^{\tilde{u}_n(t-h,0)}\beta(h(t-h)-H\xi)d\xi \\
&- \frac{1}{h}\frac{1}{s_n(t)-a}\int_0^{\tilde{u}_n(t-h,0)}\Big(\beta(h(t-h)-H\xi) - \beta(h(t)-H\xi)\Big)d\xi
\end{aligned}
$$

The term I_3 can be dealt with as follows

$$\begin{aligned}
I_3 \geq& \frac{1}{h}\frac{k}{2(s_n(t)-a)^2}\left(\int_0^1 |\tilde{u}_{ny}(t)|^2 dy - \int_0^1 |\tilde{u}_{ny}(t-h)|^2 dy\right)\\
=& \frac{1}{h}\left(\frac{k}{2(s_n(t)-a)^2}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy - \frac{k}{2(s_n(t-h)-a)^2}\int_0^1 |\tilde{u}_{ny}(t-h)|^2 dy\right)\\
&+ \frac{1}{h}\left(\frac{k}{2(s_n(t-h)-a)^2} - \frac{k}{2(s_n(t)-a)^2}\right)\int_0^1 |\tilde{u}_{ny}(t-h)|^2 dy
\end{aligned}$$

Combining all these lower bounds and using the fact that $t \to k/(s_n(t)-a)^2|\tilde{u}_{ny}(t)|^2$ is continuous on $[0,T]$, we obtain

$$\begin{aligned}
&\liminf_{h\to 0}(I_1+I_2+I_3)\\
\geq& \frac{d}{dt}\psi^t(\tilde{u}_n(t)) + \frac{s_{nt}(t)}{(s_n(t)-a)^2}\int_0^{\tilde{u}_n(t,1)} a_0\sigma(\xi)(\sigma(\xi)-\varphi(s_n(t)))d\xi\\
&+ \frac{a_0\varphi'(s_n(t))s_{nt}(t)}{s_n(t)-a}\int_0^{\tilde{u}_n(t,1)}\sigma(\xi)d\xi + \frac{s_{nt}(t)}{(s_n(t)-a)^2}\int_0^{\tilde{u}_n(t,0)}\beta(h(t)-H\xi)d\xi\\
&- \frac{1}{s_n(t)-a}\int_0^{\tilde{u}_n(t,0)}\beta'(h(t)-H\xi)h_t(t)d\xi + \frac{ks_{nt}(t)}{(s_n(t)-a)^3}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy.
\end{aligned}$$

Applying this result to (3.19) and letting $h \to 0$, we observe

$$\begin{aligned}
&|\tilde{u}_{nt}(t)|^2_{L^2(0,1)} + \frac{d}{dt}\psi^t(\tilde{u}_n(t))\\
\leq& \int_0^1 \frac{ys_{nt}(t)}{s_n(t)-a}\tilde{u}_{ny}(t)\tilde{u}_{nt}(t)dy + \frac{|s_{nt}(t)|}{(s_n(t)-a)^2}\left|\int_0^{\tilde{u}_n(t,1)} a_0\sigma(\xi)(\varphi(s_n(t))-\sigma(\xi))d\xi\right|\\
&+ a_0\frac{|\varphi'(s_n(t))||s_{nt}(t)|}{s_n(t)-a}\int_0^{\tilde{u}_n(t,1)}\sigma(\xi)d\xi + \frac{|s_{nt}(t)|}{(s_n(t)-a)^2}\int_0^{\tilde{u}_n(t,0)}\beta(h(t)-H\xi)d\xi\\
&+ \frac{1}{s_n(t)-a}\left|\int_0^{\tilde{u}_n(t,0)}\beta'(h(t)-H\xi)h_t(t)d\xi\right| + \frac{k|s_{nt}(t)|}{(s_n(t)-a)^3}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy.
\end{aligned}$$

Using Lemma 3.1, we estimate now from above each of the terms J_i for $1 \leq i \leq 6$ that pinpoint each term from the the right-hand side of the above inequality. By using $\sigma(r) \leq |r| + \varphi(a)$ for $r \in \mathbb{R}$ the following upper bounds hold true:

$$\begin{aligned}
J_1 \leq& \frac{1}{2}|\tilde{u}_{nt}(t)|^2_{L^2(0,1)} + \frac{1}{2}\frac{|s_{nt}(t)|^2}{(s_n(t)-a)^2}|\tilde{u}_{ny}(t)|^2_{L^2(0,1)}\\
\leq& \frac{1}{2}|\tilde{u}_{nt}(t)|^2_{L^2(0,1)} + \frac{|s_{nt}(t)|^2}{k}\left(C_0\psi^t(\tilde{u}_n(t)) + C_1\right),\\
J_2 \leq& \frac{a_0|s_{nt}(t)|\varphi(s_n(t))}{2\delta^2}\left(\frac{|\tilde{u}_n(t,1)|^2}{2} + \tilde{u}_n(t,1)\varphi(a)\right),\\
\leq& \frac{a_0|s_{nt}(t)|\varphi(s_n(t))}{2\delta^2}\left(|\tilde{u}_n(t,1)|^2 + \frac{\varphi^2(a)}{2}\right),\\
J_3 \leq& \frac{a_0c_\varphi}{\delta}|s_{nt}(t)|\left(\frac{|\tilde{u}_n(t,1)|^2}{2} + \tilde{u}_n(t,1)\varphi(a)\right)\\
\leq& \frac{a_0c_\varphi}{\delta}|s_{nt}(t)|\left(|\tilde{u}_n(t,1)|^2 + \frac{\varphi^2(a)}{2}\right),
\end{aligned}$$

$$J_4 \le \frac{|s_{nt}(t)|c_\beta}{\delta^2}|\tilde{u}_n(t,0)| \le \frac{c_\beta}{\delta^2}\left(\frac{|s_{nt}(t)|^2}{2} + \frac{|\tilde{u}_n(t,0)|^2}{2}\right),$$

$$J_5 \le \frac{c_\beta}{\delta}|h_t(t)||\tilde{u}_n(t,0)| \le \frac{c_\beta}{\delta}\left(\frac{|h_t(t)|^2}{2} + \frac{|\tilde{u}_n(t,0)|^2}{2}\right),$$

$$J_6 \le \frac{k|s_{nt}(t)|}{(s_n(t)-a)^3}\int_0^1 |\tilde{u}_{ny}(t)|^2 dy \le \frac{2|s_{nt}(t)|}{\delta}\left(C_0\psi^t(\tilde{u}_n(t)) + C_1\right).$$

