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LONG TIME BEHAVIOR FOR THE VISCO-ELASTIC DAMPED WAVE EQUATION IN $\mathbb{R}^n_+$ AND THE BOUNDARY EFFECT

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Abstract. In this paper, we investigate the existence and long time behavior of the solution for the nonlinear visco-elastic damped wave equation in $\mathbb{R}^n_+$, provided that the initial data is sufficiently small. It is shown that for the long time, one can use the convected heat kernel to describe the hyperbolic wave transport structure and damped diffusive mechanism. The Green's function for the linear initial boundary value problem can be described in terms of the fundamental solution (for the full space problem) and reflected fundamental solution coupled with the boundary operator. Using the Duhamel's principle, we get the L^p decaying rate for the nonlinear solution $\partial^\alpha_{\mathbf{x}} u$ for $|\alpha| \le 1$.

1. **Introduction.** In this paper, we study the global existence and the L^p estimate of the solution for the nonlinear visco-elastic damped wave equation

$$\begin{cases} \partial_t^2 u - c^2\Delta u - \nu\partial_t\Delta u = div f(u),\ t>0, \\ u|_{t=0} = u_0(\mathbf{x}), \\ u_t|_{t=0} = u_1(\mathbf{x}), \end{cases} \tag{1}$$

in multi-dimensional half space $\mathbb{R}^n_+ := \mathbb{R}_+ \times \mathbb{R}^{n-1}$, with absorbing and radiative boundary condition

$$(a_1\partial_{x_1} u + a_2 u)(x_1 = 0, \mathbf{x}', t) = 0. \tag{2}$$

$\mathbb{R}_+ = (0,\infty)$, $\mathbf{x} = (x_1, \mathbf{x}')$ is the space variable with $x_1 \in \mathbb{R}_+$, $\mathbf{x}' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$. ν is the viscosity, a_1 and a_2 are constants. The Laplacian $\Delta = \sum_{j=1}^n \partial^2_{x_j}$, $div f(u)$ is the smooth nonlinear term and $f(u) = O(|u|^k)$ when $|u| \le 1$, k is some positive integer.

Equation (1) is an essential nonlinear damped wave equation and attracts the attention of many mathematicians and engineers. In multi-dimensional case, it can model the formation tracking control (formation flight, air traffic control, etc.) and has been seen as the error position equation of the agents with the velocity feedback term [27].

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The Cauchy problem for the visco-elastic damped wave equation has been investigated by many authors and the solution has the diffusive structure for the long time, see [26] for the decaying rate of the linear solution, [12, 13] for the asymptotic profiles of the linear problem, [4, 24] for the nonlinear equation, etc. In this paper we restudy the Cauchy problem and give the pointwise estimate for the linear problem. Our results show that for the long time, the hyperbolic wave transport mechanism and the visco-elastic damped mechanism interact with each other so that the solution behaves like the convented heat kernel. The solution exhibits the generalized Huygens principle.

For the initial-boundary value problem, Webb [30] first studied the global well-posedness for the general bounded domain. Later, many authors studied existence, long time behaviors, global attractors and decaying rate estimates of some elementary wave for the initial-boundary value problem by using delicate energy estimate method, for example [1, 14, 31, 32, 33]. It seems that the pure energy estimates are not enough to get the sharp decaying rates for the solutions, one needs to find out other ways to understand the interaction mechanism between the boundary effect and interior waves. A good approach was given by [17]. When studying three dimensional Navier-Stokes equations with Dirichlet boundary condition, the authors used the solution for the linearized problem to get the $L^p(2 \le p \le \infty)$ convergence rate for the nonlinear solution and L^2 convergence rate for its first derivative.

In this paper, we consider the multi-dimensional nonlinear half space problem with mixed boundary condition and aim to get the decaying rate for the solution of nonlinear problem. The accurate expression of the Green's functions for the linear case problem are needed. The Green's functions for the multi-dimensional linear initial-boundary value problem are defined to satisfy the following system

$$\begin{cases} \partial_t^2 \mathbb{G}_1 - c^2 \Delta \mathbb{G}_1 - \nu \partial_t \Delta \mathbb{G}_1 = 0, x_1, y_1 > 0, \mathbf{x}' \in \mathbb{R}^{n-1}, t > 0, \\ \mathbb{G}_1 (x_1, \mathbf{x}', 0; y_1) = \delta(x_1 - y_1)\delta(\mathbf{x}'), \mathbb{G}_{1t} (x_1, \mathbf{x}', 0; y_1) = 0, \\ a_1 \partial_{x_1} \mathbb{G}_1(0, \mathbf{x}', t; y_1) + a_2 \mathbb{G}_1(0, \mathbf{x}', t; y_1) = 0; \end{cases} \tag{3}$$

