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HOMOGENIZATION OF NONLINEAR HYPERBOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH NONLINEAR DAMPING AND FORCING

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Dedicated to the memory of Professor Salah-Eldin A. Mohammed
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Abstract. In this paper we deal with the homogenization of stochastic nonlinear hyperbolic equations with periodically oscillating coefficients involving nonlinear damping and forcing driven by a multi-dimensional Wiener process. Using the two-scale convergence method and crucial probabilistic compactness results due to Prokhorov and Skorokhod, we show that the sequence of solutions of the original problem converges in suitable topologies to the solution of a homogenized problem, which is a nonlinear damped stochastic hyperbolic partial differential equation. More importantly, we also prove the convergence of the associated energies and establish a crucial corrector result.

1. **Introduction and setting of the problem.** Homogenization theory has become an important tool in the investigation of processes taking place in highly heterogenous media ranging from soil to the most advanced aircraft the construction of which uses composite materials. So far, the problems solved by means of homogenization have mainly involved deterministic partial differential equations (PDEs) and the homogenization of PDEs with randomly oscillating coefficients; the great wealth of results obtained over several decades on problems of diverse classes and methodologies can be found for instance in [9, 6, 40, 41, 23, 34, 22, 49, 31, 17, 4, 32, 36, 46, 50, 33], for the deterministic case and [13, 14, 18, 20, 24, 37, 19, 47, 48]. for the random case. Fundamental methods were subsequently developed such as the method of asymptotic expansions ([9], [6], [40], [41]), the two scale-convergence ([4], [32]), Tartar method of oscillating test functions and H-convergence ([49]),

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the asymptotic method for non periodically perforated domains ([23], [46]), G-convergence ([36]) and Γ-convergence developed by De Giorgi and his students; relevant extensions of some of these methods, including their random counterparts, have also emerged in recent times. One rapidly developping important branch of homogenization is that of numerical homogenization; see [1], [2].

However physical processes under random fluctuations are better modelled by stochastic partial differential equations (SPDEs). It was therefore natural to consider homogenization of this very important class of PDEs. Research in this direction is still at its infancy, despite the importance of such problems in both applied and fundamental sciences. Some relevant interesting work have recently been undertaken, mainly for parabolic SPDEs; see for instance [3, 8, 10, 11, 21, 43, 44]. We also note the closely related work [3, 25, 15, 16] dealing with stochastic homogenization for SPDEs with small parameters. The list of references is of course not exhaustive, but a representation of the main trends in the field.

The homogenization of hyperbolic SPDEs was initiated in [27], [28, 29], [30] where the authors studied the homogenization of Dirichlet problems for linear hyperbolic equation with rapidly oscillating coefficients using the method of the two-scale convergence pioneered by Nguetseng in [32] and developed by Allaire in [4] and [5]; they also dealt with the linear Neumann problem by means of Tartar's method and obtained the corresponding corrector results within these settings; [30] deals with a semilinear hyperbolic SPDE by Tartar's method.

In the present work, following the two-scale convergence method, we investigate the homogenization of a non-linear hyperbolic equation with nonlinear damping, where the intensity of the noise is also nonlinear and is assumed to satisfy Lipschitz's condition. Our investigation relies on crucial compactness results of analytic (Aubin-Lions-Simon's type) and probabilistic (Prokhorov and Skorokhod fundamental theorems) nature. It should be noted that these methods extend readily to the case when Lipschitz condition on the intensity of the noise is replaced by a mere continuity. In contrast to the linear and the semilinear cases considered in previous papers, the type of nonlinear damping and nonlinear noise in the present paper leads to new challenges in obtaining uniform a priori estimates as well as in the passage to the limit. It should be noted that the process of damping in mechanical systems is a crucial stabilizing factor when the system is subjected to very extreme tasks; mathematically this translates in some regularizing effects on the solution of the governing equations.

We are concerned with the homogenization of the initial boundary value problem with oscillating data, referred to throughout the paper as problem (P_ϵ):

$$\begin{aligned} du_t^\epsilon - div\left(A_\epsilon\left(x\right)\nabla u^\epsilon\right)dt + B(t,u_t^\epsilon)dt \\ = f(t,x,x/\varepsilon,\nabla u^\epsilon)dt + g(t,x,x/\varepsilon,u_t^\epsilon)dW \text{ in } (0,T)\times Q \\ u^\epsilon = 0 \text{ on } (0,T)\times\partial Q, \\ u^\epsilon(0,x) = a^\epsilon(x),\ u_t^\epsilon(0,x) = b^\epsilon(x) \text{ in } Q, \end{aligned}$$

where u_t^ϵ denotes the partial derivative $\partial u^\epsilon/\partial t$ of u^ϵ with respect to t, $\epsilon > 0$ is a sufficiently small parameter which ultimately tends to zero, $T > 0$, Q is an open and bounded (at least Lipschitz) subset of $\mathbb{R}^n$, $W = (W(t))$ $(t \in [0,T])$ an m-dimensional standard Wiener process defined on a given filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T})$; $\mathbb{E}$ denotes the corresponding mathematical expectation. For a physical motivation, we refer to [27, 28], where the authors discussed real life processes of vibrational nature subjected to random fluctuations; for instance the nonlinear term $B(t, u_t^\epsilon)$ stands for damping effects, the term $f(t, x, x/\varepsilon, \nabla u^\epsilon)$ is the oscillating regular part of the force acting on the system and depending linearly on ∇u^ϵ, while the term $g(t, x, x/\varepsilon, u_t^\epsilon)dW$ represents the oscillating random component of the force; it depends on u_t^ε. More precise assumptions on the data will be provided shortly.

Few words about the difference between the current work and previous works by the authors on homogenization of SPDEs. Compared to [27, 28, 29, 30], the structure of problem (P_ε) is dominated by nonlinear terms such as the damping $B(t, u_t^\epsilon)$, leading to $L^p(Q)$-like norms whose combination with the predominently L^2-like norms coming from the other terms requires special care, both in the derivation of the a priori estimates, as well as in the passage to the limit. Though, two-scale convergence method is also used in the paper [27], the model there is essentially linear. The works [43, 44] deal with stochastic parabolic equations in domains with fine grained boundaries, where no conditions of periodicity hold and the methodology implemented there is a stochastic counterpart of Kruslov-Marchenko's [23] and Skrypnik's [46] homogenization theories based on potiential theory; for instance the homogenized problems in [43, 44] involve an additional term of capacitary type. The investigation of a hyperbolic counterpart of these works has still not been undertaken and is somehow overdue. Finally, compared with the above mentioned works, the current paper involves a simpler proof of the convergence of the stochastic nonlinear term (its integral) thanks to a blending of two-scale convergence with a regularizing argument and a result on convergence of stochastic integrals due to Rozovskii [39, Theorem 4, P 63].

We now introduce some functions spaces needed in the sequel.

For $2 \leq p \leq \infty$, we define the Sobolev space

$$W^{1,p}(Q) = \left\{ \phi : \phi \in L^p(Q),\ \frac{\partial \phi}{\partial x_j} \in L^p(Q),\ j = 1, ..., n \right\},$$

where the derivatives exist in the weak sense, and $L^p(Q)$ is the usual Lebesgue space. For $p = 2$, $W^{1,2}(Q)$ is denoted by $H^1(Q)$. By $W_0^{1,p}(Q)$ we denote the space of elements $\psi \in W^{1,p}(Q)$ such that $\psi|_{\partial Q} = 0$ with the $W^{1,p}(Q)$-norm. By (ϕ, ψ) we denote the inner product in $L^2(Q)$ and by $\langle ., . \rangle$ we denote the duality pairing between $W_0^{1,p}(Q)$ and $W^{-1,p'}(Q)$ ($\frac{1}{p} + \frac{1}{p'} = 1$). We also consider the following spaces, $H(Q) = \{u \in H^1(Q) | \mathcal{M}_Q(u) = 0\}$ where $\mathcal{M}_Q(u)$ is the mean value of u over Q, $C_{\text{per}}^\infty(Y)$ the subspace of $C^\infty(\mathbb{R}^n)$ of Y- periodic functions where $Y = (0, l_1) \times ... \times (0, l_n)$. Let $H_{\text{per}}^1(Y)$ be the closure of $C_{\text{per}}^\infty(Y)$ in the H^1-norm, and $H_{\text{per}}(Y)$ the subspace of $H_{\text{per}}^1(Y)$ with zero mean on Y.

For a Banach space X, and $1 \leq p \leq \infty$, we denote by $L^p(0, T; X)$ the space of measurable functions $\phi : t \in [0, T] \longrightarrow \phi(t) \in X$ and p-integrable with the norm

$$||\phi||_{L^p(0,T;X)} = \left(\int_0^T ||\phi||_X^p \, dt \right)^{\frac{1}{p}}, \quad 0 \leq p < \infty.$$

When $p = \infty$, $L^\infty(0, T; X)$ is the space of all essentially bounded functions on the closed interval $[0, T]$ with values in X equipped with the norm

$$\|\phi\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{[0,\,T]} \|\phi\|_X \; < \; \infty.$$

For $1 \leq q, p < \infty$, $L^q(\Omega, \mathcal{F}, \mathbb{P}, L^p(0,T;X))$ ($(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$) consists of all random functions $\phi : (\omega, t) \in \Omega \times [0,T] \longrightarrow \phi(\omega, t, \cdot) \in X$ such that $\phi(\omega, t, x)$ is progressively measurable with respect to (ω, t). We endow this space with the norm

$$||\phi||_{L^q(\Omega,\mathcal{F},\mathbb{P};L^p(0,T;X))} = \left(\mathbb{E}\, ||\phi||^q_{L^p(0,T;X)}\right)^{1/q}.$$

When $p = \infty$, the norm in the space $L^q(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0,T;X))$ is given by

$$||\phi||_{L^q(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;X))} = \left(\mathbb{E}\, ||\phi||^q_{L^\infty(0,T;X)}\right)^{1/q}.$$

It is well known that under the above norms, $L^q(\Omega, \mathcal{F}, \mathbb{P}, L^p(0,T;X))$ is a Banach space.

We now impose the following hypotheses on the data.

(A1) $A_\epsilon(x) = A(\frac{x}{\epsilon}) = (a_{i,j}(\frac{x}{\epsilon}))_{1\leq i,j\leq n}$ is an $n \times n$ symmetric matrix, the components $a_{i,j}$, are $Y-$periodic and there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2 \text{ for, } \xi \in \mathbb{R}^n,$$
$$a_{i,j} \in L^\infty(\mathbb{R}^n),\ i,j = 1, \ldots, n.$$

(A2) $B(t,\cdot) : u \in W_0^{1,p}(Q) \longrightarrow W^{-1,p'}(Q)$ such that
  - (i) $B(t,\cdot)$ is a hemicontinuous operator, i.e. $\lambda \longrightarrow \langle B(t, u + \lambda v), w\rangle$ is a continuous operator for all $t \in (0,T)$ and all $u, v, w \in W_0^{1,p}(Q)$;
  - (ii) There exists a constant $\gamma > 0$ such that $\langle B(t,u), u\rangle \geq \gamma \|u\|^p_{W_0^{1,p}(Q)}$ for *a.e.* $t \in (0,T)$ and all $u \in W_0^{1,p}(Q)$;
  - (iii) There exists a positive constant β such that $\|B(t,u)\|_{W^{-1,p'}(Q)} \leq \beta \|u\|^{p-1}_{W_0^{1,p}(Q)}$ for *a.e.* $t \in (0,T)$ and all $u \in W_0^{1,p}(Q)$;
  - (iv) $\langle B(t,u) - B(t,v), u - v\rangle \geq 0$, for *a.e.* $t \in (0,T)$ and all $u, v \in W_0^{1,p}(Q)$;
  - (v) The map $t \longrightarrow B(t,u)$ is Lebesgue measurable in $(0,T)$ with values in $W^{-1,p'}(Q)$ for all $u \in W_0^{1,p}(Q)$.

(A3) We assume that $f(t,x,y,w)$ is measurable with respect to (x,y) for any $(t,w) \in (0,T) \times \mathbb{R}^n$, continuous with respect to (t,w) for almost every $(x,y) \in Q \times Y$, and Y-periodic with respect to y. Also there exists an $\mathbb{R}^n$-valued function $F = (F_i(t,x,y))_{1\leq i\leq n}$ such that $f(t,x,y,w) = F(t,x,y)\cdot w$. Furthermore,

$$\left|\left|f\left(t, x, \frac{x}{\varepsilon}, w\right)\right|\right|_{L^2(Q)} \leq C\, ||w||_{L^2(Q)},$$

for any $(t, w, \varepsilon) \in (0,T) \times L^2(Q) \times (0,\infty)$, with the constant C independent of ε and t. A sufficient requirement for this condition to hold is that $F_i(t,\cdot) \in L^\infty(Q \times Y)$ for any $t \in (0,T)$.