Finally, by combining all these estimates, we obtain that

$$\begin{aligned}
&\frac{1}{2}|\tilde{u}_{nt}(t)|^2_{L^2(0,1)} + \frac{d}{dt}\psi^t(\tilde{u}_n(t)) \\
\le& \left(\frac{|s_{nt}(t)|^2}{k} + \frac{2|s_{nt}(t)|}{\delta}\right)(C_0\psi^t(\tilde{u}_n(t) + C_1) \\
&+ \frac{a_0|s_{nt}(t)|c_\varphi}{2\delta^2}\left(|\tilde{u}_n(t,1)|^2 + \frac{\varphi^2(a)}{2}\right) \\
&+ \frac{c_\beta}{\delta^2}\frac{|s_{nt}(t)|^2}{2} + \frac{a_0 c_\varphi|s_{nt}(t)|}{\delta}\left(|\tilde{u}_n(t,1)|^2 + \frac{\varphi^2(a)}{2}\right) \\
&+ \left(\frac{c_\varphi}{\delta} + \frac{c_\beta}{\delta^2}\right)\frac{|\tilde{u}_n(t,0)|^2}{2} + \frac{c_\beta}{\delta}\frac{|h_t(t)|^2}{2} \text{ for } t \in [0,T].
\end{aligned}$$

Therefore, by setting

$$\begin{aligned}
l(t) :=& \frac{|s_{nt}(t)|^2}{k} + \frac{2|s_{nt}(t)|}{\delta} + \frac{a_0|s_{nt}(t)|c_\varphi}{2\delta^2} + \frac{a_0 c_\varphi|s_{nt}(t)|}{\delta} + \frac{1}{2}\left(\frac{c_\varphi}{\delta} + \frac{c_\beta}{\delta^2}\right) \\
&+ \frac{\varphi^2(a)}{2}\left(\frac{a_0|s_{nt}(t)|c_\varphi}{2\delta^2} + \frac{a_0 c_\varphi|s_{nt}(t)|}{\delta}\right)
\end{aligned}$$

and using Gronwall's lemma, we have that

$$\begin{aligned}
&\frac{1}{2}\int_0^t |\tilde{u}_{nt}(\tau)|^2_{L^2(0,1)}d\tau + \psi^t(\tilde{u}_n(t)) \\
\le&\Big[\psi^0(\tilde{u}(0)) + \frac{c_\beta}{2\delta^2}\int_0^t |s_{nt}(t)|^2 d\tau + \frac{c_\beta}{2\delta}\int_0^t |h_t(\tau)|^2 d\tau \\
&+ (C_1+1)\int_0^t l(\tau)d\tau\Big]e^{C_0\int_0^t l(\tau)d\tau} \text{ for } t \in [0,T]. \qquad (3.20)
\end{aligned}$$

Therefore, by $l \in L^2(0,T)$ and combining the latter inequality with (A2) we see that the right hand side of (3.20) is bounded. From this result, we infer that the sequence $\{\tilde{u}_n\}$ is bounded in $W^{1,2}(0,T;L^2(0,1))$ and the sequence $\{\psi^{(\cdot)}(\tilde{u}_n(\cdot))\}$ is bounded in $L^\infty(0,T)$. Finally, this result in combination with Lemma 3.1, (3.18) and (3.20) means that the sequence $\{\tilde{u}_n\}$ is bounded in $W^{1,2}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1(0,1))$. Therefore, we can take a sequence $\{n_k\} \subset \{n\}$ such that for some $\tilde{u} \in V(T) := W^{1,2}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1(0,1))$, $\tilde{u}_{n_k} \to \tilde{u}$ weakly in $W^{1,2}(0,T;L^2(0,1))$, weakly -* in $L^\infty(0,T;H^1(0,1))$ and in $C(\overline{Q(T)})$ as $k \to \infty$. By letting $k \to \infty$, we get that $\tilde{u}$ is a solution of $(\mathrm{AP})^\sigma_{\tilde{u}_0,s,h}$ on $[0,T]$. □

4. **Local existence.** In this section, using the results obtained in Section 3, we establish the existence of a solution $(\mathrm{P})^\sigma_{\tilde{u}_0,s_0,h}$ which leads to clarifying Theorem 2.4. Throughout the rest of this section, we assume (A1)-(A5). For $T > 0$ and $L > s_0$, we define the set

$$M(T,s_0,a') := \{s \in W^{1,2}(0,T) | a' \le s < L \text{ on } [0,T], s(0) = s_0\}.$$

Also, for given $s \in M(T, s_0, a')$, we define the operator $\Phi : M(T, s_0, a') \to V(T)$ by $\Phi(s) = \tilde{u}$, where $\tilde{u}$ is a solution of $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s,h}$, and the operator $\Gamma_T : M(T, s_0, a') \to W^{1,2}(0, T)$ by $\Gamma_T(s) = s_0 + \int_0^t a_0(\sigma(\Phi(s)(\tau, 1)) - \varphi(s(\tau)))d\tau$ for $t \in [0, T]$. Moreover, for any $K > 0$ we put

$$M_K(T) := \{s \in M(T, s_0, a')|\ |s|_{W^{1,2}(0,T)} \leq K\}.$$

The construction of a solution of $(\mathrm{P})^{\sigma}_{\tilde{u}_0,s_0,h}$ is done in a couple of steps: First, by the continuous dependence of a solution $\tilde{u}$ of $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s,h}$ for given s in a suitable subspace of $W^{1,2}(0, T)$ we show that Γ_{T_1} is a contraction mapping on $M_K(T_1)$ in $W^{1,2}(0, T_1)$ for some $T_1 < T$. Next, by Banach's fixed point theorem, we prove the existence of a locally in time solution of $(\mathrm{P})^{\sigma}_{\tilde{u}_0,s_0,h}$ (Lemma 4.2). The above setting is constructed such that, relying on (3.18) and (3.20) in Lemma 3.4, the inequality in the next Lemma holds true.

Finally, by using (2.7), the solution of $(\mathrm{P})^{\sigma}_{\tilde{u}_0,s_0,h}$ is a solution of $(\mathrm{P})^{\sigma}_{u_0,s_0,h}$, and by the maximum principle, we observe that a solution (s, u) of $(\mathrm{P})^{\sigma}_{u_0,s_0,h}$ on $[0, T]$ satisfies $\varphi(a) \leq u \leq |h|_{L^\infty(0,T)}H^{-1}$ on $Q(T)$ (Lemma 4.3), and remove σ.

