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ON BOUNDARY OPTIMAL CONTROL PROBLEM FOR AN ARTERIAL SYSTEM: FIRST-ORDER OPTIMALITY CONDITIONS

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Abstract. We discuss a control constrained boundary optimal control problem for the Boussinesq-type system arising in the study of the dynamics of an arterial network. We suppose that the control object is described by an initial-boundary value problem for $1D$ system of pseudo-parabolic nonlinear equations with an unbounded coefficient in the principle part and the Robin-type of boundary conditions. The main question we study in this part of the paper is about the existence of optimal solutions and first-order optimality conditions.

1. **Introduction.** The main goal of this paper is to study one class of optimal control problems (OCPs) for a viscous Boussinesq system arising in the study of the dynamics of cardiovascular networks. We consider the boundary control problem for a $1D$ system of coupled PDEs with the Robin-type boundary conditions, describing the dynamics of pressure and flow in the arterial segment. We discuss in this part of paper the existence of optimal solutions and provide a substantial analysis of the first-order optimality conditions. Namely, we deal with the following minimization

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problem:

$$\text{Minimize}\quad J(g,h,\eta,u) := \frac{1}{2}\int_\Omega \alpha_\Omega \left(u(T)-u_\Omega\right)^2\,dx + \frac{\nu}{2}\int_0^T\int_\Omega \left(\eta_{xx}\right)^2\,dxdt$$
$$+\frac{1}{2}\int_0^T \left|\int_\Omega \alpha_Q\Big(\eta(t)+r_0 u_{xt}(t)-\eta_Q\Big)\,dx\right|^2 dt + \frac{1}{2}\int_0^T \Big(\beta_g\,|g|^2+\beta_h\,|h|^2\Big)\,dt \qquad (1)$$

subject to the constraints

$$\begin{cases} \eta_t+\eta_x u+\eta u_x+\dfrac{1}{2}r_0 u_x-\nu\eta_{xx}=0 & \text{in } Q,\\ \left[u-(\delta u_x)_x\right]_t+\dfrac{1}{2}\left(u^2\right)_x+\mu\eta_x=f & \text{in } Q, \end{cases} \qquad (2)$$

$$\begin{cases} \eta(0,\cdot)=\eta_0 & \text{in } \Omega,\\ u(0,\cdot)-\left(\delta(\cdot)u_x(0,\cdot)\right)_x=u_0 & \text{in } \Omega, \end{cases} \qquad (3)$$

$$\begin{cases} \eta(\cdot,0)=\eta(\cdot,L)=\eta^* & \text{in } (0,T),\\ \delta(0)\dot u_x(\cdot,0)+\sigma_0 u(\cdot,0)=g, & \text{in } (0,T),\\ \delta(L)\dot u_x(\cdot,L)+\sigma_1 u(\cdot,L)=h, & \text{in } (0,T),\\ \delta(L)u_x(0,L)=\delta(0)u_x(0,0)=0 \end{cases} \qquad (4)$$

and

$$(g,h)\in G_{ad}\times H_{ad}\subset L^2(0,T)\times L^2(0,T). \qquad (5)$$

Here, β_g, β_h, and η^* are positive constants, and G_{ad} and H_{ad} are the sets of admissible boundary controls. These sets and the rest of notations will be specified in the next section.

Optimal control problem (1)–(5) comes from the fluid dynamic models of blood flows in arterial systems. It is well known that the cardiovascular system consists of a pump that propels a viscous liquid (the blood) through a network of flexible tubes. The heart is one key component in the complex control mechanism of maintaining pressure in the vascular system. The aorta is the main artery originating from the left ventricle and then bifurcates to other arteries, and it is identified by several segments (ascending, thoracic, abdominal). The functionality of the aorta, considered as a single segment, is worth exploring from a modeling perspective, in particular in relationship to the presence of the aortic valve.

In the first part of our investigation (see [5]) we make use of the standard viscous hyperbolic system (see [2, 21]) which models cross-section area $S(x,t)$ and average velocity $u(x,t)$ in the spatial domain:

$$\frac{\partial S}{\partial t}+\frac{\partial (Su)}{\partial x}-\nu\frac{\partial^2 S}{\partial x^2}=0, \qquad (6)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}=f, \qquad (7)$$

where $(t,x)\in Q=(0,L)\times(0,T)$, $f=f(x,t)$ is a friction force, usually taken to be $f=-22\mu\pi u/S$, μ is the fluid viscosity, $P(x,t)$ is the hydrodynamic pressure, L is the length of an arterial segment, and $T=T_{pulse}=60/(HartRate)$ is the duration of an entire heartbeat. Here we include the inertial effects of the wall motion, described by the wall displacement $\eta=\eta(x,t)$:

$$\eta = r - r_0 = \frac{1}{\sqrt{\pi}}(\sqrt{S} - \sqrt{S_0}) \simeq \frac{S - S_0}{2\sqrt{\pi S_0}}, \tag{8}$$

where $r(x,t)$ is the radius, $r_0 = r(x,0)$, $S_0 = S(x,0)$.

The fluid structure interaction is modeled using inertial forces, which gives the pressure law

$$P = P_{ext} + \frac{\beta}{r_0^2}\eta + \rho_\omega h \frac{\partial^2 \eta}{\partial t^2}. \tag{9}$$

Here, P_{ext} is the external pressure, $\beta = \frac{E}{1-\sigma^2}h$, σ is the Poisson ratio (usually $\sigma^2 = \frac{1}{2}$), E is Young modulus, h is the wall thickness, $m = \frac{\rho_\omega h}{2\sqrt{\pi S_0}}$, ρ_ω is the density of the wall.

This leads to the following Boussinesq system:

$$\begin{cases} \eta_t + \eta_x u + \eta u_x + \dfrac{1}{2} r_0 u_x - \nu \eta_{xx} = 0, \\ u_t + u u_x + \dfrac{2Eh}{\rho r_0^2}\eta_x + \dfrac{\rho_\omega h}{\rho}\eta_{xtt} = f, \end{cases}$$

where ρ is the blood density. Considering the relation $\eta_t = -\frac{1}{2}r_0 u_x$ and rearranging terms in u we get the system in the form (2)–(3). It remains to furnish the system by corresponding initial and boundary conditions which we propose to take in the form (3)–(4).

As for the OCP that is related with the arterial system, we are interested in finding the optimal heart rate (HR) which leads to the minimization of the following cost functional

$$J = \int_{t_0}^{t_0+T_{pulse}} |P_{avg}(t) - P_{ref}|^2 \, dt = \int_{t_0}^{t_0+T_{pulse}} \left| \frac{1}{L}\int_0^L P(x,t)\, dx - P_{ref} \right|^2 dt. \tag{10}$$

The systolic period is taken to be consistently one quarter of T_{pulse}, and $P_{ref} = 100$ mmHg.

It is easy to note that relations (8)–(9) lead to the following representation for the cost functional (10)

$$\begin{aligned} J &= \int_{t_0}^{t_0+T_{pulse}} \left| \frac{1}{L}\int_0^L P(x,t)\, dx - P_{ref} \right|^2 dt \\ &= \frac{1}{L^2}\int_{t_0}^{t_0+T_{pulse}} \left| \int_0^L \left(P_{ext}(t) + \frac{2Eh}{r_0^2}\eta(t,x) + \rho_\omega h\, \eta_{tt}(t,x) - L\, P_{ref} \right) dx \right|^2 dt. \end{aligned} \tag{11}$$

Since $\eta_t \approx -\frac{1}{2}r_0 u_x$ (see [3]) and we suppose that $\nu\eta_{xx}$ should be small enough, it easily follows from (11) that the given cost functional (10) can be reduced to the tracking type (1).

The research in the field of the cardiovascular system is very active (see, for instance the literature describing the dynamics of the vascular network coupled with a heart model, [2, 9, 10, 12, 15, 16, 17, 18, 19, 20, 21]). However, there seems to be no studies that focus on both aspects at the same time: a detailed description of the four chambers of the heart and on the spatial dynamics in the aorta. Some numerical aspects of optimizing the dynamics of the pressure and flow in the aorta as well as the heart rate variability, taking into account the elasticity of the aorta together with an aortic valve model at the inflow and a peripheral resistance model at the outflow, based on the discontinuous Galerkin method and a two-step time integration scheme of Adam-Bashfort, were recently treated in [3] for the Boussinesq system like (2). More broadly, theory and applications of optimization and control in spatial networks, basing on the different types of conservation laws have been extensively developed in literature, have been successfully applied to telecommunications, transportation or supply networks ([6, 7]).

From mathematical point of view, the characteristic feature of the Boussinesq system (2) is the fact that it involves a pseudo-parabolic operator with unbounded coefficient in its principle part. In the first part of this paper it was shown that for any pair of boundary controls $g \in G_{ad}$ and $h \in H_{ad}$, and for given $f \in L^\infty(0,T;L^2(\Omega))$, $\mu \in L^\infty(0,T;L^2(\Omega))$, $\sigma_0 \in L^\infty(0,T)$, $\sigma_1 \in L^\infty(0,T)$, $u_0 \in V_\delta$, $\eta_0 \in H^1_0(\Omega)$, $r_0 \in H^1(\Omega)$, and $\delta \in L^1(\Omega)$, the set of feasible solutions to optimal control problem (1)–(5) is non-empty and the corresponding weak solution $(\eta(t), u(t))$ of the viscous Boussinesq system (2)–(4) possesses the extra regularity properties $\eta_{xx}, u_{xt} \in L^2(0,T;L^2(\Omega))$, which play a crucial role in the proof of solvability of OCP (1)–(5). In this paper we deal with the existence of optimal solutions and derive the corresponding optimality conditions for the problem (1)–(5). It should be mentioned, that application of Lagrange principle requires even higher smoothness of solutions to the initial Boussinesq system (2)–(4). In order to avoid such limitations, we deal with a simplified version of the initial optimal control problem (2)–(4) (see (39), argumentation above and [3, 5] for physical description of the considered model). Also, in the second part of the paper, in order to provide the thorough substantiation of the first-order optimality conditions to the considered OCP, we make the special assumption for δ to be an element of the class $H^1(\Omega)$. Since the coefficient δ depends on such indicators as wall thickness, density of the wall and blood density, i.e. indicators varying slowly and smoothly, such assumption seems justified.

2. **Preliminaries.**

Let $T > 0$ and $L > 0$ be given values. We set $\Omega = (0, L)$, $Q = (0,T) \times \Omega$, and $\Sigma = (0,T) \times \partial\Omega$. Let $\delta \in H^1(\Omega)$ be a given function such that $\delta(x) \geq \delta_0 > 0$ for a.e. $x \in \Omega$. We use the standard notion $L^2(\Omega, \delta\, dx)$ for the set of measurable functions u on Ω such that

$$\|u\|_{L^2(\Omega,\delta\, dx)} = \left(\int_\Omega u^2 \delta\, dx \right)^{1/2} < +\infty.$$

We set $H = L^2(\Omega)$, $V_0 = H^1_0(\Omega)$, $V = H^1(\Omega)$, and identify the Hilbert space H with its dual H^*. On H we use the common natural inner product $(\cdot,\cdot)_H$, and endow the Hilbert spaces V_0 and V with the inner products

$$(\varphi,\psi)_{V_0} = (\varphi',\psi')_H \;\; \forall\, \varphi,\psi \in V_0$$

and

$$(\varphi,\psi)_V = (\varphi,\psi)_H + (\varphi',\psi')_H \;\; \forall\, \varphi,\psi \in V,$$

respectively.

