

## HOMOGENIZATION AND EXACT CONTROLLABILITY FOR PROBLEMS WITH IMPERFECT INTERFACE

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**ABSTRACT.** The first aim of this paper is to study, by means of the periodic unfolding method, the homogenization of elliptic problems with source terms converging in a space of functions less regular than the usual  $L^2$ , in an  $\varepsilon$ -periodic two component composite with an imperfect transmission condition on the interface. Then we exploit this result to describe the asymptotic behaviour of the exact controls and the corresponding states of hyperbolic problems set in composites with the same structure and presenting the same condition on the interface. The exact controllability is developed by applying the Hilbert Uniqueness Method, introduced by J. -L. Lions, which leads us to the construction of the exact controls as solutions of suitable transposed problem.

**1. Introduction.** Let  $\Omega$  be a domain of  $\mathbb{R}^n$ ,  $n \geq 2$ , made up of a connected set  $\Omega_1^\varepsilon$  and a disconnected one,  $\Omega_2^\varepsilon$ , consisting of  $\varepsilon$ -periodic connected inclusions of size  $\varepsilon$ . Let  $\Gamma^\varepsilon = \partial\Omega_2^\varepsilon$  denote the interface separating the two sub-domains of  $\Omega$  and suppose that  $\partial\Omega \cap \Gamma_\varepsilon = \emptyset$  (see Figure 1).

In the first part of the paper, we consider the stationary heat equation in the two component composite modeled by  $\Omega$ , assuming that on the interface  $\Gamma^\varepsilon$  the heat flux is proportional to the jump of the temperature field, by means of a function of order  $\varepsilon^\gamma$  (see Section 3, problem (3.1)). The order of magnitude of the parameter  $\gamma$ , with respect to the period  $\varepsilon$ , determines the influence of the thermal resistance in the heat exchange between the two materials (see [5] for the physical justification of the model). As observed by H.C. Hummel in [41], it is natural to suppose  $\gamma \leq 1$ , otherwise one cannot expect to have boundedness of the solutions.

This interface problem was studied in [28, 49, 50] in the case of fixed source term in  $L^2$  by the classical method of oscillating test functions due to L. Tartar (see also [9], Section 8.5). The authors proved that, as long as the interfacial resistance

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increases, one gets, at the limit, a composite where the two components become more and more isolated. More precisely, asymptotically, the composite behaves as in presence of just one temperature field. However, the effective thermal conductivity of the homogenized material changes according to  $\gamma$ . Indeed,

- for  $\gamma < -1$ , it is the one obtained in the case of a classical composite without barrier resistance;
- for  $\gamma = -1$ , it also takes into account the contact barrier;
- for  $-1 < \gamma < 1$ , it is the one obtained in the case of a perforated composite with no material occupying the inclusions;
- for  $\gamma = 1$ , it is the same of the previous case, but an additional term depending on the interface resistance appears in the limit behaviour of the solution. This means that the heat exchange is not sufficient to spread out the interfacial contribution and the heat source inside the inclusions.

Later on, in [26], the above results were recovered and completed by specifying the convergences of the flux by means of the periodic unfolding method, introduced for the first time by D. Cioranescu, A. Dalmlamian and G. Griso in [6].

In [35], with the further assumption of symmetry of the coefficients' matrix, these results were extended, only for  $-1 < \gamma \leq 1$ , to the case of source terms converging in a space of functions less regular than the usual  $L^2$ , by using the classical method of oscillating test functions due to L. Tartar (see also [9], Section 8.5). Some difficulties arose when considering the remaining values of  $\gamma$ .

In this paper, our first aim is to overcome these difficulties by means of the periodic unfolding method and to conclude the asymptotic analysis started in [35] by considering the remaining cases  $\gamma < -1$  and  $\gamma = -1$ .

More precisely, in Theorems 3.14 and 3.18 (see also Corollaries 3.15 and 3.19), we prove that also in this framework, at the limit one gets the same effective thermal conductivities of [26, 49]. Nevertheless, due to the less regularity of the source terms, a relevant difference appears. Indeed, here the heat source in the limit problem depends on subsequences of the heat sources at  $\varepsilon$ -level (see Remarks 3.16 and 3.20). We remark that, if fixed right-hand members are considered, the homogenization results of this paper exactly recover the ones of [26, 49]. Moreover, we point out that the arguments used in this work can be easily adapted to the cases  $-1 < \gamma < 1$  and  $\gamma = 1$ . In fact we improve the results of [35] since we don't require the coefficients' matrix to be symmetric anymore.

Physically speaking, the weak data may model two different wiry heat sources positioned in the two components of the material, for  $n = 2$ , or two heat sources that can be represented as  $n - 1$ -dimensional varieties, for  $n \geq 3$ .

The above mentioned homogenization results with less regular source terms, interesting in itself, have as relevant application the study of the exact controllability of hyperbolic problems set in composites with the same structure and presenting the same jump condition on the interface, that cannot be performed at all using the results of [26, 49].

For an evolution problem, given a time interval  $[0, T]$ , the exact controllability issue consists in asking if it is possible to act on the solutions, by means of a suitable control, in order to drive the system to a desired state at time  $T$ , for all initial data. When homogenization processes are involved, a further interesting question arises: provided the exact controllabilities of the  $\varepsilon$ -problems and of the homogenized one, do the exact controls and the corresponding states at  $\varepsilon$ -level converge to the ones of the homogenized problem? Having in mind this question, the second aim

of this paper is to study the asymptotic behaviour of the exact controls and the corresponding states of the wave equation in a medium made up of two components with very different coefficients of propagation, giving rise to the jump condition on the interface depending on  $\gamma$  (see Section 4, problem (4.1)). Taking into account the homogenization results of Section 3, in Theorem 4.3 we give a positive answer to the above question, for  $\gamma \leq -1$ . For the remaining cases of  $\gamma$  we refer the reader to [36].

The plan of the paper is the following one. In Section 2, we describe in details the two component domain  $\Omega$ . In Section 3, at first, we recall the definitions and the properties of specific functional spaces, suitable for the solutions of these kinds of interface problems, introduced in [21, 23, 28, 49]. Then, we remind the definitions and the main properties of two unfolding operators for the two component domain  $\Omega$ , defined for the first time in [7, 26]. Finally, we develop the homogenization of the stationary imperfect transmission problem with less regular source term, by means of the periodic unfolding method. Section 4, is devoted to the study of the exact controllability of the hyperbolic imperfect transmission problem. Here we use a constructive method, known as Hilbert Uniqueness Method, introduced for the first time by Lions in [44, 45]. The idea is to build the exact controls as the solutions of transposed problems associated to suitable initial conditions obtained by calculating at zero time the solutions of related backward problems. These controls, obtained by HUM, are also energy minimizing controls. More precisely, in Theorem 4.3, we describe the asymptotic behavior of the  $\varepsilon$ -controllability problem. To this aim, at first, we recall the homogenization results of [21] for the wave equation in the same two component domain  $\Omega$  (cf. Theorem 4.5). Then, having in mind the transposed problem at  $\varepsilon$ -level given by HUM method, we prove a homogenization result for the wave equation but with less regular initial data and zero right-hand member (cf. Theorem 4.7). This requires the asymptotic analysis of a stationary  $\varepsilon$ -problem, with right-hand member converging in a space of functions less regular than the usual  $L^2$ , which is possible thanks to the results of Section 3. Finally we prove that the exact control of the problem at  $\varepsilon$ -level and the corresponding state, converge, as  $\varepsilon \rightarrow 0$ , to the exact control and to the solution of the homogenized problem respectively.

Similar elliptic homogenization problems and corrector results can be found in [1, 3, 19, 20, 28, 41, 47, 48, 49, 50, 51]. Different homogenization results for stationary problems in  $\varepsilon$ -periodic perforated domains have been studied in [4, 29, 37]. For previous homogenization results in the case of weakly converging data, we quote Tartar (see [9], Proposition 8.17, Remark 8.18 and Theorem 8.19) and [13, 52]. As regards evolution problems in domains with imperfect interface, we refer to [21, 22, 23, 54, 55].

The exact controllability of hyperbolic problems with oscillating coefficients in fixed domains is treated in [44] and, in the case of perforated domains, in [8, 11]. In [14]÷[18], [31]÷[33] and [53], the authors study the optimal control and exact controllability problems in domains with highly oscillating boundary. We refer the reader to [38, 39] for the optimal control of hyperbolic problems in composites with imperfect interface and to [42] for the optimal control of rigidity parameters of thin inclusions in composite materials. We quote [23]÷[25] and [34] for the correctors and the approximate control for a class of parabolic equations with interfacial contact resistance. In [30], the approximate controllability of linear parabolic equations in perforated domains is considered. In [57, 58], the author treats the approximate

controllability of a parabolic problem with highly oscillating coefficients in a fixed domain. Null controllability results for semilinear heat equations in a fixed domain can be found in [40], while the exact internal controllability and exact boundary controllability for semilinear wave equations are considered in [43] and [56], respectively.

**2. The  $\varepsilon$ -periodic two component domain.** Let  $Y := \prod_{i=1}^n ]0, l_i[$ ,  $n \geq 2$ , be the reference cell, where  $l_i$ , for  $i = 1, \dots, n$ , are positive real numbers. Then, let  $Y_1$  and  $Y_2$  be two nonempty open and disjoint subsets of  $Y$  such that

$$\bar{Y}_2 \subset Y \quad Y := Y_1 \cup \bar{Y}_2.$$

Moreover we suppose that  $Y_1$  is connected and  $\Gamma := \partial Y_2$  is Lipschitz continuous.

For any  $k \in \mathbb{Z}^n$ , we denote by  $Y_i^k$  and  $\Gamma_k$  the following translated sets

$$Y_i^k := k_l + Y_i, \quad i = 1, 2, \quad \Gamma_k := k_l + \Gamma,$$

where  $k_l = (k_1 l_1, \dots, k_n l_n)$ . Moreover, for any given  $\varepsilon$ , we set

$$K^\varepsilon := \{k \in \mathbb{Z}^n \mid \varepsilon \Gamma_k \cap \Omega \neq \emptyset\},$$

where  $\varepsilon$  is a sequence of positive real numbers converging to zero.

Let  $\Omega$  be a connected open bounded subset of  $\mathbb{R}^n$ , we define

$$\Omega_i^\varepsilon := \Omega \cap \left\{ \bigcup_{k \in K^\varepsilon} \varepsilon Y_i^k \right\}, \quad i = 1, 2, \quad \Gamma^\varepsilon := \partial \Omega_2^\varepsilon$$

and assume that

$$\partial \Omega \cap \left( \bigcup_{k \in \mathbb{Z}^n} (\varepsilon \Gamma_k) \right) = \emptyset. \tag{2.1}$$

We explicitly observe that, by construction, the set  $\Omega$  is decomposed into two components  $\Omega = \Omega_1^\varepsilon \cup \bar{\Omega}_2^\varepsilon$  with  $\Omega_1^\varepsilon$  connected and  $\Omega_2^\varepsilon$  a disconnected union of  $\varepsilon$ -periodic disjoint translated sets of  $\varepsilon Y_2$ . In view of (2.1), the interface separating the two components,  $\Gamma^\varepsilon$ , is such that  $\partial \Omega \cap \Gamma^\varepsilon = \emptyset$  (see Figure 1).

Throughout the paper we denote by

- $\tilde{u}$ : the zero extension to the whole  $\Omega$  of a function  $u$  defined on  $\Omega_1^\varepsilon$  or  $\Omega_2^\varepsilon$ ,
- $\chi_E$ : the characteristic function of any measurable set  $E \subseteq \mathbb{R}^n$ ,
- $\mathcal{M}_E(f) := \frac{1}{|E|} \int_E f \, dx$ , the average on  $E$  of any function  $f \in L^1(E)$ .

Let us recall (see for instance [9]) that, as  $\varepsilon \rightarrow 0$ ,

$$\chi_{\Omega_i^\varepsilon} \rightharpoonup \theta_i := \frac{|Y_i|}{|Y|} \quad \text{weakly in } L^2(\Omega), \quad \text{for } i = 1, 2, \tag{2.2}$$

$\theta_i$  being the proportion of the material occupying  $\Omega_i^\varepsilon$ .

**3. Homogenization of an elliptic imperfect transmission problem with weakly converging data.** Our first goal is to describe, for  $\gamma \leq -1$ , the asymptotic behavior, as  $\varepsilon \rightarrow 0$ , of the following stationary problem

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} & \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon, \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where  $n_{i\varepsilon}$  is the unitary outward normal to  $\Omega_i^\varepsilon$ ,  $i=1,2$ .

We suppose that

$$A \in M(\alpha, \beta, Y) \quad (3.2)$$

for some  $\alpha, \beta \in \mathbb{R}$ ,  $0 < \alpha < \beta$ , where  $M(\alpha, \beta, Y)$  is the set of the  $n \times n$   $Y$ -periodic matrix-valued functions with bounded coefficients such that, for any  $\lambda \in \mathbb{R}^n$ ,

$$\begin{cases} (A\lambda, \lambda) \geq \alpha |\lambda|^2 & \text{a.e. in } Y, \\ |A\lambda| \leq \beta |\lambda| & \text{a.e. in } Y. \end{cases} \quad (3.3)$$

We assume that

$$\begin{cases} h \text{ is a } Y\text{-periodic function in } L^\infty(\Gamma) \text{ and} \\ \exists h_0 \in \mathbb{R} \text{ such that } 0 < h_0 < h(y) \text{ a.e. in } \Gamma. \end{cases} \quad (3.4)$$

Moreover, for any fixed  $\varepsilon$ ,  $A^\varepsilon, h^\varepsilon$  are given by

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega, \quad (3.5)$$

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \Gamma^\varepsilon. \quad (3.6)$$

**3.1. The functional space  $H_\gamma^\varepsilon$  and its dual  $(H_\gamma^\varepsilon)'$ .** In this subsection, we recall the definition and some useful properties of a class of functional spaces introduced for the first time in [49], and successively in [28], when studying the analogous stationary problem but with regular data (see also [19, 23]). These spaces take into account the geometry of the domain where the material is confined as well as the boundary and interfacial conditions, hence they are suitable for the solutions of this particular kind of interface problems.

**Definition 3.1.** [[49]] For every  $\gamma \in \mathbb{R}$ , the Banach space  $H_\gamma^\varepsilon$  is defined by

$$H_\gamma^\varepsilon := \{u = (u_1, u_2) \mid u_1 \in V^\varepsilon, u_2 \in H^1(\Omega_2^\varepsilon)\} \quad (3.7)$$

equipped with the norm

$$\|u\|_{H_\gamma^\varepsilon}^2 = \|\nabla u_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla u_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon^\gamma \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)}^2 \quad (3.8)$$

where

$$V^\varepsilon := \{v \in H^1(\Omega_1^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}$$

is a Banach space endowed with the norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}, \quad (3.9)$$

see [12].

The condition on  $\partial\Omega$  in the definition of  $V^\varepsilon$  has to be understood in a density sense, since we don't require any regularity on  $\partial\Omega$ . Namely,  $V^\varepsilon$  is the closure, with respect to the  $H^1(\Omega_1^\varepsilon)$ -norm, of the set of the functions in  $C^\infty(\Omega_1^\varepsilon)$  with a compact support contained in  $\Omega$ . This can be done in view of (2.1).

**Proposition 3.2** ([23, 26]). *There exists a positive constant  $C_1$ , independent of  $\varepsilon$ , such that*

$$\|u\|_{H_\gamma^\varepsilon}^2 \leq C_1(1 + \varepsilon^{\gamma-1})\|u\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 \quad \forall \gamma \in \mathbb{R}, \forall u \in H_\gamma^\varepsilon. \quad (3.10)$$

If  $\gamma \leq 1$ , then there exists another positive constant  $C_2$ , independent of  $\varepsilon$ , such that

$$C_2\|u\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 \leq \|u\|_{H_\gamma^\varepsilon}^2 \leq C_1(1 + \varepsilon^{\gamma-1})\|u\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 \quad \forall u \in H_\gamma^\varepsilon. \quad (3.11)$$

**Corollary 3.3** ([26]). *Let  $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$  be a bounded sequence of  $H_\gamma^\varepsilon$ . Then, if  $\gamma \leq 1$ , there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\|u_{2\varepsilon}\|_{H^1(\Omega_2^\varepsilon)} \leq C. \quad (3.12)$$

We denote by  $(H_\gamma^\varepsilon)'$  the dual of  $H_\gamma^\varepsilon$ . As proved in [23], for any fixed  $\varepsilon$ , the norms of  $(H_\gamma^\varepsilon)'$  and  $(V^\varepsilon)' \times (H^1(\Omega_2^\varepsilon))'$  are equivalent. Moreover, if  $v = (v_1, v_2) \in (V^\varepsilon)' \times (H^1(\Omega_2^\varepsilon))'$  and  $u = (u_1, u_2) \in V^\varepsilon \times H^1(\Omega_2^\varepsilon)$ , then

$$\langle v, u \rangle_{(H_\gamma^\varepsilon)', H_\gamma^\varepsilon} = \langle v_1, u_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle v_2, u_2 \rangle_{(H^1(\Omega_2^\varepsilon))', H^1(\Omega_2^\varepsilon)}. \quad (3.13)$$

For sake of simplicity, throughout this paper, we denote by  $L_\varepsilon^2(\Omega) := L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon)$ . The space  $L_\varepsilon^2(\Omega)$  will be equipped with the usual product norm, that is,

$$\|(w_1, w_2)\|_{L_\varepsilon^2(\Omega)}^2 = \|w_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|w_2\|_{L^2(\Omega_2^\varepsilon)}^2 \quad \forall (w_1, w_2) \in L_\varepsilon^2(\Omega).$$

Since the homogenization results proved in this section will be applied to study the exact controllability of the wave equation in composites with the same structure, we need to recall some further properties of the space  $H_\gamma^\varepsilon$ .