Now, we start this section from noting the following estimates, which is already obtained in Section 3:

Lemma 4.1. *Let $T > 0$ and $K > 0$. It holds that*

$$|\Phi(s)|_{W^{1,2}(0,T;L^2(0,1))} + |\Phi(s)|_{L^\infty(0,T;H^1(0,1))} \leq C \text{ for } s \in M_K(T),$$

where $C = C(T, \tilde{u}_0, K, L, h)$ depending on T, $\tilde{u}_0$, K, L and h.

By using Lemma 4.1 we show that for some $T > 0$, the mapping Γ_T is a contraction mapping on the closed set of $M_K(T)$ for any $K > 0$.

Lemma 4.2. *Let $a < a' \leq s_0$ and $K > 0$. There exists a positive constant $T_1 \leq T$ such that the mapping $\Gamma_{T_1} : M_K(T_1) \to M_K(T_1)$ is well defined. Furthermore, the mapping Γ_{T_1} is a contraction on the closed set $M_K(T_1)$ in $W^{1,2}(0, T)$.*

Proof. For $T > 0$ and $L > s_0$, let $s \in M(T, s_0, a')$ and $\tilde{u} = \Phi(s)$. Then, $\tilde{u}$ is a solution of $(\mathrm{AP})^{\sigma}_{\tilde{u}_0,s,h}$ so that $\sigma(\Phi(s)(t, 1)) \geq \varphi(a)$ for $t \in [0, T]$, and

$$\begin{aligned}\Gamma_T(s)(t) &= s_0 + \int_0^t a_0(\sigma(\Phi(s)(\tau, 1)) - \varphi(s(\tau)))d\tau \\ &\geq s_0 + a_0(\varphi(a) - c_\varphi)t \text{ for } t \in [0, T]. \end{aligned} \tag{4.1}$$

Here, by (3.13) and Lemma 4.1, it follows that

$$\begin{aligned}&\int_0^t |\tilde{u}(\tau, 1)|^2 d\tau \leq C_e \int_0^t (|\tilde{u}_y|_{L^2(0,1)}|\tilde{u}|_{L^2(0,1)} + |\tilde{u}|^2_{L^2(0,1)})d\tau \\ \leq& C_e \left(|\tilde{u}|_{L^\infty(0,T;L^2(0,1))}\sqrt{t} \left(\int_0^t |\tilde{u}_y|^2_{L^2(0,1)} d\tau \right)^{1/2} + t|\tilde{u}|^2_{L^\infty(0,T;L^2(0,1))} \right) \\ \leq& \sqrt{t} C_e (1 + \sqrt{T}) C^2. \end{aligned}$$

Then, we have that

$$\begin{aligned}\Gamma_T(s) &\leq s_0 + a_0\sqrt{t}\left(\int_0^t |\Phi(s)(\tau, 1)|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq s_0 + a_0 t^{\frac{3}{4}}(C_e(1 + \sqrt{T})C^2)^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

Hence, we obtain that

$$\int_0^t |\Gamma_T(s)|^2 d\tau \le 2s_0^2 t + 2a_0^2 t T^{\frac{3}{2}} \left(C_e(1+\sqrt{T})C^2 \right) \tag{4.3}$$

and

$$\int_0^t |\Gamma'_T(s)|^2 d\tau \le a_0^2 \int_0^t |\Phi(s)(\tau,1))|^2 d\tau$$
$$\le a_0^2 \sqrt{t} C_e (1+\sqrt{T}) C^2. \tag{4.4}$$

Therefore, by (4.1)-(4.4) we see that there exists $T_0 < T$ such that $\Gamma_{T_0}(s) \in M_K(T_0)$. Next, let $\tilde{u}_1$ and $\tilde{u}_2$ for s_1 and $s_2 \in M_K(T_0)$, respectively, and set $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, $s = s_1 - s_2$ and $\delta = a' - a$. Then, we have that

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 - \int_0^1 \left(\frac{k}{(s_1(t)-a)^2}\tilde{u}_{1yy}(t) - \frac{k}{(s_2(t)-a)^2}\tilde{u}_{2yy}(t) \right) \tilde{u}(t) dy$$
$$= \int_0^1 \left(\frac{y s_{1t}(t)}{s_1(t)-a}\tilde{u}_{1y}(t) - \frac{y s_{2t}(t)}{s_2(t)-a}\tilde{u}_{2y}(t) \right) \tilde{u}(t) dy. \tag{4.5}$$

Regarding the second term of the left hand side of (4.5), we write

$$- \int_0^1 \left(\frac{k}{(s_1(t)-a)^2}\tilde{u}_{1yy}(t) - \frac{k}{(s_2(t)-a)^2}\tilde{u}_{2yy}(t) \right) \tilde{u}(t) dy$$
$$= \int_0^1 \left(\frac{k}{(s_1(t)-a)^2}\tilde{u}_{1y}(t) - \frac{k}{(s_2(t)-a)^2}\tilde{u}_{2y}(t) \right) \tilde{u}_y(t) dy$$
$$- \left(\frac{k}{(s_1(t)-a)^2}\tilde{u}_{1y}(t)(t,1) - \frac{k}{(s_2(t)-a)^2}\tilde{u}_{2y}(t)(t,1) \right) \tilde{u}(t,1)$$
$$+ \left(\frac{k}{(s_1(t)-a)^2}\tilde{u}_{1y}(t,0) - \frac{k}{(s_2(t)-a)^2}\tilde{u}_{2y}(t,0) \right) \tilde{u}(t,0)$$
$$=: I_1 + I_2 + I_3.$$