We also make use of the weighted Sobolev space V_δ as the set of functions $u \in V$ for which the norm

$$\|u\|_{V_\delta} = \left(\int_\Omega \left(u^2 + \delta (u')^2 \right) \, dx \right)^{1/2}$$

is finite. Note that due to the following estimate, V_δ is complete with respect to the norm $\|\cdot\|_{V,\delta}$:

$$\|u\|_V^2 := \int_\Omega \left(u^2 + (u')^2\right)\, dx \le \max\left\{1, \delta_0^{-1}\right\} \int_\Omega \left(u^2 + \delta(u')^2\right)\, dx \\ = \max\left\{1, \delta_0^{-1}\right\} \|u\|_{V_\delta}^2. \tag{12}$$

Recall that V_0, V, and, hence, V_δ are continuously embedded into $C(\overline{\Omega})$, see [1, 14] for instance. Since $\delta, \delta^{-1} \in L^1(\Omega)$, it follows that V_δ is a uniformly convex separable Banach space [14]. Moreover, in view of the estimate (12), the embedding $V_\delta \hookrightarrow H$ is continuous and dense. Hence, $H = H^*$ is densely and continuously embedded in V_δ^*, and, therefore, $V_\delta \hookrightarrow H \hookrightarrow V_\delta^*$ is a Hilbert triplet (see [11] for the details).

Let us recall some well-known inequalities, that will be useful in the sequel (see [5]).

- $\|u\|_{L^\infty(\Omega)} \le \sqrt{2\max\{L, L^{-1}\}}\|u\|_V$, $\forall\, u \in V$ and $\|u\|_{L^\infty(\Omega)} \le 2\sqrt{L}\,\|u\|_{V_0}$, $\forall\, u \in V_0$.
- (*Friedrich's Inequality*) For any $u \in V_0$, we have

$$\|u\|_H \le L\|u_x\|_H = L\|u\|_{V_0}. \tag{13}$$

By $L^2(0,T;V_0)$ we denote the space of measurable abstract functions (equivalence classes) $u : [0,T] \to V$ such that

$$\|u\|_{L^2(0,T;V_0)} := \left(\int_0^T \|u(t)\|_{V_0}^2\, dt\right)^{1/2} < +\infty.$$

By analogy we can define the spaces $L^2(0,T;V_\delta)$, $L^\infty(0,T;H)$, $L^\infty(0,T;V_\delta)$, and $C([0,T];H)$ (for the details, we refer to [8]). In what follows, when t is fixed, the expression $u(t)$ stands for the function $u(t,\cdot)$ considered as a function in Ω with values into a suitable functional space. When we adopt this convention, we write $u(t)$ instead of $u(t,x)$ and $\dot{u}$ instead of u_t for the weak derivative of u in the sense of distribution

$$\int_0^T \varphi(t)\left\langle \dot{u}(t), v\right\rangle_{V^*;V}\, dt = -\int_0^T \dot{\varphi}(t)\left\langle u(t), v\right\rangle_{V^*;V}\, dt, \quad \forall\, v \in V,$$

where $\langle\cdot,\cdot\rangle_{V^*;V}$ denotes the pairing between V^* and V.

We also make use of the following Hilbert spaces

$$W_0(0,T) = \left\{u \in L^2(0,T;V_0) :\ \dot{u} \in L^2(0,T;V_0^*)\right\}, \\ W_\delta(0,T) = \left\{u \in L^2(0,T;V_\delta) :\ \dot{u} \in L^2(0,T;V_\delta^*)\right\},$$

supplied with their common inner product, see [8, p. 473], for instance.

Remark 1. The following result is fundamental (see [8]): Let (V,H,V^*) be a Hilbert triplet, $V \hookrightarrow H \hookrightarrow V^*$, with V separable, and let $u \in L^2(0,T;V)$ and $\dot{u} \in L^2(0,T;V^*)$. Then

(i) $u \in C([0,T];H)$ and $\exists\, C_E > 0$ such that

$$\max_{1\le t\le T} \|u(t)\|_H \le C_E\left(\|u\|_{L^2(0,T;V)} + \|\dot{u}\|_{L^2(0,T;V^*)}\right);$$

(ii) if $v \in L^2(0,T;V)$ and $\dot{v} \in L^2(0,T;V^*)$, then the following integration by parts formula holds:

$$\int_s^t \left(\left\langle \dot{u}(\gamma), v(\gamma)\right\rangle_{V^*;V} + \left\langle u(\gamma), \dot{v}(\gamma)\right\rangle_{V^*;V}\right)\, d\gamma = (u(t),v(t))_H - (u(s),v(s))_H \tag{14}$$

for all $s, t \in [0, T]$.

The similar assertions are valid for the Hilbert triplet $V_\delta \hookrightarrow H \hookrightarrow V_\delta^*$.

3. **On solvability of optimal control problem** (1)–(5)**.** Let $\nu > 0$ be a viscosity parameter, and let

$$f \in L^\infty(0,T;H),\ \mu \in L^\infty(0,T;V),\ \sigma_0 \in L^\infty(0,T),\ \sigma_1 \in L^\infty(0,T), \tag{15}$$

$$\alpha_\Omega \in L^\infty(\Omega),\ \alpha_Q \in L^\infty(Q),\ u_\Omega \in L^2(\Omega),\ \eta_Q \in L^2(0,T;H), \tag{16}$$

$$u_0 \in V_\delta,\ \eta_0 \in H_0^1(\Omega),\ r_0 \in H^1(\Omega), \tag{17}$$

be given distributions. In particular, f stands for a fixed forcing term, u_Ω and η_Q are given desired states for the wall displacement and average velocity, respectively, α_Ω and α_Q are non-negative weights (without loss of generality we suppose that α_Q is a nonnegative constant function on $[0,T] \times [0,L]$), u_0 and η_0 are given initial states, and δ is a singular (possibly locally unbounded) weight function such that $\delta(x) \geq \delta_0 > 0$ for a.e. $x \in \Omega$.

We assume that the sets of admissible boundary controls G_{ad} and H_{ad} are given as follows

$$\begin{aligned} G_{ad} &= \left\{ g \in L^2(0,T) :\ g_0 \leq g \leq g_1 \text{ a.e. in } (0,T) \right\}, \\ H_{ad} &= \left\{ h \in L^2(0,T) :\ h_0 \leq h \leq h_1 \text{ a.e. in } (0,T) \right\}, \end{aligned} \tag{18}$$

where $g_0, h_0, g_1, h_1 \in L^\infty(0,T)$ with $g_0(t) \leq g_1(t)$ and $h_0(t) \leq h_1(t)$ almost everywhere in $(0,T)$.

The optimal control problem we consider in this paper is to minimize the discrepancy between the given distributions $(u_\Omega, \eta_Q) \in L^2(\Omega) \times L^2(Q)$ and the pair of distributions $(u(T), \eta(t) + \eta_{tt}(t))$ (see, for instance, [5] for the physical interpretation), where $(\eta(t), u(t))$ is the solution of a viscous Boussinesq system, by an appropriate choice of boundary controls $g \in G_{ad}$ and $h \in H_{ad}$. Namely, we deal with the minimization problem (1)–(5).

Definition 3.1. We say that, for given boundary controls $g \in G_{ad}$ and $h \in H_{ad}$, a couple of functions $(\eta(t), u(t))$ is a weak solution to the initial-boundary value problem (2)–(4) if

$$\eta(t) = w(t) + \eta^*, \quad w(\cdot) \in W_0(0,T), \quad u(\cdot) \in W_\delta(0,T), \tag{19}$$

$$\delta(L) u_x(0,L) = 0, \qquad \delta(0) u_x(0,0) = 0, \tag{20}$$

$$(w(0), \chi)_H = (\eta_0 - \eta^*, \chi)_H \quad \text{for all } \chi \in H, \tag{21}$$

$$(u(0) - (\delta u_x(0))_x, \chi)_{V_\delta} = (u_0, \chi)_{V_\delta} \quad \text{for all } \chi \in V_\delta, \tag{22}$$

and the following relations

$$\begin{aligned} &\langle \dot{w}(t), \varphi \rangle_{V_0^*; V_0} + ((w(t)u(t))_x, \varphi)_H + \nu (w_x(t), \varphi_x)_H \\ &\qquad + \frac{1}{2} \left(r_0 u_x(t) + 2\eta^* u_x(t), \varphi \right)_H = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &\langle \dot{u}(t), \psi \rangle_{V_\delta^*; V_\delta} + \int_\Omega \delta \dot{u}_x(t) \psi_x \, dx + (u(t)u_x(t), \psi)_H + (\mu(t) w_x(t), \psi)_H \\ &\qquad + \sigma_1(t) u(t,L) \psi(L) - \sigma_0(t) u(t,0) \psi(0) \\ &\qquad = (f(t), \psi)_H + h(t)\psi(L) - g(t)\psi(0) \end{aligned} \tag{24}$$

hold true for all $\varphi \in V_0$ and $\psi \in V_\delta$ and a.e. $t \in [0,T]$.