**Remark 3.4.** We point out that  $H_\gamma^\varepsilon$  is a separable and reflexive Hilbert space dense in  $L_\varepsilon^2(\Omega)$ . Furthermore,  $H_\gamma^\varepsilon \subseteq L_\varepsilon^2(\Omega)$  with continuous imbedding. On the other hand, one has that  $L_\varepsilon^2(\Omega) \subseteq (H_\gamma^\varepsilon)'$ , where  $L_\varepsilon^2(\Omega)$  is a separable Hilbert space. This means that the triple  $(H_\gamma^\varepsilon, L_\varepsilon^2(\Omega), (H_\gamma^\varepsilon)')$  is an evolution triple. We refer the reader to [21, 22] for an in-depth analysis on this aspect.

**3.2. Periodic unfolding operators in two-component domains.** In this subsection, we recall the definitions and the main properties of two unfolding operators. The first one,  $\mathcal{T}_1^\varepsilon$ , concerning functions defined in  $\Omega_1^\varepsilon$ , is exactly that introduced in [7] for perforated domains. The second one,  $\mathcal{T}_2^\varepsilon$ , acts on functions defined in  $\Omega_2^\varepsilon$  and was defined for the first time in [26]. These operators map functions defined on the oscillating domains  $\Omega_1^\varepsilon, \Omega_2^\varepsilon$  into functions defined on the fixed domains  $\Omega \times Y_1$  and  $\Omega \times Y_2$ , respectively. Consequently, there is no need to introduce extension operators to pass to the limit in the problem.

Using the notations of Section 2, let us introduce the following sets (see Figure 2)

- $\widehat{K}^\varepsilon = \{k \in \mathbb{Z}^n \mid \varepsilon Y^k \subset \Omega\}$
- $\widehat{\Omega}^\varepsilon = \text{int} \bigcup_{k \in \widehat{K}^\varepsilon} \varepsilon(k_l + \overline{Y})$ ,  $\Lambda^\varepsilon = \Omega \setminus \widehat{\Omega}^\varepsilon$ ,
- $\widehat{\Omega}_i^\varepsilon = \bigcup_{k \in \widehat{K}^\varepsilon} \varepsilon Y_i^k$ ,  $\Lambda_i^\varepsilon = \Omega_i^\varepsilon \setminus \widehat{\Omega}_i^\varepsilon$ ,  $i = 1, 2$ ,  $\widehat{\Gamma}^\varepsilon = \partial \widehat{\Omega}_2^\varepsilon$ .

In the sequel, for  $z \in \mathbb{R}^n$ , we use  $[z]_Y$  to denote its integer part  $k_l$ , such that  $z - [z]_Y \in Y$  and set

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. in } \mathbb{R}^n.$$

Then, for a.e.  $x \in \mathbb{R}^n$ , one has

$$x = \varepsilon \left( \left[ \frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right).$$

**Definition 3.5.** [[7, 26]] For any Lebesgue-measurable function  $\phi$  on  $\Omega_i^\varepsilon$ ,  $i = 1, 2$ , the periodic unfolding operator  $\mathcal{T}_i^\varepsilon$  is defined by

$$\mathcal{T}_i^\varepsilon(\phi)(x, y) = \begin{cases} \phi \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{a.e. } (x, y) \in \widehat{\Omega}^\varepsilon \times Y_i, \\ 0 & \text{a.e. } (x, y) \in \Lambda^\varepsilon \times Y_i. \end{cases}$$

**Remark 3.6.** In order to simplify the presentation, in the sequel if  $\Phi$  is a function defined in  $\Omega$ , we simply denote  $\mathcal{T}_i^\varepsilon(\Phi|_{\Omega_i^\varepsilon})$  by  $\mathcal{T}_i^\varepsilon(\Phi)$ , for  $i = 1, 2$ .

Let us collect the following results which are proved in [7, 10, 26].

**Proposition 3.7** ([7, 10, 26]). *Let  $p \in [1, +\infty[$  and  $i = 1, 2$ . The operators  $\mathcal{T}_i^\varepsilon$  are linear and continuous from  $L^p(\Omega_i^\varepsilon)$  to  $L^p(\Omega \times Y_i)$ . Moreover,*

- i)  $\mathcal{T}_i^\varepsilon(\varphi\psi) = \mathcal{T}_i^\varepsilon(\varphi)\mathcal{T}_i^\varepsilon(\psi)$ , for every  $\varphi, \psi$  Lebesgue-measurable on  $\Omega_i^\varepsilon$ .
- ii) For every  $\varphi \in L^1(\Omega_i^\varepsilon)$ , one has

$$\frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(\varphi)(x, y) \, dx \, dy = \int_{\widehat{\Omega}^\varepsilon} \varphi(x) \, dx = \int_{\Omega_i^\varepsilon} \varphi(x) \, dx - \int_{\Lambda_i^\varepsilon} \varphi(x) \, dx.$$

- iii) For every  $\varphi \in L^p(\Omega_i^\varepsilon)$ , one has

$$\|\mathcal{T}_i^\varepsilon(\varphi)\|_{L^p(\Omega \times Y_i)} \leq |Y|^{1/p} \|\varphi\|_{L^p(\Omega_i^\varepsilon)}.$$

- iv) For every  $\varphi \in L^p(\Omega)$ , one has

$$\mathcal{T}_i^\varepsilon(\varphi) \longrightarrow \varphi \text{ strongly in } L^p(\Omega \times Y_i).$$

- v) Let  $\varphi_\varepsilon$  be a sequence in  $L^p(\Omega)$  such that  $\varphi_\varepsilon \longrightarrow \varphi$  strongly in  $L^p(\Omega)$ . Then,

$$\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi \text{ strongly in } L^p(\Omega \times Y_i).$$

- vi) Let  $\varphi \in L^p(Y_i)$  be a  $Y$ -periodic function and set  $\varphi^\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$ . Then

$$\mathcal{T}_i^\varepsilon(\varphi^\varepsilon)(x, y) = \varphi(y) \quad \text{a.e. in } \widehat{\Omega}^\varepsilon \times Y_i.$$

- vii) Let  $\varphi \in W^{1,p}(\Omega_i^\varepsilon)$ . Then

$$\nabla_y [\mathcal{T}_i^\varepsilon(\varphi)] = \varepsilon \mathcal{T}_i^\varepsilon(\nabla \varphi) \text{ and } \mathcal{T}_i^\varepsilon(\varphi) \in L^2(\Omega, W^{1,p}(Y_i)).$$

The following convergence result holds:

**Proposition 3.8** ([6, 7, 10, 26]). *Let  $p \in ]1, +\infty[$  and  $i = 1, 2$ .*

*If  $\varphi_\varepsilon \in L^p(\Omega_i^\varepsilon)$  satisfies  $\|\varphi_\varepsilon\|_{L^p(\Omega_i^\varepsilon)} \leq C$  and  $\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi}$  weakly in  $L^p(\Omega \times Y_i)$ , then*

$$\widetilde{\varphi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}(\widehat{\varphi}) \text{ weakly in } L^p(\Omega),$$

where  $\theta_i$  is given in (2.2).

We now give a result concerning the jump on the interface proved in [26].

**Lemma 3.9** ([26]). *Let  $\varphi \in \mathcal{D}(\Omega)$ ,  $h$  satisfy (3.4) and  $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \in H_\gamma^\varepsilon$ . Then, for  $\varepsilon$  small enough, we have*

$$\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon} - u_{2\varepsilon}) \varphi \, d\sigma_x = \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\mathcal{T}_1^\varepsilon(u_{1\varepsilon}) - \mathcal{T}_2^\varepsilon(u_{2\varepsilon})) \mathcal{T}_1^\varepsilon(\varphi) \, dx \, d\sigma_y,$$

with  $h^\varepsilon$  given by (3.6)

Let us finally recall a known result about the convergences of the unfolding operators, previously introduced, applied to bounded sequences in  $H_\gamma^\varepsilon$ . We restrict our attention to the case we are interested in,  $\gamma \leq -1$ .

**Theorem 3.10** ([26, 27]). *Let  $\gamma \leq -1$  and  $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$  be a bounded sequence in  $H_\gamma^\varepsilon$ , then there exist a subsequence, still denoted  $\varepsilon$ ,  $u \in H_0^1(\Omega)$ ,  $\widehat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$  a.e. in  $\Omega$  and  $\widehat{u}_2 \in L^2(\Omega, H^1(Y_2))$  such that*

$$\begin{cases} \mathcal{T}_1^\varepsilon(u_{1\varepsilon}) \rightarrow u & \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u + \nabla_y \widehat{u}_1 & \text{weakly in } L^2(\Omega \times Y_1), \end{cases} \quad (3.14)$$

$$\begin{cases} \mathcal{T}_2^\varepsilon(u_{2\varepsilon}) \rightharpoonup u & \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \rightharpoonup \nabla u + \nabla_y \widehat{u}_2 & \text{weakly in } L^2(\Omega \times Y_2). \end{cases} \quad (3.15)$$

Furthermore,

i) if  $\gamma < -1$ , we have

$$\widehat{u}_1 = \widehat{u}_2 + \xi_\Gamma \quad \text{on } \Omega \times \Gamma,$$

for some function  $\xi_\Gamma \in L^2(\Omega)$ ;

ii) if  $\gamma = -1$ , the following convergence holds

$$\frac{\mathcal{T}_1^\varepsilon(u_{1\varepsilon}) - \mathcal{T}_2^\varepsilon(u_{2\varepsilon})}{\varepsilon} \rightharpoonup \widehat{u}_1 - \widehat{u}_2 \quad \text{weakly in } L^2(\Omega \times \Gamma).$$

**3.3. Homogenization with weakly converging data.** Let  $f_\varepsilon \in (H_\gamma^\varepsilon)'$ , by (3.13), the variational formulation of problem (3.1) is the following

$$\begin{cases} \text{Find } (u_{1\varepsilon}, u_{2\varepsilon}) \in H_\gamma^\varepsilon \text{ s. t.} \\ \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_{1\varepsilon} \nabla v_1 \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_{2\varepsilon} \nabla v_2 \, dx \\ + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon} - u_{2\varepsilon})(v_1 - v_2) \, d\sigma_x \\ = \langle f_{1\varepsilon}, v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle f_{2\varepsilon}, v_2 \rangle_{(H^1(\Omega_2^\varepsilon))', H^1(\Omega_2^\varepsilon)} \quad \forall (v_1, v_2) \in H_\gamma^\varepsilon. \end{cases} \quad (3.16)$$

The existence and uniqueness of a solution  $u_\varepsilon := (u_{1\varepsilon}, u_{2\varepsilon})$  of (3.1), for every fixed  $\varepsilon$ , is a result of the Lax-Milgram theorem, together with Proposition 3.2.

In order to describe the asymptotic behaviour, as  $\varepsilon$  tends to zero, of the solution  $u_\varepsilon$  of problem (3.1), we suppose that there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that

$$\|f_\varepsilon\|_{(H_\gamma^\varepsilon)'} \leq C. \quad (3.17)$$

**Remark 3.11.** Let us observe that, if  $(u_1, u_2) \in H_0^1(\Omega) \times H^1(\Omega)$ , then the couple  $(u_{1|\Omega_1^\varepsilon}, u_{2|\Omega_2^\varepsilon}) \in V^\varepsilon \times H^1(\Omega_2^\varepsilon)$ . Then it is easily seen that the functionals

$$\begin{aligned} \bar{f}_{1\varepsilon} &: H_0^1(\Omega) \rightarrow \mathbb{R}, \\ \bar{f}_{2\varepsilon} &: H^1(\Omega) \rightarrow \mathbb{R} \end{aligned}$$

defined as

$$\bar{f}_{1\varepsilon}(u_1) = \left\langle f_{1\varepsilon}, u_{1|\Omega_1^\varepsilon} \right\rangle_{(V^\varepsilon)', V^\varepsilon} \quad (3.18)$$

$$\bar{f}_{2\varepsilon}(u_2) = \left\langle f_{2\varepsilon}, u_{2|\Omega_2^\varepsilon} \right\rangle_{(H^1(\Omega_2^\varepsilon))', H^1(\Omega_2^\varepsilon)}, \quad (3.19)$$



are linear and continuous. Therefore (3.18) and (3.19) can be rewritten as

$$\langle \bar{f}_{1\varepsilon}, u_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle f_{1\varepsilon}, u_{1|\Omega_1^\varepsilon} \rangle_{(V^\varepsilon)', V^\varepsilon} \tag{3.20}$$

$$\langle \bar{f}_{2\varepsilon}, u_2 \rangle_{(H^1(\Omega))', H^1(\Omega)} = \langle f_{2\varepsilon}, u_{2|\Omega_2^\varepsilon} \rangle_{(H^1(\Omega_2^\varepsilon))', H^1(\Omega_2^\varepsilon)}. \tag{3.21}$$

Moreover, due to (3.17), one has

$$\begin{aligned} \bar{f}_{1\varepsilon} &\rightharpoonup f_1 && \text{in } H^{-1}(\Omega), \\ \bar{f}_{2\varepsilon} &\rightharpoonup f_2 && \text{in } (H^1(\Omega))', \end{aligned} \tag{3.22}$$

up to a subsequence, still denoted  $\varepsilon$ .

In the sequel, for sake of simplicity and where no ambiguity arises, in view of (3.20) and (3.21) we will still denote by  $f_{1\varepsilon}$  and  $f_{2\varepsilon}$  the functionals  $\bar{f}_{1\varepsilon}$  and  $\bar{f}_{2\varepsilon}$  respectively.

Let us first recall an a priori estimate proved in [28, 49] in the case of fixed datum in  $L^2(\Omega)$  and extended in [35] to the case of weakly converging ones.

**Proposition 3.12.** *Let  $u_\varepsilon$  be the solution of problem (3.1). Then, under assumptions (3.2)÷(3.6) and (3.17),  $u_\varepsilon$  is a bounded sequence in  $H_\gamma^\varepsilon$ .*

We describe the homogenized problems for every  $\gamma \leq -1$  by treating separately the two cases  $\gamma < -1$ ,  $\gamma = -1$ . In the case  $\gamma = -1$ , when passing to the limit in problem (3.16), we meet an additional difficulty to treat the integral over the interface. In order to overcome that, we use Theorem 3.10 ii).

Now, let us consider an auxiliary problem related to problem (3.1), already introduced in [35], i.e.

$$\begin{cases} -\Delta \rho_{1\varepsilon} = f_{1\varepsilon} & \text{in } \Omega_1^\varepsilon, \\ -\Delta \rho_{2\varepsilon} = f_{2\varepsilon} & \text{in } \Omega_2^\varepsilon, \\ \nabla \rho_{1\varepsilon} \cdot n_{1\varepsilon} = -\nabla \rho_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon, \\ \nabla \rho_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (\rho_{1\varepsilon} - \rho_{2\varepsilon}) & \text{on } \Gamma^\varepsilon, \\ \rho_{1\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.23}$$

where  $f_\varepsilon$ ,  $h^\varepsilon$  and  $n_{i\varepsilon}$ ,  $i = 1, 2$ , are the same of problem (3.1). The variational formulation of (3.23) is

$$\begin{cases} \text{Find } (\rho_{1\varepsilon}, \rho_{2\varepsilon}) \in H_\gamma^\varepsilon \text{ s. t.} \\ \int_{\Omega_1^\varepsilon} \nabla \rho_{1\varepsilon} \nabla v_1 \, dx + \int_{\Omega_2^\varepsilon} \nabla \rho_{2\varepsilon} \nabla v_2 \, dx \\ + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (\rho_{1\varepsilon} - \rho_{2\varepsilon}) (v_1 - v_2) \, d\sigma_x \\ = \langle f_{1\varepsilon}, v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle f_{2\varepsilon}, v_2 \rangle_{(H^1(\Omega_2^\varepsilon))', H^1(\Omega_2^\varepsilon)} \quad \forall (v_1, v_2) \in H_\gamma^\varepsilon. \end{cases} \tag{3.24}$$

Observe that, clearly, also for the solution  $\rho_\varepsilon := (\rho_{1\varepsilon}, \rho_{2\varepsilon})$  of problem (3.23), under assumptions (3.4), (3.6) and (3.17), the same result as in Proposition 3.12 hold as well as those in Theorem 3.10.