For the term I_1, it holds that

$$I_1 = \frac{k}{(s_1(t)-a)^2}|\tilde{u}_y(t)|^2_{L^2(0,1)} + \int_0^1 \left(\frac{k}{(s_1(t)-a)^2} - \frac{k}{(s_2(t)-a)^2} \right) \tilde{u}_{2y}(t)\tilde{u}_y(t) dy$$
$$\ge \frac{k}{(s_1(t)-a)^2}|\tilde{u}_y(t)|^2_{L^2(0,1)} - \frac{2Lk|s(t)|}{\delta^3(s_1(t)-a)}|\tilde{u}_{2y}(t)|_{L^2(0,1)}|\tilde{u}_y(t)|_{L^2(0,1)}$$
$$\ge \left(1 - \frac{\eta}{2}\right) \frac{k}{(s_1(t)-a)^2}|\tilde{u}_y(t)|^2_{L^2(0,1)} - \frac{k}{2\eta}\left(\frac{2L}{\delta^3}\right)^2 |s(t)|^2 |\tilde{u}_{2y}|^2_{L^2(0,1)},$$

where η is arbitrary positive number. The term I_2 is handled as follows:

$$- \left(\frac{k}{(s_1(t)-a)^2}\tilde{u}_{1y}(t,1) - \frac{k}{(s_2(t)-a)^2}\tilde{u}_{2y}(t,1) \right) \tilde{u}(t,1)$$
$$= a_0 \left(\frac{\sigma(\tilde{u}_1(t,1))}{s_1(t)-a}(\sigma(\tilde{u}_1(t,1)) - \varphi(s_1(t))) - \frac{\sigma(\tilde{u}_2(t,1))}{s_2(t)-a}\sigma(\tilde{u}_2(t,1)) - \varphi(s_2(t))) \right) \tilde{u}(t,1)$$
$$= \frac{a_0}{s_1(t)-a} \times$$
$$\left(\sigma(\tilde{u}_1(t,1))(\sigma(\tilde{u}_1(t,1)) - \varphi(s_1(t))) - \sigma(\tilde{u}_2(t,1))(\sigma(\tilde{u}_2(t,1)) - \varphi(s_2(t))) \right) \tilde{u}(t,1)$$

$$
\begin{aligned}
&+\left(\frac{1}{s_1(t)-a}-\frac{1}{s_2(t)-a}\right)a_0\sigma(\tilde u_2(t,1))(\sigma(\tilde u_2(t,1))-\varphi(s_2(t)))\tilde u(t,1)\\
=&\frac{a_0}{s_1(t)-a}\Big(\sigma(\tilde u_1(t,1))-\sigma(\tilde u_2(t,1))\Big)(\sigma(\tilde u_1(t,1))-\varphi(s_1(t)))\tilde u(t,1)\\
&+\frac{a_0}{s_1(t)-a}\sigma(\tilde u_2(t,1))\Big(\sigma(\tilde u_1(t,1))-\varphi(s_1(t))-\sigma(\tilde u_2(t,1))+\varphi(s_2(t))\Big)\tilde u(t,1)\\
&+\left(\frac{1}{s_1(t)-a}-\frac{1}{s_2(t)-a}\right)a_0\sigma(\tilde u_2(t,1))(\sigma(\tilde u_2(t,1))-\varphi(s_2(t)))\tilde u(t,1)\\
=:&\, I_{21}+I_{22}+I_{23}.
\end{aligned}
$$

By using (3.13) and (A4), the following inequalities hold:

$$
\begin{aligned}
|I_{21}|\le&\frac{a_0C_e}{s_1(t)-a}|\sigma(\tilde u_1(t,1))-\varphi(s_1(t))||\tilde u(t)|_{H^1(0,1)}|\tilde u(t)|_{L^2(0,1)}\\
|I_{22}|\le&\frac{a_0}{s_1(t)-a}\sigma(\tilde u_2(t,1))\Big(|\tilde u(t,1)|^2+|\varphi(s_1(t))-\varphi(s_2(t))|\tilde u(t,1)|\Big)\\
\le&\frac{a_0C_e}{s_1(t)-a}\sigma(\tilde u_2(t,1))|\tilde u(t)|_{H^1(0,1)}|\tilde u(t)|_{L^2(0,1)}\\
&+\frac{a_0^2C_e}{2(s_1(t)-a)^2}(\sigma(\tilde u_2(t,1))^2|\tilde u(t)|_{H^1(0,1)}|\tilde u(t)|_{L^2(0,1)}+\frac{c_\varphi^2}{2}|s(t)|^2\\
|I_{23}|=&\left(\frac{s(t)}{(s_1(t)-a)(s_2(t)-a)}\right)a_0\sigma(\tilde u_2(t,1))(\sigma(\tilde u_2(t,1))-\varphi(s_2(t)))\tilde u(t,1)\\
\le&\frac{C_e\Big(a_0\sigma(\tilde u_2(t,1))(\sigma(\tilde u_2(t,1))-\varphi(s_2(t)))\Big)^2}{2\delta^2(s_1(t)-a)^2}|\tilde u(t)|_{H^1(0,1)}|\tilde u(t)|_{L^2(0,1)}\\
&+\frac12|s(t)|^2.
\end{aligned}
$$

Accordingly, by adding the above three estimates, for $t\in[0,T_0]$ we obtain:

$$
\begin{aligned}
&\sum_{k=1}^{3}|I_{2k}|\\
\le&\left(\frac{L_1(t)}{s_1(t)-a}+\frac{L_2(t)}{(s_1(t)-a)^2}\right)|\tilde u(t)|_{H^1(0,1)}|\tilde u(t)|_{L^2(0,1)}+\frac{(c_\varphi^2+1)}{2}|s(t)|^2, \qquad (4.6)
\end{aligned}
$$

where $L_1(t)=a_0C_e(|\tilde u_1(t,1)|+\varphi(a)+c_\varphi)+a_0C_e(|\tilde u_2(t,1)|+\varphi(a))$ and $L_2(t)=a_0^2C_e(|\tilde u_2(t,1)|^2+\varphi^2(a))+C_e(a_0^2(|\tilde u_1(t,1)|+\varphi(a))^4)/2\delta^2$. As for I_2, we split the term I_3 as follows:

$$
\begin{aligned}
&\left(\frac{k}{(s_1(t)-a)^2}\tilde u_{1y}(t,0)-\frac{k}{(s_2(t)-a)^2}\tilde u_{2y}(t,0)\right)\tilde u(t,0)\\
=&-\left(\frac{1}{s_1(t)-a}\beta(h(t)-H\tilde u_1(t,0))-\frac{1}{s_2(t)-a}\beta(h(t)-H\tilde u_2(t,0))\right)\tilde u(t,0)\\
=&-\frac{1}{s_1(t)-a}\Big(\beta(h(t)-H\tilde u_1(t,0))-\beta(h(t)-H\tilde u_2(t,0))\Big)\tilde u(t,0)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{1}{s_1(t)-a} - \frac{1}{s_2(t)-a} \right) \beta(h(t) - H\tilde{u}_2(t,0))\tilde{u}(t,0) \\
&=: I_{31} + I_{32}.
\end{aligned}
$$