$$\begin{cases} \partial_t^2 \mathbb{G}_2 - c^2 \Delta \mathbb{G}_2 - \nu \partial_t \Delta \mathbb{G}_2 = 0, x_1, y_1 > 0, \mathbf{x}' \in \mathbb{R}^{n-1}, t > 0, \\ \mathbb{G}_2 (x_1, \mathbf{x}', 0; y_1) = 0, \mathbb{G}_{2t} (x_1, \mathbf{x}', 0; y_1) = \delta(x_1 - y_1)\delta(\mathbf{x}'), \\ a_1 \partial_{x_1} \mathbb{G}_2(0, \mathbf{x}', t; y_1) + a_2 \mathbb{G}_2(0, \mathbf{x}', t; y_1) = 0. \end{cases} \tag{4}$$

A general way of considering such above initial-boundary value problem in any spatial dimension was initiated by [19]. A master relationship is established for the boundary data, and they can be computed exactly in the transformed variables. The fundamental solution for the Cauchy problem, together with the boundary relation, yields the explicit expression of the Green's function for the initial-boundary value problem. The explicit expression of the Green's function is essential for many quantitative and qualitative understanding of physical phenomena that these problems aim to model. However, it is quite difficult to inverse the master relationship especially for the system in high dimensional case. Series of applications and improvements of this method were given by [3, 5, 6, 8, 10, 20, 29] and the reference therein. They improved the procedure to simplify the complicated computations of inverse Fourier and Laplace transforms with the help of a deep understanding of the connections of the different symbols. Hence the Green's function for the initial boundary value problem of systems can be described in terms of the fundamental solution for the Cauchy problem and the boundary surface operator. However, for the high dimensional case the boundary surface operator could be very complicated for system of equations, see [5, 20, 29]. In paper [10], by comparing symbols in the transformed tangential-spatial and time space, the author showed that one can simplify the Green's function for the half space problem to essential part. In this paper, we use the same approach to get the simplified expression of the Green's functions, and they benefit the follow-up nonlinear analysis.

Another similar model called the frictional damped wave equation is given as follows

$$\begin{cases} \partial_t^2 u - c^2\Delta u + \nu\partial_t u = div f(u), \\ u|_{t=0} = u_0(\mathbf{x}), \\ u_t|_{t=0} = u_1(\mathbf{x}). \end{cases} \tag{5}$$

Many mathematicians have concentrated on solving it, [21, 23, 28] etc. Quite recently, Du et al. [9] showed that for the long time, the fundamental solution for the linear system (5) with the given initial data $u|_{t=0} = 0, u_t|_{t=0} = \delta(\mathbf{x})$ behaves like the Gauss kernel $\frac{e^{-\frac{\mathbf{x}^2}{C(t+1)}}}{(t+1)^{\frac{n}{2}}}$. However, for the visco-elastic damped wave equation system (1), the fundamental solution with the same initial data behaves like the convented heat kernel $\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-3}{4}}}$ for the odd dimensional case. For other related damped wave model, one can refer to [15, 16] for the wave equation with two types of damping terms, [2] for the damped abstract wave equation.

By the accurate expression of Green's functions for the linear half space problem and the Duhamel's principle, we get the $L^p(2 \le p \le \infty)$ decaying rate for the nonlinear solution $\partial_{\mathbf{x}}^\alpha u$, $|\alpha| \le 1$. We only treat the case $a_1a_2 < 0$. The boundary condition of Dirichlet type ($a_1 = 0$) and Neumann type ($a_2 = 0$) are much simpler. For the case of $a_1a_2 > 0$, the linearized problem is unstable. The estimates indicate that due to the boundary effect, we can not gain the extra decaying rate from initial data as the Cauchy problem. The main theorem is stated as follows:

Theorem 1.1. *Assume that the initial data $u_0(\mathbf{x})$ and $u_1(\mathbf{x})$ satisfy*

$$\|u_0, u_1\|_{H^l\cap L^1} \le O(1)\varepsilon \tag{6}$$

for ε sufficiently small and $l = [\frac{n}{2}] + 2$, $n \ge 3$, there exists a unique global classical solution to the nonlinear problem (1) for $k > [\frac{n+1}{n-1}]$ with the mixed boundary condition (2) where $a_1a_2 \le 0$. The solution has the following $L^p(\mathbb{R}^n_+)$ estimates for $|\alpha| \le 1$,

$$\|\partial_{\mathbf{x}}^\alpha u(\cdot,t)\|_{L^p(\mathbb{R}^n_+)} \le O(1)\varepsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}, \quad p \in [2,\infty].$$