3.3.1. *Homogenization results by periodic unfolding method for  $\gamma < -1$ .* Let us start by using the unfolding method to prove a preliminary convergence result for a subsequence of the solutions of problem (3.23).

**Lemma 3.13.** *Let  $\gamma < -1$  and  $\rho_\varepsilon$  be the solution of problem (3.23). Then, under the assumptions (3.4), (3.6) and (3.17), there exist a subsequence, still denoted  $\varepsilon$ ,  $\rho \in H_0^1(\Omega)$  and  $\widehat{\rho} \in L^2(\Omega; H_{per}^1(Y))$  with  $\mathcal{M}_\Gamma(\widehat{\rho}) = 0$  a.e. in  $\Omega$  such that*

$$\begin{cases} \mathcal{T}_1^\varepsilon(\rho_{1\varepsilon}) \longrightarrow \rho & \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla \rho_{1\varepsilon}) \rightharpoonup \nabla \rho + \nabla_y \widehat{\rho}|_{\Omega \times Y_1} & \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(\rho_{2\varepsilon}) \rightharpoonup \rho & \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla \rho_{2\varepsilon}) \rightharpoonup \nabla \rho + \nabla_y \widehat{\rho}|_{\Omega \times Y_2} & \text{weakly in } L^2(\Omega \times Y_2) \end{cases} \quad (3.25)$$

and

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y} (\nabla \rho + \nabla_y \widehat{\rho}) \nabla_y \Phi \, dx \, dy \\ &= \lim_{n \rightarrow +\infty} \left( \lim_{\varepsilon \rightarrow 0} (\langle f_{1\varepsilon}, \varepsilon \omega_n \psi_n^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega_n \psi_n^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}) \right), \end{aligned} \quad (3.26)$$

for every  $\Phi \in L^2(\Omega, H_{per}^1(Y))$  and where  $w_n \in \mathcal{D}(\Omega)$  and  $\psi_n^\varepsilon(x) = \psi_n(x/\varepsilon)$ , with  $\psi_n \in H_{per}^1(Y)$ , for any  $n \in \mathbb{N}$ , are such that

$$w_n \psi_n \rightarrow \Phi \text{ strongly in } L^2(\Omega, H_{per}^1(Y)). \quad (3.27)$$

*Proof.* From Theorem 3.10 and Proposition 3.12 we deduce there exist a subsequence, still denoted  $\varepsilon$ ,  $\rho \in H_0^1(\Omega)$ ,  $\widehat{\rho}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{\rho}_1) = 0$  a.e. in  $\Omega$  and  $\widehat{\rho}_2 \in L^2(\Omega, H^1(Y_2))$  such that the convergences (3.25)<sub>1,3</sub> hold and

$$\begin{cases} \mathcal{T}_1^\varepsilon(\nabla \rho_{1\varepsilon}) \rightharpoonup \nabla \rho + \nabla_y \widehat{\rho}_1 & \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(\nabla \rho_{2\varepsilon}) \rightharpoonup \nabla \rho + \nabla_y \widehat{\rho}_2 & \text{weakly in } L^2(\Omega \times Y_2). \end{cases} \quad (3.28)$$

Let us take  $v_1 = v_2 = v_\varepsilon = \varepsilon \omega \psi^\varepsilon$  as test functions in (3.24), where  $\omega \in \mathcal{D}(\Omega)$ ,  $\psi \in H_{per}^1(Y)$  and  $\psi^\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$ .

The term concerning the interface vanishes and, in view of Remark 3.11, we get

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} \nabla \rho_{1\varepsilon} \nabla v_\varepsilon \, dx + \int_{\Omega_2^\varepsilon} \nabla \rho_{2\varepsilon} \nabla v_\varepsilon \, dx \\ &= \langle f_{1\varepsilon}, v_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, v_\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}. \end{aligned} \quad (3.29)$$

In view of the definitions of  $\Lambda_i^\varepsilon$ ,  $i = 1, 2$ , and  $v_\varepsilon$ , by Proposition 3.7 ii), via unfolding, we get that, for  $\varepsilon$  sufficiently small, (3.29) can be rewritten as

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} \mathcal{T}_1^\varepsilon(\nabla \rho_{1\varepsilon}) \mathcal{T}_1^\varepsilon(\nabla v_\varepsilon) \, dx \, dy \\ &+ \frac{1}{|Y|} \int_{\Omega \times Y_2} \mathcal{T}_2^\varepsilon(\nabla \rho_{2\varepsilon}) \mathcal{T}_2^\varepsilon(\nabla v_\varepsilon) \, dx \, dy \\ &= \langle f_{1\varepsilon}, v_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, v_\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}, \end{aligned} \quad (3.30)$$

where we also used Proposition 3.7 i).

Since  $\nabla v_\varepsilon(x) = \varepsilon \psi\left(\frac{x}{\varepsilon}\right) \nabla \omega(x) + \omega(x) \nabla_y \psi\left(\frac{x}{\varepsilon}\right)$ , by Proposition 3.7 i), iv) and vi), it is easily seen that, for  $i = 1, 2$ ,

$$\mathcal{T}_i^\varepsilon (\nabla v_\varepsilon) = \varepsilon \psi \mathcal{T}_i^\varepsilon (\nabla \omega) + \nabla_y \psi \mathcal{T}_i^\varepsilon (\omega) \longrightarrow \nabla_y (\omega \psi) \text{ strongly in } L^2 (\Omega \times Y_i). \quad (3.31)$$

From (3.28) and (3.31), passing to the limit as  $\varepsilon \rightarrow 0$  in (3.30) we obtain, up to a subsequence, still denoted  $\varepsilon$ ,

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \widehat{\rho}_1) \nabla_y (\omega \psi) \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \widehat{\rho}_2) \nabla_y (\omega \psi) \, dx \, dy \\ & = \lim_{\varepsilon \rightarrow 0} (\langle f_{1\varepsilon}, \varepsilon \omega \psi^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega \psi^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}). \end{aligned} \quad (3.32)$$

According to Theorem 3.10 i) we have  $\widehat{\rho}_1 = \widehat{\rho}_2 + \xi_\Gamma$  on  $\Omega \times \Gamma$  for some function  $\xi_\Gamma \in L^2(\Omega)$ .

Thus, if we set

$$\widehat{\rho}(\cdot, y) = \begin{cases} \widehat{\rho}_1(\cdot, y) & y \in Y_1, \\ \widehat{\rho}_2(\cdot, y) + \xi_\Gamma & y \in Y_2, \end{cases}$$

a.e. in  $\Omega$ , and extend this function by periodicity to a function still denoted by  $\widehat{\rho}$ , we get that

$$\widehat{\rho} \in L^2(\Omega, H_{per}^1(Y))$$

and  $m_\Gamma(\widehat{\rho}) = 0$  for a.e.  $x \in \Omega$ . Also note that

$$\begin{cases} \nabla_y \widehat{\rho}|_{\Omega \times Y_1} = \nabla_y \widehat{\rho}_1, \\ \nabla_y \widehat{\rho}|_{\Omega \times Y_2} = \nabla_y \widehat{\rho}_2. \end{cases} \quad (3.33)$$

Therefore (3.28) and (3.33) give us (3.25)<sub>2,4</sub> and (3.32) can be rewritten as

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y} (\nabla \rho + \nabla_y \widehat{\rho}) \nabla_y (\omega \psi) \, dx \, dy \\ & = \lim_{\varepsilon \rightarrow 0} (\langle f_{1\varepsilon}, \varepsilon \omega \psi^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega \psi^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}). \end{aligned} \quad (3.34)$$

Now let us take  $\Phi \in L^2(\Omega, H_{per}^1(Y))$ . By density there exist  $w_n \in \mathcal{D}(\Omega)$  and  $\psi_n \in H_{per}^1(Y)$ , for any  $n \in \mathbb{N}$ , such that

$$w_n \psi_n \rightarrow \Phi \text{ strongly in } L^2(\Omega, H_{per}^1(Y)).$$

Hence, (3.34) gives, for any fixed  $n \in \mathbb{N}$ ,

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y} (\nabla \rho + \nabla_y \widehat{\rho}) \nabla_y (\omega_n \psi_n) \, dx \, dy \\ & = \lim_{\varepsilon \rightarrow 0} (\langle f_{1\varepsilon}, \varepsilon \omega_n \psi_n^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega_n \psi_n^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}), \end{aligned}$$

where  $\psi_n^\varepsilon(x) = \psi_n(x/\varepsilon)$ , for any  $n \in \mathbb{N}$ . Passing to the limit as  $n \rightarrow +\infty$ , we get (3.26).  $\square$

Now we are able to prove the homogenization result for problem (3.1) when  $\gamma < -1$ .

**Theorem 3.14.** *Let  $\gamma < -1$  and  $u_\varepsilon$  be the solution of problem (3.1). Then, under the assumptions (3.2)÷(3.6) and (3.17), there exist a subsequence, still denoted  $\varepsilon$ ,  $u \in H_0^1(\Omega)$  and  $\widehat{u} \in L^2(\Omega, H_{per}^1(Y))$ , with  $\mathcal{M}_\Gamma(\widehat{u}) = 0$  a.e. in  $\Omega$ , such that*

$$\begin{cases} \widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i u & \text{weakly in } L^2(\Omega), \, i = 1, 2, \\ \mathcal{T}_1^\varepsilon(u_{1\varepsilon}) \longrightarrow u & \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u + \nabla_y \widehat{u}|_{\Omega \times Y_1} & \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(u_{2\varepsilon}) \rightharpoonup u & \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \rightharpoonup \nabla u + \nabla_y \widehat{u}|_{\Omega \times Y_2} & \text{weakly in } L^2(\Omega \times Y_2), \end{cases} \quad (3.35)$$

where the pair  $(u, \hat{u})$  is the unique solution of the following problem

$$\left\{ \begin{array}{l} \text{Find } u \in H_0^1(\Omega), \hat{u} \in L^2(\Omega, H_{per}^1(Y)), \\ \text{with } \mathcal{M}_\Gamma(\hat{u}) = 0 \text{ a.e. } x \in \Omega, \text{ s.t.} \\ \frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u + \nabla_y \hat{u}) (\nabla \varphi + \nabla_y \Phi) \, dx \, dy \\ = \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ + \frac{1}{|Y|} \int_{\Omega \times Y} (\nabla \rho + \nabla_y \hat{\rho}) \nabla_y \Phi \, dx \, dy \\ \forall \varphi \in H_0^1(\Omega), \forall \Phi \in L^2(\Omega, H_{per}^1(Y)), \end{array} \right. \quad (3.36)$$

where  $\rho$  and  $\hat{\rho}$  are as in Lemma 3.13, hence the term  $\int_{\Omega \times Y} (\nabla \rho + \nabla_y \hat{\rho}) \nabla_y \Phi \, dx \, dy$  depends only on a subsequence of  $f_\varepsilon$ .

*Proof.* Arguing as in the proof of Lemma 3.13, we get that there exist a subsequence, still denoted  $\varepsilon$ ,  $u \in H_0^1(\Omega)$ ,  $\hat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\hat{u}_1) = 0$  a.e. in  $\Omega$  and  $\hat{u}_2 \in L^2(\Omega, H^1(Y_2))$  such that the convergences (3.35)<sub>2,4</sub> hold and

$$\left\{ \begin{array}{l} \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u + \nabla_y \hat{u}_1 \text{ weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \rightharpoonup \nabla u + \nabla_y \hat{u}_2 \text{ weakly in } L^2(\Omega \times Y_2). \end{array} \right. \quad (3.37)$$

Then, from (3.12) of Corollary 3.3, (3.35)<sub>2,4</sub> and Proposition 3.8 we obtain that, for  $i = 1, 2$ ,

$$\tilde{u}_{i\varepsilon} \rightharpoonup \theta_i \mathcal{M}_{Y_i}(u) \text{ weakly in } L^2(\Omega)$$

and, since  $u$  is constant with respect to  $y$ , we deduce (3.35)<sub>1</sub>.

In order to get the limit problem, let  $v_\varepsilon = \varepsilon \omega \psi^\varepsilon$  as in the proof of Lemma 3.13 and  $\varphi \in \mathcal{D}(\Omega)$ . If we take  $v_1 = v_2 = \varphi + v_\varepsilon$  as test functions in (3.16), in view of Remark 3.11, we get

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_{1\varepsilon} \nabla(\varphi + v_\varepsilon) \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_{2\varepsilon} \nabla(\varphi + v_\varepsilon) \, dx \\ & = \langle f_{1\varepsilon}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ & \quad + \langle f_{1\varepsilon}, v_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, v_\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}. \end{aligned} \quad (3.38)$$

Then if we take  $v_1 = v_2 = v_\varepsilon$  as test functions in (3.24), (3.38) can be rewritten as

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_{1\varepsilon} \nabla(\varphi + v_\varepsilon) \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_{2\varepsilon} \nabla(\varphi + v_\varepsilon) \, dx \\ & = \langle f_{1\varepsilon}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ & \quad + \int_{\Omega_1^\varepsilon} \nabla \rho_{1\varepsilon} \nabla v_\varepsilon \, dx + \int_{\Omega_2^\varepsilon} \nabla \rho_{2\varepsilon} \nabla v_\varepsilon \, dx. \end{aligned} \quad (3.39)$$

In view of the definitions of  $\Lambda_i^\varepsilon$ ,  $i = 1, 2$ , and  $v_\varepsilon$ , by Proposition 3.7 ii), via unfolding, we get that, for  $\varepsilon$  sufficiently small, (3.39) can be rewritten as

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y) \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \mathcal{T}_1^\varepsilon(\nabla \varphi + \nabla v_\varepsilon) \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y) \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \mathcal{T}_2^\varepsilon(\nabla \varphi + \nabla v_\varepsilon) \, dx \, dy \end{aligned}$$

$$\begin{aligned}
 &= \langle f_{1\varepsilon}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\
 &\quad + \frac{1}{|Y|} \int_{\Omega \times Y_1} \mathcal{T}_1^\varepsilon (\nabla \rho_{1\varepsilon}) \mathcal{T}_1^\varepsilon (\nabla v_\varepsilon) \, dx \, dy \\
 &\quad + \frac{1}{|Y|} \int_{\Omega \times Y_2} \mathcal{T}_2^\varepsilon (\nabla \rho_{2\varepsilon}) \mathcal{T}_2^\varepsilon (\nabla v_\varepsilon) \, dx \, dy,
 \end{aligned} \tag{3.40}$$

where we also used Proposition 3.7 i) and vi).

From (3.22), (3.25)<sub>2,4</sub>, (3.31) and (3.37), passing to the limit as  $\varepsilon \rightarrow 0$  in the previous identity we obtain, up to a subsequence,

$$\begin{aligned}
 &\frac{1}{|Y|} \int_{\Omega \times Y_1} A(y) (\nabla u + \nabla_y \widehat{u}_1) (\nabla \varphi + \nabla_y (\omega \psi)) \, dx \, dy \\
 &+ \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y) (\nabla u + \nabla_y \widehat{u}_2) (\nabla \varphi + \nabla_y (\omega \psi)) \, dx \, dy \\
 &= \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\
 &\quad + \frac{1}{|Y|} \int_{\Omega \times Y} (\nabla \rho + \nabla_y \widehat{\rho}) \nabla_y (\omega \psi) \, dx \, dy.
 \end{aligned} \tag{3.41}$$

Arguing as in Lemma 3.13, by Theorem 3.10 i), if we set

$$\widehat{u}(\cdot, y) = \begin{cases} \widehat{u}_1(\cdot, y) & y \in Y_1, \\ \widehat{u}_2(\cdot, y) + \zeta_\Gamma & y \in Y_2, \end{cases} \tag{3.42}$$

where  $\zeta_\Gamma \in L^2(\Omega)$ , and extend it by periodicity to a function still denoted by  $\widehat{u}$ , we get

$$\widehat{u} \in L^2(\Omega, H_{per}^1(Y))$$

and  $m_\Gamma(\widehat{u}) = 0$  a.e. in  $\Omega$ . Moreover,

$$\begin{cases} \nabla_y \widehat{u}|_{\Omega \times Y_1} = \nabla_y \widehat{u}_1, \\ \nabla_y \widehat{u}|_{\Omega \times Y_2} = \nabla_y \widehat{u}_2. \end{cases} \tag{3.43}$$

Therefore, (3.37) and (3.43) give us (3.35)<sub>3,5</sub> and (3.41) can be rewritten as

$$\begin{aligned}
 &\frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u + \nabla_y \widehat{u}) (\nabla \varphi + \nabla_y (\omega \psi)) \, dx \, dy \\
 &= \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\
 &\quad + \frac{1}{|Y|} \int_{\Omega \times Y} (\nabla \rho + \nabla_y \widehat{\rho}) \nabla_y (\omega \psi) \, dx \, dy,
 \end{aligned} \tag{3.44}$$

for every  $\varphi, \omega \in \mathcal{D}(\Omega)$  and  $\psi \in H_{per}^1(Y)$ .