Then, by using (3.13) and (A3), we notice that

$$
\begin{aligned}
&\sum_{k=1}^{2} |I_{3k}| \\
&\leq \left(\frac{c_\beta C_e H}{s_1(t)-a} + \frac{c_\beta^2 C_e}{2\delta^2 (s_1(t)-a)^2} \right) |\tilde{u}(t)|_{H^1(0,1)} |\tilde{u}(t)|_{L^2(0,1)} + \frac{1}{2}|s(t)|^2 \text{ for } t \in [0, T_0].
\end{aligned}
\tag{4.7}
$$

What concerns the right-hand side of (4.5), we obtain that

$$
\begin{aligned}
&\int_0^1 \left(\frac{y s_{1t}(t)}{s_1(t)-a} \tilde{u}_{1y}(t) - \frac{y s_{2t}(t)}{s_2(t)-a} \tilde{u}_{2y}(t) \right) \tilde{u}(t) dy \\
&= \int_0^1 \frac{y s_{1t}(t)}{s_1(t)-a} \tilde{u}_y(t) \tilde{u}(t) dy + \int_0^1 \frac{y s_t(t)}{s_1(t)-a} \tilde{u}_{2y}(t) \tilde{u}(t) dy \\
&\quad + \int_0^1 \left(\frac{1}{s_1(t)-a} - \frac{1}{s_2(t)-a} \right) y s_{2t}(t) \tilde{u}_{2y}(t) \tilde{u}(t) dy,
\end{aligned}
$$

while the three terms are controlled from above in the following way:

$$
\begin{aligned}
I_{41} &\leq \frac{\eta k}{2(s_1(t)-a)^2} |\tilde{u}_y(t)|^2_{L^2(0,1)} + \frac{1}{2\eta k} |s_{1t}(t)|^2 |\tilde{u}(t)|^2_{L^2(0,1)}, \\
I_{42} &\leq \frac{1}{2\delta} \left(|s_t(t)|^2 + |\tilde{u}_{2y}(t)|^2_{L^2(0,1)} |\tilde{u}(t)|^2_{L^2(0,1)} \right), \\
I_{43} &\leq \frac{1}{2\delta^2} \left(|s(t)|^2 |\tilde{u}_2(t)|^2_{L^2(0,1)} + |s_{2t}(t)|^2 |\tilde{u}(t)|^2_{L^2(0,1)} \right),
\end{aligned}
$$

Then, by (4.6) and (4.7) we have

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} |\tilde{u}(t)|^2_{L^2(0,1)} + (1-\eta) \frac{k}{(s_1(t)-a)^2} |\tilde{u}_y(t)|^2_{L^2(0,1)} \\
&\leq (L_1(t) + c_\beta C_e H) \frac{1}{s_1(t)-a} |\tilde{u}(t)|_{H^1(0,1)} |\tilde{u}(t)|_{L^2(0,1)} \\
&\quad + \left(L_2(t) + \frac{c_\beta^2 C_e}{2\delta^2} \right) \frac{1}{(s_1(t)-a)^2} |\tilde{u}(t)|_{H^1(0,1)} |\tilde{u}(t)|_{L^2(0,1)} \\
&\quad + \left(\frac{1}{2\eta k} |s_{1t}(t)|^2 + \frac{1}{2\delta} |\tilde{u}_{2y}(t)|^2_{L^2(0,1)} + \frac{1}{2\delta^2} |s_{2t}(t)|^2 \right) |\tilde{u}(t)|^2_{L^2(0,1)} \\
&\quad + \left(\frac{c_\varphi^2}{2} + 1 + \frac{1}{2\delta^2} |\tilde{u}_2(t)|^2_{L^2(0,1)} + \frac{k}{2\eta} \left(\frac{2L}{\delta^3} \right)^2 |\tilde{u}_{2y}|^2_{L^2(0,1)} \right) |s(t)|^2 + \frac{1}{2\delta} |s_t(t)|^2.
\end{aligned}
\tag{4.8}
$$

Young's inequality together with (3.13) ensure

$$\begin{aligned}
&(L_1(t)+c_\beta C_e H)\frac{1}{s_1(t)-a}|\tilde u(t)|_{H^1(0,1)}|\tilde u(t)|_{L^2(0,1)}\\
\le{}&(L_1(t)+c_\beta C_e H)\frac{1}{s_1(t)-a}\left(|\tilde u_y(t)|_{L^2(0,1)}|\tilde u(t)|_{L^2(0,1)}+|\tilde u(t)|^2_{L^2(0,1)}\right)\\
\le{}&(L_1(t)+c_\beta C_e H)\left(\frac{\eta k}{2(s_1(t)-a)^2}|\tilde u_y(t)|^2_{L^2(0,1)}+(\frac{1}{2\eta k}+\frac{1}{\delta})|\tilde u(t)|^2_{L^2(0,1)}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(L_2(t)+\frac{c_\beta^2 C_e}{2\delta^2}\right)\frac{1}{(s_1(t)-a)^2}|\tilde u(t)|_{H^1(0,1)}|\tilde u(t)|_{L^2(0,1)}\\
\le&\left(L_2(t)+\frac{c_\beta^2 C_e}{2\delta^2}\right)\frac{1}{(s_1(t)-a)^2}(|\tilde u_y(t)|_{L^2(0,1)}|\tilde u(t)|_{L^2(0,1)}+|\tilde u(t)|^2_{L^2(0,1)})\\
\le&\left(L_2(t)+\frac{c_\beta^2 C_e}{2\delta^2}\right)\frac{1}{(s_1(t)-a)^2}\frac{\eta k}{2}|\tilde u_y(t)|^2_{L^2(0,1)}+\frac{1}{\delta^2}(\frac{1}{2\eta k}+1)|\tilde u(t)|^2_{L^2(0,1)},
\end{aligned}$$