(A4) $a^\epsilon(x) \in H_0^1(Q)$, $b^\epsilon(x) \in L^2(Q)$, for any $\epsilon > 0$.

(A5) $g(t,x,y,\phi)$ is an m-dimensional vector-function whose components $g_j(t,x,y,\phi)$ satisfy the following conditions:
  - $g_j(t,x,y,\phi)$ is Y-periodic with respect to y, measurable with respect to x and y for almost all $t \in (0,T)$ and for all $\phi \in L^2(Q)$,
  - $g_j(t,x,y,\phi)$ is continuous with respect to ϕ for almost all $(t,x,y) \in (0,T) \times Q \times Y$, and there exists a positive constant C independent of t, x and y, such

that

$$||g_j(t,x,y,\phi)||_{L^2(Q)} \leq C\left(1+||\phi||_{L^2(Q)}\right), \tag{1}$$

and

- $g_j(t,x,y,\cdot)$ satisfies Lipschitz's condition

$$|g_j(t,x,y,s_1) - g_j(t,x,y,s_2)| \leq L|s_1 - s_2|, \tag{2}$$

with the constant L independent of t, x and y.

If $||g_j(t,x,y,0)||_{L^2(Q\times Y)} < \infty$ for any $i = 1,...,m$ and any $t \in (0,T)$, the condition (1) is redundant since it follows from the Lipschitz condition (2).

From now on we use the following oscillating functions

$$f^\epsilon(t,x,w) = f\left(t,x,\frac{x}{\varepsilon},w\right),\ g_j^\varepsilon(t,x,\phi) = g_i\left(t,x,\frac{x}{\varepsilon},\phi\right).$$

We now introduce our notion of solution; namely the strong probabilistic one.

Definition 1.1. We define the strong probabilistic solution of the problem (P_ϵ) on the prescribed filtered probability space $\left(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\in[0,T]}\right)$ as a process

$$u^\epsilon : \Omega\times[0,T] \longrightarrow H_0^1(Q),$$

satisfying the following conditions:

(1) u^ϵ, u_t^ϵ are $\mathcal{F}_t-$measurable,

(2)

$$u^\epsilon \in L^2\left(\Omega,\mathcal{F},\mathbb{P};C(0,T;H_0^1(Q))\right)$$
$$u_t^\epsilon \in L^2\left(\Omega,\mathcal{F},\mathbb{P};C(0,T;L^2(Q))\right)\cap L^p\left(\Omega,\mathcal{F},\mathbb{P};L^p(0,T;W_0^{1,p}(Q))\right),$$

(3) $\forall t\in[0,T]$, $u^\epsilon(t,.)$ satisfies the identity

$$(u_t^\epsilon(t,.),\phi) - (u_t^\epsilon(0,.),\phi) + \int_0^t (A_\epsilon\nabla u^\epsilon(s,.),\nabla\phi)ds + \int_0^t \langle B^\epsilon(s,u_t^\epsilon),\phi\rangle ds$$
$$= \int_0^t (f^\epsilon(s,.,\nabla u^\epsilon),\phi)\,ds + \left(\int_0^t g^\epsilon(s,.,u_t^\epsilon)\,dW(s),\phi\right),$$

$\forall\phi\in C_c^\infty(Q)$.

The problem of existence and uniqueness of a strong probabilistic solution of (P_ϵ) was dealt with in [38]. The relevant result is

Theorem 1.2. *Suppose that the assumptions* $(A1)-(A5)$ *hold and let* $p\geq 2$. *Then for fixed* $\epsilon>0$, *the problem* (P_ϵ) *has a unique strong probabilistic solution* u^ϵ *in the sense of Definition 1.1.*

Our goal is to show that as ϵ tends to zero the sequence of solutions (u^ϵ) converge in a suitable sense to a solution u of the following SPDE

$$(P)\begin{cases} du_t - \mathrm{div}A_0\nabla u dt + B(t,u_t)dt = \tilde{f}(t,x,\nabla u)dt + \tilde{g}(t,x,u_t)d\tilde{W} \text{ in } Q\times(0,T), \\ u = 0 \text{ on } \partial Q\times(0,T), \\ u(0,x) = a(x)\in H_0^1(Q), u_t(0,x) = b(x)\in L^2(Q), \end{cases}$$

where A_0 is a constant elliptic matrix defined by

$$A_0 = \frac{1}{|Y|}\int_Y (A(y) - A(y)\chi(y))dy,$$

$\chi(y) \in H_{\rm per}(Y)$ is the unique solution of the following boundary value problem:

$$\left\{ \begin{array}{c} {\rm div}_y(A(y)\nabla_y \chi(y)) = \nabla_y \cdot A(y) \text{ in } Y \\ \chi \text{ is } Y \text{ periodic,} \end{array} \right.$$

for any $\lambda \in \mathbb{R}^n$ and $Y = (0, l_1) \times ... \times (0, l_n)$,

$$\tilde{f}(t, x, \nabla u) = \frac{1}{|Y|} \int_Y F(t, x, y) \cdot [\nabla_x u(t, x) + \nabla_y u_1(t, x, y)] dy,$$

$$\tilde{g}\left(t, x, u_t\right) = \frac{1}{|Y|} \int_Y g\left(t, x, y, u_t\right) dy,$$

a and b are suitable limits of the oscillating initial conditions a^ϵ and b^ϵ, respectively, $\tilde{W}$ is an m-dimensional Wiener process

2. **A priori estimates.** Here and in the sequel, C will denote a constant independent of ϵ. In the following lemma, we obtain the energy estimates associated to problem (P_ϵ).

Lemma 2.1. *Under the assumptions* $(A1)$-$(A5)$*, the solution* u^ϵ *of* (P_ϵ) *satisfies the following estimates:*

$$\mathbb{E} \sup_{0 \le t \le T} \|u^\epsilon(t)\|^2_{H^1_0(Q)} \le C, \mathbb{E} \sup_{0 \le t \le T} \|u^\epsilon_t(t)\|^2_{L^2(Q)} \le C, \tag{3}$$

and

$$\mathbb{E} \int_0^T \|u^\epsilon_t(t)\|^p_{W^{1,p}_0(Q)} \le C. \tag{4}$$

Proof. The following arguments are used modulo appropriate stopping times. Itô's formula and the symmetry of A give

$$d[\|u^\epsilon_t\|^2_{L^2(Q)} + (A_\epsilon \nabla u^\epsilon, \nabla u^\epsilon)] + 2\langle B(t, u^\epsilon_t), u^\epsilon_t)\rangle dt$$
$$= 2(f^\epsilon(t, x, \nabla u^\epsilon)), u^\epsilon_t)dt + 2(g^\epsilon(t, x, u^\epsilon_t), u^\epsilon_t)dW + \sum_{j=0}^m \|g^\epsilon_j(t, x, u^\epsilon_t)\|^2_{L^2(Q)} dt.$$

Integrating over $(0, t)$, $t \le T$, we get

$$\begin{aligned} &\|u^\epsilon_t(t)\|^2_{L^2(Q)} + (A_\epsilon \nabla u^\epsilon(t), \nabla u^\epsilon(t)) + 2\int_0^t \langle B(s, u^\epsilon_t(s)), u^\epsilon_t(s))\rangle ds \\ &= \|u^\epsilon_1\|^2_{L^2(Q)} + (A_\epsilon \nabla u^\epsilon_0, \nabla u^\epsilon_0) \\ &+ 2\int_0^t (f^\epsilon(s, x, \nabla u^\epsilon), u^\epsilon_t) ds + 2\int_0^t (g^\epsilon(s, x, u^\epsilon_t), u^\epsilon_t) dW \\ &+ \sum_{j=0}^m \int_0^t \|g^\epsilon_j(s, x, u^\epsilon_t)\|^2_{L^2(Q)} ds. \end{aligned}$$

Using the assumptions $(A1)$, $(A2)(ii)$, $(A5)$ and taking the supremum over $t \in [0, T]$ and the expectation on both sides of the resulting relation, we have

$$\mathbb{E}\left[\sup_{0 \le t \le T} \|u^\epsilon_t(t)\|^2_{L^2(Q)} + \sup_{0 \le t \le T} \|u^\epsilon(t)\|^2_{H^1_0(Q)} + 2\gamma \int_0^t \|u^\epsilon_t(s)\|^p_{W^{1,p}_0(Q)} ds\right] \tag{5}$$
$$\le C\left[C_1 + \int_0^t \|u^\epsilon_t(t)\|^2_{L^2(Q)} dt + 2\int_0^t |(f^\epsilon(s, x, \nabla u^\epsilon), u^\epsilon_t)| ds\right.$$

$$+2\sup_{0\le s\le t}\left|\int_0^s (g^\epsilon(\sigma,x,u_t^\epsilon),u_t^\epsilon)dW\right|\Bigg],$$

where

$$C_1 = C(T) + \|u_1^\epsilon\|^2_{L^2(Q)} + \|u_0^\epsilon\|^2_{H_0^1(Q)}.$$

Using assumptions (A3), thanks to Cauchy-Schwarz's and Young's inequalities, we have

$$\begin{aligned}\mathbb{E}\int_0^T |(f^\epsilon(s,x,\nabla u^\epsilon),u_t^\epsilon)|dt &\le \mathbb{E}\int_0^T \|\nabla u^\epsilon\|_{L^2(Q)}\|u_t^\epsilon\|_{L^2(Q)}dt \\ &\le \mathbb{E}\sup_{0\le t\le T}\|u_t^\epsilon(t)\|_{L^2(Q)}\int_0^T \|\nabla u^\epsilon\|_{L^2(Q)}dt \qquad (6)\\ &\le \varrho\mathbb{E}\sup_{0\le t\le T}\|u_t^\epsilon(t)\|^2_{L^2(Q)} + C(\varrho)T\left(\int_0^T \|\nabla u^\epsilon\|^2_{L^2(Q)}dt\right),\end{aligned}$$

where $\varrho > 0$. Thanks to Burkholder-Davis-Gundy's inequality, followed by Cauchy-Schwarz's inequality, the last term in 5 can be estimated as

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s (g^\epsilon(\sigma,x,u_t^\epsilon(\sigma)),u_t^\epsilon(\sigma))dW(\sigma)\right| \\ &\le C\mathbb{E}\left(\int_0^t (g^\epsilon(\sigma,x,u_t^\epsilon(\sigma)),u_t^\epsilon(\sigma))^2 d\sigma\right)^{\frac12} \\ &\le C\mathbb{E}\left(\sup_{0\le s\le t}\|u_t^\epsilon(s)\|_{L^2(Q)}\int_0^t \|g^\epsilon(\sigma,x,u_t^\epsilon(\sigma))\|^2_{L^2(Q)}d\sigma\right)^{\frac12}.\end{aligned}$$

Again using Young's inequality and the assumptions $(A5)$, we get

$$\begin{aligned}&2\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s (g^\epsilon(\sigma,x,u_t^\epsilon(\sigma)),u_t^\epsilon(\sigma))dW\right| \\ &\le \varrho\mathbb{E}\sup_{0\le s\le t}\|u_t^\epsilon(s)\|^2_{L^2(Q)} + C(\varrho)\int_0^T \|g^\epsilon(\sigma,u_t^\epsilon(\sigma))\|^2_{L^2(Q)}d\sigma \\ &\le \varrho\mathbb{E}\sup_{0\le s\le t}\|u_t^\epsilon(s)\|^2_{L^2(Q)} + C(\varrho)(T) + C(\varrho)\int_0^T \|u_t^\epsilon(\sigma)\|^2_{L^2(Q)}d\sigma, \qquad (7)\end{aligned}$$

for $\varrho > 0$. Combining the estimates 6, 7, 5 and assumption $(A5)$ and taking ϱ sufficiently small, we infer that

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}\|u_t^\epsilon(t)\|^2_{L^2(Q)} + \mathbb{E}\sup_{0\le t\le T}\|u^\epsilon(t)\|^2_{H_0^1(Q)} + C\mathbb{E}\int_0^t \|u_t^\epsilon(s)\|^p_{W_0^{1,p}(Q)}ds \\ &\le C(T,C_1,C_2) + C\mathbb{E}\int_0^t \left[\|u_t^\epsilon(s)\|^2_{L^2(Q)} + \|u^\epsilon(s)\|^2_{H_0^1(Q)}\right]dt, \qquad (8)\end{aligned}$$

Using Gronwall's inequality, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}\|u_t^\epsilon(t)\|^2_{L^2(Q)} + \sup_{0\le t\le T}\|u^\epsilon(t)\|^2_{H_0^1(Q)}\right] \le C,$$

and subsequently

$$\mathbb{E}\int_0^t \|u_t^\epsilon(s)\|^p_{W_0^{1,p}(Q)}ds \le C.$$

The proof is complete. □

The following lemma will be of great importance in proving the tightness of probability measures generated by the solution of problem (P_ϵ) and its time derivative.