Finally, by density we get (3.36). □

In the following result we point out that the limit problem (3.36) is equivalent to an elliptic problem set in the fixed domain  $\Omega$  whose homogenized matrix is the same obtained in [49] for  $\gamma < -1$ , i.e. that of the classical elliptic homogenization in the fixed domain  $\Omega$  (see [2]).

**Corollary 3.15.** *Let  $\gamma < -1$  and  $u_\varepsilon$  be the solution of problem (3.1). Then, under the assumptions (3.2)÷(3.6) and (3.17), there exist a subsequence, still denoted  $\varepsilon$ , and  $u \in H_0^1(\Omega)$  such that*

$$\begin{cases} \widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i u & \text{weakly in } L^2(\Omega), \, i = 1, 2, \\ A^\varepsilon \widetilde{\nabla u}_{1\varepsilon} \rightharpoonup A_\gamma^1 \nabla u + \theta_1 \mathcal{M}_{Y_1} (A \nabla_y \widehat{\chi} |_{Y_1}) & \text{weakly in } L^2(\Omega), \\ A^\varepsilon \widetilde{\nabla u}_{2\varepsilon} \rightharpoonup A_\gamma^2 \nabla u + \theta_2 \mathcal{M}_{Y_2} (A \nabla_y \widehat{\chi} |_{Y_2}) & \text{weakly in } L^2(\Omega). \end{cases} \tag{3.45}$$

In (3.45) the constant matrices  $A_\gamma^l = (a_{ij}^l)_{n \times n}$ ,  $l = 1, 2$ , are defined by

$$a_{ij}^l = \theta_l \mathcal{M}_{Y_l} \left( a_{ij} - \sum_{k=1}^n a_{ik} \frac{\partial \chi^j}{\partial y_k} \right), \quad (3.46)$$

where the functions  $\chi^j$ ,  $j = 1, \dots, n$ , are the unique solutions of the cell problems

$$\begin{cases} -\operatorname{div}(A \nabla (\chi^j - y_j)) = 0 & \text{in } Y, \\ \chi^j \text{ } Y\text{-periodic, } \mathcal{M}_Y(\chi^j) = 0 \end{cases} \quad (3.47)$$

and the function  $\widehat{\chi}$ , for a.e.  $x \in \Omega$ , is the unique solution of the following problem

$$\begin{cases} \text{Find } \widehat{\chi} \in L^2(\Omega; H_{per}^1(Y)) \text{ s. t.} \\ \int_Y A(y) \nabla_y \widehat{\chi} \nabla_y \psi \, dy = \int_Y (\nabla \rho + \nabla_y \widehat{\rho}) \nabla_y \psi \, dy, \forall \psi \in H_{per}^1(Y), \end{cases} \quad (3.48)$$

where  $\rho$  and  $\widehat{\rho}$  are the same functions as in Lemma 3.13.

Moreover the limit function  $u$  is the unique solution of the problem

$$\begin{cases} -\operatorname{div}(A_\gamma^0 \nabla u) = f_1 + f_2 + \operatorname{div}(\mathcal{M}_Y(A(y) \nabla_y \widehat{\chi})) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.49)$$

where the homogenized matrix is given by

$$A_\gamma^0 := A_\gamma^1 + A_\gamma^2. \quad (3.50)$$

*Proof.* Choosing  $\varphi = 0$  in (3.36), we get

$$\frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u + \nabla_y \widehat{u}) \nabla_y \Phi \, dx \, dy = \frac{1}{|Y|} \int_{\Omega \times Y} (\nabla \rho + \nabla_y \widehat{\rho}) \nabla_y \Phi \, dx \, dy,$$

for all  $\Phi \in L^2(\Omega, H_{per}^1(Y))$ .

By following some classical arguments as in the two-scale method (see [9], ch. 9), this gives

$$\widehat{u}(x, y) = \widehat{\chi}(x, y) - \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) \chi^j(y), \quad (3.51)$$

where  $\chi^j$ ,  $j = 1, \dots, n$  are the solutions of the cell problems (3.47) and  $\widehat{\chi}$  satisfies (3.48).

We now choose  $\Phi = 0$  in (3.36), obtaining

$$\frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u + \nabla_y \widehat{u}) \nabla \varphi \, dx \, dy = \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)},$$

for all  $\varphi \in H_0^1(\Omega)$ .

Replacing  $\widehat{u}$ , given by (3.51), in the previous equality we obtain

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{|Y|} \int_Y \left( a_{ij}(y) - \sum_{k=1}^n a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) dy \right) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, dx \\ &= \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ & \quad - \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{|Y|} \int_Y a_{ij}(y) \frac{\partial \widehat{\chi}}{\partial y_j}(y) \, dy \right) \frac{\partial \varphi}{\partial x_i} \, dx, \end{aligned}$$

for all  $\varphi \in H_0^1(\Omega)$  which means that  $u$  satisfies the following problem

$$\begin{cases} -\sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( \frac{1}{|Y|} \int_Y \left( a_{ij}(y) - \sum_{k=1}^n a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) dy \right) \frac{\partial u}{\partial x_j} \\ = f_1 + f_2 + \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( \frac{1}{|Y|} \int_Y a_{ij}(y) \frac{\partial \widehat{\chi}}{\partial y_j}(y) dy \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This implies that  $u$  is the unique solution of problem (3.49) where  $A_\gamma^0$  is the matrix defined by (3.50).

From (3.42) and (3.51), we have

$$\begin{cases} \widehat{u}_1 = \widehat{u}|_{\Omega \times Y_1} = \widehat{\chi}|_{\Omega \times Y_1} - \sum_{j=1}^n \frac{\partial u}{\partial x_j} \chi^j|_{Y_1}, \\ \widehat{u}_2 = \widehat{u}|_{\Omega \times Y_2} - \zeta_\Gamma = \widehat{\chi}|_{\Omega \times Y_2} - \sum_{j=1}^n \frac{\partial u}{\partial x_j} \chi^j|_{Y_2} - \zeta_\Gamma, \end{cases} \quad (3.52)$$

where  $\zeta_\Gamma$  is a function in  $L^2(\Omega)$ . On the other hand, from Proposition 3.7 i) and vi) and convergences (3.37), we have

$$\begin{cases} \mathcal{T}_1^\varepsilon(A^\varepsilon \nabla u_{1\varepsilon}) \rightharpoonup A(y)(\nabla u + \nabla_y \widehat{u}_1) \quad \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(A^\varepsilon \nabla u_{2\varepsilon}) \rightharpoonup A(y)(\nabla u + \nabla_y \widehat{u}_2) \quad \text{weakly in } L^2(\Omega \times Y_2). \end{cases}$$

Then, using Proposition 3.8, we deduce that

$$\begin{cases} A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \rightharpoonup \theta_1 \mathcal{M}_{Y_1}[A(y)(\nabla u + \nabla_y \widehat{u}_1)] \quad \text{weakly in } L^2(\Omega), \\ A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup \theta_2 \mathcal{M}_{Y_2}[A(y)(\nabla u + \nabla_y \widehat{u}_2)] \quad \text{weakly in } L^2(\Omega). \end{cases} \quad (3.53)$$

After some computations, by using (3.52), convergences (3.53) give (3.45)<sub>2,3</sub>.  $\square$

**Remark 3.16.** Let us observe that in problem (3.49) the right-hand side of the limit equation is not exactly the sum of the weak limits of  $f_{1\varepsilon}$  and  $f_{2\varepsilon}$  as in the case of more regular data, but it is a more complicated function depending on a subsequence of  $f_{i\varepsilon}$ ,  $i = 1, 2$  (see Lemma 3.13 and (3.48) of Corollary 3.15).

3.3.2. *Homogenization results by periodic unfolding method for  $\gamma = -1$ .* As in the previous case, let us start by using the unfolding method to prove a preliminary convergence result for a subsequence of the solutions of problem (3.23).

**Lemma 3.17.** *Let  $\gamma = -1$  and  $\rho^\varepsilon$  be the solution of problem (3.23). Then, under the assumptions (3.4), (3.6) and (3.17), there exist a subsequence, still denoted  $\varepsilon$ ,  $\rho \in H_0^1(\Omega)$ ,  $\widehat{\rho}_1 \in L^2(\Omega; H_{per}^1(Y_1))$ , with  $\mathcal{M}_\Gamma(\widehat{\rho}_1) = 0$  a.e. in  $\Omega$ ,  $\widehat{\rho}_2 \in L^2(\Omega; H^1(Y_2))$  such that*

$$\begin{cases} \mathcal{T}_1^\varepsilon(\rho_{1\varepsilon}) \longrightarrow \rho & \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla \rho_{1\varepsilon}) \rightharpoonup \nabla \rho + \nabla_y \widehat{\rho}_1 & \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(\rho_{2\varepsilon}) \rightharpoonup \rho & \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla \rho_{2\varepsilon}) \rightharpoonup \nabla \rho + \nabla_y \widehat{\rho}_2 & \text{weakly in } L^2(\Omega \times Y_2), \end{cases} \quad (3.54)$$

and

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \widehat{\rho}_1) \nabla_y \Phi_1 \, dx \, dy + \frac{1}{|Y|} \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \widehat{\rho}_2) \nabla_y \Phi_2 \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\widehat{\rho}_1 - \widehat{\rho}_2) (\Phi_1 - \Phi_2) \, dx \, d\sigma_y \end{aligned} \quad (3.55)$$

$$= \lim_{n \rightarrow +\infty} \left( \lim_{\varepsilon \rightarrow 0} (\langle f_{1\varepsilon}, \varepsilon \omega_{1n} \psi_{1n}^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega_{2n} \psi_{2n}^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}) \right),$$

for every  $\Phi_1 \in L^2(\Omega, H_{per}^1(Y_1))$ ,  $\Phi_2 \in L^2(\Omega, H^1(Y_2))$  and where, for  $i = 1, 2$ ,  $w_{in} \in \mathcal{D}(\Omega)$ ,  $\psi_{1n}^\varepsilon(x) = \psi_{1n}(x/\varepsilon)$ , with  $\psi_{1n} \in H_{per}^1(Y_1)$  and  $\psi_{2n}^\varepsilon(x) = \psi_{2n}(x/\varepsilon)$ , with  $\psi_{2n} \in H^1(Y_2)$ , for any  $n \in \mathbb{N}$ , are such that

$$\begin{aligned} w_{1n} \psi_{1n} &\rightarrow \Phi_1 \text{ strongly in } L^2(\Omega, H_{per}^1(Y_1)), \\ w_{2n} \psi_{2n} &\rightarrow \Phi_2 \text{ strongly in } L^2(\Omega, H^1(Y_2)). \end{aligned}$$

*Proof.* Arguing as in Lemma 3.13, we deduce there exist a subsequence, still denoted  $\varepsilon$ ,  $\rho \in H_0^1(\Omega)$ ,  $\widehat{\rho}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{\rho}_1) = 0$  a.e. in  $\Omega$  and  $\widehat{\rho}_2 \in L^2(\Omega, H^1(Y_2))$  such that the convergences (3.54) hold.

For  $i = 1, 2$ , let us take  $v_i = v_{i\varepsilon} = \varepsilon \omega_i \psi_i^\varepsilon$  as test functions in (3.24), where  $\omega_i \in \mathcal{D}(\Omega)$ ,  $\psi_1 \in H_{per}^1(Y_1)$ ,  $\psi_1^\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}\right)$ ,  $\psi_2 \in H^1(Y_2)$  and  $\psi_2^\varepsilon(x) = \psi_2\left(\frac{x}{\varepsilon}\right)$ .

In view of Remark 3.11, we get

$$\begin{aligned} &\int_{\Omega_1^\varepsilon} \nabla \rho_{1\varepsilon} \nabla v_{1\varepsilon} \, dx + \int_{\Omega_2^\varepsilon} \nabla \rho_{2\varepsilon} \nabla v_{2\varepsilon} \, dx \\ &+ \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(\rho_{1\varepsilon} - \rho_{2\varepsilon})(v_{1\varepsilon} - v_{2\varepsilon}) \, d\sigma_x \\ &= \langle f_{1\varepsilon}, v_{1\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, v_{2\varepsilon} \rangle_{(H^1(\Omega))', H^1(\Omega)}. \end{aligned} \tag{3.56}$$

Following the same argument as in Lemma 3.13, we have that, for  $i = 1, 2$ ,

$$\mathcal{T}_i^\varepsilon(\nabla v_{i\varepsilon}) \longrightarrow \nabla_y(\omega_i \psi_i) \text{ strongly in } L^2(\Omega \times Y_i). \tag{3.57}$$

In view of the definitions of  $\Lambda_i^\varepsilon$  and  $v_{i\varepsilon}$ ,  $i = 1, 2$ , by Proposition 3.7 *ii*), via unfolding, we get that, for  $\varepsilon$  sufficiently small, (3.56) can be rewritten as

$$\begin{aligned} &\frac{1}{|Y|} \int_{\Omega \times Y_1} \mathcal{T}_1^\varepsilon(\nabla \rho_{1\varepsilon}) \mathcal{T}_1^\varepsilon(\nabla v_{1\varepsilon}) \, dx \, dy + \frac{1}{|Y|} \int_{\Omega \times Y_2} \mathcal{T}_2^\varepsilon(\nabla \rho_{2\varepsilon}) \mathcal{T}_2^\varepsilon(\nabla v_{2\varepsilon}) \, dx \, dy \\ &+ \frac{1}{\varepsilon |Y|} \int_{\Omega \times \Gamma} h(y) (\mathcal{T}_1^\varepsilon(\rho_{1\varepsilon}) - \mathcal{T}_2^\varepsilon(\rho_{2\varepsilon})) (\psi_1(y) \mathcal{T}_1^\varepsilon(\omega_1) - \psi_2(y) \mathcal{T}_2^\varepsilon(\omega_2)) \, dx \, d\sigma_y \\ &= \langle f_{1\varepsilon}, \varepsilon \omega_1 \psi_1^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega_2 \psi_2^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}, \end{aligned} \tag{3.58}$$

where we also used Proposition 3.7 *i*), *vi*) and Lemma 3.9.

From (3.54)<sub>2,4</sub>, (3.57), Proposition 3.7 *iv*) and Theorem 3.10 *ii*) passing to the limit as  $\varepsilon \rightarrow 0$  in the previous identity we obtain, up to as subsequence, still denoted  $\varepsilon$ ,

$$\begin{aligned} &\frac{1}{|Y|} \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \widehat{\rho}_1) \nabla_y(\omega_1 \psi_1) \, dx \, dy \\ &+ \frac{1}{|Y|} \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \widehat{\rho}_2) \nabla_y(\omega_2 \psi_2) \, dx \, dy \\ &+ \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\widehat{\rho}_1 - \widehat{\rho}_2) (\omega_1 \psi_1 - \omega_2 \psi_2) \, dx \, d\sigma_y \\ &= \lim_{\varepsilon \rightarrow 0} (\langle f_{1\varepsilon}, \varepsilon \omega_1 \psi_1^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega_2 \psi_2^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}). \end{aligned} \tag{3.59}$$

Now let us take  $\Phi_1 \in L^2(\Omega, H_{per}^1(Y_1))$ ,  $\Phi_2 \in L^2(\Omega, H^1(Y_2))$ . By density there exist, for  $i = 1, 2$ ,  $w_{in} \in \mathcal{D}(\Omega)$ ,  $\psi_{1n} \in H_{per}^1(Y_1)$ ,  $\psi_{2n} \in H^1(Y_2)$ , for any  $n \in \mathbb{N}$ , such that

$$\begin{aligned} w_{1n} \psi_{1n} &\rightarrow \Phi_1 \text{ strongly in } L^2(\Omega, H_{per}^1(Y_1)), \\ w_{2n} \psi_{2n} &\rightarrow \Phi_2 \text{ strongly in } L^2(\Omega, H^1(Y_2)). \end{aligned}$$



Hence, (3.59) gives, for any fixed  $n \in \mathbb{N}$ ,

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \hat{\rho}_1) \nabla_y (w_{1n} \psi_{1n}) \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \hat{\rho}_2) \nabla_y (w_{2n} \psi_{2n}) \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\hat{\rho}_1 - \hat{\rho}_2) (w_{1n} \psi_{1n} - w_{2n} \psi_{2n}) \, dx \, d\sigma_y \\ & = \lim_{\varepsilon \rightarrow 0} (\langle f_{1\varepsilon}, \varepsilon \omega_{1n} \psi_{1n}^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varepsilon \omega_{2n} \psi_{2n}^\varepsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}) \end{aligned} \quad (3.60)$$

where, for  $i = 1, 2$ ,  $\psi_{in}^\varepsilon(x) = \psi_{in}(x/\varepsilon)$ .