Remark 2. Let us mention that if we multiply the left- and right-hand sides of equations (23)–(24) by function $\chi \in L^2(0,T)$ and integrate the result over the interval $(0,T)$, all integrals are finite. Moreover, closely following the arguments of Korpusov and Sveshnikov (see [13]), it can be shown that the weak solution to (2)–(4) in the sense of Definition 3.1 is equivalent to the following one: $(\eta(t), u(t))$ is a weak solution to the initial-boundary value problem (2)–(4) if the conditions (19)-(22) hold true and

$$\int_0^T \langle A_1(w(t),u(t)), \varphi(t)\rangle_{V_0^*;V_0}\, dt = 0, \quad \forall\, \varphi(\cdot) \in L^2(0,T;V_0), \tag{25}$$

$$\int_0^T \langle A_2(w(t),u(t)), \psi(t)\rangle_{V_\delta^*;V_\delta}\, dt = 0, \quad \forall\, \psi(\cdot) \in L^2(0,T;V_\delta), \tag{26}$$

where

$$A_1(w,u) = \frac{\partial w}{\partial t} - \nu w_{xx} + w_x u + w u_x + \frac{1}{2} r_0 u_x + \eta^* u_x \in V_0^*, \tag{27}$$

$$A_2(w,u) = \begin{bmatrix} \frac{\partial}{\partial t}\left(u - (\delta u_x)_x\right) + \frac{1}{2}\left(u^2\right)_x + \mu w_x - f \\ \delta(0)\dot u_x(\cdot,0) + \sigma_0 u(\cdot,0) - g \\ \delta(L)\dot u_x(\cdot,L) + \sigma_1 u(\cdot,L) - h \end{bmatrix} \in V_\delta^*. \tag{28}$$

Lemma 3.2 ([5]). *Assume that the conditions (15)–(17) hold true. Let $g \in G_{ad}$ and $h \in H_{ad}$ be an arbitrary pair of admissible boundary controls. Then there exists a unique solution $(\eta(\cdot), u(\cdot))$ of the system (2)–(4) in the sense of Definition 3.1 such that*

$$\begin{gathered} (\eta(\cdot), u(\cdot)) \in (W_0(0,T) + \eta^*) \times W_\delta(0,T), \\ w \in L^\infty(0,T;H) \cap L^2(0,T;H^2(\Omega) \cap V_0), \\ \dot w \in L^2(0,T;H),\ u \in W^{1,\infty}(0,T;V_\delta) \end{gathered} \tag{29}$$

and there exists a constant $D_* > 0$ depending only on initial data (15),(17) and control constrains h_1, g_1, satisfying the estimates

$$\|w\|^2_{L^2(0,T;H^2(\Omega))} + \|w\|^2_{L^\infty(0,T;H)} + \|\dot w\|^2_{L^2(0,T;H)} \le D_*, \tag{30}$$

$$\|u\|^2_{L^\infty(0,T;V_\delta)} + \|\dot u\|^2_{L^\infty(0,T;V_\delta)} \le D_*. \tag{31}$$

We also define the feasible set to the problem (1)–(5), (18) as follows:

$$\Xi = \left\{ (g,h,\eta,u) \;\middle|\; \begin{array}{c} g \in G_{ad}, \quad h \in H_{ad}, \\ \eta(t) = w(t) + \eta^*, \quad w \in W_0(0,T), \quad u \in W_\delta(0,T), \\ (w(t),u(t)) \text{ satisfies relations (19)–(24)} \\ \text{for all } \varphi \in V_0,\ \psi \in V_\delta, \text{ and a.e. } t \in [0,T], \\ J(g,h,\eta,u) < +\infty. \end{array} \right\} \tag{32}$$

We say that a tuple $\left(g^0, h^0, \eta^0, u^0\right) \in \Xi$ is an optimal solution to the problem (1)–(5), (18) if

$$J\left(g^0, h^0, \eta^0, u^0\right) = \inf_{(g,h,\eta,u)\in\Xi} J(g,h,\eta,u).$$

In [5] it was shown that $\Xi \neq \emptyset$ and $\Xi_\lambda = \{(g,h,\eta,u) \in \Xi : J(g,h,\eta,u) \le \lambda\}$ is a bounded set in $L^2(0,T) \times L^2(0,T) \times (W_0(0,T) + \eta^*) \times W_\delta(0,T)$ for every $\lambda > 0$.

While proving these hypotheses, the authors in [5] obtained a series of useful estimates for the weak solutions to initial-boundary value problem (2)–(4).

Lemma 3.3. [5, Lemmas 6.3 and 6.5 along with Remark 6.5] *Let $g \in G_{ad}$ and $h \in H_{ad}$ be an arbitrary pair of admissible boundary controls. Let $(\eta(\cdot), u(\cdot)) = (w(\cdot) + \eta^*, u(\cdot))$ be the corresponding weak solution to the system (2)–(4) in the sense of Definition 3.1. Under assumptions (15)–(17), there exist positive constants C_1, C_2, C_3 depending on the initial data only such that for a.a. $t \in [0, T]$*

$$\|w(t)\|_H^2 + \|u(t)\|_{V_\delta}^2 \le C_1, \quad \|\dot{w}(t)\|_{V_0^*} \le C_2, \quad \|\dot{u}(t)\|_{V_\delta} \le C_3. \tag{33}$$

In the context of solvability of OCP (18)–(5), the regularity of the solutions of the corresponding initial-boundary value problem (2)–(4) plays a crucial role.

Theorem 3.4 ([5]). *The set of feasible solutions Ξ to the problem (1)–(5), (18) is nonempty provided the initial data satisfy the conditions (15)-(17).*

Now we proceed with the result concerning existence of optimal solutions to OCP (1)–(5), (18).

Theorem 3.5. *For each*

$$f \in L^\infty(0, T; L^2(\Omega)),\ \mu \in L^\infty(0, T; V),\ \sigma_0 \in L^\infty(0, T),\ \sigma_1 \in L^\infty(0, T),$$
$$\alpha_\Omega \in L^\infty(\Omega),\ \alpha_Q \in \mathbb{R}_+,\ u_\Omega \in L^2(\Omega),\ \eta_Q \in W(0, T; H),$$
$$u_0 \in V_\delta,\ \eta_0 \in V_0,\ r_0 \in H^1(\Omega),\ \delta \in L^1(\Omega)$$

the optimal control problem (1)–(5), (18) admits at least one solution (g^0, h^0, η^0, u^0).

Proof. We apply for the proof the direct method of the calculus of variations. Let us take $\lambda \in \mathbb{R}_+$ large enough, such that

$$\Xi_\lambda = \{(g, h, \eta, u) \in \Xi \ : \ J(g, h, \eta, u) \le \lambda\} \ne \emptyset.$$

Since the cost functional (1) is bounded below on Ξ, this implies the existence of a minimizing sequence $\{(g_n, h_n, \eta_n, u_n)\}_{n \ge \mathbb{N}} \subset \Xi_\lambda$, where $\eta_n = w_n + \eta^*$. In [5], the authors have proved that this sequence is bounded in $L^2(0, T) \times L^2(0, T) \times (W_0(0, T) + \eta^*) \times W_\delta(0, T)$. Moreover, using (30)–(31), we get

$$\|\eta_{xx}\|_{L^2(0,T;L^2(\Omega))}^2 = \|w_{xx}\|_{L^2(0,T;L^2(\Omega))}^2 \le \|w\|_{L^2(0,T;H^2(\Omega))}^2 \le D_*,$$
$$\|u_{xt}\|_{L^2(0,T;H)}^2 \le \max\{1, \delta_0^{-1}\}\|\dot{u}\|_{L^\infty(0,T;V_\delta)}^2 \le D_*.$$

Therefore, within a subsequence, still denoted by the same index, we can suppose that

$$g_n \rightharpoonup g^0 \text{ in } L^2(0, T),\ h_n \rightharpoonup h^0 \text{ in } L^2(0, T),$$
$$u_n \to u^0 \text{ strongly in } L^2(0, T; H),$$
$$u_n \overset{*}{\rightharpoonup} u^0 \text{ weakly-}* \text{ in } L^\infty(0, T; V_\delta),$$
$$\dot{u}_n \rightharpoonup v \text{ weakly in } L^2(0, T; V_\delta) \text{ and weakly-}* \text{ in } L^\infty(0, T; V_\delta),$$

where $v = \dot{u}^0$ in the sense of distributions $\mathcal{D}'(0, T; V_\delta)$. Also, by Lemma 3.3 (see relation (33)), we get

$$\|u_n(t)\|_{V_\delta}^2 \le C_1 \quad \text{for all } n \in \mathbb{N} \text{ and for all } t \in [0, T],$$

whence, passing to a subsequence, if necessary, we obtain

$$u_n(T, \cdot) \rightharpoonup u^0(T, \cdot) \text{ in } V_\delta,$$
$$u_n(T, \cdot) \to u^0(T, \cdot) \text{ strongly in } H$$

due to the continuity of embedding $V_\delta \hookrightarrow V$ and the compactness of embedding $V \hookrightarrow H$. In view of this, lower semicontinuity of norms in $L^2(0,T)$, $L^2(\Omega)$ with respect to the weak convergence and the fact that

$$\eta_n(t,x) \rightharpoonup \eta^0(t,x) \text{ in } V_0,\ \dot{u}(t,x) \rightharpoonup \dot{u}^0(t,x) \text{ in } V_\delta \text{ for a.e. } t\in[0,T],$$
$$(\eta_n(t,x)+r_0(x)u_{nxt}(t,x)-\eta_Q) \rightharpoonup (\eta^0(t,x)+r_0(x)u^0_{xt}(t,x)-\eta_Q)) \text{ in } L^1(\Omega)$$
$$\text{for a.e. } t\in[0,T],$$
$$\int_\Omega a_Q(\eta_n(t,x)+r_0(x)u_{nxt}(t,x)-\eta_Q)\,dx \to$$
$$\to \int_\Omega a_Q(\eta^0(t,x)+r_0(x)u_{nxt}(t,x)-\eta_Q))\,dx \text{ for a.e. } t\in[0,T],$$
$$\lim_{n\to\infty}\int_0^T\left(\int_\Omega a_Q(\eta_n(t,x)+r_0(x)u_{nxt}(t,x)-\eta_Q)dx\right)^2 dt$$
$$=\int_0^T\left(\int_\Omega a_Q(\eta^0(t,x)+r_0(x)u_{nxt}(t,x)-\eta_Q))\right)^2 dt,$$

we have $J(g^0,h^0,\eta^0,u^0) \le \inf_{n\in\mathbb{N}} J(g_n,h_n,\eta_n,u_n)$. □

4. **Auxiliary results.** This section aims to prove a range of auxiliary results that will be used in the sequel. Throughout this section the tuple (g^0,h^0,η^0,u^0), where $\eta^0=w^0+\eta^*$ denotes an optimal solution to initial OCP problem (1)–(5).

The following proposition aims to prove rather technical result, however it is useful for substantiation of the first-order optimality conditions to the initial OCP (1)–(5).

Proposition 1. *Let $\delta\in H^1(\Omega)$. Then, for the initial data (15)-(17), the following inclusions take place*

$$u^0\left[u^0_{xx}\eta^0+2u^0_x\eta^0_x+\eta^0_{xx}u^0\right]-(\alpha_Q)^2\int_\Omega\left(\eta^0-\eta_Q\right)dx\in L^2(0,T;V^*),$$
$$\eta^0\left[u^0_{xx}\eta^0+2u^0_x\eta^0_x+\eta^0_{xx}u^0\right]\in L^2(0,T;V^*).$$

Proof. To begin with, let us prove that

$$\eta^0\left[u^0_{xx}\eta^0+2u^0_x\eta^0_x+\eta^0_{xx}u^0\right]\in L^2(0,T;V^*).$$

Obviously, in order to show that

$$u^0\left[u^0_{xx}\eta^0+2u^0_x\eta^0_x+\eta^0_{xx}u^0\right]-(\alpha_Q)^2\int_\Omega\left(\eta^0-\eta_Q\right)\,dx\in L^2(0,T;V^*)$$

it would be enough to apply the similar arguments. Since $\eta^0\in W(0,T;V)\hookrightarrow C(Q)$, it is enough to show that there exists $\widetilde{C}$ such that

$$\left\|u^0_{xx}\eta^0+2u^0_x\eta^0_x+\eta^0_{xx}u^0\right\|_{V^*}\le\widetilde{C} \text{ for a.a. } t\in[0,T].$$

It should be noticed that as far as

$$u^0_x\in L^2(\Omega;\delta\,dx)\hookrightarrow L^2(\Omega)\quad\text{for a.a. } t\in[0;T],$$

then $u^0_{xx}\in(H^1(\Omega))^*=V^*$.