The Fourier transform of the Green's function $\mathbb{G}_2(\mathbf{x}, t)$ in (3) is singular near the origin $\boldsymbol{\xi} = 0$ (the same as the Cauchy problem [4]). Here the condition (6) and the nonlinear term $div f(u)$ can ensure the closure of the nonlinearity. For the Cauchy problem, one can get extra $\frac{1}{2}$ decaying rate if more assumptions are imposed for the initial boundary data. One can also get the sharp estimates for the two dimensional Cauchy problem. These are explained in Section 3. However, we cannot improve the decaying rate for the initial-boundary value problem in the same way. Based on the assumption of the nonlinear term, we can only get the decaying rate for the case $n \ge 3$.

The rest of paper is arranged as follows: in Section 2, some notations are introduced. We restudy the fundamental solutions for the linear Cauchy problem and give a description of the hyperbolic wave transport structure and damped diffusive mechanism in Section 3. In Section 4, the Green's functions for the half space problem are constructed by using the comparing method. We also prove the decaying rate of the solution for the nonlinear problem. Some useful lemmas are given in Appendix.

2. **Notations.** Let C and $O(1)$ be denoted as generic positive constants. For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\partial_{\mathbf{x}}^{\alpha} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Let L^p denote the usual L^p space on $\mathbf{x} \in \mathbb{R}_+^n$. For nonnegative integer l, we denote by $W^{l,p}(1 \le p < \infty)$ the usual L^p- Soblev space of order l: $W^{l,p} = \{u \in L^p : \partial_{\mathbf{x}}^{\alpha} u \in L^p(|\alpha| \le l)\}(l \ge 1), W^{0,p} = L^p$. The norm is denoted by $\|\cdot\|_{W^{l,p}} =: \|u\|_{W^{l,p}} = \sum_{|\alpha| \le l} \|\partial_{\mathbf{x}}^{\alpha} u\|_{L^p}$. When $p = 2$, we define $W^{l,2} = H^l$ for all $l \ge 0$ by $H^l = \{u \in L^2, \|u\|_{H^l} < \infty\}$.

Define $k_p(|\alpha|)$ $2 \le p \le \infty$ by

$$k_p(|\alpha|) = \begin{cases} \max\left\{0, |\alpha| + n(\frac{1}{2} - \frac{1}{p})\right\}, & \text{when } \ p = 2, \\ \max\left\{0, \left[|\alpha| + n(\frac{1}{2} - \frac{1}{p})\right]\right\} + 1, & \text{when } \ 2 < p \le \infty. \end{cases}$$

$[a] = \max\{b, b \text{ is an integer}, \ b \le a\}$.

Introduce the Fourier transform and Laplace transform of $f(\mathbf{x}, t)$:

$$\begin{aligned} f(\boldsymbol{\xi}, t) &:= \mathcal{F}[f](\boldsymbol{\xi}, t) = \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x}, t) d\mathbf{x}, \\ f(\mathbf{x}, s) &:= \mathcal{L}[f](\mathbf{x}, s) = \int_0^{\infty} e^{-st} f(\mathbf{x}, t) dt. \end{aligned}$$

We denote $\mathcal{D}_\delta := \{\boldsymbol{\xi} \in \mathbb{C}^n | |Im(\xi_i)| \le \delta, i = 1, 2, \cdots, n\}$.

3. Long time behavior of solution for the Cauchy problem in $\mathbb{R}^n$.

3.1. Fundamental solutions for the Cauchy problem. The necessary preliminaries for the Green's functions of the initial-boundary value problem are the fundamental solutions of the Cauchy problem. The fundamental solutions for the linear damped wave equations are defined

$$\begin{cases} \partial_t^2 G_1 - c^2 \Delta G_1 - \nu \partial_t \Delta G_1 = 0 \\ G_1(\mathbf{x}, 0) = \delta(\mathbf{x}), G_{1t}(\mathbf{x}, 0) = 0, \end{cases} \quad \begin{cases} \partial_t^2 G_2 - c^2 \Delta G_2 - \nu \partial_t \Delta G_2 = 0 \\ G_2(\mathbf{x}, 0) = 0, G_{2t}(\mathbf{x}, 0) = \delta(\mathbf{x}). \end{cases} \tag{7}$$