Passing to the limit as  $n \rightarrow +\infty$  in (3.60) we get (3.55).  $\square$

Now we are able to prove the homogenization result for problem (3.1) when  $\gamma = -1$ .

**Theorem 3.18.** *Let  $\gamma = -1$  and  $u_\varepsilon$  be the solution of problem (3.1). Then, under the assumptions (3.2)÷(3.6) and (3.17), there exist a subsequence, still denoted  $\varepsilon$ ,  $u \in H_0^1(\Omega)$ ,  $\hat{u}_1 \in L^2(\Omega, W_{per}(Y_1))$  and  $\hat{u}_2 \in L^2(\Omega, H^1(Y_2))$  such that*

$$\left\{ \begin{array}{ll} \tilde{u}_{i\varepsilon} \rightharpoonup \theta_i u & \text{weakly in } L^2(\Omega), \, i = 1, 2, \\ \mathcal{T}_1^\varepsilon(u_{1\varepsilon}) \rightarrow u & \text{strongly in } L^2(\Omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u + \nabla_y \hat{u}_1 & \text{weakly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(u_{2\varepsilon}) \rightharpoonup u & \text{weakly in } L^2(\Omega, H^1(Y_2)), \\ \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \rightharpoonup \nabla u + \nabla_y \hat{u}_2 & \text{weakly in } L^2(\Omega \times Y_2), \end{array} \right. \quad (3.61)$$

where  $(u, \hat{u}_1, \hat{u}_2)$  is the unique solution of the following problem

$$\left\{ \begin{array}{l} \text{Find } u \in H_0^1(\Omega), \hat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1)) \text{ with } \mathcal{M}_\Gamma(\hat{u}_1) = 0 \text{ a.e. } x \in \Omega, \\ \hat{u}_2 \in L^2(\Omega, H^1(Y_2)), \text{ s. t.} \\ \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y) (\nabla u + \nabla_y \hat{u}_1) (\nabla \varphi + \nabla_y \Phi_1) \, dx \, dy \\ + \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y) (\nabla u + \nabla_y \hat{u}_2) (\nabla \varphi + \nabla_y \Phi_2) \, dx \, dy \\ + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\hat{u}_1 - \hat{u}_2) (\Phi_1 - \Phi_2) \, dx \, d\sigma_y \\ = \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ + \frac{1}{|Y|} \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \hat{\rho}_1) \nabla_y \Phi_1 \, dx \, dy \\ + \frac{1}{|Y|} \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \hat{\rho}_2) \nabla_y \Phi_2 \, dx \, dy \\ + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\hat{\rho}_1 - \hat{\rho}_2) (\Phi_1 - \Phi_2) \, dx \, d\sigma_y, \\ \forall \varphi \in H_0^1(\Omega), \Phi_1 \in L^2(\Omega, H_{per}^1(Y_1)), \Phi_2 \in L^2(\Omega, H^1(Y_2)), \end{array} \right. \quad (3.62)$$

where the functions  $\rho$ ,  $\hat{\rho}_1$  and  $\hat{\rho}_2$  are as in Lemma 3.17, hence the term

$$\begin{aligned} & \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \hat{\rho}_1) \nabla_y \Phi_1 \, dx \, dy + \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \hat{\rho}_2) \nabla_y \Phi_2 \, dx \, dy \\ & + \int_{\Omega \times \Gamma} h(y) (\hat{\rho}_1 - \hat{\rho}_2) (\Phi_1 - \Phi_2) \, dx \, d\sigma_y \end{aligned}$$

depends only on a subsequence of  $f_\varepsilon$ .

*Proof.* Convergences (3.61) hold as in the proof of Theorem 3.14.

In order to get the limit problem satisfied by  $(u, \widehat{u}_1, \widehat{u}_2)$ , for  $i = 1, 2$ , let  $v_{i\varepsilon} = \varepsilon\omega_i\psi_i^\varepsilon$  be as in the proof of Lemma 3.17 and  $\varphi \in \mathcal{D}(\Omega)$ . If we take  $v_i = \varphi + v_{i\varepsilon}$  as test functions in (3.16), in view of Remark 3.11, we get

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_{1\varepsilon} \nabla(\varphi + v_{1\varepsilon}) \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_{2\varepsilon} \nabla(\varphi + v_{2\varepsilon}) \, dx \\ & \quad + \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) (v_{1\varepsilon} - v_{2\varepsilon}) \, d\sigma_x \\ & = \langle f_{1\varepsilon}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ & \quad + \langle f_{1\varepsilon}, v_{1\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, v_{2\varepsilon} \rangle_{(H^1(\Omega))', H^1(\Omega)}. \end{aligned} \quad (3.63)$$

Then if we take  $v_i = v_{i\varepsilon}$ ,  $i = 1, 2$ , as test functions in (3.24), (3.63) can be rewritten as

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_{1\varepsilon} \nabla(\varphi + v_{1\varepsilon}) \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_{2\varepsilon} \nabla(\varphi + v_{2\varepsilon}) \, dx \\ & \quad + \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) (v_{1\varepsilon} - v_{2\varepsilon}) \, d\sigma_x \\ & = \langle f_{1\varepsilon}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ & \quad + \int_{\Omega_1^\varepsilon} \nabla \rho_{1\varepsilon} \nabla v_{1\varepsilon} \, dx + \int_{\Omega_2^\varepsilon} \nabla \rho_{2\varepsilon} \nabla v_{2\varepsilon} \, dx \\ & \quad + \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon (\rho_{1\varepsilon} - \rho_{2\varepsilon}) (v_{1\varepsilon} - v_{2\varepsilon}) \, d\sigma_x. \end{aligned} \quad (3.64)$$

In view of the definitions of  $\Lambda_i^\varepsilon$  and  $v_{i\varepsilon}$ ,  $i = 1, 2$ , by Proposition 3.7 ii), via unfolding, we get that, for  $\varepsilon$  sufficiently small, (3.64) can be rewritten as

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y) \mathcal{T}_1^\varepsilon (\nabla u_{1\varepsilon}) \mathcal{T}_1^\varepsilon (\nabla \varphi + \nabla v_{1\varepsilon}) \, dx \, dy \\ & \quad + \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y) \mathcal{T}_2^\varepsilon (\nabla u_{2\varepsilon}) \mathcal{T}_2^\varepsilon (\nabla \varphi + \nabla v_{2\varepsilon}) \, dx \, dy \\ & + \frac{1}{\varepsilon |Y|} \int_{\Omega \times \Gamma} h(y) (\mathcal{T}_1^\varepsilon (u_{1\varepsilon}) - \mathcal{T}_2^\varepsilon (u_{2\varepsilon})) (\psi_1(y) \mathcal{T}_1^\varepsilon (\omega_1) - \psi_2(y) \mathcal{T}_2^\varepsilon (\omega_2)) \, dx \, d\sigma_y \\ & = \langle f_{1\varepsilon}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_{2\varepsilon}, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ & \quad + \frac{1}{|Y|} \int_{\Omega \times Y_1} \mathcal{T}_1^\varepsilon (\nabla \rho_{1\varepsilon}) \mathcal{T}_1^\varepsilon (\nabla v_{1\varepsilon}) \, dx \, dy \\ & \quad + \frac{1}{|Y|} \int_{\Omega \times Y_2} \mathcal{T}_2^\varepsilon (\nabla \rho_{2\varepsilon}) \mathcal{T}_2^\varepsilon (\nabla v_{2\varepsilon}) \, dx \, dy \\ & + \frac{1}{\varepsilon |Y|} \int_{\Omega \times \Gamma} h(y) (\mathcal{T}_1^\varepsilon (\rho_{1\varepsilon}) - \mathcal{T}_2^\varepsilon (\rho_{2\varepsilon})) (\psi_1(y) \mathcal{T}_1^\varepsilon (\omega_1) - \psi_2(y) \mathcal{T}_2^\varepsilon (\omega_2)) \, dx \, d\sigma_y. \end{aligned} \quad (3.65)$$

where we also used Proposition 3.7 i), vi) and Lemma 3.9.

From (3.22), (3.61)<sub>3,5</sub>, (3.54)<sub>2,4</sub>, (3.57), Proposition 3.7 iv) and Theorem 3.10 ii), passing to the limit as  $\varepsilon \rightarrow 0$  in (3.65), we obtain

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y) (\nabla u + \nabla_y \widehat{u}_1) (\nabla \varphi + \nabla_y (\omega_1 \psi_1)) \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y) (\nabla u + \nabla_y \widehat{u}_2) (\nabla \varphi + \nabla_y (\omega_2 \psi_2)) \, dx \, dy \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\widehat{u}_1 - \widehat{u}_2) (\omega_1 \psi_1 - \omega_2 \psi_2) \, dx \, d\sigma_y \\
& = \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\
& + \frac{1}{|Y|} \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \widehat{\rho}_1) \nabla_y (\omega_1 \psi_1) \, dx \, dy \\
& + \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \widehat{\rho}_2) \nabla_y (\omega_2 \psi_2) \, dx \, dy \\
& + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (\widehat{\rho}_1 - \widehat{\rho}_2) (\omega_1 \psi_1 - \omega_2 \psi_2) \, dx \, d\sigma_y.
\end{aligned}$$

Then, by density we get the limit problem (3.62).

Let us finally show that (3.62) admits a unique solution  $(u, \widehat{u}_1, \widehat{u}_2) \in H_0^1(\Omega) \times L^2(\Omega, W_{per}(Y_1)) \times L^2(\Omega, H^1(Y_2))$ .

To this aim, let

$$\mathcal{B} := H_0^1(\Omega) \times L^2(\Omega, W_{per}(Y_1)) \times L^2(\Omega, H^1(Y_2)),$$

where the space  $W_{per}(Y_1)$  is defined by

$$W_{per}(Y_1) := \{g \in H_{per}^1(Y_1) \mid \mathcal{M}_\Gamma(g) = 0\}.$$

For  $V = (v_1, v_2, v_3) \in \mathcal{B}$ , we define

$$\begin{aligned}
\|V\|_{\mathcal{B}}^2 & := \int_{\Omega \times Y_1} |\nabla v_1 + \nabla_y v_2|^2 \, dx \, dy + \int_{\Omega \times Y_2} |\nabla v_1 + \nabla_y v_3|^2 \, dx \, dy \\
& + \int_{\Omega \times \Gamma} |v_2 - v_3|^2 \, dx \, d\sigma_y.
\end{aligned}$$

As proved in [27], this last application is a norm on  $\mathcal{B}$ .

Now, for any  $V = (v_1, v_2, v_3), W = (w_1, w_2, w_3) \in \mathcal{B}$ , consider the bilinear form on  $\mathcal{B}$  defined by

$$\begin{aligned}
a(V, W) & = \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y) (\nabla v_1 + \nabla_y v_2) (\nabla w_1 + \nabla_y w_2) \, dx \, dy \\
& + \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y) (\nabla v_1 + \nabla_y v_3) (\nabla w_1 + \nabla_y w_3) \, dx \, dy \\
& + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) (v_2 - v_3) (w_2 - w_3) \, dx \, d\sigma_y
\end{aligned}$$

and the map

$$\begin{aligned}
F : V = (v_1, v_2, v_3) \in \mathcal{B} & \longrightarrow \langle f_1, v_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, v_1 \rangle_{(H^1(\Omega))', H^1(\Omega)} + \\
& \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \widehat{\rho}_1) \nabla_y v_2 \, dx \, dy + \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \widehat{\rho}_2) \nabla_y v_3 \, dx \, dy \\
& + \int_{\Omega \times \Gamma} h(y) (\widehat{\rho}_1 - \widehat{\rho}_2) (v_2 - v_3) \, dx \, d\sigma_y.
\end{aligned}$$

It is easily seen that  $a$  is continuous and coercive, and  $F$  is linear and continuous on  $\mathcal{B}$ . Hence, applying the Lax-Milgram theorem, we obtain that problem (3.62) has a unique solution.  $\square$

As for the previous case, in the following result we point out that the limit problem (3.62) is equivalent to an elliptic problem set in the fixed domain  $\Omega$  whose homogenized matrix is the same obtained in [49] for  $\gamma = -1$ .

**Corollary 3.19.** *Let  $\gamma = -1$  and  $u_\varepsilon$  be the solution of problem (3.1). Then, under the assumptions (3.2)÷(3.6) and (3.17), there exist a subsequence, still denoted  $\varepsilon$ , and  $u \in H_0^1(\Omega)$  such that*

$$\begin{cases} \tilde{u}_{i\varepsilon} \rightharpoonup \theta_i u & \text{weakly in } L^2(\Omega), \quad i = 1, 2, \\ A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \rightharpoonup A_\gamma^1 \nabla u + \theta_1 \mathcal{M}_{Y_1}(A \nabla_y \widehat{\chi}_1) & \text{weakly in } L^2(\Omega), \\ A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup A_\gamma^2 \nabla u + \theta_2 \mathcal{M}_{Y_2}(A \nabla_y \widehat{\chi}_2) & \text{weakly in } L^2(\Omega). \end{cases} \quad (3.66)$$

In (3.66), the constant matrices  $A_\gamma^l = (a_{ij}^l)_{n \times n}$ ,  $l = 1, 2$ , are defined by

$$\begin{cases} a_{ij}^1 = \theta_1 \mathcal{M}_{Y_1} \left( a_{ij} - \sum_{k=1}^n a_{ik} \frac{\partial \chi_1^j}{\partial y_k} \right), \\ a_{ij}^2 = \theta_2 \mathcal{M}_{Y_2} \left( a_{ij} - \sum_{k=1}^n a_{ik} \frac{\partial \chi_2^j}{\partial y_k} \right), \end{cases} \quad (3.67)$$

where the couples  $(\chi_1^j, \chi_2^j)$ ,  $j = 1, \dots, n$ , are the unique solutions of the cell problems,

$$\begin{cases} -\operatorname{div} \left( A \nabla \left( \chi_1^j - y_j \right) \right) = 0 & \text{in } Y_1, \\ -\operatorname{div} \left( A \nabla \left( \chi_2^j - y_j \right) \right) = 0 & \text{in } Y_2, \\ A \nabla \left( \chi_1^j - y_j \right) \cdot n_1 = -A \nabla \left( \chi_2^j - y_j \right) \cdot n_2 & \text{on } \Gamma, \\ A \nabla \left( \chi_1^j - y_j \right) \cdot n_1 = -h \left( \chi_1^j - \chi_2^j \right) & \text{on } \Gamma, \\ \chi_1^j \text{ } Y \text{-periodic, } \mathcal{M}_{Y_1} \left( \chi_1^j \right) = 0. \end{cases} \quad (3.68)$$

The couple  $(\widehat{\chi}_1, \widehat{\chi}_2)$ , for a.e.  $x \in \Omega$ , is the unique solution of the following problem

$$\begin{cases} \text{Find } (\widehat{\chi}_1, \widehat{\chi}_2) \in L^2(\Omega, H_{per}^1(Y_1) \times H^1(Y_2)) \text{ s. t.} \\ \int_{Y_1} A(y) \nabla_y \widehat{\chi}_1 \nabla_y \psi_1 \, dy + \int_{Y_2} A(y) \nabla_y \widehat{\chi}_2 \nabla_y \psi_2 \, dy \\ + \int_{\Gamma} h(y) (\widehat{\chi}_1 - \widehat{\chi}_2) (\psi_1 - \psi_2) \, d\sigma_y = \int_{Y_1} (\nabla \rho + \nabla_y \widehat{\rho}_1) \nabla_y \psi_1 \, dy \\ + \int_{Y_2} (\nabla \rho + \nabla_y \widehat{\rho}_2) \nabla_y \psi_2 \, dy + \int_{\Gamma} h(y) (\widehat{\rho}_1 - \widehat{\rho}_2) (\psi_1 - \psi_2) \, d\sigma_y, \\ \forall (\psi_1, \psi_2) \in H_{per}^1(Y_1) \times H^1(Y_2), \end{cases} \quad (3.69)$$

where  $\rho$  and  $\widehat{\rho}_i$ ,  $i = 1, 2$ , are the same functions as in Lemma 3.17.