Also the fact that $\eta^0\in H^2(\Omega)$ gives $\eta^0_{xx}\in L^2(\Omega)$ and $\eta^0_x\in H^1(\Omega)\hookrightarrow C(\overline{\Omega})$ for a.a. $t\in[0;T]$. Therefore, we have

$$
\begin{aligned}
&\left\|u^0_{xx}(t)\eta^0(t)+2u^0_x(t)\eta^0_x(t)+\eta^0_{xx}(t)u^0(t)\right\|_{V^*}\\
&\quad=\sup_{\|v\|_V\le 1}\langle u^0_{xx}(t)\eta^0(t)+2u^0_x(t)\eta^0_x(t)+\eta^0_{xx}(t)u^0(t),v\rangle_{V^*;V}\\
&\quad=\int_\Omega u^0_{xx}(t)\eta^0(t)v\,dx+2\int_\Omega u^0_x(t)\eta^0_x(t)v\,dx+\int_\Omega \eta^0_{xx}(t)u^0(t)v\,dx\\
&\quad\le \|\eta^0(t)\|_{C(\overline{\Omega})}\|v\|_V\|u^0_{xx}(t)\|_{V^*}+\|\eta^0_x(t)\|_{L^\infty(\Omega)}\|u^0_x(t)\|_H\|v\|_H\\
&\qquad+\|u^0\|_{C(\overline{\Omega})}\|\eta_{xx}(t)\|_H\|v\|_H\\
&\quad\le \|v\|_V\\
&\quad\times\underbrace{\left(\|\eta^0(t)\|_{C(\overline{\Omega})}\|u^0_{xx}\|_{V^*}+\|\eta^0_x(t)\|_{L^\infty(\Omega)}\|u^0_x(t)\|_{L^2(\Omega)}+\|u^0\|_{C(\overline{\Omega})}\|\eta_{xx}(t)\|_{L^2(\Omega)}\right)}_{C(t)}.
\end{aligned}
$$

It is clear that if only $\eta^0\in(W_0(0,T)+\eta^*)\cap L^2(0,T;H^2(\Omega)\cap V)$, then we have $\eta^0\in C(0,T;V)$, $\eta^0\in C(\overline{\Omega})$, and $\eta^0_x\in L^2(0,T;V)$. Moreover, from $(\delta u^0_x)_x=\delta_x u^0_x+\delta u^0_{xx}$ we can deduce that

$$
\|u^0_{xx}\|_{V^*}=\left\|\frac{1}{\delta}\left((\delta u^0_x)_x-\delta_x u^0_x\right)\right\|_{V^*}\le\frac{1}{\delta_0}\left(\|(\delta u^0_x)_x\|_{V^*}+\|\delta_x u^0_x\|_{V^*}\right) \tag{34}
$$

and

$$
\begin{aligned}
\|C(t)\|^2_{L^2(0;T)}\le{}&\frac{2}{\delta_0^2}\|\eta^0\|^2_{C(0,T;H)}\int_0^T\left(\|(\delta u^0_x)_x\|^2_{V^*}+\|\delta_x u^0_x\|^2_{V^*}\right)dt\\
&+\frac{2\max\{L,L^{-1}\}}{\delta_0}\int_0^T\|\eta^0_x\|^2_V\|u^0\|^2_{V_\delta}\,dt+\|u^0\|^2_{C(0,T;H)}\int_0^T\|\eta^0_{xx}\|^2_H\,dt\\
\le{}&\frac{2}{\delta_0^2}\|\eta^0\|^2_{C(0,T;H)}\int_0^T\left(\|(\delta u^0_x)_x\|^2_{V^*}+\|\delta_x u^0_x\|^2_{V^*}\right)dt\\
&+\frac{2\max\{L,L^{-1}\}}{\delta_0}\|u^0\|^2_{W^{1,\infty}(0,T;V_\delta)}\|\eta^0\|^2_{L^2(0,T;H^2)}\\
&+\|u^0\|^2_{C(0,T;H)}\|\eta^0\|^2_{L^2(0,T;H^2)}.
\end{aligned} \tag{35}
$$

Let us show that the integrals $\int_0^T\|\delta_x u^0_x\|^2_{V^*}\,dt$ and $\int_0^T\|(\delta u^0_x)_x\|^2_{V^*}\,dt$ are finite. We take into account the continuous embedding $V\hookrightarrow C(\overline{\Omega})$. Then $\exists\, c(E)$ such that $\|v\|_{C(\overline{\Omega})}\le c(E)\|v\|_V$, for all $v\in V$. As for the first integral, we have

$$
\begin{aligned}
\int_0^T\|\delta_x u^0_x(t)\|^2_{V^*}\,dt&=\int_0^T\left(\sup_{\|v\|_V\le 1}\int_\Omega|\delta_x||u^0_x(t)||v|\,dx\right)^2dt\\
&\le\int_0^T\left(\sup_{\|v\|_V\le 1}\|v\|_{C(\overline{\Omega})}\|\delta\|_V\|u(t)\|_V\right)^2dt\\
&\le\frac{c^2(E)}{\delta_0}\|v\|^2_V\|\delta\|^2_V\|u\|^2_{L^2(0,T;V_\delta)}\le\frac{c^2(E)T}{\delta_0}\|\delta\|^2_V\|u\|^2_{L^\infty(0,T;V_\delta)}.
\end{aligned}
$$

Now, to estimate the second integral, we make use of the equation $(2)_2$ and the well known inequality $(a+b+c)^2\le 3(a^2+b^2+c^2)$.

$$\int_0^T \|(\delta u_x^0)_x\|_{V^*}^2\,dt = \int_0^T \left(\sup_{\|v\|_V\le 1}\int_\Omega |(\delta u_x^0)_x v|\,dx\right)^2 dt$$
$$= \int_0^T \Big(\sup_{\|v\|_V\le 1}\int_\Omega \Big|\Big[\int_0^t (f(s)-u^0(s)u_x^0(s)-\mu(s)\eta_x^0(s))ds + u^0(t)$$
$$+ u_0 + (\delta(u_0)_x)_x\Big]v\Big|\,dx\Big)^2 dt$$
$$\le \int_0^T 2\left(\sup_{\|v\|_V\le 1}\int_\Omega \left|\int_0^t (f(s)v-u^0(s)u_x^0(s)v-\mu(s)\eta_x^0(s)v)ds\right|\,dx\right)^2 dt$$
$$+ \int_0^T 2\left(\sup_{\|v\|_V\le 1}\int_\Omega \left|(u^0(t)+u_0+(\delta(u_0)_x)_x)v\right|\,dx\right)^2 dt$$
$$\le \int_0^T 2\left(\sup_{\|v\|_V\le 1}\int_\Omega\int_0^T |f(s)v-u^0(s)u_x^0(s)v-\mu(s)\eta_x^0(s)v)|ds\,dx\right)^2 dt$$
$$+ \int_0^T 2\left(\sup_{\|v\|_V\le 1}\left[\|u^0(t)\|_V\|v\|_V + \|u_0\|_V\|v\|_V + \|(\delta(u_0)_x)_x\|_{V^*}\|v\|_V\right]\right)^2 dt$$
$$\le \int_0^T 2\left(\sup_{\|v\|_V\le 1}\int_0^T\int_\Omega \left[|f(s)v|+|u^0(s)u_x^0(s)v|+|\mu(s)\eta_x^0(s)v|\right]\,dx\,ds\right)^2 dt$$
$$+ \int_0^T 6\left(\left[\|u^0(t)\|_V^2 + \|u_0\|_V^2 + \|(\delta(u_0)_x)_x\|_{V^*}^2\right]\right)^2 dt$$
$$\le \int_0^T 2\Big(\sup_{\|v\|_V\le 1}\int_0^T \Big(\|f(t)\|_H\|v\|_V + \|u^0(t)\|_{C(\overline{\Omega})}\|u^0(t)\|_V\|v\|_V$$
$$+ \|\mu(t)\|_H\|\eta^0(t)\|_V\|v\|_{C(\overline{\Omega})}\Big)\,ds\Big)^2 dt$$
$$+ \frac{6T}{\delta_0}\|u^0\|_{L^\infty(0,T;V_\delta)}^2 + 6T\|u_0\|_V^2 + 6T\|(\delta(u_0)_x)_x\|_{V^*}^2$$
$$\le 6T\Big[T\|f\|_{L^2(0,T;H)}^2 + (c(E))^2\max\{1,\delta_0^{-1}\}T\|u^0\|_{L^\infty(0,T;V_\delta)}^4$$
$$+ (c(E))^2\|\mu\|_{L^2(0,T;H)}^2\|\eta^0\|_{L^2(0,T;V)}^2\Big]$$
$$+ \frac{6T}{\delta_0}\|u^0\|_{L^\infty(0,T;V_\delta)}^2 + 6T\|u_0\|_V^2 + 6T\|(\delta(u_0)_x)_x\|_{V^*}^2 < +\infty.$$

It is worth to mention here that, in fact, $(\delta(u_0)_x)_x \in (H^1(\Omega))^*$ because the element $\delta(u_0)_x$ belongs to $L^2(\Omega)$. Indeed,

$$\int_\Omega (\delta(u_0)_x)^2\,dx \le \|\delta\|_{C(\overline{\Omega})}\int_\Omega \delta((u_0)_x)^2\,dx \le c(E)\|\delta\|_V\|u_0\|_{V_\delta}.$$

It remains to note that the property $\int_0^T \left(\int_\Omega \left(\eta^0-\eta_Q\right)\,dx\right)^2 dt < \infty$ can be rewritten as follows $\int_\Omega \left(\eta^0-\eta_Q\right)\,dx \in L^2(0,T)$. □

Let us consider two operators γ_1 and γ_2 that define the restriction of the functions from $V = H^1(\Omega)$ to the boundary $\partial\Omega = \{x=L, x=0\}$, respectively (i.e. $\gamma_1[u(t,\cdot)] = u(t,L)$ and $\gamma_2[u(t,\cdot)] = u(t,0)$). Also we put into consideration two operators

$$A, B : L^2(0,T;V_0) \times L^2(0,T;V_\delta) \to \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2,$$

defined on the set of vector functions $\mathbf{p} = (p,q)^t \in L^2(0,T;V_0) \times L^2(0,T;V_\delta)$ by the rule

$$(A\mathbf{p})(t) := A(t)\mathbf{p}(t) = \begin{pmatrix} p(t) \\ q(t) - (\delta q_x(t))_x \\ \gamma_1[\delta q_x(t)] \\ -\gamma_2[\delta q_x(t)] \end{pmatrix}, \tag{36}$$

$$(B\mathbf{p})(t) := B(t)\mathbf{p}(t) = \begin{pmatrix} u^0 p_x(t) + \nu p_{xx}(t) + (\mu q)_x(t) \\ \left(\eta^0 + \dfrac{1}{2} r_0\right) p_x(t) + \frac{1}{2}(r_0)_x\, p(t) + u^0 q_x(t) \\ -(\sigma_1(t) + \gamma_1[u^0])\gamma_1[q(t)] \\ (\sigma_0(t) + \gamma_2[u^0])\gamma_2[q(t)] \end{pmatrix}. \tag{37}$$

Here, we use the fact that $V_\delta^* = V_0^* \oplus H^{-1/2}(\partial\Omega)$, which in one-dimensional case obviously turns to $V^* = V_0^* \oplus \mathbb{R} \oplus \mathbb{R}$ and, hence, $L^2(0,T;V_\delta^*) = L^2(0,T;V_0) \oplus L^2(0,T) \oplus L^2(0,T)$. Then the following result holds true.