Lemma 2.2. *Let the conditions of Lemma 2.1 be satisfied and let* $p \geq 2$. *Then there exists a constant* $C > 0$ *such that*

$$\mathbb{E} \sup_{|\theta| \leq \delta} \int_0^T \|u_t^\epsilon(t+\theta) - u_t^\epsilon(t)\|_{W^{-1,p'}(Q)}^{p'} dt \leq C\delta^{p'/p},$$

for any $\epsilon > 0$ *and* $0 < \delta < 1$.

Proof. . We consider that $div\,(A_\epsilon \nabla \phi)$ has been restricted to the space $W^{-1,p'}(Q)$ and that the restriction induces a bounded mapping from $W_0^{1,p}(Q)$ to $W^{-1,p'}(Q)$.

Assume that u_t^ϵ is extended by zero outside the interval $[0, T]$ and that $\theta > 0$. We write

$$\begin{aligned} u_t^\epsilon(t+\theta) - u_t^\epsilon(t) &= \int_t^{t+\theta} div(A_\epsilon \nabla u^\epsilon) ds - \int_t^{t+\theta} B(s, u_t^\epsilon(s)) ds \\ &\quad + \int_t^{t+\theta} f^\epsilon(s, x, \nabla u^\epsilon) ds + \int_t^{t+\theta} g^\epsilon(s, u_t^\epsilon(s)) dW(s). \end{aligned}$$

Then

$$\begin{aligned} \|u_t^\epsilon(t+\theta) - u_t^\epsilon(t)\|_{W^{-1,p'}(Q)} \leq &\left\| \int_t^{t+\theta} \text{div}(A_\epsilon \nabla u^\epsilon) ds \right\|_{W^{-1,p'}(Q)} \\ &+ \left\| \int_t^{t+\theta} B(s, u_t^\epsilon(s)) ds \right\|_{W^{-1,p'}(Q)} \\ &+ \left\| \int_t^{t+\theta} f^\epsilon(s, x, \nabla u^\epsilon) ds \right\|_{W^{-1,p'}(Q)} \\ &+ \left\| \int_t^{t+\theta} g^\epsilon(s, u_t^\epsilon(s)) dW(s) \right\|_{W^{-1,p'}(Q)}. \end{aligned} \tag{9}$$

Firstly, thanks to assumption $(A1)$, we have

$$\begin{aligned} &\left\| \int_t^{t+\theta} \text{div}(A_\epsilon \nabla u^\epsilon) ds \right\|_{W^{-1,p'}(Q)} \\ &\leq \sup_{\phi \in W_0^{1,p}(Q): \|\phi\|=1} \left| \langle \int_t^{t+\theta} \text{div}(A_\epsilon \nabla u^\epsilon) ds, \phi \rangle_{W^{-1,p'}(Q), W_0^{1,p}(Q)} \right| \\ &= \sup_{\phi \in W_0^{1,p}(Q): \|\phi\|=1} \int_Q \int_t^{t+\theta} A_\epsilon \nabla u^\epsilon \nabla \phi dx ds \\ &\leq C \sup_{\phi \in W_0^{1,p}(Q): \|\phi\|=1} \int_t^{t+\theta} \|\nabla u^\epsilon\|_{L^{p'}(Q)} \|\nabla \phi\|_{L^p(Q)} ds \\ &\leq C \int_t^{t+\theta} \|\nabla u^\epsilon\|_{L^2(Q)} ds \leq C\theta^{1/2} \left(\int_t^{t+\theta} \|\nabla u^\epsilon\|_{L^2(Q)}^2 ds \right)^{1/2}, \end{aligned} \tag{10}$$

where we have used the fact that $p' \leq 2$.

Secondly, we use assumption $(A2)(iii)$, estimate 4 and Hölder's inequality to get

$$\begin{aligned}
&\left\|\int_t^{t+\theta} B(s,u_t^\epsilon(s))ds\right\|_{W^{-1,p'}(Q)} \\
&\qquad \le \sup_{\phi\in W_0^{1,p}(Q):\|\phi\|=1}\left|\langle\int_t^{t+\theta} B(s,u_t^\epsilon(s))ds,\phi\rangle_{W^{-1,p'}(Q),W_0^{1,p}(Q)}\right| \\
&\qquad \le \sup_{\phi\in W_0^{1,p}(Q):\|\phi\|=1}\int_t^{t+\theta}\|B(s,u_t^\epsilon(s))\|_{W^{-1,p'}(Q)}\|\phi\|_{W_0^{1,p}(Q)}ds \qquad (11)\\
&\qquad \le C\theta^{1/p}\left(\int_t^{t+\theta}\|u_t^\epsilon\|^p_{W_0^{1,p}(Q)}ds\right)^{1/p'}.
\end{aligned}$$

Thirdly,

$$\begin{aligned}
&\left\|\int_t^{t+\theta} f^\epsilon(s,x,\nabla u^\epsilon)ds\right\|_{W^{-1,p'}(Q)} \\
&\le \left\|\int_t^{t+\theta} f^\epsilon(s,x,\nabla u^\epsilon)ds\right\|_{L^2(Q)} \\
&\le C\int_t^{t+\theta}\|\nabla u^\epsilon\|_{L^2(Q)} \le \theta^{1/2}\left(\int_t^{t+\theta}\|\nabla u^\epsilon\|^2_{L^2(Q)}ds\right)^{1/2}, \qquad (12)
\end{aligned}$$

where we have used assumption (A3).

Using 10, 11 and 12 in 9 raised to the power p', for fixed $\delta>0$, we get

$$\begin{aligned}
&\mathbb{E}\sup_{0<\theta\le\delta}\int_0^T\|u_t^\epsilon(t+\theta)-u_t^\epsilon(t)\|^{p'}_{W^{-1,p'}(Q)}dt \qquad (13)\\
&\qquad \le C\mathbb{E}\sup_{0<\theta\le\delta}\theta^{p'/2}\int_0^T\left(\int_t^{t+\theta}\|\nabla u^\epsilon\|^2_{L^2(Q)}ds\right)^{p'/2}dt \\
&\qquad + C\mathbb{E}\sup_{0<\theta\le\delta}\theta^{p'/p}\int_0^T\int_t^{t+\theta}\|u_t^\epsilon\|^p_{W_0^{1,p}(Q)}dsdt \\
&\qquad + \mathbb{E}\sup_{0<\theta\le\delta}\int_0^T\left\|\int_t^{t+\theta}g^\epsilon(s,u_t^\epsilon(s)dW(s)\right\|^{p'}_{W^{-1,p'}(Q)}dt.
\end{aligned}$$

We now estimate the term involving the stochastic integral.

We use the embedding

$$W_0^{1,p}(Q)\hookrightarrow L^2(Q)\hookrightarrow W^{-1,p'}(Q)$$

to get the estimate

$$\begin{aligned}
&\mathbb{E}\sup_{0<\theta\le\delta}\int_0^T\left\|\int_t^{t+\theta}g^\epsilon(s,u_t^\epsilon(s)dW(s)\right\|^{p'}_{W^{-1,p'}}dt \\
&\qquad \le \mathbb{E}\sup_{0<\theta\le\delta}\int_0^T\left\|\int_t^{t+\theta}g^\epsilon(s,u_t^\epsilon(s)dW(s)\right\|^{p'}_{L^2(Q)}dt. \qquad (14)
\end{aligned}$$

Thanks to Fubini's theorem and Hölder's inequality, we have

$$\mathbb{E}\int_0^T \sup_{0<\theta\le\delta}\left|\left|\int_t^{t+\theta} g^\epsilon(s,u_t^\epsilon(s)dW(s)\right|\right|_{L^2(Q)}^{p'} dt$$

$$\le \int_0^T\left(\int_Q \mathbb{E}\sup_{0<\theta\le\delta}\left(\int_t^{t+\theta} g^\epsilon\left(s,u_t^\epsilon(s)\right)dW(s)\right)^2 dx\right)^{p'/2} dt \tag{15}$$

$$\le \int_0^T\left(\mathbb{E}\int_t^{t+\delta}||g^\epsilon(s,u_t^\epsilon(s)||_{L^2(Q)}^2\,ds\right)^{p'/2} dt,$$

where we have used Burkholder-Davis-Gundys inequality. We now invoke assumption $(A5)$ and estimate 3 to deduce from 14 and 15 that

$$\mathbb{E}\sup_{0<\theta\le\delta}\int_0^T\left|\left|\int_t^{t+\theta} g^\epsilon(s,u_t^\epsilon(s)dW(s)\right|\right|_{W^{-1,p'}}^{p'} dt \tag{16}$$

$$\le \int_0^T\left[\mathbb{E}\int_t^{t+\delta}\left(1+||u_t^\epsilon(s)||_{L^2(Q)}^2\right)ds\right]^{p'/2} dt \le CT\delta^{p'/2}.$$

For the first term in the right-hand side of 13, we use Fubini's theorem, Hölder's inequality and estimate 3 to get

$$\mathbb{E}\sup_{0<\theta\le\delta}\theta^{p'/2}\int_0^T\left(\int_t^{t+\theta}\|\nabla u^\epsilon\|_{L^2(Q)}^2 ds\right)^{p'/2}$$

$$\le \delta^{p'/2}\int_0^T\left(\mathbb{E}\int_t^{t+\delta}\|\nabla u^\epsilon\|_{L^2(Q)}^2 ds\right)^{p'/2} \tag{17}$$

$$\le CT\delta^{p'}.$$

The second term on the right hand side of 13 is estimated using 4 and we get

$$\mathbb{E}\sup_{0<\theta\le\delta}\theta^{p'/p}\int_0^T\int_t^{t+\theta}\|u_t^\epsilon\|_{W_0^{1,p}(Q)}^p dsdt$$

$$\le \delta^{p'/p}\int_0^T\mathbb{E}\int_0^T\|u_t^\epsilon\|_{W_0^{1,p}(Q)}^p dsdt \le C\delta^{p'/p}. \tag{18}$$

Combining 13, 16, 17 and 18, and taking into account the fact that the similar estimates hold for $\theta<0$, we conclude that

$$\mathbb{E}\sup_{|\theta|\le\delta}\int_0^T\|u_t^\epsilon(t+\theta)-u_t^\epsilon(t)\|_{W^{-1,p'}(Q)}^{p'} dt \le C\delta^{p'/p}.$$

This completes the proof. □

3. **Tightness property of probability measures.** The following Lemmas are needed in the proof of the tightness and the study of the properties of the probability measures generated by the sequence $(W,u^\epsilon,u_t^\epsilon)$.

We have from [45]

Lemma 3.1. *Let B_0, B and B_1 be some Banach spaces such that $B_0\subset B\subset B_1$ and the injection $B_0\subset B$ is compact. For any $1\le p,q\le\infty$, and $0<s\le1$ let E be a set bounded in $L^q(0,T;B_0)\cap N^{s,p}(0,T;B_1)$, where*

$$N^{s,p}(0,T;B_1)=\left\{v\in L^p(0,T;B_1):\sup_{h>0}h^{-s}\|v(t+h)-v(t)\|_{L^p(0,T-\theta,B_1)}<\infty\right\}.$$

Then E is relatively compact in $L^p(0,T;B)$

The following two lemmas are collected from [12]. Let $\mathcal{S}$ be a separable Banach space and consider its Borel σ-field to be $\mathcal{B}(\mathcal{S})$. We have

Lemma 3.2. ***(Prokhorov)*** *A sequence of probability measures $(\Pi_n)_{n\in\mathbb{N}}$ on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ is tight if and only if it is relatively compact.*

Lemma 3.3. ***(Skorokhod)*** *Suppose that the probability measures $(\mu_n)_{n\in\mathbb{N}}$ on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ weakly converge to a probability measure μ. Then there exist random variables $\xi, \xi_1, \dots \xi_n, \dots,$ defined on a common probability space $(\Omega, \mathcal{F}, P)$, such that $\mathcal{L}(\xi_n) = \mu_n$ and $\mathcal{L}(\xi) = \mu$ and*

$$\lim_{n\to\infty} \xi_n = \xi, \quad P - a.s.;$$

the symbol $\mathcal{L}(\cdot)$ stands for the law of $\cdot$.