Moreover, the limit function  $u$  is the unique solution of the problem

$$\begin{cases} -\operatorname{div} (A_\gamma^0 \nabla u) = f_1 + f_2 + \theta_1 \operatorname{div} (\mathcal{M}_{Y_1} (A \nabla_y \widehat{\chi}_1)) \\ + \theta_2 \operatorname{div} (\mathcal{M}_{Y_2} (A \nabla_y \widehat{\chi}_2)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.70)$$

where the homogenized matrix is defined by

$$A_\gamma^0 := A_\gamma^1 + A_\gamma^2. \quad (3.71)$$

*Proof.* Choosing  $\varphi \equiv 0$  in (3.62) yields

$$\begin{aligned} & \int_{\Omega \times Y_1} A(y) (\nabla u + \nabla_y \hat{u}_1) \nabla_y \Phi_1 \, dx \, dy + \int_{\Omega \times Y_2} A(y) (\nabla u + \nabla_y \hat{u}_2) \nabla_y \Phi_2 \, dx \, dy \\ & + \int_{\Omega \times \Gamma} h(y) (\hat{u}_1 - \hat{u}_2) (\Phi_1 - \Phi_2) \, dx \, d\sigma_y = \int_{\Omega \times Y_1} (\nabla \rho + \nabla_y \hat{\rho}_1) \nabla_y \Phi_1 \, dx \, dy \\ & + \int_{\Omega \times Y_2} (\nabla \rho + \nabla_y \hat{\rho}_2) \nabla_y \Phi_2 \, dx \, dy + \int_{\Omega \times \Gamma} h(y) (\hat{\rho}_1 - \hat{\rho}_2) (\Phi_1 - \Phi_2) \, dx \, d\sigma_y, \end{aligned}$$

for all  $\Phi_1 \in L^2(\Omega, H_{per}^1(Y_1))$ ,  $\Phi_2 \in L^2(\Omega, H^1(Y_2))$ .

By standard arguments, as in the two scale method (see [9], ch. 9), this gives

$$\begin{cases} \hat{u}_1(x, y) = \hat{\chi}_1(x, y) - \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) \chi_1^j(y), \\ \hat{u}_2(x, y) = \hat{\chi}_2(x, y) - \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) \chi_2^j(y), \end{cases} \quad (3.72)$$

where  $\chi_1^j$ ,  $\chi_2^j$ ,  $j = 1, \dots, n$ , are the solutions of the cell problems (3.68) and  $\hat{\chi}_1$ ,  $\hat{\chi}_2$  satisfy (3.69).

We now choose  $\Phi_1 = \Phi_2 \equiv 0$  in (3.62) obtaining

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y) (\nabla u + \nabla_y \hat{u}_1) \nabla \varphi \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y) (\nabla u + \nabla_y \hat{u}_2) \nabla \varphi \, dx \, dy \\ & = \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)}, \end{aligned} \quad (3.73)$$

for all  $\varphi \in H_0^1(\Omega)$ .

Replacing (3.72) in (3.73), we easily deduce, after some computations,

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_1} \left( a_{ij}(y) - \sum_{k=1}^n a_{ik}(y) \frac{\partial \chi_1^j}{\partial y_k}(y) \right) dy \right) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, dx \\ & + \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_2} \left( a_{ij}(y) - \sum_{k=1}^n a_{ik}(y) \frac{\partial \chi_2^j}{\partial y_k}(y) \right) dy \right) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, dx \\ & = \langle f_1, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \\ & - \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_1} a_{ij}(y) \frac{\partial \hat{\chi}_1}{\partial y_j}(y) \, dy \right) \frac{\partial \varphi}{\partial x_i} \, dx \\ & - \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_2} a_{ij}(y) \frac{\partial \hat{\chi}_2}{\partial y_j}(y) \, dy \right) \frac{\partial \varphi}{\partial x_i} \, dx, \end{aligned}$$

for all  $\varphi \in H_0^1(\Omega)$  which means that  $u$  satisfies the following problem

$$\begin{cases} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_1} \left( a_{ij}(y) - \sum_{k=1}^n a_{ik}(y) \frac{\partial \chi_1^j}{\partial y_k}(y) \right) dy \right) \frac{\partial u}{\partial x_j} \\ - \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_2} \left( a_{ij}(y) - \sum_{k=1}^n a_{ik}(y) \frac{\partial \chi_2^j}{\partial y_k}(y) \right) dy \right) \frac{\partial u}{\partial x_j} \\ = f_1 + f_2 + \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_1} a_{ij}(y) \frac{\partial \hat{\chi}_1}{\partial y_j}(y) \, dy \right) \\ + \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left( \frac{1}{|Y|} \int_{Y_2} a_{ij}(y) \frac{\partial \hat{\chi}_2}{\partial y_j}(y) \, dy \right) \end{cases} \quad \begin{array}{l} \text{in } \Omega, \\ \text{on } \partial\Omega. \end{array}$$

$u = 0$

This implies that  $u$  is the unique solution of problem (3.70) where  $A_\gamma^0$  is the matrix defined by (3.71).

Arguing as in the last part of the proof of Corollary 3.15, when proving (3.53), but taking into account that in this case  $\widehat{u}_1$  and  $\widehat{u}_2$  are given by (3.72), we get (3.66)<sub>2,3</sub>.  $\square$

**Remark 3.20.** As in the previous case, in problem (3.70) the right-hand side of the limit equation is not exactly the sum of the weak limits of  $f_{1\varepsilon}$  and  $f_{2\varepsilon}$  as in the case of more regular data, but it is a more complicated function depending on a subsequence of  $f_{i\varepsilon}$ ,  $i = 1, 2$  (see Lemma 3.17 and (3.69) of Corollary 3.19).

**4. Exact controllability of an imperfect transmission problem.** The second issue we deal with concerns the study of the exact controllability of a hyperbolic imperfect transmission problem posed in the domain  $\Omega$  described in Section 2. More precisely, let  $\zeta_\varepsilon := (\zeta_{1\varepsilon}, \zeta_{2\varepsilon}) \in L^2(0, T; L_\varepsilon^2(\Omega))$  be a control. For any fixed  $T > 0$  and  $\gamma \leq -1$ , let us consider the following problem

$$\begin{cases} u''_{1\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = \zeta_{1\varepsilon} & \text{in } \Omega_1^\varepsilon \times ]0, T[, \\ u''_{2\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = \zeta_{2\varepsilon} & \text{in } \Omega_2^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0, T[, \\ u_{1\varepsilon}(0) = U_{1\varepsilon}^0, \quad u'_{1\varepsilon}(0) = U_{1\varepsilon}^1 & \text{in } \Omega_1^\varepsilon, \\ u_{2\varepsilon}(0) = U_{2\varepsilon}^0, \quad u'_{2\varepsilon}(0) = U_{2\varepsilon}^1 & \text{in } \Omega_2^\varepsilon, \end{cases} \quad (4.1)$$

where  $n_{i\varepsilon}$  is the unitary outward normal to  $\Omega_{i\varepsilon}$ ,  $i = 1, 2$ , and

$$\begin{cases} \text{i) } U_\varepsilon^0 := (U_{1\varepsilon}^0, U_{2\varepsilon}^0) \in H_\gamma^\varepsilon, \\ \text{ii) } U_\varepsilon^1 := (U_{1\varepsilon}^1, U_{2\varepsilon}^1) \in L_\varepsilon^2(\Omega). \end{cases} \quad (4.2)$$

Moreover  $A^\varepsilon$  and  $h^\varepsilon$  are as in (3.2)÷(3.6) but, as usual when dealing with hyperbolic problems, in this section we require the additional symmetry assumption on  $A$

$$a_{ij} = a_{ji}, \quad i, j = 1, \dots, n. \quad (4.3)$$

For clearness sake, throughout the paper, we denote by  $u_\varepsilon(\zeta_\varepsilon) := (u_{1\varepsilon}(\zeta_\varepsilon), u_{2\varepsilon}(\zeta_\varepsilon))$  the solution of problem (4.1) and where no ambiguity arises, we omit the explicit dependence on the control.

**Definition 4.1.** System (4.1) is exactly controllable at time  $T > 0$ , if for every  $(U_\varepsilon^0, U_\varepsilon^1)$ ,  $(\overline{U}_\varepsilon^0, \overline{U}_\varepsilon^1)$  in  $H_\gamma^\varepsilon \times L_\varepsilon^2(\Omega)$ , there exists a control  $\zeta_\varepsilon^{ex} := (\zeta_{1\varepsilon}^{ex}, \zeta_{2\varepsilon}^{ex})$  belonging to  $L^2(0, T; L_\varepsilon^2(\Omega))$  such that the corresponding solution  $u_\varepsilon$  of problem (4.1) satisfies

$$u_\varepsilon(T) = \overline{U}_\varepsilon^0, \quad u'_\varepsilon(T) = \overline{U}_\varepsilon^1.$$

**Remark 4.2.** It is well known that for a linear system, driving it to any state is equivalent to driving it to the null state and this is known as null controllability. Hence, in the sequel we study the null controllability of the considered systems, namely we take  $(\overline{U}_\varepsilon^0, \overline{U}_\varepsilon^1) = (0, 0)$ .

We will prove that the system (4.1) is null controllable. We use a constructive method known as the Hilbert Uniqueness Method introduced by Lions (see [44, 45]). The idea is to build a control as the solution of a transposed problem associated to some suitable initial conditions. These initial conditions are obtained by calculating at zero time the solution of a backward problem. Let us underline that the control

obtained by HUM is unique being the one minimizing the norm in  $L^2(0, T; L^2_\varepsilon(\Omega))$ . In [21], the asymptotic behaviour, as  $\varepsilon \rightarrow 0$ , of the solutions of problem (4.1) has already been studied. Whence, a natural question arises: provided the exact controllability of the homogenized problem, do the exact control and its corresponding solution converge, as  $\varepsilon$  goes to zero, to the exact control of the homogenized problem and to the corresponding solution, respectively?

We give a positive answer to this question by proving the following main result:

**Theorem 4.3.** *Let  $T > 0$ ,  $\gamma \leq -1$  and  $(U_\varepsilon^0, U_\varepsilon^1) \in H_\gamma^\varepsilon \times L^2_\varepsilon(\Omega)$  satisfy*

$$\begin{cases} \text{i) } \widetilde{U}_\varepsilon^0 \rightharpoonup U^0 := (U_1^0, U_2^0) \text{ weakly in } [L^2(\Omega)]^2, \text{ with } U_2^0 \in H_0^1(\Omega), \\ \text{ii) } \widetilde{U}_\varepsilon^1 \rightharpoonup U^1 := (U_1^1, U_2^1) \text{ weakly in } [L^2(\Omega)]^2, \\ \text{iii) } \|\widetilde{U}_\varepsilon^0\|_{H_\gamma^\varepsilon} \leq C, \end{cases} \tag{4.4}$$

with  $C$  positive constant independent of  $\varepsilon$ . Further, assume that (3.2) ÷ (3.6) and (4.3) hold.

Let  $\zeta_\varepsilon^{ex} = (\zeta_{1\varepsilon}^{ex}, \zeta_{2\varepsilon}^{ex}) \in L^2(0, T; L^2_\varepsilon(\Omega))$  be the exact control of problem (4.1) minimizing the norm in  $L^2(0, T; L^2_\varepsilon(\Omega))$ . Then

$$\begin{cases} \widetilde{\zeta}_{1\varepsilon}^{ex} \rightharpoonup \theta_1 \zeta_1^{ex} \text{ weakly in } L^2(0, T; L^2(\Omega)), \\ \widetilde{\zeta}_{2\varepsilon}^{ex} \rightharpoonup \theta_2 \zeta_1^{ex} \text{ weakly in } L^2(0, T; L^2(\Omega)), \end{cases} \tag{4.5}$$

where  $\theta_i$ ,  $i = 1, 2$ , is given in (2.2) and  $\zeta_1^{ex}$  is the exact control, minimizing the norm in  $L^2(0, T; L^2(\Omega))$ , of the homogenized system

$$\begin{cases} u_1'' - \operatorname{div} (A_\gamma^0 \nabla u_1) = \zeta_1^{ex} & \text{in } \Omega \times ]0, T[, \\ u_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ u_1(0) = U_1^0 + U_2^0 & \text{in } \Omega, \\ u_1'(0) = U_1^1 + U_2^1 & \text{in } \Omega. \end{cases} \tag{4.6}$$

The homogenized matrix  $A_\gamma^0$  is given by (3.46) and (3.50), for  $\gamma < -1$ , while, for  $\gamma = -1$ , is given by (3.67) and (3.71).

Moreover denoted by  $u_1 := u_1(\zeta_1^{ex}) \in L^2(0, T; H_0^1(\Omega))$ , with  $u_1' := u_1'(\zeta_1^{ex}) \in L^2(0, T; L^2(\Omega))$  the unique solution of problem (4.6), there exists an extension operator

$$P_1^\varepsilon \in \mathcal{L}(L^\infty(0, T; H^k(\Omega_1^\varepsilon)); L^\infty(0, T; H^k(\Omega))),$$

for  $k = 1, 2$ , such that

$$\begin{cases} P_1^\varepsilon \widetilde{u}_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup u_1(\zeta_1^{ex}) \text{ weakly* in } L^\infty(0, T; H_0^1(\Omega)), \\ \widetilde{u}_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup \theta_1 u_1(\zeta_1^{ex}) \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{u}_{2\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup \theta_2 u_1(\zeta_1^{ex}) \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \end{cases} \tag{4.7}$$

and

$$\begin{cases} P_1^\varepsilon \widetilde{u}'_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup u_1'(\zeta_1^{ex}) \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{u}'_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup \theta_1 u_1'(\zeta_1^{ex}) \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{u}'_{2\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup \theta_2 u_1'(\zeta_1^{ex}) \text{ weakly* in } L^\infty(0, T; L^2(\Omega)). \end{cases} \tag{4.8}$$

Let us observe that by (4.4),  $U_1^0$  is in fact in  $H_0^1(\Omega)$  (see [21], Remark 2.7 for details).

**4.1. Asymptotic behaviour of two types of evolution imperfect transmission problems.** In this subsection, for reader’s convenience, we start by recalling some properties of the solution of the evolution imperfect transmission problem already studied in [21]. Although these results hold for  $\gamma \leq 1$ , we restrict our attention to the case we are interested in.

Hence, for  $T > 0$  and  $\gamma \leq -1$ , let  $z_\varepsilon := (z_{1\varepsilon}, z_{2\varepsilon})$  satisfy

$$\begin{cases} z''_{1\varepsilon} - \operatorname{div}(A^\varepsilon \nabla z_{1\varepsilon}) = g_{1\varepsilon} & \text{in } \Omega_1^\varepsilon \times ]0, T[, \\ z''_{2\varepsilon} - \operatorname{div}(A^\varepsilon \nabla z_{2\varepsilon}) = g_{2\varepsilon} & \text{in } \Omega_2^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla z_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla z_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla z_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(z_{1\varepsilon} - z_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ z_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0, T[, \\ z_{1\varepsilon}(0) = Z_{1\varepsilon}^0, \quad z'_{1\varepsilon}(0) = Z_{1\varepsilon}^1 & \text{in } \Omega_1^\varepsilon, \\ z_{2\varepsilon}(0) = Z_{2\varepsilon}^0, \quad z'_{2\varepsilon}(0) = Z_{2\varepsilon}^1 & \text{in } \Omega_2^\varepsilon, \end{cases} \quad (4.9)$$

where  $n_{i\varepsilon}$  is the unitary outward normal to  $\Omega_i^\varepsilon$ ,  $i = 1, 2$  and

$$\begin{cases} \text{i) } g_\varepsilon := (g_{1\varepsilon}, g_{2\varepsilon}) \in L^2(0, T; L^2_\varepsilon(\Omega)), \\ \text{ii) } Z_\varepsilon^0 := (Z_{1\varepsilon}^0, Z_{2\varepsilon}^0) \in H^\gamma_\varepsilon, \\ \text{iii) } Z_\varepsilon^1 := (Z_{1\varepsilon}^1, Z_{2\varepsilon}^1) \in L^2_\varepsilon(\Omega). \end{cases} \quad (4.10)$$

For any  $\varepsilon > 0$ , we set

$$W^\varepsilon := \{v = (v_1, v_2) \in L^2(0, T; H^\gamma_\varepsilon) \text{ s. t. } v' = (v'_1, v'_2) \in L^2(0, T; L^2_\varepsilon(\Omega))\}, \quad (4.11)$$

which is a Hilbert space if equipped with the norm

$$\begin{aligned} \|v\|_{W^\varepsilon} &= \|v_1\|_{L^2(0, T; V^\varepsilon)} + \|v_2\|_{L^2(0, T; H^1(\Omega_2^\varepsilon))} + \|v'_1\|_{L^2(0, T; L^2(\Omega_1^\varepsilon))} + \|v'_2\|_{L^2(0, T; L^2(\Omega_2^\varepsilon))}, \end{aligned}$$

(see [21]).