Lemma 4.1. *The operator $A(t) : V_0 \times V_\delta \to [V_0^*]^2 \times \mathbb{R} \times \mathbb{R}$, defined by (36), satisfies the following conditions:*

$A(t)$ is radially continuous, i.e. for any fixed $\mathbf{v}_1,\, \mathbf{v}_2 \in V_0 \times V_\delta := \widetilde{V}$ and almost each $t \in (0,T)$ the real-valued function $s \to \langle A(t)(\mathbf{v}_1 + s\mathbf{v}_2), \mathbf{v}_2\rangle_{\widetilde{V}^*;\widetilde{V}}$ is continuous in $[0,1]$;

for some constant C and some function $g \in L^2(0,T)$

$$\|A(t)\mathbf{v}\|_{\widetilde{V}^*} \le C\|\mathbf{v}\|_{\widetilde{V}} + g(t), \quad \text{for a.e.} \quad t \in [0,T], \ \forall\, \mathbf{v} \in \widetilde{V};$$

it is strictly monotone uniformly with respect to $t \in [0,T]$ in the following sense: there exists a constant $m > 0$, independent of t, such that

$$\langle A(t)\mathbf{v}_1 - A(t)\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\rangle_{\widetilde{V}^*;\widetilde{V}} \ge \|\mathbf{v}_1^1 - \mathbf{v}_2^1\|_H^2 + m\|\mathbf{v}_1^2 - \mathbf{v}_2^2\|_{V_\delta}^2,$$
$$\forall\, \mathbf{v}_1,\, \mathbf{v}_2 \in \widetilde{V} \text{ and for a.e. } t \in [0,T].$$

Moreover, the operator $B : L^2(0,T;V_0) \times L^2(0,t;V_\delta) \to \left[L^2(0,T;V_0^*)\right]^2 \times L^2(0,T) \times L^2(0,T)$ possesses the Lipschitz property, i.e. there exists a constant $L > 0$ such that

$$\|B\mathbf{v}_1 - B\mathbf{v}_2\|_{L^2(0,T;\widetilde{V}^*)} \le L\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(0,T;\widetilde{V})}, \text{ for all } \mathbf{v}_1,\, \mathbf{v}_2 \in L^2(0,T;\widetilde{V}).$$

Proof. Since the radial continuity of operator A is obvious, we begin with the proof of the second property. Let $\mathbf{v} = (v,w)$, $\mathbf{z} = (z,y) \in \widetilde{V}$ be arbitrary elements. Then

$$\begin{aligned} \|A(t)\mathbf{v}\|_{\widetilde{V}^*} &= \sup_{\|\mathbf{z}\|_{\widetilde{V}} \le 1} |\langle A(t)\mathbf{v}, \mathbf{z}\rangle_{\widetilde{V}^*;\widetilde{V}}| \\ &= \sup_{\|z\|_{V_0} + \|y\|_{V_\delta} \le 1} \Big| \int_\Omega (vz + wy)\, dx - \int_\Omega (\delta w_x)_x y\, dx \\ &\qquad + \delta(L) w_x(L) y(L) - \delta(0) w_x(0) y(0) \Big| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|\mathbf{z}\|_{\widetilde{V}}\le 1} \left| \int_\Omega (vz + wy)\,dx + \int_\Omega \delta w_x y_x\,dx \right| \\
&\le \sup_{\|\mathbf{z}\|_{\widetilde{V}}\le 1} (\|v\|_H \|z\|_H + \|w\|_H \|y\|_H + \|w\|_{V_\delta} \|y\|_{V_\delta}) \\
&\le 2(\|v\|_{V_0} + \|y\|_{V_\delta}) = 2\|\mathbf{v}\|_{\widetilde{V}}.
\end{aligned}
$$

As for the monotonicity property, for every $\mathbf{p}_1, \mathbf{p}_2 \in V_0 \times V_\delta$, we have

$$
\begin{aligned}
\langle A(t)\mathbf{p}_1 - A(t)\mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2\rangle_{\widetilde{V}^*;\widetilde{V}} &= \int_\Omega (p_1 - p_2)^2\,dx + \int_\Omega (q_1 - q_2)^2\,dx \\
&- \int_\Omega [(\delta(q_1)_x)_x - (\delta(q_2)_x)_x]\,(q_1 - q_2)\,dx \\
&+ [\delta(L)(q_1(\cdot,L))_x - \delta(L)(q_2(\cdot,L))_x]\,(q_1(\cdot,L) - q_2(\cdot,L)) \\
&- [\delta(0)(q_1(\cdot,0))_x - \delta(0)(q_2(\cdot,0))_x]\,(q_1(\cdot,0) - q_2(\cdot,0)) \\
&= \|p_1 - p_2\|_H + \|q_1 - q_2\|_H + \|q_1 - q_2\|^2_{L^2(\Omega,\delta\,dx)}.
\end{aligned}
$$

It remains to show the Lipschitz continuity of operator $B(t)$. With that in mind we consider three vector-valued functions $\mathbf{v} = (v_1, v_2)^t$, $\mathbf{w} = (w_1, w_2)^t$ and $\mathbf{z} = (z_1, z_2)^t$. Then

$$
\begin{aligned}
&\|B\mathbf{v} - B\mathbf{w}\|_{L^2(0,T;\widetilde{V}^*)} = \sup_{\|\mathbf{z}\|_{\widetilde{V}\le 1}} \left| \langle B\mathbf{v} - B\mathbf{w}, \mathbf{z}\rangle_{\widetilde{V}^*;\widetilde{V}} \right| \\
&= \int_0^T \Big[\left| (u^0(t)(v_{1x}(t) - w_{1x}(t)), z_1(t))_H \right| + \nu \left| (v_{1x}(t) - w_{1x}(t), z_{1x}(t))_H \right| \\
&\quad + |(\mu_x(v_2(t) - w_2(t)), z_1(t))_H| + |(\mu(v_{2x}(t) - w_{2x}(t)), z_1(t))_H| \\
&\quad + \frac{1}{2} \left| \left((r_0 + 2\eta^0)(v_{1x}(t) - w_{1x}(t)), z_2(t) \right)_H \right| \\
&\quad + \frac{1}{2} \left| ((r_0)_x(v_1(t) - w_1(t)), z_2(t))_H \right| + \left| (u^0(t)(v_{2x}(t) - w_{2x}(t)), z_2(t))_H \right| \\
&\quad + \left| (\sigma_1(t) + u^0(t,L))(v_2(t,L) - w_2(t,L)) z_2(t,L) \right| \\
&\quad + \left| (\sigma_0(t) + u^0(t,0))(v_2(t,0) - w_2(t,0)) z_2(t,0) \right| \Big]\,dt \\
&\le \|u^0\|_{C(Q)} \|v_1 - w_1\|_{L^2(0,T;V_0)} \|z_1\|_{L^2(0,T;V_0)} + \nu \|v_1 - w_1\|_{L^2(0,T;V_0)} \|z_1\|_{L^2(0,T;V_0)} \\
&+ \int_0^T \Big(2\|z\|_{C(\overline{\Omega})} \delta_0^{-1/2} \|\mu\|_V \|v_2 - w_2\|_{V_\delta} + \frac{1}{2} (\|r_0 + 2\eta^0\|_H \\
&\quad + \|r_0\|_V) \|v_1 - w_1\|_V \|z_2\|_{C(\overline{\Omega})} \Big)\,dt + \|u^0\|_{C(Q)} \delta_0^{-1} \|v_2 - w_2\|_{V_\delta} \|z_2\|_{V_\delta} \\
&\quad + \int_0^T \Big(|\sigma_1(t)| + |\sigma_0(t)| + 2\|u^0(t)\|_{C(\overline{\Omega})} \Big) \|v_2(t) - w_2(t)\|_{C(\overline{\Omega})}\,dt.
\end{aligned}
$$

Taking into account the continuous embedding $V_\delta, V_0 \hookrightarrow C(\overline{\Omega})$ and the corresponding inequality

$$
\|v\|_{C(\overline{\Omega})} \le c(E)\|v\|_V \le c(E)\delta_0^{-1/2}\|v\|_{V_\delta},
$$

we finally have

$$
\|B\mathbf{v} - B\mathbf{w}\|_{L^2(0,T;\widetilde{V}^*)} \le L\|\mathbf{v} - \mathbf{w}\|_{L^2(0,T;\widetilde{V})},
$$

where $L = \max\{C_1; C_2\}$ and

$$C_1 = \|u^0\|_{C(Q)} + \nu + c(E)(\|r_0\|_V + \|\eta^0\|_{C(0,T;H)}),$$
$$C_2 = 2c(E)\delta_0^{-1}\|\mu\|_{L^\infty(0,T;V)} + \|u^0\|_{C(Q)}\delta_0^{-1} + c(E)(\|\sigma_1\|_{L^2(0,T)} + \|\sigma_2\|_{L^2(0,T)} + 2\|u^0\|_{C(Q)}).$$

This concludes the proof. □

Lemma 4.2. *Operator*

$$A : L^2(0,T;V_0) \times L^2(0,T;V_\delta) \to \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2,$$

which is defined by (36), is radially continuous, strictly monotone and there exists an inverse Lipschitz-continuous operator

$$A^{-1} : \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2 \to L^2(0,T;V_0) \times L^2(0,T;V_\delta)$$

such that

$$(A^{-1}f)(t) = A^{-1}(t)f(t) \text{ for a.e. } t \in [0,T]$$
$$\text{and for all } f \in \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2,$$

where $A^{-1}(t) : [V_0^*]^2 \times \mathbb{R} \times \mathbb{R} \to V_0 \times V_\delta$ *is an inverse operator to*

$$A(t) : V_0 \times V_\delta \to [V_0^*]^2 \times \mathbb{R} \times \mathbb{R}.$$