Let us introduce the space $Z = Z_1 \times Z_2$, where

$$Z_1 = \left\{ \phi : \sup_{0\le t\le T} \|\phi(t)\|^2_{H^1_0(Q)} \le C_1, \quad \sup_{0\le t\le T} \|\phi'(t)\|^2_{L^2(Q)} \le C_1 \right\},$$

and

$$Z_2 = \Bigg\{ \psi : \sup_{0\le t\le T} \|\psi(t)\|^2_{L^2(Q)} \le C_3, \ \int_0^T \|\psi(t)\|^p_{W^{1,p}_0(Q)} dt \le C_4 \ ,$$
$$\int_0^T \|\psi(t+\theta) - \psi(t)\|^{p'}_{W^{-1,p'}(Q)} \le C_5 \theta^{1/p} \Bigg\}.$$

We endow Z with the norm

$$\begin{aligned} \|(\phi,\psi)\|_Z = \|\phi\|_{Z_1} + \|\psi\|_{Z_2} &= \sup_{0\le t\le T} \|\phi'(t)\|_{L^2(Q)} + \sup_{0\le t\le T} \|\phi\|_{H^1_0(Q)} \\ &+ \sup_{0\le t\le T} \|\psi(t)\|^2_{L^2(Q)} + \left(\int_0^T \|\psi(t)\|^p_{W^{1,p}_0(Q)} dt \right)^{\frac{1}{p}} \\ &+ \left(\sup_{\theta>0} \frac{1}{\theta^{1/p}} \int_0^T \|\psi(t+\theta) - \psi(t)\|^{p'}_{W^{-1,p'}(Q)} \right)^{\frac{1}{p'}}. \end{aligned}$$

Lemma 3.4. *The above constructed space Z is a compact subset of $L^2(0,T;L^2(Q)) \times L^2(0,T;L^2(Q))$.*

Proof. Lemma 3.1 together with suitable arguments due to Bensoussan [7] give the compactness of Z_1 and Z_2 in $L^2(0,T;L^2(Q))$. □

We now consider the space $\mathcal{X} = C(0,T;\mathbb{R}^m) \times L^2(0,T;L^2(Q)) \times L^2(0,T;L^2(Q))$ and $\mathcal{B}(\mathcal{X})$ the $\sigma-$algebra of its Borel sets. Let Ψ_ϵ be the $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued measurable map defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Psi_\epsilon : \omega \mapsto (W(\omega), u^\epsilon(\omega), u^\epsilon_t(\omega)).$$

Define on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ the family of probability measures (Π_ϵ) by

$$\Pi_\epsilon(A) = \mathbb{P}(\Psi_\epsilon^{-1}(A)) \ \text{ for all } \ A \in \mathcal{B}(\mathcal{X}).$$

Lemma 3.5. *The family of probability measures $\{\Pi_\epsilon : \epsilon > 0\}$ is tight in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$.*

Proof. We carry out the proof following a long the lines of the proof of [27, lemma 7]. For $\delta > 0$, we look for compact subsets

$$W_\delta \subset C(0,T;\mathbb{R}^m),\ \ D_\delta \subset L^2(0,T;L^2(Q)),\ \ E_\delta \subset L^2(0,T;L^2(Q))$$

such that

$$\Pi_\epsilon\big\{(W,u^\epsilon,u^\epsilon_t) \in W_\delta \times D_\delta \times E_\delta\big\} \geq 1-\delta.$$

This is equivalent to

$$\mathbb{P}\big\{\omega : W(\cdot,\omega) \in W_\delta, u^\epsilon(\cdot,\omega) \in D_\delta, u^\epsilon_t)(\cdot,\omega) \in E_\delta\big\} \geq 1-\delta,$$

which can be proved if we can show that

$$\mathbb{P}\big\{\omega : W(\cdot,\omega) \notin W_\delta\big\} \leq \delta,\ \ \mathbb{P}\{u^\epsilon(\cdot,\omega) \notin D_\delta\} \leq \delta,\ \ \mathbb{P}\{u^\epsilon_t)(\cdot,\omega). \notin E_\delta\} \leq \delta.$$

Let L_δ be a positive constant and $n \in \mathbb{N}$. Then we deal with the set

$$W_\delta = \{W(\cdot) \in C(0,T;\mathbb{R}^m) : \sup_{t,s\in[0,T]} n|W(s)-W(t)| \leq L_\delta : |s-t| \leq Tn^{-1}\}.$$

Using Arzela's theorem and the fact that W_δ is closed in $C(0,T;\mathbb{R}^m)$, we ensure the compactness of W_δ in $C(0,T;\mathbb{R}^m)$. From Markov's inequality

$$\mathbb{P}(\omega : \eta(\omega) \geq \alpha) \leq \frac{\mathbb{E}|\eta(\omega)|^k}{\alpha^k}, \tag{19}$$

where η is a nonnegative random variable and k a positive real number, we have

$$\mathbb{P}\big\{\omega : W(\cdot,\omega) \notin W_\delta\big\} \leq \mathbb{P}\Bigg[\bigcup_{n=1}^{\infty}\Bigg(\sup_{t,s\in[0,T]} |W(s)-W(t)| \geq \frac{L_\delta}{n} : |s-t| \leq Tn^{-1}\Bigg)\Bigg]$$

$$\leq \sum_{n=0}^{\infty}\mathbb{P}\Bigg[\bigcup_{j=1}^{n^6}\Bigg(\sup_{Tjn^{-6}\leq t\leq T(j+1)n^{-6}} |W(s)-W(t)| \geq \frac{L_\delta}{n}\Bigg)\Bigg].$$

But

$$\mathbb{E}\left(W_i(t)-W_i(s)\right)^{2k} = (2k-1)!!(t-s)^k,\ \ k=1,2,3,\dots,$$

where $(2k-1)!! = 1\cdot 3\cdots(2k-1)$ and W_i denotes the i-th component of W. For $k=4$, we have

$$\begin{aligned}
&\mathbb{P}\left\{\omega : W(.,\omega) \notin W_\delta\right\} \\
&\leq \sum_{n=0}^{\infty}\sum_{j=1}^{n^6}\left(\frac{n}{L_\delta}\right)^4 \mathbb{E}\left(\sup_{Tjn^{-6}\leq t\leq T(j+1)n^{-6}} \left|W(t)-W(jTn^{-6})\right|^4\right) \\
&\leq C\sum_{n=0}^{\infty}\sum_{j=1}^{n^6}\left(\frac{n}{L_\delta}\right)^4 \left(Tn^{-6}\right)^2 = \frac{CT^2}{(L_\delta)^4}\sum_{n=0}^{\infty} n^{-2}.
\end{aligned}$$

Choosing $(L_\delta)^4 = \dfrac{(\sum n^{-2})^{-1}}{3CT^2\delta}$, we have

$$\mathbb{P}\left\{\omega : W(.,\omega) \notin W_\delta\right\} \leq \frac{\delta}{3}.$$

Now, let $K_\delta,\ M_\delta$ be positive constants. We define

$$D_\delta = \big\{z : \sup_{0\leq t\leq T}\|z(t)\|^2_{H^1_0(Q)} \leq K_\delta,\ \ \sup_{0\leq t\leq T}\|z'(t)\|^2_{L^2(Q)} \leq M_\delta\big\}.$$

Lemma 3.4 shows that D_δ is compact subset of $L^2(0,T;L^2(Q))$, for any $\delta>0$. It is therefore easy to see that

$$\mathbb{P}\{u^\epsilon\notin D_\delta\}\le\mathbb{P}\big\{\sup_{0\le t\le T}\|u^\epsilon(t)\|^2_{H^1_0(Q)}\ge K_\delta\big\}+\mathbb{P}\big\{\sup_{0\le t\le T}\|u^\epsilon_t(t)\|^2_{L^2(Q)}\ge M_\delta\big\}.$$

Markov's inequality 19 gives

$$\mathbb{P}\{u^\epsilon\notin D_\delta\}\le\frac{1}{K_\delta}\mathbb{E}\sup_{0\le t\le T}\|u^\epsilon(t)\|^2_{H^1_0(Q)}+\frac{1}{M_\delta}\mathbb{E}\sup_{0\le t\le T}\|u^\epsilon_t(t)\|^2_{L^2(Q)}\le\frac{C}{K_\delta}+\frac{C}{M_\delta}=\frac{\delta}{3}.$$

for $K_\delta=M_\delta=\frac{6C}{\delta}$.

Similarly, we let $\mu_n,\ \nu_m$ be sequences of positive real numbers such that $\mu_n,\ \nu_n\to 0$ as $n\to\infty$, $\sum_n\frac{\mu_n^{p'/p}}{\nu_n}<\infty$ (for the series to converge we can choose $\nu_n=1/n^2$, $\mu_n=1/n^\alpha$, with $\alpha p'/p>4$) and define

$$B_\delta=\Bigg\{v:\sup_{0\le t\le T}\|v(t)\|^2_{L^2(Q)}\le K'_\delta,\ \int_0^T\|v(t)\|^p_{W^{1,p}_0(Q)}dt\le L'_\delta,$$
$$\sup_{\theta\le\mu_n}\int_0^T\|v(t+\theta)-v(t)\|^{p'}_{W^{-1,p'}(Q)}dt\le\nu_nM'_\delta\Bigg\}.$$

Owing to Proposition 3.1 in [7], B_δ is a compact subset of $L^2(0,T;L^2(Q))$ for any $\delta>0$. We have

$$\mathbb{P}\{u^\epsilon_t\notin B_\delta\}\le\mathbb{P}\left\{\sup_{0\le t\le T}\|u^\epsilon_t(t)\|^2_{L^2(Q)}\ge K'_\delta\right\}+\mathbb{P}\left\{\int_0^T\|u^\epsilon_t(t)\|^p_{W^{1,p}_0(Q)}dt\ge L'_\delta\right\}$$
$$+\,\mathbb{P}\left\{\sup_{\theta\le\mu_n}\int_0^T\|u^\epsilon_t(t+\theta)-u^\epsilon_t(t)\|^{p'}_{W^{-1,p}(Q)}dt\ge\nu_nM'_\delta\right\}.$$

Again thanks to 19, we obtain

$$\begin{aligned}&\mathbb{P}\{u^\epsilon_t\notin B_\delta\}\\&\quad\le\tfrac{1}{K'_\delta}\mathbb{E}\sup\nolimits_{0\le t\le T}\|u^\epsilon_t(t)\|^2_{L^2(Q)}+\tfrac{1}{L'_\delta}\mathbb{E}\int_0^T\|u^\epsilon_t(t)\|^p_{W^{1,p}_0(Q)}dt\\&\quad+\sum\nolimits_{n=0}^\infty\tfrac{1}{\nu_nM'_\delta}\mathbb{E}\Big\{\sup\nolimits_{\theta\le\mu_n}\int_0^T\|u^\epsilon_t(t+\theta)-u^\epsilon_t(t)\|^{p'}_{W^{-1,p}(Q)}dt\Big\}\\&\quad\le\tfrac{C}{K'_\delta}+\tfrac{C}{L'_\delta}+\tfrac{C}{M'_\delta}\sum\tfrac{\mu_n^{p'/p}}{\nu_n}=\tfrac{\delta}{3}\delta,\end{aligned}$$

for $K'_\delta=\frac{9C}{\delta}$, $L'_\delta=\frac{9C}{\delta}$ and $M'_\delta=\dfrac{9C\sum\frac{\mu_n^{p'/p}}{\nu_n}}{\delta}$. This completes the proof. □

From Lemmas 3.2 and 3.5, there exist a subsequence $\{\Pi_{\epsilon_j}\}$ and a measure Π such that

$$\Pi_{\epsilon_j}\rightharpoonup\Pi$$

weakly. From lemma 3.3, there exist a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and $\mathcal{X}$-valued random variables $(W_{\epsilon_j},u^{\epsilon_j},u^{\epsilon_j}_t)$, $(\tilde W,u,u_t)$ such that the probability law of $(W_{\epsilon_j},u^{\epsilon_j},u^{\epsilon_j}_t)$ is Π_{ϵ_j} and that of $(\tilde W,u,u_t)$ is Π. Furthermore, we have

$$(W_{\epsilon_j},u^{\epsilon_j},u^{\epsilon_j}_t)\to(\tilde W,u,u_t)\text{ in }\mathcal{X},\ \tilde{\mathbb{P}}-a.s.. \tag{20}$$

Let us define the filtration

$$\tilde{\mathcal{F}}_t=\sigma\{\tilde W(s),u(s),u_t(s)\}_{0\le s\le t}.$$

We show that $\tilde{W}(t)$ is an $\tilde{\mathcal{F}}_t$-wiener process following [7] and [42]. Arguing as in [42], we get that $(W_{\epsilon_j}, u^{\epsilon_j}, u_t^{\epsilon_j})$ satisfies $\tilde{\mathbb{P}} - a.s.$ the problem $\left(P_{\epsilon_j}\right)$ in the sense of distributions.

4. **Two-scale convergence.** In this section, we state some key facts about the powerful two-scale convergence invented by Nguetseng [32].

Definition 4.1. A sequence $\{v^\epsilon\}$ in $L^p(0,T;L^p(Q))(1 < p < \infty)$ is said to be two-scale converge to $v = v(t,x,y)$, $v \in L^p(0,T;L^p(Q\times Y))$, as $\epsilon \to 0$ if for any $\psi = \psi(t,x,y) \in L^p((0,T)\times Q; C^\infty_{\text{per}}(Y))$, one has

$$\lim_{\epsilon\to 0}\int_0^T\int_Q v^\epsilon\psi^\epsilon dxdt = \frac{1}{|Y|}\int_0^T\int_{Q\times Y} v(t,x,y)\psi(t,x,y)dydxdt, \tag{21}$$

where $\psi^\epsilon(t,x) = \psi(t,x,\frac{x}{\epsilon})$. We denote this by $\{v^\epsilon\} \to v$ 2-s in $L^p(0,T;L^p(Q))$.