Thanks to Remark 3.4, by using an approach to evolutionary problems based on evolution triples, we assume as variational formulation of the formal problem (4.9) the following one

$$\begin{cases} \text{Find } z_\varepsilon = (z_{1\varepsilon}, z_{2\varepsilon}) \in W^\varepsilon \text{ s. t.} \\ \langle z''_{1\varepsilon}, v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle z''_{2\varepsilon}, v_2 \rangle_{(H^1(\Omega_2^\varepsilon))', H^1(\Omega_2^\varepsilon)} \\ + \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla z_{1\varepsilon} \nabla v_1 \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla z_{2\varepsilon} \nabla v_2 \, dx \\ + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon(z_{1\varepsilon} - z_{2\varepsilon})(v_1 - v_2) \, d\sigma_x = \int_{\Omega_1^\varepsilon} g_{1\varepsilon} v_1 \, dx + \int_{\Omega_2^\varepsilon} g_{2\varepsilon} v_2 \, dx, \\ \forall (v_1, v_2) \in H^\gamma_\varepsilon \text{ in } \mathcal{D}'(0, T), \\ z_{1\varepsilon}(0) = Z_{1\varepsilon}^0, \quad z'_{1\varepsilon}(0) = Z_{1\varepsilon}^1 \quad \text{in } \Omega_1^\varepsilon, \\ z_{2\varepsilon}(0) = Z_{2\varepsilon}^0, \quad z'_{2\varepsilon}(0) = Z_{2\varepsilon}^1 \quad \text{in } \Omega_2^\varepsilon. \end{cases} \quad (4.12)$$

As observed in [21], an abstract Galerkin’s method provides the existence and uniqueness result for the solution of problem (4.9) and also some a priori estimates for any  $\varepsilon > 0$ .

**Theorem 4.4** ([21]). *Under the assumptions (3.2)–(3.6), (4.3) and (4.10), problem (4.9) admits a unique weak solution  $z_\varepsilon \in W_\varepsilon$ . Moreover, there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\|z_\varepsilon\|_{L^\infty(0, T; H^\gamma_\varepsilon)} + \|z'_\varepsilon\|_{L^\infty(0, T; L^2_\varepsilon(\Omega))} \leq C \left( \|Z_\varepsilon^0\|_{H^\gamma_\varepsilon} + \|Z_\varepsilon^1\|_{L^2_\varepsilon(\Omega)} + \|g_\varepsilon\|_{L^2(0, T; L^2_\varepsilon(\Omega))} \right).$$



Let us point out that, for any fixed  $\varepsilon$ , the solution of problem (4.9) has some further regularity properties (see [46], Chapter 3, Theorem 8.2). In fact, under the same hypotheses of Theorem 4.4, the unique solution  $z_\varepsilon$  of problem (4.9) is such that

$$z_\varepsilon \in C([0, T]; H_\gamma^\varepsilon), z'_\varepsilon \in C([0, T]; L_\varepsilon^2(\Omega)).$$

Now, let us recall the homogenization result for problem (4.9), proved in [21].

**Theorem 4.5** ([21]). *Let  $(Z_\varepsilon^0, Z_\varepsilon^1) \in H_\gamma^\varepsilon \times L_\varepsilon^2(\Omega)$  satisfy*

$$\begin{cases} \text{i) } \widetilde{Z_\varepsilon^0} \rightharpoonup Z^0 := (Z_1^0, Z_2^0) \text{ weakly in } [L^2(\Omega)]^2, \text{ with } Z_2^0 \in H_0^1(\Omega), \\ \text{ii) } \widetilde{Z_\varepsilon^1} \rightharpoonup Z^1 := (Z_1^1, Z_2^1) \text{ weakly in } [L^2(\Omega)]^2, \\ \text{iii) } \|Z_\varepsilon^0\|_{H_\gamma^\varepsilon} \leq C, \end{cases} \quad (4.13)$$

with  $C$  positive constant independent of  $\varepsilon$ , and  $g_\varepsilon \in L^2(0, T; L_\varepsilon^2(\Omega))$  be such that

$$(\widetilde{g_{1\varepsilon}}, \widetilde{g_{2\varepsilon}}) \rightharpoonup (g_1, g_2) \text{ weakly in } L^2\left(0, T; [L^2(\Omega)]^2\right). \quad (4.14)$$

Under the assumptions (3.2) ÷ (3.6) and (4.3), there exists an extension operator

$$P_1^\varepsilon \in \mathcal{L}(L^\infty(0, T; H^k(\Omega_1^\varepsilon)); L^\infty(0, T; H^k(\Omega))),$$

for  $k = 1, 2$ , such that the solution  $z_\varepsilon$  of problem (4.9) satisfies the following convergences

$$\begin{cases} P_1^\varepsilon z_{1\varepsilon} \rightharpoonup z_1 & \text{weakly* in } L^\infty(0, T; H_0^1(\Omega)), \\ \widetilde{z_{1\varepsilon}} \rightharpoonup \theta_1 z_1 & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{z_{2\varepsilon}} \rightharpoonup \theta_2 z_1 & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \\ P_1^\varepsilon z'_{1\varepsilon} \rightharpoonup z'_1 & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{z'_{1\varepsilon}} \rightharpoonup \theta_1 z'_1 & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{z'_{2\varepsilon}} \rightharpoonup \theta_2 z'_1 & \text{weakly* in } L^\infty(0, T; L^2(\Omega)) \end{cases}$$

where  $\theta_i, i=1,2$ , is given in (2.2) and  $z_1 \in L^2(0, T; H_0^1(\Omega))$ , with  $z'_1 \in L^2(0, T; L^2(\Omega))$ , is the unique solution of the following homogenized problem

$$\begin{cases} z_1'' - \operatorname{div}(A_\gamma^0 \nabla z_1) = g_1 + g_2 & \text{in } \Omega \times ]0, T[, \\ z_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ z_1(0) = Z_1^0 + Z_2^0 & \text{in } \Omega, \\ z'_1(0) = Z_1^1 + Z_2^1 & \text{in } \Omega. \end{cases}$$

Moreover

$$A^\varepsilon \widetilde{\nabla z_{1\varepsilon}} + A^\varepsilon \widetilde{\nabla z_{2\varepsilon}} \rightharpoonup A_\gamma^0 \nabla z_1 \text{ weakly* in } L^\infty(0, T; L^2(\Omega)).$$

The homogenized matrix  $A_\gamma^0$  is given by (3.46) and (3.50), for  $\gamma < -1$ , while, for  $\gamma = -1$ , is given by (3.67) and (3.71).

**Remark 4.6.** Let us observe that (see for instance [9])  $A_\gamma^0$  is a symmetric constant matrix such that

$$A_\gamma^0 \in M(\alpha, \beta, \Omega), \quad (4.15)$$

where  $\alpha$  and  $\beta$  are defined in (3.3).

In order to prove Theorem 4.3, we need to study the homogenization of another evolution imperfect transmission problem with less regular initial data (see Subsection 4.2).

More precisely, for  $T > 0$  and  $\gamma \leq -1$ , let  $\varphi_\varepsilon := (\varphi_{1\varepsilon}, \varphi_{2\varepsilon})$  be the solution of the following problem

$$\begin{cases} \varphi_{1\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla \varphi_{1\varepsilon}) = 0 & \text{in } \Omega_1^\varepsilon \times ]0, T[, \\ \varphi_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla \varphi_{2\varepsilon}) = 0 & \text{in } \Omega_2^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla \varphi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \varphi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla \varphi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (\varphi_{1\varepsilon} - \varphi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ \varphi_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \varphi_{1\varepsilon}(0) = \varphi_{1\varepsilon}^0, \quad \varphi_{1\varepsilon}'(0) = \varphi_{1\varepsilon}^1 & \text{in } \Omega_1^\varepsilon, \\ \varphi_{2\varepsilon}(0) = \varphi_{2\varepsilon}^0, \quad \varphi_{2\varepsilon}'(0) = \varphi_{2\varepsilon}^1 & \text{in } \Omega_2^\varepsilon, \end{cases} \quad (4.16)$$

where  $n_{i\varepsilon}$  is the unitary outward normal to  $\Omega_i^\varepsilon$ ,  $i = 1, 2$  and

$$\begin{cases} \text{i) } \varphi_\varepsilon^0 := (\varphi_{1\varepsilon}^0, \varphi_{2\varepsilon}^0) \in L_\varepsilon^2(\Omega), \\ \text{ii) } \varphi_\varepsilon^1 := (\varphi_{1\varepsilon}^1, \varphi_{2\varepsilon}^1) \in (H_\gamma^\varepsilon)'. \end{cases} \quad (4.17)$$

Since the initial data are in a weak space, in order to give an appropriate definition of weak solution of problem (4.16), one needs to apply the so called transposition method (see [46], Chapter 3, Section 9, Theorems 9.3 and 9.4) to obtain a unique solution  $\varphi_\varepsilon \in C([0, T]; L_\varepsilon^2(\Omega)) \cap C^1([0, T]; (H_\gamma^\varepsilon)')$  satisfying the estimate

$$\|\varphi_\varepsilon\|_{L^\infty(0, T; L_\varepsilon^2(\Omega))} + \|\varphi_\varepsilon'\|_{L^\infty(0, T; (H_\gamma^\varepsilon)')} \leq C(\|\varphi_\varepsilon^0\|_{L_\varepsilon^2(\Omega)} + \|\varphi_\varepsilon^1\|_{(H_\gamma^\varepsilon)'}), \quad (4.18)$$

with  $C$  positive constant independent of  $\varepsilon$ .

Assume that the initial data satisfy

$$\begin{cases} \text{i) } \widetilde{\varphi_\varepsilon^0} \rightharpoonup \varphi^0 := (\varphi_1^0, \varphi_2^0) \text{ weakly in } (L^2(\Omega))^2, \\ \text{ii) } \|\varphi_\varepsilon^1\|_{(H_\gamma^\varepsilon)'} \leq C, \end{cases} \quad (4.19)$$

with  $C$  positive constant independent of  $\varepsilon$ .

The results of Theorem 4.5 can't be applied directly to problem (4.16), hypotheses (4.17) and (4.19) being too weak, but, thanks to the homogenization results of Section 3, we overcome the difficulty and prove the following new result.

**Theorem 4.7.** *Let  $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)'$  satisfy (4.19). Under the assumptions (3.2) ÷ (3.6) and (4.3), there exist a subsequence, still denoted  $\varepsilon$ , and a function  $\varphi^* \in H^{-1}(\Omega)$  such that for the solution  $\varphi_\varepsilon$  of problem (4.16) it holds*

$$\begin{aligned} \widetilde{\varphi_{1\varepsilon}} &\rightharpoonup \theta_1 \varphi_1 \text{ in } L^2(0, T; L^2(\Omega)) \\ \widetilde{\varphi_{2\varepsilon}} &\rightharpoonup \theta_2 \varphi_1 \text{ in } L^2(0, T; L^2(\Omega)), \end{aligned} \quad (4.20)$$

where  $\theta_i$ ,  $i=1, 2$ , is given in (2.2) and the function  $\varphi_1 \in L^2(0, T; L^2(\Omega))$ , with  $\varphi_1' \in L^2(0, T; L^2(\Omega))$ , is the unique solution of the following homogenized problem

$$\begin{cases} \varphi_1'' - \operatorname{div}(A_\gamma^0 \nabla \varphi_1) = 0 & \text{in } \Omega \times ]0, T[, \\ \varphi_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \varphi_1(0) = \varphi_1^0 + \varphi_2^0 & \text{in } \Omega, \\ \varphi_1'(0) = \varphi^* & \text{in } \Omega. \end{cases} \quad (4.21)$$

The homogenized matrix  $A_\gamma^0$  is given by (3.46) and (3.50), for  $\gamma < -1$ , while, for  $\gamma = -1$ , is given by (3.67) and (3.71).

*Proof.* Estimate (4.18) and hypothesis (4.19) provide the existence of two functions  $\bar{\varphi} \in L^2(0, T; L^2(\Omega))$  and  $\varphi_2 \in L^2(0, T; L^2(\Omega))$  such that in particular, up to a subsequence,

$$\begin{aligned} \widetilde{\varphi_{1\varepsilon}} &\rightharpoonup \bar{\varphi} \text{ in } L^2(0, T; L^2(\Omega)), \\ \widetilde{\varphi_{2\varepsilon}} &\rightharpoonup \varphi_2 \text{ in } L^2(0, T; L^2(\Omega)). \end{aligned} \quad (4.22)$$

Let  $\xi_\varepsilon := (\xi_{1\varepsilon}, \xi_{2\varepsilon})$  be the unique solution of the following system

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla \xi_{1\varepsilon}) = -\varphi_{1\varepsilon}^1 & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon \nabla \xi_{2\varepsilon}) = -\varphi_{2\varepsilon}^1 & \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla \xi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \xi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla \xi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(\xi_{1\varepsilon} - \xi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon, \\ \xi_{1\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.23)$$

By hypotheses (3.2)÷(3.6) and estimate (4.19) ii) the results of Corollary 3.15 and Corollary 3.19 apply obtaining that there exists a function  $\varphi^* \in H^{-1}(\Omega)$  such that, up to a subsequence, still denoted  $\varepsilon$ ,

$$\begin{cases} \text{i) } \widetilde{\xi_{1\varepsilon}} \rightharpoonup \theta_1 \xi_1 & \text{weakly in } L^2(\Omega), \\ \text{ii) } \widetilde{\xi_{2\varepsilon}} \rightharpoonup \theta_2 \xi_1 & \text{weakly in } L^2(\Omega), \end{cases} \quad (4.24)$$

with  $\theta_i$   $i=1,2$  given in (2.2) and  $\xi_1 \in H_0^1(\Omega)$  unique solution of

$$\begin{cases} -\operatorname{div}(A_\gamma^0 \nabla \xi_1) = -\varphi^* & \text{in } \Omega, \\ \xi_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.25)$$

where  $A_\gamma^0$  is the matrix defined in (3.46) and (3.50) if  $\gamma < -1$  or (3.67) and (3.71) if  $\gamma = -1$ . Denote

$$\sigma_{i\varepsilon}(x, t) := \int_0^t \varphi_{i\varepsilon}(x, s) ds + \xi_{i\varepsilon}(x), \quad i = 1, 2. \quad (4.26)$$

We do observe that this transformation leads to a system whose initial data are more regular than  $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$ . Indeed,  $\sigma_\varepsilon := (\sigma_{1\varepsilon}, \sigma_{2\varepsilon})$  satisfies

$$\begin{cases} \sigma_{1\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla \sigma_{1\varepsilon}) = 0 & \text{in } \Omega_1^\varepsilon \times ]0, T[, \\ \sigma_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla \sigma_{2\varepsilon}) = 0 & \text{in } \Omega_2^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla \sigma_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \sigma_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla \sigma_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(\sigma_{1\varepsilon} - \sigma_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ \sigma_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \sigma_{1\varepsilon}(0) = \xi_{1\varepsilon}, \quad \sigma_{1\varepsilon}'(0) = \varphi_{1\varepsilon}^0 & \text{in } \Omega_1^\varepsilon, \\ \sigma_{2\varepsilon}(0) = \xi_{2\varepsilon}, \quad \sigma_{2\varepsilon}'(0) = \varphi_{2\varepsilon}^0 & \text{in } \Omega_2^\varepsilon. \end{cases} \quad (4.27)$$

Since  $\varphi_\varepsilon^1 \in (H_\gamma^\varepsilon)'$ , one has  $\xi_\varepsilon \in H_\gamma^\varepsilon$ , hence the initial data  $(\xi_\varepsilon, \varphi_\varepsilon^0) \in H_\gamma^\varepsilon \times L_\varepsilon^2(\Omega)$ . Moreover, by (4.19) ii) and (4.23) we get

$$\|\xi_\varepsilon\|_{H_\gamma^\varepsilon} \leq C \quad (4.28)$$

with  $C$  positive constant independent of  $\varepsilon$ .

By (4.19) i), (4.24) and (4.28) we can apply Theorem 4.5 to system (4.27) obtaining in particular

$$\begin{cases} \text{i) } \widetilde{\sigma_{1\varepsilon}} \rightharpoonup \theta_1 \sigma_1 & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \text{ii) } \widetilde{\sigma_{1\varepsilon}'} \rightharpoonup \theta_1 \sigma_1' & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \text{iii) } \widetilde{\sigma_{2\varepsilon}} \rightharpoonup \theta_2 \sigma_1 & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \text{iv) } \widetilde{\sigma_{2\varepsilon}'} \rightharpoonup \theta_2 \sigma_1' & \text{weakly in } L^2(0, T; L^2(\Omega)), \end{cases} \quad (4.29)$$

where  $\sigma_1$  is the unique solution of the homogenized system

$$\begin{cases} \sigma_1'' - \operatorname{div}(A_\gamma^0 \nabla \sigma_1) = 0 & \text{in } \Omega \times ]0, T[, \\ \sigma_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \sigma_1(0) = \xi_1 & \text{in } \Omega, \\ \sigma_1'(0) = \varphi_1^0 + \varphi_2^0 & \text{in } \Omega. \end{cases} \quad (4.30)$$

By (4.26) it results

$$\widetilde{\sigma}'_{i\varepsilon} = \widetilde{\varphi}_{i\varepsilon}, \quad i = 1, 2. \tag{4.31}$$

Hence (4.22), (4.29) ii) and (4.29) iv), by passing to the limit in (4.31), provide  $\bar{\varphi} = \theta_1 \sigma'_1$  and  $\varphi_2 = \theta_2 \sigma'_1$ .