Proof. It is easy to see that the action of operator $A(t)$ on element $\mathbf{p} = (p,q)^t$ can be also given by the rule:

$$A(t)\mathbf{p}(t) = \begin{pmatrix} A_1(t)p(t) \\ A_2(t)q(t) \end{pmatrix},$$
$$A_1 : L^2(0,T;V_0) \to L^2(0,T;V_0^*),$$
$$A_2 : L^2(0,T;V_\delta) \to L^2(0,T;V_0^*) \times L^2(0,T) \times L^2(0,T),$$

where

$$A_1(t)p(t) = p(t) \text{ and } A_2(t)q(t) = \begin{pmatrix} q(t) - (\delta q_x(t))_x \\ \gamma_1[\delta q_x(t)] \\ -\gamma_2[\delta q_x(t)] \end{pmatrix}.$$

It is easy to see, that $A_1(t)$ is the identity operator. Therefore, $A_1^{-1}(t) \equiv A_1(t)$. As for the operator $A_2(t)$, it is strongly monotone for all $t \in [0,T]$ because

$$\langle (A_2q_1)(t) - (A_2q_2)(t), q_1(t) - q_2(t)\rangle_{V_\delta^*;V_\delta} = \|q_1 - q_2\|_{V_\delta}.$$

Moreover, $A_2(t)$ satisfies all preconditions of [11, Lemma 2.2] that establishes the existence of a Lipschitz continuous inverse operator

$$A_2^{-1} : L^2(0,T;V_0^*) \times L^2(0,T) \times L^2(0,T) \to L^2(0,T;V_\delta)$$

such that

$$(A_2^{-1}f)(t) = A_2^{-1}(t)f(t) \text{ for a.e. } t \in [0,T] \text{ and } \forall f \in \left[L^2(0,T;V_0^*)\right] \times \left[L^2(0,T)\right]^2,$$

where $A_2^{-1}(t) : [V_0^*] \times \mathbb{R} \times \mathbb{R} \to V_\delta$ is an inverse operator to $A_2(t) : V_\delta \to V_0^* \times \mathbb{R} \times \mathbb{R}$. The proof is complete. □

Before proceeding further, we make use of the following result concerning the solvability of Cauchy problems for pseudoparabolic equations (for the proof we refer to [11, Theorem 2.4]).

Theorem 4.3. *For operators*

$$A, B : L^2(0,T;V_0) \times L^2(0,T;V_\delta) \to \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2$$

defined in (36),(37)*, and for any*

$$F \in \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2 \quad \text{and} \quad b \in V_0^* \times V_\delta^*,$$

the Cauchy problem

$$(A(t)\mathbf{p})'_t + B(t)\mathbf{p} = F(t),$$
$$A(T)\mathbf{p}(T) = b$$

admits a unique solution.

5. **First-order optimality conditions.** In this section we focus on the derivation of the first-order optimality conditions for optimization problem (1)–(5). The Lagrange functional

$$\mathcal{L} : \Big(W_0(0,T) \cap L^2(0,T;H^2(\Omega)\cap V_0)\Big) \times W^{1,\infty}(0,T;V_\delta) \times L^2(0,T) \times L^2(0,T) \times \mathbb{R}$$
$$\times \Big(W_0(0,T) \cap L^2(0,T;H^2(\Omega)\cap V_0)\Big) \times W^{1,\infty}(0,T;V_\delta) \to \mathbb{R},$$

associated to problem (1)–(5) (see also Remark 2) is defined by

$$\begin{aligned}
&\mathcal{L}(w,u,g,h,\lambda,p,q) = \lambda J(g,h,w,u)\\
&\quad - \int_0^T \left[\langle A_1(w,u), p\rangle_{V_0^*;V_0} + \langle A_2(w,u), q\rangle_{V_\delta^*;V_\delta}\right] dt\\
&\quad = \lambda J(g,h,w,u)\\
&\quad - \int_0^T \left[\langle \dot w, p\rangle_{V_0^*;V_0} - \nu\langle w_{xx}, p\rangle_{V_0^*;V_0} + ((wu)_x, p)_H + \frac12((r_0+2\eta^*)u_x, p)_H\right] dt\\
&\quad - \int_0^T \left[\langle \dot u - (\delta \dot u_x)_x, q\rangle_{V_\delta^*;V_\delta} + \frac12\left((u^2)_x, q\right)_H + (\mu w_x, q)_H - (f,q)_H\right] dt\\
&\quad - \int_0^T \Big[\left(\delta(L)\dot u_x(t,L) + \sigma_1(t)u(t,L) - h\right) q(t,L)\\
&\qquad - \left(\delta(0)\dot u_x(t,0) + \sigma_0(t)u(t,0) - g\right) q(t,0)\Big] dt\\
&\quad = \lambda J(g,h,w,u)\\
&\quad - \int_0^T \left[\langle \dot w, p\rangle_{V_0^*;V_0} - \nu\langle w_{xx}, p\rangle_{V_0^*;V_0} + ((wu)_x, p)_H + \frac12((r_0+2\eta^*)u_x, p)_H\right] dt\\
&\quad - \int_0^T \left[\langle \dot u, q\rangle_{V_\delta^*;V_\delta} + \int_\Omega \delta \dot u_x q_x\,dx + \frac12\left((u^2)_x, q\right)_H + (\mu w_x, q)_H - (f,q)_H\right] dt\\
&\quad - \int_0^T \left[\sigma_1(t)u(t,L)q(t,L) - h(t)q(t,L) - \sigma_0(t)u(t,0)q(t,0) + g(t)q(t,0)\right] dt.
\end{aligned}$$

Let us shift the correspondent derivatives from w and u to Lagrange multipliers p and q, taking into account the initial and boundary conditions (3)–(4):

$$\begin{aligned}
&\mathcal{L}(w,u,g,h,\lambda,p,q) = \lambda J(g,h,w,u)\\
&\quad + \int_0^T \left[\langle w, \dot p\rangle_{V_0^*;V_0} + \nu\langle w, p_{xx}\rangle_{V_0^*;V_0} + (wu, p_x)_H + \frac12(u, ((r_0+2\eta^*)p)_x)_H\right] dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega p(T)w(T)\,dx + \int_\Omega p(0)w(0)\,dx \\
&+ \int_0^T \left[\langle u, \dot q\rangle_{V_\delta^*;V_\delta} + \int_\Omega \delta u_x \dot q_x\,dx + \frac{1}{2}\left(u^2, q_x\right)_H + (w,(\mu q)_x)_H + (f,q)_H \right] dt \\
&- \langle u(T,\cdot), q(T,\cdot)\rangle_{V_\delta^*;V_\delta} - \int_\Omega \delta u_x(T) q_x(T)\,dx \\
&+ \langle u(0,\cdot), q(0,\cdot)\rangle_{V_\delta^*;V_\delta} + \int_\Omega \delta u_x(0) q_x(0)\,dx \\
&- \int_0^T \left[\sigma_1(t)u(t,L)q(t,L) - h(t)q(t,L) - \sigma_0(t)u(t,0)q(t,0) + g(t)q(t,0)\right] dt \\
=&\lambda J(g,h,w,u) \\
&+ \int_0^T \left[\langle w, \dot p\rangle_{V_0^*;V_0} + \nu\langle w, p_{xx}\rangle_{V_0^*;V_0} + (wu, p_x)_H + \frac{1}{2}(u,((r_0+2\eta^*)p)_x)_H \right] dt \\
&- \int_\Omega p(T)w(T)\,dx + \int_\Omega p(0)w(0)\,dx \\
&+ \int_0^T \left[\langle u, \dot q\rangle_{V_\delta^*;V_\delta} + \int_\Omega \delta u_x \dot q_x\,dx + \frac{1}{2}\left(u^2, q_x\right)_H + (w,(\mu q)_x)_H + (f,q)_H \right] dt \\
&- \langle u(T,\cdot), q(T,\cdot) - (\delta q_x(T,\cdot))_x\rangle_{V_\delta^*;V_\delta} - \delta(L)u(T,L)q_x(T,L) \\
&+ \delta(0)u(T,0)q_x(T,0) + \langle u(0,\cdot) - (\delta u_x(0,\cdot))_x, q(0,\cdot)\rangle_{V_\delta^*;V_\delta} \\
&- \int_0^T \left[\sigma_1(t)u(t,L)q(t,L) - h(t)q(t,L) - \sigma_0(t)u(t,0)q(t,0) + g(t)q(t,0)\right] dt \\
&- \frac{1}{2}\int_0^T (u^2(t,L)q(t,L) - u^2(t,0)q(t,0))\,dt \\
=&\lambda J(g,h,w,u) \\
&+ \int_0^T \left[\langle w, \dot p\rangle_{V_0^*;V_0} + \nu\langle w, p_{xx}\rangle_{V_0^*;V_0} + (wu, p_x)_H + \frac{1}{2}(u,((r_0+2\eta^*)p)_x)_H \right] dt \\
&- \int_\Omega p(T)w(T)\,dx + \int_\Omega p(0)w(0)\,dx \\
&+ \int_0^T \left[\langle u, \dot q - (\delta \dot q_x)_x\rangle_{V_\delta^*;V_\delta} + \frac{1}{2}\left(u^2, q_x\right)_H + (w,(\mu q)_x)_H + (f,q)_H \right] dt \\
&- \langle u(T,\cdot), q(T,\cdot) - (\delta q_x(T,\cdot))_x\rangle_{V_\delta^*;V_\delta} \\
&- \int_0^T \left[(\sigma_1(t)q(t,L) - (\delta(L)\dot q_x(t,L))u(t,L) - h(t)q(t,L)\right] dt \\
&- \int_0^T \left[\sigma_0(t)(q(t,0) - (\delta(0)\dot q_x(t,0))u(t,0) - g(t)q(t,0)\right] dt \\
&- \frac{1}{2}\int_0^T (u^2(t,L)q(t,L) - u^2(t,0)q(t,0))\,dt.
\end{aligned}
$$

For each fixed $(p,q) \in \Big(W_0(0,T) \cap L^2(0,T;H^2(\Omega)\cap V_0)\Big) \times W^{1,\infty}(0,T;V_\delta)$ the Lagrangian is continuously Frechet-differentiable with respect to

$$
(w,u,g,h) \in \Big(W_0(0,T)\cap L^2(0,T;H^2(\Omega)\cap V_0)\Big)\times W^{1,\infty}(0,T;V_\delta)\times L^2(0,T)\times L^2(0,T).
$$

Notice that, for a fixed t, we have $u \in V \subset C(\overline{\Omega})$ and $w \in V_0 \subset C(\overline{\Omega})$, hence, the inner products $(w_x(t)u(t) + w(t)u_x(t), p(t))_H$ and $(u(t)u_x(t), q(t))_H$ are correctly defined almost everywhere in $[0, T]$.

Further we make use of the following relation $\eta_t = -\frac{1}{2} r_0 u_x$ that was introduced in [3]. Substituting this one to (2), we have $\nu \eta_{xx} = (\eta u)_x = \eta_x u + u_x \eta$.