They can be obtained by studying the Fourier transform $G_i(\boldsymbol{\xi}, t)$ $(i = 1, 2)$,

$$\begin{aligned} & G_1(\boldsymbol{\xi}, t) = \frac{\sigma_+ e^{\sigma_- t} - \sigma_- e^{\sigma_+ t}}{\sigma_+ - \sigma_-}, \quad G_2(\boldsymbol{\xi}, t) = \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-}, \\ & \sigma_{\pm} = -\frac{\nu}{2}|\boldsymbol{\xi}|^2 \pm \frac{1}{2}\sqrt{\nu^2|\boldsymbol{\xi}|^4 - 4|\boldsymbol{\xi}|^2 c^2}. \end{aligned} \tag{8}$$

The L^p estimates of $G_i(\mathbf{x}, t)$ $(i = 1, 2)$ are given by [4] using the Fourier transform and real analysis. For the pointwise estimate of $G_i(\mathbf{x}, t)$, when studying the compressible Navier-Stokes system in two and three dimensional case, paper [7] developed the local analysis method to achieve algebraic wave structure for the long wave components, and also adopted local analysis method for the short wave components to give a description about all types of singular functions. We can apply the same method to estimate the fundamental solutions for the general multi-dimensional case. These results are crucial for understanding our later description of the Green's functions for the half space problem and getting the uniform decaying rates. The results are given as follows:

Lemma 3.1. *Assume that the spatial space dimension $n \geq 2$, we have the following pointwise estimates of $G_i(\mathbf{x},t)$ $(i=1,2)$ for all $\mathbf{x} \in \mathbb{R}^n$, $|\alpha| \geq 0$,*

$$\begin{aligned}
&\Big|\partial_{\mathbf{x}}^{\alpha}(G_1(\mathbf{x},t) - j_n(\mathbf{x},t) - e^{-c^2t/\nu}\delta_n(\mathbf{x}) - e^{-t/C}Y_n(\mathbf{x}))\,,\\
&\quad \partial_{\mathbf{x}}^{\alpha}\partial_{x_i}\left(G_2(\mathbf{x},t) - j_n(\mathbf{x},t) - e^{-t/C}Y_n(\mathbf{x})\right)\Big|\\
&\leq O(1)\Bigg(e^{-\frac{(|\mathbf{x}|+t)}{C}}\\
&\qquad +(1+t)^{-\frac{|\alpha|}{2}}\begin{cases} \dfrac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}} + \dfrac{H(ct-|\mathbf{x}|)}{(1+t)^{\frac{3n-2}{4}}(ct-|\mathbf{x}|+\sqrt{t})^{\frac{1}{2}}}, & \textit{for } n \textit{ even}\\[2ex] \dfrac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}}, & \textit{for } n \textit{ odd}\end{cases}\Bigg),
\end{aligned}$$

while

$$\begin{aligned}
&|j_n(\mathbf{x},t)| \leq O(1)L_n(t)e^{-(|\mathbf{x}|+t)/C}, \quad L_2(t) = \log(t), \quad L_n(t) = t^{-\frac{n-2}{2}} \quad \textit{for} \quad n \geq 3;\\
&Y_2(\mathbf{x}) = O(1)\frac{1}{2\pi}BesselK_0(|\mathbf{x}|), \quad Y_n(\mathbf{x}) = O(1)\frac{e^{-|\mathbf{x}|}}{|\mathbf{x}|^{n-2}} \quad \textit{for} \quad n \geq 3.
\end{aligned}$$

$BesselK_0(|\mathbf{x}|)$ is the modified Bessel function of the second kind with degree 0.

Moreover, one can obtain the following L^p $(p \geq 1)$ estimates

$$\begin{aligned}
\|\partial_{\mathbf{x}}^{\alpha}G_1 * u_0\|_{L^p} &\leq O(1)\left((1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\alpha|}{2}}\|u_0\|_{L^1} + e^{-\frac{t}{C}}\|u_0\|_{H^{k_p(|\alpha|)}}\right),\\
\|\partial_{\mathbf{x}}^{\alpha}\partial_{x_i}G_2 * u_1\|_{L^p} &\leq O(1)\left((1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\alpha|}{2}}\|u_1\|_{L^1} + e^{-\frac{t}{C}}\|u_1\|_{H^{k_p(|\alpha|)}}\right).
\end{aligned}$$

Proof. Outside the finite Mach number region, one can use the weighted energy estimates to get the exponentially decaying rate in time and space (see [5, 8]), here we omit the details.