Here, by (3.13) and Lemma 4.1, we have that

$$\begin{aligned}
|\tilde u_i(t,1)|^2&\le C_e(|\tilde u_{iy}(t)|_{L^2(0,1)}|\tilde u_i(t)|_{L^2(0,1)}+|\tilde u_i(t)|^2_{L^2(0,1)})\\
&\le 2C_eC^2 \text{ for } t\in[0,T_0],
\end{aligned}\tag{4.9}$$

where C is the same constant as in Lemma 4.1. Then, by (4.9) we notice that L_1 and L_2 are bounded in $L^\infty(0,T_0)$. Accordingly, by applying these results to (4.8) and taking a suitable $\eta=\eta_0$, we have

$$\begin{aligned}
&\frac12\frac{d}{dt}|\tilde u(t)|^2_{L^2(0,1)}+\frac12\frac{k}{(s_1(t)-a)^2}|\tilde u_y(t)|^2_{L^2(0,1)}\\
\le{}&(L_1(t)+c_\beta C_e H)\left(\frac{1}{2\eta_0 k}+\frac{1}{\delta}\right)|\tilde u(t)|^2_{L^2(0,1)}\\
&+\left(L_2(t)+\frac{c_\beta^2C_e}{2\delta^2}\right)\frac{1}{\delta^2}\left(\frac{1}{2\eta_0 k}+1\right)|\tilde u(t)|^2_{L^2(0,1)}\\
&+\left(\frac{1}{2\eta_0 k}|s_{1t}(t)|^2+\frac{1}{2\delta}|\tilde u_{2y}(t)|^2_{L^2(0,1)}+\frac{1}{2\delta^2}|s_{2t}(t)|^2\right)|\tilde u(t)|^2_{L^2(0,1)}\\
&+\left(\frac{c_\varphi^2}{2}+1+\frac{1}{2\delta^2}|\tilde u_2(t)|^2_{L^2(0,1)}+\frac{k}{2\eta_0}\left(\frac{2L}{\delta^3}\right)^2|\tilde u_{2y}(t)|^2_{L^2(0,1)}\right)|s(t)|^2+\frac{1}{2\delta}|s_t(t)|^2.
\end{aligned}\tag{4.10}$$

Now, we put the summation of all coefficient of $|\tilde u|^2_{L^2(0,1)}$ by $L_3(t)$ for $t\in[0,T_0]$, and take $L_4(t)=c_\varphi^2/2+1+|\tilde u_2(t)|^2_{L^2(0,1)}/2\delta^2+k(4L^4|\tilde u_{2y}(t)|^2_{L^2(0,1)})/2\eta_0\delta^6+1/2\delta$. Then, we have

$$\begin{aligned}
&\frac12\frac{d}{dt}|\tilde u(t)|^2_{L^2(0,1)}+\frac12\frac{k}{(s_1(t)-a)^2}|\tilde u_y(\tau)|^2_{L^2(0,1)}\\
\le{}&L_3(t)|\tilde u(t)|^2_{L^2(0,1)}+L_4(t)(|s(t)|^2+|s_t(t)|^2) \text{ for } t\in[0,T_0].
\end{aligned}\tag{4.11}$$

Here, using Lemma 4.1, (4.2) and $s_i \in M_K(T_0)$ for $i = 1, 2$, we see that $L_3 \in L^1(0, T_0)$ and $L_4 \in L^\infty(0, T_0)$. Therefore, Gronwall's inequality guarantees that

$$\begin{aligned} &\frac{1}{2}|\tilde{u}(t)|^2_{L^2(0,1)} + \frac{1}{2}\frac{k}{(s_1(t) - a)^2}\int_0^t |\tilde{u}_y(\tau)|^2_{L^2(0,1)} d\tau \\ &\leq \left(|L_4|_{L^\infty(0,T_0)}|s|^2_{W^{1,2}(0,T)}\right) e^{\int_0^t L_3(\tau)d\tau} \text{ for } t \in [0, T_0]. \end{aligned} \tag{4.12}$$

By using (4.12) we show that there exists $T^* < T_0$ such that Γ_{T^*} is a contraction mapping on the closed subset of $M_K(T^*)$. To do so, from the subtraction of the time derivatives of $\Gamma_{T_0}(s_1)$ and $\Gamma_{T_0}(s_2)$ and relying on (3.13) and (4.12), we have for $T_1 < T_0$ the following estimate:

$$\begin{aligned} &|(\Gamma_{T_1}(s_1))_t - (\Gamma_{T_1}(s_2))_t|_{L^2(0,T_1)} \\ =&a_0|\sigma(\tilde{u_1}(\cdot, 1)) - \varphi(s_1(\cdot)) - \sigma((\tilde{u_2}(\cdot, 1)) - \varphi(s_2(\cdot))|_{L^2(0,T_1)} \\ \leq& a_0\left(|\tilde{u}_1(\cdot, 1) - \tilde{u}_2(\cdot, 1)|_{L^2(0,T_1)} + c_\varphi|s|_{L^2(0,T_1)}\right) \\ \leq& a_0c_\varphi T_1|s_t|_{L^2(0,T_1)} + a_0\sqrt{C_e}\left(\int_0^{T_1}(|\tilde{u}_y|_{L^2(0,1)}|\tilde{u}|_{L^2(0,1)} + |\tilde{u}|^2_{L^2(0,1)})dt\right)^{1/2} \\ \leq& a_0c_\varphi T_1|s_t|_{L^2(0,T_1)} \\ &+ C_3\left(\varepsilon|s|_{W^{1,2}(0,T_1)} + \frac{1}{\varepsilon}\sqrt{T_1}|s|_{W^{1,2}(0,T_1)} + \sqrt{T_1}|s|_{W^{1,2}(0,T_1)}\right), \end{aligned} \tag{4.13}$$

where C_3 is a positive constant and ε is an arbitrary positive number. We obtain

$$\begin{aligned} &|\Gamma_{T_1}(s_1) - \Gamma_{T_1}(s_2)|_{L^2(0,T_1)} \\ \leq& T_1\left(a_0c_\varphi T_1|s_t|_{L^2(0,T_1)} + C_3\left(\varepsilon|s|_{W^{1,2}(0,T_1)} + (\frac{1}{\varepsilon} + 1)\sqrt{T_1}|s|_{W^{1,2}(0,T_1)}\right)\right). \end{aligned} \tag{4.14}$$

Therefore, by (4.13) and (4.14) and taking a sufficiently small number ε we see that there exists $T^* < T_0$ such that Γ_{T^*} is a contraction mapping on a closed subset of $M_K(T^*)$. □

From Lemma 4.2, by applying Banach's fixed point theorem, there exists $s \in M_K(T^*)$, where T^* is the same as in Lemma 4.2 such that $\Gamma_{T^*}(s) = s$. This implies that $(\mathrm{P})^\sigma_{\tilde{u}_0,s_0,h}$ has a unique solution $(s, \tilde{u})$ on $[0, T^*]$. Thus, we can prove the existence and uniqueness of a locally in time solution of $(\mathrm{P})^\sigma_{\tilde{u}_0,s_0,h}$ and see that Theorem 2.4 holds. Moreover, this shows that by the change of variable (2.7) a pair of the function (s, u) is a solution of $(\mathrm{P})^\sigma_{u_0,s_0,h}$ on $[0, T^*]$.