The following result deals with some of the properties of the test functions which we are considering; it is a modification of Lemma 9.1 from [17, p.174].

Lemma 4.2. *(i) Let $\psi \in L^p((0,T)\times Q; C_{per}(Y))$, $1 < p < \infty$. Then $\psi(\cdot,\cdot,\frac{\cdot}{\epsilon}) \in L^p(0,T;L^p(Q))$ with*

$$\left\|\psi(\cdot,\cdot,\frac{\cdot}{\epsilon})\right\|_{L^p(0,T;L^p(Q))} \leq \|\psi(\cdot,\cdot,\cdot)\|_{L^p((0,T)\times Q;C_{per}(Y))} \tag{22}$$

and

$$\psi(\cdot,\cdot,\frac{\cdot}{\epsilon}) \rightharpoonup \frac{1}{|Y|}\int_Y \psi(\cdot,\cdot,y)dy \text{ weakly in } L^p(0,T;L^p(Q)).$$

Furthermore if $\psi \in L^2((0,T)\times Q; C_{per}(Y))$, then

$$\lim_{\epsilon\to 0}\int_0^T\int_Q \left[\psi(t,x,\frac{x}{\epsilon})\right]^2 dxdt = \frac{1}{|Y|}\int_0^T\int_{Q\times Y}\left[\psi(t,x,y)\right]^2 dtdxdy. \tag{23}$$

(ii) If $\psi(t,x,y) = \psi_1(t,x)\psi_2(y)$, $\psi_1 \in L^p(0,T;L^s(Q))$, $\psi_2 \in L^r_{per}(Y)$, $1 \leq s,r < \infty$ are such that

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{p},$$

then $\psi(\cdot,\cdot,\frac{\cdot}{\epsilon}) \in L^p(0,T;L^p(Q))$ and

$$\psi(\cdot,\cdot,\frac{\cdot}{\epsilon}) \rightharpoonup \frac{\psi_1(\cdot,\cdot)}{|Y|}\int_Y \psi_2(y)dy \text{ weakly in } L^p(0,T;L^p(Q)).$$

The following theorems are of great importance in obtaining the homogenization result; for their proofs, we refer to [4], [17] and [26].

Theorem 4.3. *Let $\{u^\epsilon\}$ be a sequence of functions in $L^2\left(0,T;L^2(Q)\right)$ such that*

$$\|u^\epsilon\|_{L^2(0,T;L^2(Q))} < \infty. \tag{24}$$

Then up to a subsequence u^ϵ is two-scale convergent in $L^2\left(0,T;L^2(Q)\right)$.

Theorem 4.4. *Let $\{u^\epsilon\}$ be a sequence satisfying the assumptions of Theorem 4.3. Furthermore, let $\{u^\epsilon\} \subset L^2\left(0,T;H^1_0(Q)\right)$ be such that*

$$\|u^\epsilon\|_{L^2\left(0,T;H^1_0(Q)\right)} < \infty.$$

Then, up to a subsequence, there exists a couple of functions (u,u_1) with $u \in L^2(0,T;H^1_0(Q))$ and $u_1 \in L^2((0,T)\times Q; H_{\text{per}}(Y))$ such that

$$u^\epsilon \to u \text{ 2-s in } L^2(0,T;L^2(Q)), \tag{25}$$

$$\nabla u^{\epsilon} \to \nabla_x u + \nabla_y u_1 \quad \textit{2-s in } L^2(0,T;L^2(Q)). \tag{26}$$

The following lemma is crucial in obtaining the convergence of the stochastic integral in the next section

Lemma 4.5. *The oscillating data given in (A5) satisfies the following convergence*

$$g\left(t,x,\frac{x}{\varepsilon},u_t^{\varepsilon}\right) \rightharpoonup \tilde{g}\left(t,x,u_t\right) =: \frac{1}{|Y|}\int_Y g\left(t,x,y,u_t\right) dy \textit{ weakly in } L^2\left((0,T)\times Q\right), \quad \tilde{\mathbb{P}} - a.s.. \tag{27}$$

Proof. Test with $\psi\left(t,x,\frac{x}{\varepsilon}\right)$, where $\psi\left(t,x,y\right) \in L^2\left((0,T)\times Q, C^{\infty}_{per}\left(Y\right)\right)$, as follows:

$$\int_0^T \int_Q g\left(t,x,\frac{x}{\varepsilon},u_t^{\varepsilon}\right)\psi\left(t,x,\frac{x}{\varepsilon}\right) dxdt = I_1^{\varepsilon} + I_2^{\varepsilon},$$

where

$$\begin{aligned} I_1^{\varepsilon} &= \int_0^T \int_Q \left[g\left(t,x,\frac{x}{\varepsilon},u_t^{\varepsilon_j}\right) - g\left(t,x,\frac{x}{\varepsilon},u_t\right)\right]\psi\left(t,x,\frac{x}{\varepsilon}\right) dxdt, \\ I_2^{\varepsilon} &= \int_0^T \int_Q g\left(t,x,\frac{x}{\varepsilon},u_t\right)\psi\left(t,x,\frac{x}{\varepsilon}\right) dxdt. \end{aligned}$$

Then

$$\begin{aligned} I_1^{\varepsilon} &\leq \left|\left|\psi\left(t,x,\tfrac{x}{\varepsilon}\right)\right|\right|_{L^2((0,T)\times Q)} \left|\left|g\left(t,x,\tfrac{x}{\varepsilon},u_t^{\varepsilon}\right) - g\left(t,x,\tfrac{x}{\varepsilon},u_t\right)\right|\right|_{L^2((0,T)\times Q)} \\ &\leq C\left|\left|u_t^{\varepsilon} - u_t\right|\right|_{L^2((0,T)\times Q)}, \end{aligned}$$

thanks to the Lipschitz condition on $g\left(t,x,\cdot\right)$. Now due to the strong convergence 20 of $u_t^{\varepsilon} - u_t$ to zero in $L^2\left((0,T)\times Q\right)$, $\tilde{\mathbb{P}}$-a.s., we get that $I_1^{\varepsilon} \to 0$, $\tilde{\mathbb{P}} - a.s.$

Now we can apply 2-scale convergence for the limit of I_2^{ε} and indeed

$$\lim_{\varepsilon\to 0} I_2^{\varepsilon} = \int_0^T \int_Q \int_Y g\left(t,x,y,u_t\right)\psi\left(t,x,y\right) dxdt, \qquad \tilde{\mathbb{P}} - a.s.$$

Therefore

$$g\left(t,x,\frac{x}{\varepsilon},u_t^{\varepsilon}\right) \overset{2-s}{\to} g\left(t,x,y,u_t\right), \qquad \tilde{\mathbb{P}} - a.s. \tag{28}$$

and this implies the result. □

Remark 1. From the assumption (A5), 28 and 23, we have the following strong convergence

$$\lim_{\epsilon\to 0}\int_0^T \int_Q \left[g(t,x,\frac{x}{\epsilon},u_t^{\epsilon})\right]^2 dxdt = \frac{1}{|Y|}\int_0^T \int_{Q\times Y} \left[g(t,x,y,u_t)\right]^2 dtdxdy. \tag{29}$$

5. **The homogenization result.** We will now study the asymptotic behaviour of the problem (P_{ϵ_j}), when $\epsilon_j \to 0$.

Theorem 5.1. *Suppose that the assumptions on the data are satisfied. Let*

$$a^{\epsilon_j} \rightharpoonup a, \quad \textit{weakly in } H_0^1(Q), \tag{30}$$

$$b^{\epsilon_j} \rightharpoonup b, \quad \textit{weakly in } L^2(Q). \tag{31}$$

Then there exist a probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \left(\tilde{\mathcal{F}}_t\right)_{0\le t\le T}\right)$ *and random variables* $(u^{\epsilon_j}, u_t^{\epsilon_j}, W_{\epsilon_j})$ *and* $(u, u_t, \tilde{W})$ *such that the convergences 20 and 26 hold. Furthermore* $(u, u_t, \tilde{W})$ *satisfies the homogenized problem* (P).

Proof. From estimates 3 and 4 and assumption $(A2)(iii)$, we have the following convergences

$$u^{\epsilon_j} \rightharpoonup u \text{ weakly in } L^\infty(0,T;H_0^1(Q)) \quad \widehat{\mathbb{P}} - a.s, \tag{32}$$

$$u_t^{\epsilon_j} \rightharpoonup u_t \text{ weakly in } L^\infty(0,T;L^2(Q)) \quad \widehat{\mathbb{P}} - a.s, \tag{33}$$

$$u_t^{\epsilon_j} \rightharpoonup u_t \text{ weakly in } L^p(0,T;W_0^{1,p}(Q)) \quad \widehat{\mathbb{P}} - a.s, \tag{34}$$

$$B(t,u_t^{\epsilon_j}) \rightharpoonup \chi \text{ weakly in } L^{p'}(0,T;W^{-1,p'}(Q)) \quad \widehat{\mathbb{P}} - a.s.. \tag{35}$$

Now let us identify the limit in 35. By arguing as in [38, Lemma 2.6, p. 51], we get

$$\int_0^t \left\langle B(s,u_t^{\epsilon_j}), u_t^{\epsilon_j}\right\rangle ds \to \int_0^t \langle \chi, u_t\rangle ds, \text{ weakly in } L^1(\Omega), \ \forall t\in[0,T]. \tag{36}$$

Having this in hand, let $v \in L^p(0,T;W_0^{1,p}(Q))$ and define

$$\chi_{\epsilon_j} = \widehat{\mathbb{E}}\int_0^T \left\langle B(t,u_t^{\epsilon_j}) - B(t,v), u_t^{\epsilon_j} - v\right\rangle dt. \tag{37}$$

From the monotonicity assumption $(A2)(iv)$, we have $\chi_{\epsilon_j} \ge 0$. Now using 34, 35 and 36 to pass to the limit in 37, we get

$$\widehat{\mathbb{E}}\int_0^T \langle \chi - B(t,v), u_t - v\rangle dt \ge 0.$$

For $\lambda > 0$ and $w \in L^p(0,T;W_0^{1,p}(Q))$, we can chose $v(t) = u_t(t) - \lambda w(t)$. Hence

$$\widehat{\mathbb{E}}\int_0^T \langle \chi - B(t,u_t(t) - \lambda w(t)), w(t)\rangle dt \ge 0. \tag{38}$$

Using the hemicontinuty assumption $(A2)(i)$, we have

$$\langle \chi - B(t,u_t(t) - \lambda w(t)), w(t)\rangle \longrightarrow \langle \chi - B(t,u_t(t)), w(t)\rangle, \text{ as } \lambda \longrightarrow 0, \ \widehat{\mathbb{P}} - a.s..$$

Now, from assumptions $(A2)(ii)$ and $(A2)(v)$, we use the Lebesgue dominated convergence theorem to pass to the limit in 38. This implies

$$\widehat{\mathbb{E}}\int_0^T \langle \chi - B(t,u_t(t)), w(t)\rangle dt \ge 0. \tag{39}$$

But the inequality 39 is true for all $w(t) \in L^p(0,T;W_0^{1,p}(Q)))$. Therefore

$$\chi = B(t,u_t(t)), \quad \widehat{\mathbb{P}} - a.s..$$

Testing problem (P_{ϵ_j}) by the function $\Phi \in C_c^\infty((0,T)\times Q)$ and integrating the first term in the right-hand side by parts, we have

$$-\int_0^T\int_Q u_t^{\epsilon_j}\Phi_t(t,x)dxdt + \int_0^T\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}\nabla\Phi dxdt + \int_0^T\int_Q \langle B^{\epsilon_j}(t,u_t^{\epsilon_j}),\Phi\rangle dxdt \tag{40}$$

$$= \int_0^T\int_Q f^{\epsilon_j}(t,x,\nabla u^{\epsilon_j})\Phi dxdt + \int_0^T\int_Q g^{\epsilon_j}(t,x,u_t^{\epsilon_j})\Phi dxdW_{\epsilon_j},$$

Using estimate 3, the convergence 20 and Theorems 4.3 and 4.4, we show the two-scale convergence

$$\nabla u^{\epsilon_j} \to \nabla_x u + \nabla_y u_1 \text{ 2-s in, } L^2(0,T;L^2(Q)).$$