Also, to simplify the deduction and in order to avoid the demanding of the increased smoothness on solutions of the initial Boussinesq system (2)–(5), we use (see [4] and [5]) elastic model for the hydrodynamic pressure

$$P(t,x) = P_{ext} + \frac{\beta}{r_0^2}\eta$$

instead of the inertial one

$$P = P_{ext} + \frac{\beta}{r_0^2}\eta + \rho_\omega h \frac{\partial^2 \eta}{\partial t^2} = P_{ext} + \frac{\beta}{r_0^2}\eta - \frac{1}{2}\rho_\omega h r_0 u_{xt}. \tag{38}$$

Indeed, if we suppose the wall thickness h to be small enough, the last term in the inertial model (38) appears negligible.

As a result, the cost functional $J(g, h, w, u)$, where $\eta = w + \eta^*$, takes the form

$$\begin{aligned} J(g,h,w,u) = \frac{1}{2}\int_\Omega \alpha_\Omega \left(u(T) - u_\Omega\right)^2 \, dx + \frac{1}{2}\int_0^T \int_\Omega \left((w(t)u(t))_x + u_x(t)\eta^*\right)^2 dx\, dt \\ + \frac{1}{2}\int_0^T \left| \int_\Omega \alpha_Q \Big(w(t) + \eta^* - \eta_Q\Big)\, dx \right|^2 dt \\ + \frac{1}{2}\int_0^T \Big(\beta_g |g|^2 + \beta_h |h|^2\Big)\, dt. \end{aligned} \tag{39}$$

In order to formulate the conjugate system for an optimal solution (g^0, h^0, η^0, u^0), where $\eta^0 = w^0 + \eta^*$, we have to find the Fréchet differentials $\mathcal{L}_w z$ and $\mathcal{L}_u v$, where

$$z \in W_0(0,T) \cap L^2(0,T; H^2(\Omega) \cap V_0) \quad \text{and} \quad v \in W^{1,\infty}(0,T; V_\delta) \times L^2(0,T).$$

With that in mind we emphasize the following point. Since the elements

$$w + z \in W_0(0,T) \cap L^2(0,T; H^2(\Omega) \cap V_0) \quad \text{and} \quad u + v \in W^{1,\infty}(0,T; V_\delta) \times L^2(0,T)$$

are some admissible solutions to OCP (39), (2)–(5), it follows that the increments z and v satisfy the homogeneous initial and boundary conditions, i.e.

$$\begin{cases} z(0,\cdot) = 0 \quad \text{in } \Omega, \\ v(0,\cdot) - \left(\delta(\cdot) v_x(0,\cdot)\right)_x = 0 \quad \text{in } \Omega, \end{cases} \tag{40}$$

$$\begin{cases} z(\cdot,0) = z(\cdot,L) = 0 \quad \text{in } (0,T), \\ \delta(0)\dot{v}_x(\cdot,0) + \sigma_0 v(\cdot,0) = 0, \quad \text{in } (0,T), \\ \delta(L)\dot{v}_x(\cdot,L) + \sigma_1 v(\cdot,L) = 0, \quad \text{in } (0,T), \\ \delta(L)v_x(0,L) = \delta(0)v_x(0,0) = 0. \end{cases} \tag{41}$$

Taking into account the definition of the Fréchet derivative of nonlinear mappings, we get

$$J(g,h,w+z,u) = J(g,h,w,u) + J_w z + R_0(w,z),$$

where $R_0(w,z)$ stands for the remainder, which takes the form

$$R_0(w,z)=\frac{1}{2}\int_0^T\int_\Omega ((zu)_x)^2\,dx\,dt+\int_0^T\left|\int_\Omega a_Q z(t)\right|^2\,dt, \tag{42}$$

and

$$\begin{aligned}
J_w z &= J(g,h,w+z,u)-J(g,h,w,u)-R_0(w,z)\\
&=\frac{1}{2}\int_0^T\int_\Omega \big(((w(t)+z(t))u(t))_x+u_x(t)\eta^*\big)^2\,dx\,dt\\
&\quad-\frac{1}{2}\int_0^T\int_\Omega \big((w(t)u(t))_x+u_x(t)\eta^*\big)^2\,dx\,dt\\
&\quad+\frac{1}{2}\int_0^T\left|\int_\Omega \alpha_Q\Big(w(t)+z(t)+\eta^*-\eta_Q\Big)\,dx\right|^2\,dt\\
&\quad-\frac{1}{2}\int_0^T\left|\int_\Omega \alpha_Q\Big(w(t)+\eta^*-\eta_Q\Big)\,dx\right|^2\,dt\\
&=\int_0^T\int_\Omega \big((w(t)u(t))_x+u_x(t)\eta^*\big)\big((z(t)u(t))_x\big)\,dx\,dt\\
&\quad+\int_0^T\left(\int_\Omega \alpha_Q\Big(w(t)+\eta^*-\eta_Q\Big)\,dx\right)\left(\int_\Omega \alpha_Q z(t)\,dx\right)\,dt\\
&=\int_0^T\int_\Omega (w_x u+u_x w+u_x\eta^*)(u_x z+z_x u)\,dx\,dt\\
&\quad+\alpha_Q^2\int_0^T\int_\Omega\left(\int_\Omega\Big(w(t)+\eta^*-\eta_Q\Big)\,dx\right)z(t)\,dx\,dt\\
&=\int_0^T\int_\Omega\Big[(w_x u_x u+(u_x)^2 w+(u_x)^2\eta^*)-(w_x u^2+u_x uw+u_x u\eta^*)_x\Big]z(t)\,dx\,dt\\
&\quad+\alpha_Q^2\int_0^T\int_\Omega\left(\int_\Omega\Big(w(t)+\eta^*-\eta_Q\Big)\,dx\right)z(t)\,dx\,dt.
\end{aligned}$$

It is obviously follows from (42) that

$$\frac{|R_0(w,x)|}{\|z\|_{L^2(0,T;H^2(\Omega)\cap V_0)}}\to 0\quad\text{as}\quad \|z\|_{L^2(0,T;H^2(\Omega)\cap V_0)}\to 0.$$

Hence, after some transformations, we have

$$\begin{aligned}
J_w z=\int_0^T\int_\Omega\Big(&-u\,[u_{xx}(w+\eta^*)+2u_x w_x+w_{xx}u]\\
&+\alpha_Q^2\int_\Omega\Big(w(t)+\eta^*-\eta_Q\Big)\,dx\Big)z(t)\,dx\,dt.
\end{aligned} \tag{43}$$

Treating similarly to the other derivative, we obtain

$$J(g,h,w,u+v)=J(g,h,w,u)+J_u v+\widetilde{\mathcal{R}}_0(u,v),$$

where the remainder $\widetilde{\mathcal{R}}_0(u,v)$ takes the form

$$\widetilde{\mathcal{R}}_0(u,v)=\frac{1}{2}\int_\Omega a_\Omega v^2(T)\,dx+\frac{1}{2}\int_0^T\int_\Omega ((wv)_x+v_x\eta^*)^2\,dx\,dt, \tag{44}$$

$$|\widetilde{\mathcal{R}}_0(u,v)|/\|v\|_{W^{1,\infty}(0,T;V_\delta)}\to 0\quad\text{as}\quad \|v\|_{W^{1,\infty}(0,T;V_\delta)}\to 0,$$

and

$$\begin{aligned}
J_u v &= J(g,h,w,u+v) - J(g,h,w,u) - \widetilde{\mathcal{R}}_0(u,v) \\
&= \frac{1}{2}\int_\Omega \alpha_\Omega \left(u(T)+v(T)-u_\Omega\right)^2 \, dx - \frac{1}{2}\int_\Omega \alpha_\Omega \left(u(T)-u_\Omega\right)^2 \, dx \\
&\quad + \frac{1}{2}\int_0^T \int_\Omega \left((w(t)(u(t)+v(t)))_x + (u_x(t)+v_x(t))\eta^*\right)^2 dx\, dt \\
&\quad - \frac{1}{2}\int_0^T \int_\Omega \left((w(t)u(t))_x + u_x(t)\eta^*\right)^2 dx\, dt \\
&= \int_\Omega \alpha_\Omega (u(T)-u_\Omega) v(T)\, dx \\
&\quad - \int_0^T \int_\Omega (w+\eta^*)\left[u_{xx}(w+\eta^*) + 2u_x w_x + w_{xx} u\right] \, dx\, dt \\
&\quad + \int_0^T \eta^*((w^0(t,L)u^0(t,L))_x + u_x^0 \eta^*) v(t,L)\, dt \\
&\quad - \int_0^T \eta^*((w^0(t,0)u^0(t,0))_x + u_x^0(t,0)\eta^*) v(t,0)\, dt. \qquad (45)
\end{aligned}$$

We are now in a position to identify the Fréchet derivatives $\mathcal{L}_w$ and $\mathcal{L}_v$ of the Lagrangian. Following in a similar manner, we have

$$\begin{aligned}
\mathcal{L}_w z = \lambda J_w z &+ \int_0^T \Big[\langle z, \dot{p}\rangle_{V_0^*;V_0} + \nu \langle z, p_{xx}\rangle_{V_0^*;V_0} + (zu, p_x)_H + (z, (\mu q)_x)_H\Big] \, dt \\
&- \langle z(T), p(T)\rangle_{V_0^*;V_0}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_u v = \lambda J_u v &+ \int_0^T \left[(wv, p_x)_H + \frac{1}{2}(v, ((r_0+2\eta^*)p)_x)_H\right] \, dt \\
&+ \int_0^T \left[\langle v, \dot{q} - \delta(\dot{q}_x)_x\rangle_{V_\delta^*;V_\delta} + (uv, q_x)_H\right] \, dt \\
&- \langle v(T,\cdot), q(T,\cdot) - (\delta q_x(T,\cdot))_x\rangle_{V_\delta^*;V_\delta} \\
&- \int_0^T \Big[\left(\sigma_1(t)q(t,L) - \delta(L)\dot{q}_x(t,L)\right) v(t,L) \\
&\qquad - \left(\sigma_0(t)q(t,0) - \delta(0)\dot{q}_x(t,0)\right) v(t,0)\Big] \, dt \\
&- \int_0^T \left(u(t,L)v(t,L)q(t,L) - u(t,0)v(t,0)q(t,0)\right) dt \\
&- \delta(L)v(T,L)q_x(T,L) + \delta(0)v(T,0)q_x(T,0).
\end{aligned}$$

As for the Fréchet derivatives $\mathcal{L}_g$ and $\mathcal{L}_h$, direct calculations leads us to the following representation:

$$\begin{aligned}
\mathcal{L}_g k(t) &= \mathcal{L}(w,u,g+k,h,p,q) - \mathcal{L}(w,u,g,h,p,q) - R(g,k) \\
&= \int_0^T \beta_g g(t) k(t)\, dt - \int_0^T k(t) q(t,0)\, dt - R(g,k), \\
\mathcal{L}_h l(t) &= \mathcal{L}(w,u,g,h+l,p,q) - \mathcal{L}(w,u,g,h,p,q) - R(h,l)
\end{aligned}$$

$$= \int_0^T \beta_h h(t) l(t)\, dt + \int_0^T l(t) q(t,L)\, dt - R_2(h,l),$$

where

$$R_1(g,k) = \frac{1}{2}\int_0^T \beta_g k^2(t)\, dt, \quad R_2(h,l) = \frac{1}{2}\int_0^T \beta_h l^2(t)\, dt,$$
$$|R_1(g,k)|/\|k\|_{L^2(0,T)} \to 0 \text{ as } \|k\|_{L^2(0,T)} \to 0,$$
$$\text{and } |R_2(h,l)|/\|l\|_{L^2(0,T)} \to 0 \text{ as } \|l\|_{L^2(0,T)} \to 0.$$

Taking into account the calculations given above, we arrive at the following representation of the first-order optimality conditions for OCP (2)–(5), (39).