At the end of this section, we still must ensure the boundedness of a solution to $(\mathrm{P})^\sigma_{u_0,s_0,h}$, which leads to Theorem 2.2.

Lemma 4.3. *Let $T > 0$, and (s, u) be a solution of $(P)^\sigma_{u_0,s_0,h}$ on $[0, T]$. Then, $\varphi(a) \leq u(t) \leq |h|_{L^\infty(0,T)}H^{-1}$ on $[a, s(t)]$ for $t \in [0, T]$.*

Proof. First, from (1.1), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_a^{s(t)} |[-u(t) + \varphi(a)]^+|^2 dz - \frac{s_t}{2}|[-u(t, s(t)) + \varphi(a)]^+|^2 \\ &+ k\int_a^{s(t)} u_{zz}(t)[-u(t) + \varphi(a)]^+ dz = 0 \text{ for a.e.} t \in [0, T]. \end{aligned} \tag{4.15}$$

By $a < s$ on $[0,T]$ and $\varphi' \geq 0$ in (A4), we note that $\varphi(s(t)) - \varphi(a) \geq 0$ on $[0,T]$. Hence, for the second term in the left hand side, if $u(t,s(t)) < \varphi(a)$, then $-\sigma(u(t,s(t))) + \varphi(s(t)) = -\varphi(a) + \varphi(s(t)) \geq 0$ so that

$$
\begin{aligned}
-\frac{s_t}{2}|[-u(t,s(t)) + \varphi(a)]^+|^2 &= \frac{a_0}{2}(-\sigma(u(t,s(t))) + \varphi(s(t)))|[-u(t,s(t)) + \varphi(a)]^+|^2 \\
&\geq 0.
\end{aligned}
$$

Also, by the boundary conditions (1.2) and (1.3) it follows that

$$
\begin{aligned}
&ku_z(t,s(t))[-u(t,s(t)) + \varphi(a)]^+ \\
=&-\sigma(u(t,s(t)))s_t(t)[-u(t,s(t)) + \varphi(a)]^+ \\
=&a_0\sigma(u(t,s(t)))(-\sigma(u(t,s(t)) + \varphi(s(t)))[-u(t,s(t)) + \varphi(a)]^+
\end{aligned}
$$

and

$$
-ku_z(t,a)[-u(t,a) + \varphi(s(t))]^+ = \beta(h(t) - Hu(t,a))[-u(t,a) + \varphi(s(t))]^+.
$$

Since $\sigma \geq 0$, $\varphi(s(t)) - \varphi(a) \geq 0$ and $\beta \geq 0$ we note that both expressions are positive. Therefore, we obtain that

$$
\frac{d}{dt}\int_a^{s(t)} |[-u(t) + \varphi(a)]^+|^2 dz + k\int_a^{s(t)} |[-u(t) + \varphi(a)]_z^+|^2 dz \leq 0 \text{ for a.e. } t \in [0,T]. \tag{4.16}
$$

Integrating (4.16) over $[0,T]$, we see that $|[-u(t)+\varphi(a)]^+|^2_{L^2(a,s(t))} = 0$ for $t \in [0,T]$ which implies $u(t) \geq \varphi(a)$ on $[a, s(t)]$ for $t \in [0,T]$. Next, we show that $u(t) \leq |h|_{L^\infty(0,T)}H^{-1}$ on $[a, s(t)]$ for $t \in [0,T]$. From (1.1), we first obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u(t)|^2_{L^2(a,s(t))} + \frac{1}{2}s_t(t)|u(t,s(t))|^2 \\
&+ k\int_a^{s(t)} |u_z(t)|^2 dz - \beta(h(t) - Hu(t,a))u(t,a) = 0 \text{ for a.e. } t \in [0,T].
\end{aligned} \tag{4.17}
$$

Here, by $u(t,s(t)) = \frac{s_t(t)}{a_0} + \varphi(s(t))$ and $u(t,s(t)) \geq \varphi(a)$ on $[0,T]$ it holds that

$$
\begin{aligned}
\frac{s_t(t)}{2}|u(t,s(t))|^2 &= \frac{1}{2}\left(\frac{|s_t(t)|^2}{a_0} + \varphi(s(t))s_t(t)\right)u(t,s(t)) \\
&\geq \frac{\varphi(a)}{2a_0}|s_t(t)|^2 - \frac{c_\varphi}{2}|s_t(t)|u(t,s(t)) \\
&\geq \frac{\varphi(a)}{4a_0}|s_t(t)|^2 - \frac{a_0 c_\varphi^2}{4\varphi(a)}u^2(t,s(t))
\end{aligned}
$$

and

$$
\begin{aligned}
&-\beta(h(t) - Hu(t,a))u(t,a) \\
=&\beta(h(t) - Hu(t,a))\frac{h(t) - Hu(t,a)}{H} - \beta(h(t) - Hu(t,a))\frac{h(t)}{H} \\
\geq&-\beta(h(t) - Hu(t,a))\frac{h(t)}{H}.
\end{aligned}
$$

Hence, the above two results and (4.17) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u(t)|^2_{L^2(a,s(t))} + \frac{\varphi(a)}{4a_0}|s_t(t)|^2 + k\int_a^{s(t)}|u_z(t)|^2dz \\
&\le \frac{a_0c_\varphi^2}{4\varphi(a)}u^2(t,s(t)) + \beta(h(t)-Hu(t,a))\frac{h(t)}{H} \text{ for a.e. } t\in[0,T]. \qquad (4.18)
\end{aligned}$$