Let $\Phi^{\epsilon_j}(t,x) = \phi(t,x) + \epsilon_j \phi_1(t,x,\frac{x}{\epsilon_j})$, where $\phi \in C_c^\infty((0,T)\times Q)$ and $\phi_1 \in C_c^\infty((0,T)\times Q; C_{\text{per}}^\infty(Y))$. Then we can still consider Φ^{ϵ_j} as test function in 40. Thus

$$\begin{aligned}
&-\int_0^T\int_Q u_t^{\epsilon_j}(t,x)\Big[\phi_t(t,x)+\epsilon_j\phi_{1t}(t,x,\frac{x}{\epsilon_j})\Big]dxdt \\
&+\int_0^T\int_Q A_{\epsilon_j}(x)\nabla u^{\epsilon_j}(x,t)\Big[\nabla_x\phi(t,x)+\epsilon_j\nabla_x\phi_1(t,x,\frac{x}{\epsilon_j})+\nabla_y\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt \\
&+\int_0^T\int_Q\Big\langle B(t,u_t^{\epsilon_j}),\Big[\phi_t(t,x)+\epsilon_j\phi_{1t}(t,x,\frac{x}{\epsilon_j})\Big]\Big\rangle dxdt \qquad (41)\\
&=\int_0^T\int_Q f^{\epsilon_j}(t,x,\nabla u^{\epsilon_j})\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt \\
&+\int_0^T\int_Q g^{\epsilon_j}(t,u_t^{\epsilon_j})\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdW_{\epsilon_j}.
\end{aligned}$$

Let us deal with these terms one by one, when $\epsilon_j \to 0$. Thanks to estimate 22 and convergence 33, we have

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_Q u_t^{\epsilon_j}(t,x)\Big[\phi_t(t,x)+\epsilon_j\phi_{1t}(t,x,\frac{x}{\epsilon_j})\Big]dxdt \\
&=\lim_{\epsilon_j\to 0}\int_0^T\int_Q u_t^{\epsilon_j}(t,x)\phi_t(t,x)dxdt+\lim_{\epsilon_j\to 0}\epsilon_j\int_0^T\int_Q u_t^{\epsilon_j}(t,x)\phi_{1t}(t,x,\frac{x}{\epsilon_j})dxdt \\
&=\int_0^T\int_Q u_t(t,x)\phi_t(t,x)dxdt, \quad \tilde{\mathbb{P}}-a.s..
\end{aligned}$$

The second term can be written as follows,

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_Q \nabla u^{\epsilon_j}(x,t)A_{\epsilon_j}\Big[\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt \qquad (42)\\
&\quad+\lim_{\epsilon_j\to 0}\epsilon_j\int_0^T\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}(x,t)\nabla_x\phi_1(t,x,\frac{x}{\epsilon_j})dxdt.
\end{aligned}$$

Since $A_{\epsilon_j} \in L^\infty(Y)$ and $\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,y) \in L^2_{\text{per}}(Y;C(Q\times(0,T)))$, we regard $A_{\epsilon_j}[\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,\frac{x}{\epsilon_j})]$ as a test function in the two-scale limit of the gradient in the first term in 42. Therefore

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_Q \nabla u^{\epsilon_j}(x,t)A_{\epsilon_j}\Big[\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt \\
&=\frac{1}{|Y|}\int_0^T\int_{Q\times Y}A(y)[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)][\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,y)]dydxdt.
\end{aligned}$$

Thanks to Hölder inequality, 22 and the fact that $A_{\epsilon_j}\nabla u^{\epsilon_j}$ is bounded in $L^\infty(0,T;L^2(Q)$, we have

$$\lim_{\epsilon_j\to 0}\epsilon_j\int_0^T\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}(x,t)\nabla_x\phi_1(t,x,\frac{x}{\epsilon_j})dxdt = 0, \quad \tilde{\mathbb{P}}-a.s..$$

Again, thanks to estimate 22 and convergence 35, we have

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_Q \left\langle B(t,u_t^{\epsilon_j}),\left[\phi_t(t,x)+\epsilon_j\phi_{1t}(t,x,\frac{x}{\epsilon_j})\right]\right\rangle dxdt\\
&=\lim_{\epsilon_j\to 0}\int_0^T\int_Q \left\langle B(t,u_t^{\epsilon_j}),\phi_t(t,x)\right\rangle dxdt\\
&+\lim_{\epsilon_j\to 0}\epsilon_j\int_0^T\int_Q \left\langle B(t,u_t^{\epsilon_j}),\phi_{1t}(t,x,\frac{x}{\epsilon_j})\right\rangle dxdt\\
&=\int_0^T\int_Q \langle B(t,u_t),\phi_t(t,x)\rangle dxdt,\quad \tilde{\mathbb{P}}-a.s..
\end{aligned}$$

Let us write

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_Q f^{\epsilon_j}(t,x,\nabla u^{\epsilon_j})\left[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\right]dxdt\\
&=\lim_{\epsilon_j\to 0}\int_0^T\int_Q F^{\epsilon_j}(t,x)\cdot\nabla u^{\epsilon_j}\left[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\right]dxdt \qquad (43)\\
&=\lim_{\epsilon_j\to 0}\int_0^T\int_Q F^{\epsilon_j}(t,x)\cdot\nabla u^{\epsilon_j}\phi(t,x)dxdt\\
&\quad+\lim_{\epsilon_j\to 0}\epsilon_j\int_0^T\int_Q F^{\epsilon_j}(t,x).\nabla u^{\epsilon_j}\phi_1(t,x,\frac{x}{\epsilon_j})dxdt,
\end{aligned}$$

where we have used the assumption (A3). It is easy to see that the second term in 43, converges to zero. For the first term in the right-hand side of 43, we readily have

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_Q F^{\epsilon_j}(t,x)\cdot\nabla u^{\epsilon_j}\phi(t,x)dxdt\\
&=\frac{1}{|Y|}\int_0^T\int_{Q\times Y}F(t,x,y)\cdot\left[\nabla_x u+\nabla_y u_1\right]\phi(t,x)dxdydt,\quad \tilde{\mathbb{P}}-a.s.. \qquad (44)
\end{aligned}$$

Concerning the stochastic integral, we have

$$\begin{aligned}
&\tilde{\mathbb{E}}\int_0^T\int_Q g^{\epsilon_j}(t,x,u_t^{\epsilon_j})\left[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\right]dxdW_{\epsilon_j}\\
&=\tilde{\mathbb{E}}\int_0^T\int_Q g^{\epsilon_j}(t,x,u_t^{\epsilon_j})\phi(t,x)dxdW_{\epsilon_j}+\tilde{\mathbb{E}}\epsilon_j\int_0^T\int_Q g^{\epsilon_j}(t,x,u_t^{\epsilon_j})\phi_1(t,x,\frac{x}{\epsilon_j})dxdW_{\epsilon_j}. \qquad (45)
\end{aligned}$$

We deal with the term involving $\phi(t,x)$. We have

$$\begin{aligned}
&\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\,g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dW_t^\varepsilon\\
&\quad=\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\,g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)d\left(W_t^\varepsilon-\tilde{W}_t\right) \qquad (46)\\
&\quad+\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\,g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)d\tilde{W}_t.
\end{aligned}$$

In view of the unbounded variation of $W_t^\varepsilon-\tilde{W}_t$, the convergence of the first term on the right-hand side of 46 needs appropriate care, in order to take advantage of the $\tilde{\mathbb{P}}-$a.s. uniform convergence of W_t^ε to $\tilde{W}_t$ in $C([0,T])$. We adopt the idea of regularization of $g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)$ with respect to the variable t, by means of the following sequence

$$g_\lambda^\varepsilon(u^\varepsilon)(t)=\frac{1}{\lambda}\int_0^T\rho\left(-\frac{t-s}{\lambda}\right)g\left(s,x,\frac{x}{\varepsilon},u_s^\varepsilon(s)\right)ds \text{ for } \lambda>0, \tag{47}$$

where ρ is a standard mollifier.

We have that $g_\lambda^\varepsilon(u^\varepsilon)(t)$ is a differentiable function of t and satisfies the relations

$$\tilde{\mathbb{E}}\int_0^T\|g_\lambda^\varepsilon(u^\varepsilon)(t)\|_{L_2(Q)}^2\,dt\le\tilde{\mathbb{E}}\int_0^T\left\|g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon(t)\right)\right\|_{L_2(Q)}^2dt, \text{ for any } \lambda>0, \tag{48}$$

and for any $\varepsilon>0$

$$g_\lambda^\varepsilon(u^\varepsilon)(t)\to g^\varepsilon(t,x,u_t^\varepsilon(t)) \text{ strongly in } L^2\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}},L_2((0,T)\times Q)\right) \text{ as } \lambda\to 0. \tag{49}$$

We split the first term in the right-hand side of 46 as

$$\begin{aligned}&\tilde{\mathbb{E}}\int_0^T\int_Q\phi(t,x)\,g^\varepsilon(t,x,u_t^\varepsilon(t))\,dxd\left(W_t^\varepsilon-\tilde{W}_t\right) \qquad (50)\\&=\tilde{\mathbb{E}}\int_0^T\int_Q\phi(t,x)\,g_\lambda^\varepsilon(u^\varepsilon)(t)\,dxd\left(W_t^\varepsilon-\tilde{W}_t\right)\\&+\tilde{\mathbb{E}}\int_0^T\int_Q\phi(t,x)\left[g^\varepsilon(t,x,u_t^\varepsilon(t))-g_\lambda^\varepsilon(u^\varepsilon)(t)\right]dxd\left(W_t^\varepsilon-\tilde{W}_t\right).\end{aligned}$$

Owing to 49, and Burkholder-Davis-Gundy's inequality, it readily follows that the second term in 50 is bounded by a function $\sigma_1(\lambda)$ which converges to zero as $\lambda\to 0$. In the first term in the same relation, we take advantage of the differentiability of g_λ^ε with respect to t in order to integrate by parts. As a result we get

$$\begin{aligned}&\tilde{\mathbb{E}}\int_0^T\int_Q\phi(t,x)\,g_\lambda^\varepsilon(u^\varepsilon)(t)\,d\left(W_t^\varepsilon-\tilde{W}_t\right)\\&=\tilde{\mathbb{E}}\int_0^T\int_Q\left(W_t^\varepsilon-\tilde{W}_t\right)\frac{\partial}{\partial t}\left[\phi(t,x)\,g_\lambda^\varepsilon(u^\varepsilon)(t)\right]dt \qquad (51)\\&+\tilde{\mathbb{E}}\int_Q\phi(T,x)\,g_\lambda^\varepsilon(u^\varepsilon)(T)\left(W_T^\varepsilon-\tilde{W}_T\right).\end{aligned}$$

Thanks to the conditions on ϕ and g and the uniform convergence obtained from the application of Skorokhod's compactness result, namely

$$W_t^\varepsilon\to\tilde{W}_t \text{ uniformly in } C([0,T]),\ \tilde{\mathbb{P}}-\text{a.s.}, \tag{52}$$

we get that both terms on the right-hand side of 51 are bounded by the product $\sigma_2(\lambda)\,\eta_1(\varepsilon)$ such that $\sigma_2(\lambda)$ is finite and $\eta_1(\varepsilon)$ vanishes as ε tends to zero. Summarizing these facts, we deduce from 50 that

$$\left|\tilde{\mathbb{E}}\int_0^T\int_Q\phi(t,x)\,g^\varepsilon(t,x,u_t^\varepsilon(t))\,dxd\left(W_t^\varepsilon-\tilde{W}_t\right)\right|\le\sigma_1(\lambda)+\sigma_2(\lambda)\,\eta_1(\varepsilon). \tag{53}$$

Thus, we infer from 46 that

$$\begin{aligned}&\left|\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dxdW_t^\varepsilon\right.\\&\qquad\left.-\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)d\tilde{W}_t\right|\\&\qquad\le \sigma_1(\lambda)+\sigma_2(\lambda)\,\eta_1(\varepsilon)\end{aligned} \tag{54}$$

Taking the limit in 54 as $\varepsilon\to 0$, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left|\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dxdW_t^\varepsilon\right.\\&\qquad\left.-\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)d\tilde{W}_t\right|\le\sigma_1(\lambda);\end{aligned}$$

but the left-hand side of this relation being independent of λ, we can pass to the limit on both sides as $\lambda\to 0$, to arrive at the crucial statement

$$\begin{aligned}&\lim_{\varepsilon\to 0}\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dxdW_t^\varepsilon\\&\qquad=\lim_{\varepsilon\to 0}\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)d\tilde{W}_t.\end{aligned} \tag{55}$$

Owing to 27; that is

$$g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)\rightharpoonup\tilde{g}(t,x,u_t)\ \text{ weakly in } L^2((0,T)\times Q),\ \tilde{\mathbb{P}}-\text{a.s.},$$

we can call upon the convergence theorem for stochastic integrals due to Rozovskii [39, Theorem 4, p. 63] to claim that

$$\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dW_t\to\tilde{\mathbb{E}}\int_0^T\int_Q \phi(t,x)\,\tilde{g}(t,x,u_t)\,d\tilde{W}_t.$$