Theorem 5.1. *Let (g^0, h^0, η^0, u^0), where $\eta^0 = w^0 + \eta^*$, be an optimal solution to the optimal control problem (1)–(5). Then there exists a unique pair*

$$(p,q) \in \Big[W_0(0,T) \cap L^2(0,T; H^2(\Omega) \cap V_0)\Big] \times W^{1,\infty}(0,T; V_\delta)$$

such that the following system

$$\int_0^T \Big[\langle \dot{w}^0(t), \varphi\rangle_{V_0^*;V_0} + ((w^0(t)u^0(t))_x, \varphi)_H + \nu(w_x^0(t), \varphi_x)_H + \frac{1}{2}\left(r_0 u_x^0(t) + 2\eta^* u_x^0(t), \varphi\right)_H \Big]\, dt = 0, \tag{46}$$

$$\begin{aligned}\int_0^T \Big[&\langle \dot{u}^0(t), \psi\rangle_{V_\delta^*;V_\delta} + \int_\Omega \delta \dot{u}_x^0(t)\psi_x\, dx \\ &+ (u^0(t)u_x^0(t), \psi)_H + \left(\mu(t) w_x^0(t), \psi\right)_H + \sigma_1(t) u^0(t,L)\psi(L) \\ &- \sigma_0(t) u^0(t,0)\psi(0)\Big]\, dt \\ = \int_0^T &\Big[(f(t), \psi)_H + h^0(t)\psi(L) - g^0(t)\psi(0)\Big]\, dt,\end{aligned} \tag{47}$$

$$\begin{aligned}\int_0^T &\Big[\langle \dot{p}(t), \varphi(t)\rangle_{V_0^*;V_0} + \nu\langle p_{xx}(t), \varphi(t)\rangle_{V_0^*;V_0} + \left(p_x(t)u^0(t), \varphi(t)\right)_H \\ &+ ((\mu q(t))_x, \varphi(t))_H\Big]\, dt - (p(T), \varphi(T))_H \\ = &\int_0^T \int_\Omega \left(u^0\left[u_{xx}^0 \eta^0 + 2u_x^0 \eta_x^0 + \eta_{xx}^0 u^0\right]\right)\varphi(t)\, dx\, dt \\ &- \int_0^T \int_\Omega \left(\alpha_Q^2 \int_\Omega \left(\eta^0(t) - \eta_Q(t)\right)\, dx\right)\varphi(t)\, dx\, dt,\end{aligned} \tag{48}$$

$$\begin{aligned}\int_0^T &\Big[\langle \dot{q}(t) - (\delta \dot{q}_x(t))_x, \psi(t)\rangle_{V_\delta^*;V_\delta} + (q_x(t)u^0(t), \psi(t))_H\Big]\, dt \\ &+ \int_0^T \Big[\left(p_x(t)\eta^0(t), \psi(t)\right)_H + \frac{1}{2}\left((r_0 p(t))_x, \psi(t)\right)_H\Big]\, dt \\ &- \int_0^T [(\sigma_1(t) + u^0(t,L))q(t,L) - \delta(L)\dot{q}_x(t,L)]\psi(t,L)\, dt \\ &+ \int_0^T [(\sigma_0(t) + u^0(t,0))q(t,0) - \delta(0)\dot{q}_x(t,0)]\psi(t,0)\, dt\end{aligned}$$

$$
\begin{aligned}
&- \langle v(T,\cdot), q(T,\cdot) - (\delta q_x(T,\cdot))_x \rangle_{V_\delta^*;V_\delta} \\
&- \delta(L) q_x(T,L)\psi(T,L) + \delta(0) q_x(T,0))\psi(T,0) \\
= &\int_0^T \int_\Omega \eta^0 \big[u^0_{xx}(t)\eta^0(t)) + 2u^0_x(t)\eta^0_x(t) + \eta^0_{xx}(t) u^0(t) \big] \psi(t)\, dx\, dt \\
&- \int_\Omega a_\Omega (u^0(T) - u_\Omega)\psi(T)\, dx - \int_0^T \eta^* (\eta^0_x(t,L) u^0(t,L) \\
&+ \eta^* u^0_x(t,L))\psi(t,L)\, dt + \int_0^T \eta^* (\eta^0_x(t,0) u^0(t,0) + \eta^* u^0_x(t,0))\psi(t,0)\, dt, \quad (49)
\end{aligned}
$$

$$
\int_0^T (\beta_g g^0(t) - q(t,0))(g(t) - g^0(t))\, dt \geq 0, \quad \forall\, g \in G_{ad}, \tag{50}
$$

$$
\int_0^T (\beta_h h^0(t) + q(t,L))(h(t) - h^0(t))\, dt \geq 0 \quad \forall\, h \in H_{ad}, \tag{51}
$$

$$
\eta^0(t) = w^0(t) + \eta^*, \tag{52}
$$

$$
\delta(L) u^0_x(0,L) = 0, \quad \delta(0) u^0_x(0,0) = 0, \quad \delta(L) q_x(T,L) = 0, \quad \delta(0) q_x(T,0) = 0, \tag{53}
$$

$$
w^0(0) = \eta^0_0 - \eta^*, \quad p(T) = 0, \quad p(\cdot,0) = p(\cdot,L) = 0, \tag{54}
$$

$$
u^0(0) - (\delta u^0_x(0))_x = u_0, \quad q(T) - (\delta q_x(T))_x = \lambda a_\Omega (u^0(T) - u_\Omega) \tag{55}
$$

holds true for all

$$
\varphi \in W_0(0,T) \cap L^2(0,T; H^2(\Omega) \cap V_0),\ \ \psi \in W^{1,\infty}(0,T; V_\delta),\ \ \varphi \in V_0,\ \ \psi \in V_\delta,
$$

and a.e. $t \in [0,T]$.

Proof. Since the derived optimality conditions (46)–(55) are the direct consequence of the Lagrange principle, we focus on the solvability of the variational problems (48)–(49) for the adjoint variables p and q. To do so, we represent the system (48)–(49) as the corresponding equalities in the sense of distributions, namely,

$$
\begin{aligned}
&p_t + \nu p_{xx} + p_x u^0 + (\mu q)_x \\
&\qquad = \lambda u^0 \big[u^0_{xx}\eta^0 + 2u^0_x \eta^0_x + \eta^0_{xx} u^0 \big] - \lambda (\alpha_Q)^2 \int_\Omega \left(\eta^0 - \eta_Q \right)\, dx, \quad (56)
\end{aligned}
$$

$$
[q - (\delta q_x)_x]_t + q_x u^0 + p_x \eta^0 + \frac{1}{2}(r_0 p)_x = \lambda \eta^0 \big[u^0_{xx}\eta^0 + 2u^0_x \eta^0_x + \eta^0_{xx} u^0 \big], \tag{57}
$$

$$
\delta(L)\dot{q}_x(\cdot,L) - (\sigma_1 + u^0(\cdot,L)) q(\cdot,L) = -\lambda \eta^* (\eta^0_x(\cdot,L) u^0(\cdot,L) + u^0_x(\cdot,L)\eta^*), \tag{58}
$$

$$
\delta(0)\dot{q}_x(\cdot,0) - (\sigma_0 + u^0(\cdot,0)) q(\cdot,0) = -\lambda \eta^* (\eta^0_x(\cdot,0) u^0(\cdot,0) + u^0_x(\cdot,0)\eta^*), \tag{59}
$$

$$
q(T) - (\delta q_x(T))_x = \lambda a_\Omega (u^0(T) - u_\Omega), \tag{60}
$$

$$
\delta(L) q_x(T,L) = \delta(0) q_x(T,0) = 0, \tag{61}
$$

$$
p(T) = 0, \quad p(\cdot,0) = p(\cdot,L) = 0. \tag{62}
$$

In the operator presentation, the system (56)–(62) takes the form (see [11]):

$$
(A(t)\mathbf{p})'_t + B(t)\mathbf{p} = F(t), \quad A(T)\mathbf{p}(T) = \mathbf{b},
$$

where the operators

$$
A(t), B(t) : L^2(0,T; V_0) \times L^2(0,T; V_\delta) \to \left[L^2(0,T; V_0^*) \right]^2 \times \left[L^2(0,T) \right]^2
$$

are defined in (36)–(37), and

$$\begin{aligned}
\mathbf{b} &= (0, \lambda a_\Omega (u^0(T) - u_\Omega), 0, 0) \in V_0^* \times V_0^* \times \mathbb{R} \times \mathbb{R},\\
F(t) &= (f_1, f_2, \phi_1, \phi_2)^t \in \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2,\\
f_1(t) &= \lambda u^0 \left[u^0_{xx}\eta^0 + 2u^0_x\eta^0_x + \eta^0_{xx}u^0\right] - \lambda(\alpha_Q)^2 \int_\Omega \left(\eta^0 - \eta_Q\right)\, dx,\\
f_2(t) &= \lambda \eta^0 \left[u^0_{xx}\eta^0 + 2u^0_x\eta^0_x + \eta^0_{xx}u^0\right],\\
\phi_1(t) &= -\lambda\eta^*(\eta^0_x(t,L)u^0(t,L) + u^0_x(t,L)\eta^*),\\
\phi_2(t) &= \lambda\eta^*(\eta^0_x(t,0)u^0(t,0) + u^0_x(t,0)\eta^*).
\end{aligned}$$

As a result, the existence of a unique pair $(p(t), q(t))$ satisfying the system (48)–(51) is a mere consequence of Theorem 5.1. Moreover, since the Cauchy problem has a solution for any

$$F \in \left[L^2(0,T;V_0^*)\right]^2 \times \left[L^2(0,T)\right]^2 \quad \text{and} \quad \mathbf{b} \in V_0^* \times V_0^* \times \mathbb{R} \times \mathbb{R},$$

the Lagrange multiplier λ in the definition of the Lagrange functional

$$\mathcal{L} = \mathcal{L}(w, u, g, h, \lambda, p, q)$$

can be taken equal to 1. □

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