Inside the finite Mach number region, we divide ξ into two parts: the long wave component and the short wave component

$$\begin{aligned}
&f(\mathbf{x},t) = f^L(\mathbf{x},t) + f^S(\mathbf{x},t),\\
&\mathcal{F}[f^L] = H\left(1 - \frac{|\boldsymbol{\xi}|}{\varepsilon_0}\right)\mathcal{F}[f](\boldsymbol{\xi},t),\\
&\mathcal{F}[f^S] = \left(1 - H\left(1 - \frac{|\boldsymbol{\xi}|}{\varepsilon_0}\right)\right)\mathcal{F}[f](\boldsymbol{\xi},t),
\end{aligned}$$

with the parameter $\varepsilon_0 \ll 1$, the Heaviside function $H(x)$ is defined by

$$H(x) = \begin{cases} 1, & x > 0,\\ 0, & x < 0.\end{cases}$$

For the long wave component, we use the complex analysis developed by [18] . When $|\boldsymbol{\xi}| \leq \varepsilon_0 \ll 1$, we have

$$\frac{\sigma_+e^{\sigma_-t} - \sigma_-e^{\sigma_+t}}{\sigma_+ - \sigma_-} = e^{-\frac{\nu|\boldsymbol{\xi}|^2}{2}t}\left(\frac{\nu|\boldsymbol{\xi}|^2\sin\left(c|\boldsymbol{\xi}|\sqrt{1-\frac{\nu^2|\boldsymbol{\xi}|^2}{4c^2}}t\right)}{2c|\boldsymbol{\xi}|\sqrt{1-\frac{\nu^2|\boldsymbol{\xi}|^2}{4c^2}}} + \cos\left(c|\boldsymbol{\xi}|\sqrt{1-\frac{\nu^2|\boldsymbol{\xi}|^2}{4c^2}}t\right)\right).$$

One can expand $\sqrt{1-\frac{\nu^2|\boldsymbol{\xi}|^2}{4c^2}} = 1+\mathcal{A}(|\boldsymbol{\xi}|^2)$ where $\mathcal{A}(|\boldsymbol{\xi}|^2) = O(|\boldsymbol{\xi}|^2)$ is an analytic function of $|\boldsymbol{\xi}|^2$. Therefore, we have the following expressions

$$\begin{aligned}
&\frac{\sigma_+ e^{\sigma_- t} - \sigma_- e^{\sigma_+ t}}{\sigma_+ - \sigma_-} \\
= \; & e^{-\frac{\nu|\boldsymbol{\xi}|^2}{2}t}\frac{\sin(c|\boldsymbol{\xi}|t)}{c|\boldsymbol{\xi}|} \times \left\{\frac{\nu|\boldsymbol{\xi}|^2}{2}\frac{\cos(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2))}{1+\mathcal{A}(|\boldsymbol{\xi}|^2)} - \frac{\sin(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2)}{c|\boldsymbol{\xi}|}c^2|\boldsymbol{\xi}|^2\right\} \\
& + e^{-\frac{\nu|\boldsymbol{\xi}|^2}{2}t}\cos(c|\boldsymbol{\xi}|t) \times \left\{\frac{\nu|\boldsymbol{\xi}|^2}{2}\frac{\sin(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2))}{c|\boldsymbol{\xi}|(1+\mathcal{A}(|\boldsymbol{\xi}|^2))} + \cos(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2)\right\}.
\end{aligned}$$

Set

$$\begin{aligned}
\mathbb{D}_1 &= \int_{|\boldsymbol{\xi}|\le\varepsilon_0} e^{i\mathbf{x}\cdot\boldsymbol{\xi}} e^{-\frac{\nu|\boldsymbol{\xi}|^2}{2}t}\left\{\frac{\nu|\boldsymbol{\xi}|^2}{2}\frac{\cos(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2))}{1+\mathcal{A}(|\boldsymbol{\xi}|^2)} - \frac{\sin(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2)}{c|\boldsymbol{\xi}|}c^2|\boldsymbol{\xi}|^2\right\} d\boldsymbol{\xi}, \\
\mathbb{D}_2 &= \int_{|\boldsymbol{\xi}|\le\varepsilon_0} e^{i\mathbf{x}\cdot\boldsymbol{\xi}} e^{-\frac{\nu|\boldsymbol{\xi}|^2}{2}t}\left\{\frac{\nu|\boldsymbol{\xi}|^2}{2}\frac{\sin(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2))}{c|\boldsymbol{\xi}|(1+\mathcal{A}(|\boldsymbol{\xi}|^2))} + \cos(c|\boldsymbol{\xi}|t\mathcal{A}(|\boldsymbol{\xi}|^2)\right\} d\boldsymbol{\xi}, \\
\mathbb{S}_n(\mathbf{x},t) &= \mathcal{F}^{-1}\left[\frac{\sin c|\boldsymbol{\xi}|t}{c|\boldsymbol{\xi}|}\right], \quad \mathbb{C}_n(\mathbf{x},t) = \mathcal{F}^{-1}\left[\cos c|\boldsymbol{\xi}|t\right],
\end{aligned}$$