Hence, we deduce from 55 that,

$$\int_0^T\int_Q \phi(t,x)\, g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dW_t^\varepsilon\to\int_0^T\int_Q \phi(t,x)\,\tilde{g}(t,x,u_t)\,d\tilde{W}_t,\ \tilde{\mathbb{P}}-\text{a.s.}. \tag{56}$$

For the second term in 45, thanks to Burkholder-Davis-Gundy's inequality, the assumptions on g^{ϵ_j} and 22, we have

$$\begin{aligned}&\lim_{\epsilon_j\to 0}\epsilon_j\tilde{\mathbb{E}}\sup_{t\in[0,T]}\left|\int_0^t\int_Q \phi_1\left(t,x,\frac{x}{\varepsilon}\right) g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dxdW_t^{\epsilon_j}\right|\\&\le C\lim_{\epsilon_j\to 0}\epsilon_j\tilde{\mathbb{E}}\left(\int_0^T\left(\int_Q \phi_1\left(t,x,\frac{x}{\varepsilon}\right) g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)dx\right)^2dt\right)^{\frac{1}{2}}\\&\le C\lim_{\epsilon_j\to 0}\epsilon_j\tilde{\mathbb{E}}\left(\int_0^T\|g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)\|_{L^2(Q)}\|\phi_1(t,x,\frac{x}{\epsilon_j})\|_{L^2(Q)}dt\right)^{\frac{1}{2}}\\&\le C\lim_{\epsilon_j\to 0}\epsilon_j\left(\int_0^T\|g\left(t,x,\frac{x}{\varepsilon},u_t^\varepsilon\right)\|_{L^2(Q)}dt\right)^{\frac{1}{2}}\to 0,\quad\tilde{\mathbb{P}}-a.s.\end{aligned}$$

Combining the above convergences, we obtain

$$\begin{aligned}
&-\int_0^T \int_Q u_t(t,x)\phi_t(t,x)dxdt \\
&\quad + \frac{1}{|Y|}\int_0^T \int_{Q\times Y} A(y)[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)] \\
&\quad \cdot [\nabla_x \phi(t,x) + \nabla_y \phi_1(t,x,y)]dydxdt \qquad (57)\\
&\quad + \int_0^T \int_Q \langle B(t,u_t), \phi(t,x)\rangle dxdt \\
&\quad = \frac{1}{|Y|}\int_0^T \int_{Q\times Y} F(t,x,y).[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)]\phi(t,x)dxdydt \\
&\quad + \int_0^T \int_Q \tilde{g}\left(t,x,u_t\right)\phi(t,x)\tilde{W}dx.
\end{aligned}$$

Choosing in the first stage $\phi = 0$ and after $\phi_1 = 0$, the problem 57 is equivalent to the following system of integral equations

$$\int_0^T \int_{Q\times Y} A(y)[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)][\nabla_y \phi_1(t,x,y)]dydxdt = 0, \qquad (58)$$

and

$$\begin{aligned}
&-\int_0^T \int_Q u_t(t,x)\phi_t(t,x)dxdt \qquad (59)\\
&+ \int_0^T \int_{Q\times Y} A(y)[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)][\nabla_x \phi(t,x)]dydxdt \\
&+ \int_0^T \int_Q \langle B(t,u_t), \phi(t,x)\rangle dxdt \\
&= \frac{1}{|Y|}\int_0^T \int_{Q\times Y} F(t,x,y).[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)]\phi(t,x)dxdydt \\
&+ \int_0^T \int_Q \tilde{g}\left(t,x,u_t\right)\phi(t,x)d\tilde{W}dx.
\end{aligned}$$

By standard arguments (see [17]), equation 58 has a unique solution given by

$$u_1(t,x,y) = -\chi(y)\cdot\nabla_x u(t,x) + \tilde{u_1}(t,x), \qquad (60)$$

where $\chi(y)$, known as the first order corrector, is the unique solution to the following equation:

$$\begin{cases} \operatorname{div}_y(A(y)\nabla_y\chi(y)) = \nabla_y\cdot A(y), \text{ in } Y, \\ \chi \text{ is } Y \text{ periodic.} \end{cases} \qquad (61)$$

As for the uniqueness of the solution of 59, we prove it as follows. Using 60 in 59, one obtains that 59 is the weak formulation of the equation

$$du_t - A_0\Delta u dt + B(t,u_t)dt = \tilde{f}(t,x,\nabla u)dt + \tilde{g}(t,x,u_t)d\tilde{W}, \qquad (62)$$

where

$$A_0 = \frac{1}{|Y|}\int_Y (A(y) - A(y)\nabla_y\chi(y))dy, \qquad (63)$$

$$\tilde{f}(t,x,\nabla u) = \frac{1}{|Y|}\int_Y F(t,x,y)\cdot[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)]dy,$$

and

$$\tilde{g}\left(t,x,u_t\right)=\frac{1}{|Y|}\int_Y g\left(t,x,y,u_t\right)dy.$$

But the initial boundary value problem corresponding to 62 has a unique solution by [38]. It remains to show that $u(x,0)=a(x)$ and $u_t(x,0)=b(x)$. Notice that equation 40 is valid for $\Phi^{\epsilon_j}(t,x)=\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})$ where $\phi\in C_c^\infty((0,T)\times Q)$ and $\phi_1\in C_c^\infty((0,T)\times Q;C_{\text{per}}^\infty(Y))$, such that $\phi(0,x)=v(x)$ and $\phi(T,x)=0$. Thus, we have

$$\begin{aligned}
&-\int_0^T\int_Q u_t^{\epsilon_j}(t,x)\Big[\phi_t(t,x)+\epsilon_j\phi_{1t}(t,x,\frac{x}{\epsilon_j})\Big]dxdt\\
&+\int_0^T\int_Q A_{\epsilon_j}(x)\nabla u^{\epsilon_j}(x,t)\cdot\Big[\nabla_x\phi(t,x)+\epsilon_j\nabla_x\phi_1(t,x,\frac{x}{\epsilon_j})+\nabla_y\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt\\
&+\int_0^T\int_Q\Big\langle B(t,u_t^\epsilon),\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]\Big\rangle\,dxdt\\
&=\int_0^T\int_Q f^{\epsilon_j}(t,x,\nabla u^{\epsilon_j})\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt\\
&+\int_0^T\int_Q g^{\epsilon_j}(t,x,u_t^\epsilon)\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdW_{\epsilon_j}+\int_Q u_t^{\epsilon_j}(x,0)v(x)dx,
\end{aligned}$$

where we pass to the limit, to get

$$\begin{aligned}
&-\int_0^T\int_Q u_t(t,x)\phi_t(t,x)dxdt\\
&+\int_0^T\int_{Q\times Y}A(y)[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\cdot[\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,y)]dydxdt\\
&+\int_0^T\int_Q\langle B(t,u_t),\phi(t,x)\rangle dxdt\\
&=\frac{1}{|Y|}\int_0^T\int_{Q\times Y}F(t,x,y)\cdot[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\phi(t,x)dxdydt\\
&+\int_0^T\int_Q\tilde{g}\left(t,x,u_t\right)\phi(t,x)\tilde{W}dxdt+\int_Q b(x)v(x)dx.
\end{aligned}$$

The integration by parts, in the first term gives

$$\begin{aligned}
&\int_0^T\int_Q du_t(t,x)\phi(t,x)dx+\int_Q u_t(x,0)v(x)dx\\
&+\int_0^T\int_{Q\times Y}A(y)[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\cdot[\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,y)]dydxdt\\
&+\int_0^T\int_Q\langle B(t,u_t),\phi(t,x)\rangle dxdt\\
&=\frac{1}{|Y|}\int_0^T\int_{Q\times Y}F(t,x,y)\cdot[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\phi(t,x)dxdydt
\end{aligned}$$

$$+\int_0^T\int_Q \tilde{g}(t,x,u_t)\phi(t,x)\tilde{W}dxdt+\int_Q b(x)v(x)dx.$$

In view of equation 57, we deduce that

$$\int_Q u_t(x,0)v(x)dx=\int_Q b(x)v(x)dx,$$

for any $v\in C_c^\infty(Q)$. This implies that $u_t(x,0)=b(x)$. For the other initial condition, we consider $\Phi^{\epsilon_j}(t,x)=\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})$ as a test function in 40, where $\phi\in C_c^\infty((0,T)\times Q)$ and $\phi_1\in C_c^\infty((0,T)\times Q;C_{\text{per}}^\infty(Y))$, such that $\phi(0,x)=0,\phi_t(0,x)=v(x)$ and $\phi(T,x)=0=\phi_t(T,x)$. Integration by parts in the first term of 40, gives

$$\int_0^T\int_Q u^{\epsilon_j}(t,x)\Big[\phi_{tt}(t,x)+\epsilon_j\phi_{1tt}(t,x,\frac{x}{\epsilon_j})\Big]dxdt$$
$$+\int_0^T\int_Q A_{\epsilon_j}(x)\nabla u^{\epsilon_j}(x,t)\cdot\Big[\nabla_x\phi(t,x)+\epsilon_j\nabla_x\phi_1(t,x,\frac{x}{\epsilon_j})+\nabla_y\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt$$
$$+\int_0^T\int_Q\Big\langle B(t,u_t^\epsilon),\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]\Big\rangle dxdt$$
$$=\int_0^T\int_Q f^{\epsilon_j}(t,x,\nabla u^{\epsilon_j})\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdt$$
$$+\int_0^T\int_Q g^{\epsilon_j}(t,x,u_t^\epsilon)\Big[\phi(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\Big]dxdW_{\epsilon_j}-\int_Q u^{\epsilon_j}(x,0)v(x)dx.$$

Passing to the limit in this equation, we obtain

$$\int_0^T\int_Q u(t,x)\phi_{tt}(t,x)dxdt$$
$$+\int_0^T\int_{Q\times Y}A(y)[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\cdot[\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,y)]dydxdt$$
$$+\int_0^T\int_Q\langle B(t,u_t),\phi(t,x)\rangle dxdt$$
$$=\frac{1}{|Y|}\int_0^T\int_{Q\times,Y}F(t,x,y)\cdot[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\phi(t,x)dxdydt$$
$$+\int_0^T\int_Q\tilde{g}(t,x,u_t)\phi(t,x)\tilde{W}dxdt-\int_Q a(x)v(x)dx.$$

We integrate by parts again to obtain

$$-\int_0^T\int_Q u_t(t,x)\phi_t(t,x)dxdt-\int_Q u(x,0)v(x)dx$$
$$+\int_0^T\int_{Q\times Y}A(y)[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\cdot[\nabla_x\phi(t,x)+\nabla_y\phi_1(t,x,y)]dydxdt$$
$$+\int_0^T\int_Q\langle B(t,u_t),\phi(t,x)\rangle dxdt$$
$$=\frac{1}{|Y|}\int_0^T\int_{Q\times Y}F(t,x,y)\cdot[\nabla_x u(t,x)+\nabla_y u_1(t,x,y)]\phi(t,x)dxdydt$$

$$+\int_0^T \int_Q \tilde{g}(t,x,u_t)\phi(t,x)\tilde{W}dxdt - \int_Q a(x)v(x)dx.$$

Using the same argument as before, we show that $u(x,0) = a(x)$. We note the triple $\left(\tilde{W}, u, u_t\right)$ is a probabilistic weak solution of (P) which is unique. Thus by the infinite dimensional version of Yamada-Watanabe's theorem (see [35]), we get that (W, u, u_t) is the unique strong solution of (P). Thus up to distribution (probability law) the whole sequence of solutions of (P_ϵ) converges to the solution of problem (P). Thus the proof of Theorem 5.1 is complete. □

6. **Convergence of the energy.** Let us introduce the energies associated with the problems (P_{ϵ_j}) and (P), as follows:

$$\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t) = \frac{1}{2}\tilde{\mathbb{E}}\|u_t^{\epsilon_j}(t)\|^2_{L^2(Q)} + \frac{1}{2}\tilde{\mathbb{E}}\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}(x,t)\cdot\nabla u^{\epsilon_j}(x,t)dx$$
$$+\tilde{\mathbb{E}}\int_0^t \langle B(s,u_t^{\epsilon_j}), u_t^{\epsilon_j}\rangle ds$$
$$\mathcal{E}(u)(t) = \frac{1}{2}\tilde{\mathbb{E}}\|u_t(t)\|^2_{L^2(Q)} + \frac{1}{2}\tilde{\mathbb{E}}\int_Q A_0\nabla u(x,t)\cdot\nabla u(x,t)dx$$
$$+\tilde{\mathbb{E}}\int_0^t \langle B(s,u_t), u_t\rangle ds.$$