By Sobolev's embedding theorem in one dimension, it follows that

$$\begin{aligned}
&\frac{a_0c_\varphi^2}{4\varphi(a)}u^2(t,s(t)) \le \frac{a_0c_\varphi^2}{4\varphi(a)}C'_e|u(t)|_{H^1(a,s(t))}|u(t)|_{L^2(a,s(t))} \\
&\le \frac{a_0c_\varphi^2C'_e}{4\varphi(a)}(|u_z(t)|_{L^2(a,s(t))}|u(t)|_{L^2(a,s(t))} + |u(t)|^2_{L^2(a,s(t))}) \\
&\le \frac{k}{2}|u_z(t)|^2_{L^2(a,s(t))} + \left(\frac{1}{2k}\left(\frac{a_0c_\varphi^2C'_e}{4\varphi(a)}\right)^2 + \frac{a_0c_\varphi^2C'_e}{4\varphi(a)}\right)|u(t)|^2_{L^2(a,s(t))}, \qquad (4.19)
\end{aligned}$$

where C'_e is a positive constant in Sobolev's embedding theorem in one dimension. Therefore, by (4.19), (4.18) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u(t)|^2_{L^2(a,s(t))} + \frac{\varphi(a)}{4a_0}|s_t(t)|^2 + \frac{k}{2}\int_a^{s(t)}|u_z(t)|^2dz \\
&\le \left(\frac{1}{2k}\left(\frac{a_0c_\varphi^2C'_e}{4\varphi(a)}\right)^2 + \frac{a_0c_\varphi^2C'_e}{4\varphi(a)}\right)|u(t)|^2_{L^2(a,s(t))} + c_\beta\frac{|h|_{L^\infty(0,T)}}{H}. \qquad (4.20)
\end{aligned}$$

Integrating (4.20) over $[0,T]$ we see that $s_t \in L^2(0,T)$. Now, using a similar argument as in the proof for the lower bound and $\sigma(u(t,s(t))) = u(t,s(t))$ we have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_a^{s(t)}|[u(t)-|h|_{L^\infty(0,T)}H^{-1}]^+|^2dz - \frac{s_t}{2}|[u(t,s(t))-|h|_{L^\infty(0,T)}H^{-1}]^+|^2 \\
&- k\int_a^{s(t)}u_{zz}(t)[u(t)-|h|_{L^\infty(0,T)}H^{-1}]^+dz = 0 \text{ for a.e. } t\in[0,T]. \qquad (4.21)
\end{aligned}$$

By noting from $\sup_{r\in\mathbb{R}}\varphi(r) \le |h|_{L^\infty(0,T)}H^{-1}$ in (A4) that

$$\begin{aligned}
&-ku_z(t,s(t))[u(t,s(t))-|h|_{L^\infty(0,T)}H^{-1}]^+ \\
=&u(t,s(t))s_t[u(t,s(t))-|h|_{L^\infty(0,T)}H^{-1}]^+ \\
=&a_0u(t,s(t))(u(t,s(t))-\varphi(s(t)))[u(t,s(t))-|h|_{L^\infty(0,T)}H^{-1}]^+ \\
\ge&a_0|h|_{L^\infty(0,T)}H^{-1}(|h|_{L^\infty(0,T)}H^{-1} - \sup_{r\in\mathbb{R}}\varphi(r))[u(t,s(t))-|h|_{L^\infty(0,T)}H^{-1}]^+ \ge 0,
\end{aligned}$$

and

$$\begin{aligned}
&ku_z(t,a)[u(t,a)-|h|_{L^\infty(0,T)}H^{-1}]^+ \\
=&-\beta(h(t)-Hu(t,a))[u(t,a)-|h|_{L^\infty(0,T)}H^{-1}]^+ = 0,
\end{aligned}$$

we can write (4.21) as follows:

$$\frac{1}{2}\frac{d}{dt}\int_a^{s(t)} |[u(t) - |h|_{L^\infty(0,T)}H^{-1}]^+|^2 dz + k\int_a^{s(t)} |[u(t) - |h|_{L^\infty(0,T)}H^{-1}]^+_z|^2 dz$$
$$\leq \frac{s_t(t)}{2}|[u(t,s(t)) - |h|_{L^\infty(0,T)}H^{-1}]^+|^2 \text{ for a.e. } t \in [0,T]. \tag{4.22}$$

Similarly to (4.19), we obtain

$$\begin{aligned}&\frac{s_t(t)}{2}|[u(t,s(t)) - |h|_{L^\infty(0,T)}H^{-1}]^+|^2\\ \leq&\frac{s_t(t)C_e'}{2}(|U_z(t)|_{L^2(a,s(t))}|U(t)|_{L^2(a,s(t))} + |U(t)|^2_{L^2(a,s(t))})\\ \leq&\frac{k}{2}|U_z(t)|^2_{L^2(a,s(t))} + \left(\frac{1}{2k}\left(\frac{s_t(t)C_e'}{2}\right)^2 + \frac{s_t(t)C_e'}{2}\right)|U(t)|^2_{L^2(a,s(t))},\end{aligned}$$

where $U(t,z) = [u(t,z) - |h|_{L^\infty(0,T)}H^{-1}]^+$ for $(t,z) \in Q_s(T)$. We put the coefficient of $|U(t)|^2_{L^2(a,s(t))}$ by $G(t)$. Then, $s_t \in L^2(0,T)$ so that we see that $G \in L^1(0,T)$. Therefore, by applying the above to (4.22) and using Gronwall's inequality we get

$$\frac{1}{2}|[u(t) - |h|_{L^\infty(0,T)}H^{-1}]^+|^2_{L^2(a,s(t))} + \frac{k}{2}\int_0^t |[u(t) - |h|_{L^\infty(0,T)}H^{-1}]^+_z|^2_{L^2(a,s(t))} dt$$
$$\leq \left(\frac{1}{2}|[u_0 - |h|_{L^\infty(0,T)}H^{-1}]^+|^2_{L^2(a,s_0)}\right)e^{\int_0^t G(\tau)d\tau} = 0 \text{ for } t \in [0,T].$$

This means that $u(t) \leq |h|_{L^\infty(0,T)}H^{-1}$ on $[a, s(t)]$ for $t \in [0,T]$. Thus, we see that Lemma 4.3 holds. □

By Lemma 4.3, we can remove σ from $(\mathrm{P})^\sigma_{u_0,s_0,h}$, and conclude that the solution (s,u) of $(\mathrm{P})^\sigma_{u_0,s_0,h}$ on $[0,T^*]$ is a solution of $(\mathrm{P})_{u_0,s_0,h}$ on $[0,T^*]$. Finally, by Theorem 2.4 and Lemma 4.3, Theorem 2.2 is proven.

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