By classical regularity results for hyperbolic equations we have

$$\sigma_1 \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap C^2([0, T]; H^{-1}(\Omega)).$$

Hence, by (4.25) and (4.30)

$$\sigma_1''(0) = \operatorname{div}(A_\gamma^0 \nabla \sigma_1(0)) = \operatorname{div}(A_\gamma^0 \nabla \xi_1) = \varphi^*.$$

Therefore, the function  $\varphi_1 := \sigma'_1 = \frac{\bar{\varphi}}{\theta_1}$  is the unique solution in the sense of transposition of system (4.21) and  $\varphi_2 = \theta_2 \varphi_1$ .

Now the proof is complete. □

**4.2. Proof of Theorem 4.3.** The proof of the main result of this section develops into two steps. At first we prove the null controllability (or equivalently the exact controllability, see Remark 4.2) of problem (4.1), by using HUM (Hilbert Uniqueness Method), a constructive method introduced by Lions in [44, 45]. As already observed, the idea is to build a control as the solution of a transposed problem associated to some suitable initial conditions. These initial conditions are obtained by calculating at zero time the solution of a backward problem. The crucial point is constructing an isomorphism between  $L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)'$  and its dual with constants independent of  $\varepsilon$ . This result was already proved in [36], Theorem 3.1, for the case  $-1 < \gamma \leq 1$ . The proof for the case  $\gamma \leq -1$  is exactly the same, hence here, for the reader's convenience, we detail only the noteworthy points.

In the second step, having in mind the homogenization result of the previous subsection (see Theorem 4.5), we show that the exact control of the problem at  $\varepsilon$ -level, found in the first step, and the corresponding state, converge, as  $\varepsilon \rightarrow 0$ , to the exact control and to the solution of the homogenized problem, respectively. To this aim, we need to apply the homogenization result stated in Theorem 4.7 to the transposed problem at  $\varepsilon$ -level.

**Step1.** Let us start by proving that there exists a control  $\zeta_\varepsilon^{ex} \in L^2(0, T; L_\varepsilon^2(\Omega))$  driving the corresponding solution of problem (4.1) to the null state, i.e.

$$u_\varepsilon(T) = u'_\varepsilon(T) = 0, \tag{4.32}$$

see Definition 4.1 and Remark 4.2. To this aim, let  $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)'$  and let  $\varphi_\varepsilon \in C([0, T]; L_\varepsilon^2(\Omega)) \cap C^1([0, T]; (H_\gamma^\varepsilon)')$  be the unique solution in the sense of transposition of problem (4.16). Consider the backward problem

$$\begin{cases} \psi_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla \psi_{1\varepsilon}) = -\varphi_{1\varepsilon} & \text{in } \Omega_1^\varepsilon \times ]0, T[, \\ \psi_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla \psi_{2\varepsilon}) = -\varphi_{2\varepsilon} & \text{in } \Omega_2^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla \psi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \psi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ A^\varepsilon \nabla \psi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(\psi_{1\varepsilon} - \psi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0, T[, \\ \psi_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \psi_{1\varepsilon}(T) = \psi'_{1\varepsilon}(T) = 0 & \text{in } \Omega_1^\varepsilon, \\ \psi_{2\varepsilon}(T) = \psi'_{2\varepsilon}(T) = 0 & \text{in } \Omega_2^\varepsilon, \end{cases} \tag{4.33}$$

where  $n_{i\varepsilon}$  is the unitary outward normal to  $\Omega_i^\varepsilon$ ,  $i = 1, 2$ .

As previously, for clearness sake, we denote by

$$\psi_\varepsilon(\varphi_\varepsilon) := (\psi_{1\varepsilon}(\varphi_\varepsilon), \psi_{2\varepsilon}(\varphi_\varepsilon)) \in C([0, T]; H_\gamma^\varepsilon) \cap C^1([0, T]; L_\varepsilon^2(\Omega))$$

the unique solution of problem (4.33) and, where no ambiguity arises, we omit the explicit dependence on the right hand member. Then we introduce the linear operator

$$\mathcal{L}_\varepsilon : L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)' \rightarrow L_\varepsilon^2(\Omega) \times H_\gamma^\varepsilon \tag{4.34}$$

by setting for all  $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)'$ ,

$$\mathcal{L}_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1) = (\psi'_\varepsilon(0), -\psi_\varepsilon(0)). \tag{4.35}$$

Following exactly the same argument as in [36] for the case  $-1 < \gamma \leq 1$ , the operator  $\mathcal{L}_\varepsilon$  is an isomorphism with constants independent of  $\varepsilon$  and its inverse operator  $\mathcal{L}_\varepsilon^{-1}$  satisfies the following uniform estimate

$$\|\mathcal{L}_\varepsilon^{-1}\|_{\mathcal{L}(L_\varepsilon^2(\Omega) \times H_\gamma^\varepsilon; L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)')} \leq C, \tag{4.36}$$

with  $C$  positive constant independent of  $\varepsilon$ .

Let now  $(U_\varepsilon^0, U_\varepsilon^1) \in H_\gamma^\varepsilon \times L_\varepsilon^2(\Omega)$  be the initial conditions of problem (4.1) and  $(\Phi_\varepsilon^0, \Phi_\varepsilon^1) \in L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)'$  the unique couple satisfying the equation

$$(\Phi_\varepsilon^0, \Phi_\varepsilon^1) = \mathcal{L}_\varepsilon^{-1}(U_\varepsilon^1, -U_\varepsilon^0). \tag{4.37}$$

Denote

$$\zeta_\varepsilon^{ex} := -\Phi_\varepsilon, \tag{4.38}$$

where  $\Phi_\varepsilon$  is the unique solution of problem (4.16) with initial data  $(\Phi_\varepsilon^0, \Phi_\varepsilon^1)$  given by (4.37). If  $\Psi_\varepsilon$  is the solution of problem (4.33) with the choice  $\varphi_\varepsilon = \Phi_\varepsilon$ , by (4.35) and (4.37), we get  $(\Psi'_\varepsilon(0), -\Psi_\varepsilon(0)) = (U_\varepsilon^1, -U_\varepsilon^0)$  and by uniqueness it results

$$u_\varepsilon(\zeta_\varepsilon^{ex}) = \Psi_\varepsilon, \tag{4.39}$$

which implies (4.32). Hence  $\zeta_\varepsilon^{ex}$  is the null (or equivalently exact) control at time  $T$  for system (4.1). Moreover, this control, deriving from HUM method, minimizes the norm in  $L^2(0, T; L_\varepsilon^2(\Omega))$ .

**Step2.** Let now  $\varepsilon$  tend to zero. As a consequence of (4.4) ii), (4.4) iii), (4.36) and (4.37), we get

$$\|(\Phi_\varepsilon^0, \Phi_\varepsilon^1)\|_{L_\varepsilon^2(\Omega) \times (H_\gamma^\varepsilon)'} \leq C, \tag{4.40}$$

with  $C$  positive constant independent of  $\varepsilon$ , hence we deduce the existence of  $\Phi^0 := (\Phi_1^0, \Phi_2^0) \in [L^2(\Omega)]^2$  such that, up to a subsequence, still denoted  $\varepsilon$ ,

$$\widetilde{\Phi}_\varepsilon^0 \rightharpoonup \Phi^0 \text{ weakly in } [L^2(\Omega)]^2. \tag{4.41}$$

Now we can apply Theorem 4.7 to system (4.16) for the choice  $\varphi_\varepsilon^0 = \Phi_\varepsilon^0$ ,  $\varphi_\varepsilon^1 = \Phi_\varepsilon^1$ ,  $\varphi^0 = \Phi^0$ , and get that there exist a subsequence, still denoted  $\varepsilon$ , and a function  $\Phi^* \in H^{-1}(\Omega)$  such that

$$\begin{aligned} \widetilde{\Phi}_{1\varepsilon} &\rightharpoonup \theta_1 \Phi_1 \text{ in } L^2(0, T; L^2(\Omega)) \\ \widetilde{\Phi}_{2\varepsilon} &\rightharpoonup \theta_2 \Phi_1 \text{ in } L^2(0, T; L^2(\Omega)), \end{aligned} \tag{4.42}$$

where  $\theta_i$ ,  $i=1,2$ , is given in (2.2) and the function  $\Phi_1 \in L^2(0, T; L^2(\Omega))$ , with  $\Phi'_1 \in L^2(0, T; L^2(\Omega))$ , is the unique solution of the following homogenized problem

$$\begin{cases} \Phi_1'' - \operatorname{div} (A_\gamma^0 \nabla \Phi_1) = 0 & \text{in } \Omega \times ]0, T[, \\ \Phi_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \Phi_1(0) = \Phi_1^0 + \Phi_2^0 & \text{in } \Omega, \\ \Phi_1'(0) = \Phi^* & \text{in } \Omega. \end{cases} \quad (4.43)$$

The homogenized matrix  $A_\gamma^0$  is still given by (3.46) and (3.50) for  $\gamma < -1$ , while, for  $\gamma = -1$ , is given by (3.67) and (3.71).

Observe that, as a result of (4.38) and (4.42), we get, up to a subsequence, still denoted  $\varepsilon$ ,

$$\begin{cases} \widetilde{\zeta_{1\varepsilon}^{ex}} \rightharpoonup -\theta_1 \Phi_1 & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \widetilde{\zeta_{2\varepsilon}^{ex}} \rightharpoonup -\theta_2 \Phi_1 & \text{weakly in } L^2(0, T; L^2(\Omega)). \end{cases} \quad (4.44)$$

Let now pass to the limit, as  $\varepsilon$  tends to zero, in system (4.1) with  $\zeta_\varepsilon^{ex}$  in place of  $\zeta_\varepsilon$ . In view of (4.4) and (4.44), Theorem 4.5 applies to problem (4.1), for the choice  $Z_\varepsilon^0 = U_\varepsilon^0$ ,  $Z_\varepsilon^1 = U_\varepsilon^1$ ,  $Z^0 = U^0$ ,  $Z^1 = U^1$  and  $g_\varepsilon = \zeta_\varepsilon^{ex}$  giving the following convergences,

$$\begin{cases} P_{1\varepsilon}^\varepsilon u_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup u_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; H_0^1(\Omega)), \\ \widetilde{u_{1\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup \theta_1 u_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{u_{2\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup \theta_2 u_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \end{cases} \quad (4.45)$$

$$\begin{cases} P_{1\varepsilon}^\varepsilon u'_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup u'_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{u'_{1\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup \theta_1 u'_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{u'_{2\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup \theta_2 u'_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \end{cases} \quad (4.46)$$

where  $u_1 := u_1(\Phi_1) \in L^2(0, T; H_0^1(\Omega))$ , with  $u'_1 := u'_1(\Phi_1) \in L^2(0, T; L^2(\Omega))$ , is the unique solution of the homogenized problem

$$\begin{cases} u_1'' - \operatorname{div} (A_\gamma^0 \nabla u_1) = -\Phi_1 & \text{in } \Omega \times ]0, T[, \\ u_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ u_1(0) = U_1^0 + U_2^0 & \text{in } \Omega, \\ u_1'(0) = U_1^1 + U_2^1 & \text{in } \Omega. \end{cases} \quad (4.47)$$

On the other hand, by (4.42) and Theorem 4.5, we can pass to the limit in the backward problem (4.33) with  $\varphi_\varepsilon = \Phi_\varepsilon$ , and obtain the following convergences

$$\begin{cases} P_{1\varepsilon}^\varepsilon \Psi_{1\varepsilon}(\Phi_\varepsilon) \rightharpoonup \Psi_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; H_0^1(\Omega)), \\ \widetilde{\Psi_{1\varepsilon}(\Phi_\varepsilon)} \rightharpoonup \theta_1 \psi_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{\Psi_{2\varepsilon}(\Phi_\varepsilon)} \rightharpoonup \theta_2 \psi_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \end{cases} \quad (4.48)$$

$$\begin{cases} P_{1\varepsilon}^\varepsilon \Psi'_{1\varepsilon}(\Phi_\varepsilon) \rightharpoonup \psi'_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{\Psi'_{1\varepsilon}(\Phi_\varepsilon)} \rightharpoonup \theta_1 \psi'_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{\Psi'_{2\varepsilon}(\Phi_\varepsilon)} \rightharpoonup \theta_2 \psi'_1(\Phi_1) & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \end{cases} \quad (4.49)$$

where  $\Psi_1 := \Psi_1(\Phi_1) \in L^2(0, T; H_0^1(\Omega))$ , with  $\Psi_1' := \Psi_1'(\Phi_1) \in L^2(0, T; L^2(\Omega))$ , is the unique solution of the homogenized backward problem

$$\begin{cases} \Psi_1'' - \operatorname{div}(A_\gamma^0 \nabla \Psi_1) = -\Phi_1 & \text{in } \Omega \times ]0, T[, \\ \Psi_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \Psi_1(T) = \Psi_1'(T) = 0 & \text{in } \Omega. \end{cases} \quad (4.50)$$

By (4.39), (4.45) and (4.48), we get

$$\Psi_1 = u_1 \quad (4.51)$$

and, since both  $\Psi_1$  and  $u_1$  belong to  $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  (see [46], Chapter 3, Theorem 8.2), it holds

$$u_1(T) = u_1'(T) = 0. \quad (4.52)$$

Therefore

$$\zeta_1^{ex} := -\Phi_1 \quad (4.53)$$

is an exact control for problem (4.47). On the other hand, if we apply HUM method directly to problem (4.47), in view of classical arguments about exact controllability of hyperbolic problem in fixed domains, (see [44, 45]), by considering problems (4.43) and (4.50), we construct an isomorphism  $\mathcal{L}$  between  $L^2(\Omega) \times H^{-1}(\Omega)$  and its dual such that

$$\mathcal{L}(\Phi_1^0 + \Phi_2^0, \Phi^*) = (\Psi_1(0), -\Psi_1'(0)).$$

By (4.51) we get

$$(\Phi_1^0 + \Phi_2^0, \Phi^*) = \mathcal{L}^{-1}(U_1^1 + U_2^1, -(U_1^0 + U_2^0)).$$

This identifies  $\zeta_1^{ex}$  in a unique way as the energy minimizing control of problem (4.47). Hence convergences (4.44), (4.45) and (4.46) hold for the whole sequences and by (4.53), we get (4.5), (4.7) and (4.8).

Theorem 4.3 is now completely proved.

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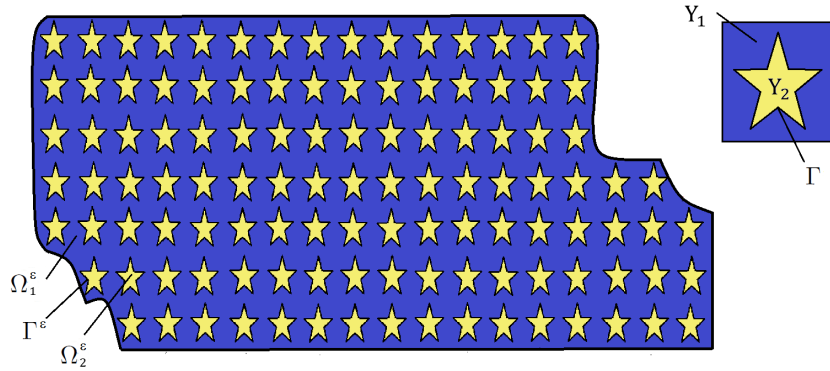
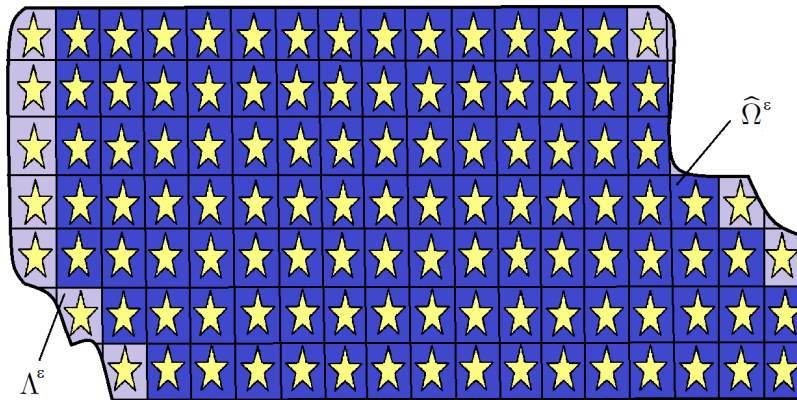


FIGURE 1. The two-component domain  $\Omega$

FIGURE 2. The sets  $\hat{\Omega}^\varepsilon$  and  $\Lambda^\varepsilon$ 

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