But from Itô's formula, we have

$$\frac{1}{2}\tilde{\mathbb{E}}\|u_t^{\epsilon_j}(t)\|^2_{L^2(Q)} + \frac{1}{2}\tilde{\mathbb{E}}\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}(t)\cdot\nabla u^{\epsilon_j}(t)dx + \tilde{\mathbb{E}}\int_0^t \langle B(s,u_t^{\epsilon_j}), u_t^{\epsilon_j}\rangle ds$$
$$= \tilde{\mathbb{E}}\left[\frac{1}{2}\|u_1^{\epsilon_j}\|^2_{L^2(Q)} + \frac{1}{2}\int_Q A_{\epsilon_j}\nabla u_0^{\epsilon_j}\cdot\nabla u_0^{\epsilon_j}dx + \int_0^t (f^{\epsilon_j}(s,x,\nabla u^{\epsilon_j}), u_t^{\epsilon_j})ds\right.$$
$$\left.+\frac{1}{2}\int_0^t \|g^{\epsilon_j}(s,u_t^{\epsilon_j})\|^2_{L^2(Q)}ds + \int_0^t (g^{\epsilon_j}(s,u_t^{\epsilon_j}), u_t^{\epsilon_j})dW_{\epsilon_j}\right].$$

Thus

$$\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t) = \frac{1}{2}\tilde{\mathbb{E}}\|u_1^{\epsilon_j}\|^2_{L^2(Q)} + \frac{1}{2}\tilde{\mathbb{E}}\int_Q A_{\epsilon_j}\nabla u_0^{\epsilon_j}\cdot\nabla u_0^{\epsilon_j}dx$$
$$+\tilde{\mathbb{E}}\int_0^t (f^{\epsilon_j}(s,x,\nabla u^{\epsilon_j}), u_t^{\epsilon_j})ds + \frac{1}{2}\tilde{\mathbb{E}}\int_0^t \|g^{\epsilon_j}(s,u_t^{\epsilon_j})\|^2_{L^2(Q)}ds, \tag{64}$$
$$\mathcal{E}(u)(t) = \frac{1}{2}\tilde{\mathbb{E}}\|u_1\|^2_{L^2(Q)} + \frac{1}{2}\tilde{\mathbb{E}}\int_Q A_0\nabla u_0\cdot\nabla u_0 dx$$
$$+\tilde{\mathbb{E}}\int_0^t (\tilde{f}(s,x,\nabla u), u_t)ds + \frac{1}{2}\tilde{\mathbb{E}}\int_0^t \|\tilde{g}(s,x,u_t)\|^2_{L^2(Q)}ds. \tag{65}$$

The vanishing of the expectation of the stochastic integrals is due to the fact that $(g^\epsilon(u_t^\epsilon), \tilde{u}_t^\epsilon)$ and $(g(u), u_t)$ are square integrable in time. We want to prove that the energy associated with the problem (P_{ϵ_j}), uniformly converges to that of the corresponding homogenized problem (P). For this purpose we need to assume some stronger assumptions on the initial data. We have the following result

Theorem 6.1. *Assume that the assumptions of Theorem 5.1 are fulfilled and*

$$-\,div(A_{\epsilon_j}\nabla a^{\epsilon_j}) \to -div(A_0\nabla a), \quad \text{strongly in } H^{-1}(Q), \tag{66}$$

$$b^{\epsilon_j} \to b, \quad \textit{strongly in } L^2(Q). \tag{67}$$

Then

$$\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t) \to \mathcal{E}(u)(t) \ \text{ in } C([0,T]),$$

where u is the solution of the homogenized problem.

Proof. Thanks to the convergences 20, 44, 29, 66 and 67, we show that

$$\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t) \to \mathcal{E}(u)(t), \ \ \forall t \in [0,T].$$

Now we need to show that $(\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t))$, is uniformly bounded and equicontinuous on $[0,T]$ and hence Arzela-Ascoli's theorem concludes the proof. We have

$$|\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t)| \le \frac{1}{2}\tilde{\mathbb{E}}\|b^{\epsilon_j}\|^2_{L^2(Q)} + \frac{\alpha}{2}\tilde{\mathbb{E}}\|a^{\epsilon_j}\|_{H^1_0} + \tilde{\mathbb{E}}\int_0^t \left|(f^{\epsilon_j}(s,x,\nabla u^{\epsilon_j}), u_t^{\epsilon_j})\right| ds$$
$$+ \frac{1}{2}\int_0^t \|g^{\epsilon_j}(s,u_t^{\epsilon_j})\|^2_{L^2(Q)} ds.$$

Thanks to the assumptions on the data $(A3)$, $(A4)$ and $(A5)$, the a priori estimates 3 and 4, we show that

$$|\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t)| \le C, \quad \forall t \in [0,T].$$

For any $h > 0$ and $t \in [0,T]$, we get

$$|\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t+h) - \mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t)|$$
$$\le \tilde{\mathbb{E}}\int_t^{t+h} |(f^{\epsilon_j}(s,x,\nabla u^{\epsilon_j}), u_t^{\epsilon_j})| ds + \frac{1}{2}\tilde{\mathbb{E}}\int_t^{t+h} \|g^{\epsilon_j}(s,u_t^{\epsilon_j})\|^2_{L^2(Q)} ds.$$

Again assumptions (A3), (A5) and Cauchy-Schwarz's inequality, give

$$|\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t+h) - \mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t)| \le C\left(h + h^{\frac{1}{2}}\right).$$

This implies the equicontinuity of the sequence $\{\mathcal{E}^{\epsilon_j}(u^{\epsilon_j})(t)\}_{\epsilon_j}$, and therefore the proof is complete. ☐

7. The corrector result.

In this section, we establish a corrector result stated in the following

Theorem 7.1. *Let the assumptions of Theorems 5.1 and 6.1 be fulfilled. Assume that $\nabla_y \chi(y) \in [L^r(Y)]^n$ and $\nabla u \in L^2(0,T;[L^s(Y)]^n)$ with $1 \le r,s < \infty$ such that*

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{2}.$$

Then

$$u_t^{\epsilon_j} - u_t - \epsilon_j u_{1t}(\cdot,\cdot,\frac{\cdot}{\epsilon_j}) \to 0 \ \textit{ strongly in } \ L^2(0,T;L^2(Q)) \quad \tilde{\mathbb{P}} - a.s., \tag{68}$$

$$u^{\epsilon_j} - u - \epsilon_j u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j}) \to 0 \ \textit{ strongly in } \ L^2(0,T;H^1(Q)) \quad \tilde{\mathbb{P}} - a.s.. \tag{69}$$

Proof. It is easy to see that

$$\lim_{\epsilon_j \to 0} \epsilon_j u_{1t}(\cdot,\cdot,\frac{\cdot}{\epsilon_j}) \to 0 \ \text{ in } \ L^2(0,T;L^2(Q)) \quad \tilde{\mathbb{P}} - a.s..$$

Then convergence 20 gives

$$u_t^{\epsilon_j} - u_t - \epsilon_j u_{1t}(\cdot,\cdot,\frac{\cdot}{\epsilon_j}) \to 0 \ \text{ in } \ L^2(0,T;L^2(Q)) \quad \tilde{\mathbb{P}} - a.s..$$

Thus 68 holds. Similarly we show that

$$u^{\epsilon_j} - u - \epsilon_j u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j}) \to 0 \text{ strongly in } L^2(0,T;L^2(Q)) \quad \tilde{\mathbb{P}}-a.s..$$

It remains to show that

$$\nabla(u^{\epsilon_j} - u - \epsilon_j u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})) \to 0 \text{ strongly in } L^2(0,T;[L^2(Q)]^n) \quad \tilde{\mathbb{P}}-a.s..$$

We have

$$\nabla(u^{\epsilon_j} - u - \epsilon_j u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})) = \nabla u^{\epsilon_j} - \nabla u - \nabla_y u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})) - \epsilon_j \nabla u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})).$$

Again

$$\lim_{\epsilon_j\to 0} \epsilon_j \nabla u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j}) \to 0 \text{ in } L^2(0,T;[L^2(Q)]^n), \quad \tilde{\mathbb{P}}-a.s..$$

Now from the ellipticity assumption on the matrix A, we have

$$\begin{aligned}
&\alpha\mathbb{E}\int_0^T \|\nabla u^{\epsilon_j} - \nabla u - \nabla_y u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})\|^2_{L^2(Q)} dt \\
&\quad\le \mathbb{E}\int_0^T\int_Q A\left(\frac{x}{\epsilon_j}\right)\left(\nabla u^{\epsilon_j} - \nabla u - \nabla_y u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})\right) \\
&\quad\cdot\left(\nabla u^{\epsilon_j} - \nabla u - \nabla_y u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})\right) dxdt \\
&\quad= \mathbb{E}\int_0^T\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}\cdot\nabla u^{\epsilon_j} dxdt \\
&\quad- 2\mathbb{E}\int_0^T\int_Q \nabla u^{\epsilon_j} A\left(\frac{x}{\epsilon_j}\right)\cdot\left(\nabla u + \nabla_y u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})\right) dxdt \\
&\quad+ \mathbb{E}\int_0^T\int_Q A\left(\frac{x}{\epsilon_j}\right)\left(\nabla u + \nabla_y u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})\right) \\
&\quad\cdot\left(\nabla u + \nabla_y u_1(\cdot,\cdot,\frac{\cdot}{\epsilon_j})\right) dxdt.
\end{aligned} \tag{70}$$

Let us pass to the limit in this inequality. We start with

$$\mathbb{E}\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}\cdot\nabla u^{\epsilon_j} dx.$$

From the convergence of the energies in Theorem 6.1 and using 63 and 60, we have

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\mathbb{E}\int_Q A_{\epsilon_j}\nabla u^{\epsilon_j}\cdot\nabla u^{\epsilon_j} dx \\
&= \mathbb{E}\int_{Q\times Y} A(y)\cdot[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)]\cdot[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)]dydx.
\end{aligned} \tag{71}$$

Next, using the two-scale convergence of ∇u^{ϵ_j}, with the test function $A\,(y)\,(\nabla u(t,x) + \nabla_y u_1(t,x,y))$, we obtain

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_Q \nabla u^{\epsilon_j}(t,x)\cdot A\left(\frac{x}{\epsilon_j}\right)\cdot\left(\nabla u + \nabla_y u_1(t,x,\frac{x}{\epsilon_j})\right) dxdt \\
&\qquad= \int_0^T\int_{Q\times Y} (\nabla u(t,x) + \nabla_y u_1(t,x,y)) \\
&\qquad\cdot A\,(y)\cdot(\nabla u(t,x) + \nabla_y u_1(t,x,y))\, dxdydt.
\end{aligned} \tag{72}$$

Now, let us write

$$\begin{aligned}\psi(t,x,y) &= A(y)\left(\nabla u(t,x)+\nabla_y u_1(t,x,y)\right)\cdot\left(\nabla u(t,x)+\nabla_y u_1(t,x,y)\right)\\ &= A(y)\,\nabla u(t,x)\cdot\nabla u(t,x)+2A(y)\,\nabla u(t,x)\cdot\nabla_y u_1(t,x,y)\\ &\quad + A(y)\,\nabla_y u_1(t,x,y)\cdot\nabla_y u_1(t,x,y).\end{aligned}$$

For u_1 given by 60, we have

$$\begin{aligned}\psi(t,x,y) =& A(y)\,\nabla u(t,x)\cdot\nabla u(t,x)-2A(y)\,\nabla u(t,x)\cdot\nabla_y[\chi(y)\cdot\nabla_x u(t,x)]\\ &+ A(y)\,\nabla_y[\chi(y)\cdot\nabla_x u(t,x)]\nabla_y[\chi(y)\cdot\nabla_x u(t,x)].\end{aligned}$$

Now using (ii) of Lemma 4.2, for $p=2$, we obtain

$$\begin{aligned}\lim_{\epsilon_j\to 0}\int_0^T\int_Q A\left(\frac{x}{\epsilon_j}\right)&\left(\nabla u(t,x)+\nabla_y u_1(t,x,\frac{x}{\epsilon_j})\right)\\ &\cdot\left(\nabla u(t,x)+\nabla_y u_1(t,x,\frac{y}{\epsilon_j})\right)dxdt\\ &=\int_0^T\int_{Q\times Y} A(y)\left(\nabla u(t,x)+\nabla_y u_1(t,x,y)\right)\\ &\cdot\left(\nabla u(t,x)+\nabla_y u_1(t,x,y)\right)dxdydt. \qquad (73)\end{aligned}$$

Combining 71, 72 and 73 with 70, we deduce that

$$\lim_{\epsilon_j\to 0}\mathbb{E}\int_0^T\|\nabla u^{\epsilon_j}-\nabla u-\nabla_y u_1(.,.,\frac{\cdot}{\epsilon_j})\|^2_{L^2(Q)}dt=0 \quad \tilde{\mathbb{P}}-a.s..$$

Thus the proof is complete. □

As a closing remark, we note that our results can readily be extended to the case of infinite dimensional Wiener processes taking values in appropriate Hilbert spaces; for instance cylindrical Wiener processes.

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