hence

$$G_1^L(\mathbf{x},t) = \mathbb{S}_n \underset{\mathbf{x}}{*} \mathbb{D}_1 + \mathbb{C}_n \underset{\mathbf{x}}{*} \mathbb{D}_2,$$

$\mathbb{D}_1$, $\mathbb{D}_2$ are parabolic waves [7] and have the following estimates

$$|D^\alpha_{\mathbf{x}}\mathbb{D}_1(\mathbf{x},t)| \le O(1)\frac{e^{-\frac{\mathbf{x}^2}{C(t+1)}}}{(1+t)^{\frac{n+2+\alpha}{2}}} + O(1)e^{-\frac{|\mathbf{x}|+t}{C}},$$

$$|D^\alpha_{\mathbf{x}}\mathbb{D}_2(\mathbf{x},t)| \le O(1)\frac{e^{-\frac{\mathbf{x}^2}{C(t+1)}}}{(1+t)^{\frac{n+\alpha}{2}}} + O(1)e^{-\frac{|\mathbf{x}|+t}{C}}.$$

When n is odd, using the Kirchoff formulas for solutions and lemma 5.1 we have

$$\begin{aligned}
|\mathbb{S}_n \underset{\mathbf{x}}{*} \mathbb{D}_1| &\le O(1)\sum_{0\le|\alpha|\le(n-3)/2} t^{|\alpha|+1}\left|\int_{|\mathbf{y}|=1} D^\alpha_{\mathbf{x}}\mathbb{D}_1(\mathbf{x}+ct\mathbf{y},t)\mathbf{y}^\alpha dS_{\mathbf{y}}\right| \\
&\le O(1)\sum_{0\le|\alpha|\le(n-3)/2} t^{|\alpha|+1}\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(1+t)^{\frac{|\alpha|+1}{2}+n}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right) \\
&\le O(1)\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right),
\end{aligned}$$

$$\begin{aligned}
|\mathbb{C}_n \underset{\mathbf{x}}{*} \mathbb{D}_2| &\le O(1)\sum_{0\le|\alpha|\le(n-1)/2} t^{|\alpha|}\left|\int_{|\mathbf{y}|=1} D^\alpha_{\mathbf{x}}\mathbb{D}_2(\mathbf{x}+ct\mathbf{y},t)\mathbf{y}^\alpha dS_{\mathbf{y}}\right| \\
&\le O(1)\sum_{0\le|\alpha|\le(n-1)/2} t^{|\alpha|}\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(1+t)^{\frac{|\alpha|-1}{2}+n}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right) \\
&\le O(1)\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right).
\end{aligned}$$

Therefore, for the odd dimensional case, we get

$$\left|G_1^L(\mathbf{x},t)\right| \le O(1)\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right).$$

Similarly when n is even, we have

$$|\mathbb{S}_n \underset{\mathbf{x}}{*} \mathbb{D}_1| \le O(1)\sum_{0\le|\alpha|\le n/2-1} t^{|\alpha|+1}\left|\int_{|\mathbf{y}|\le 1}\frac{D_{\mathbf{x}}^{\alpha}\mathbb{D}_1(\mathbf{x}+ct\mathbf{y},t)\mathbf{y}^{\alpha}}{\sqrt{1-|\mathbf{y}|^2}}d\mathbf{y}\right|$$

$$\le O(1)\sum_{0\le|\alpha|\le n/2-1}\frac{t^{|\alpha|+1}}{(1+t)^{(|\alpha|+1)/2+n}}\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(1+t)^{1/4}} + \frac{H(ct-|\mathbf{x}|)}{\sqrt{ct-|\mathbf{x}|+\sqrt{t}}}\right) + e^{-\frac{|\mathbf{x}|+t}{C}}$$

$$\le O(1)\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}} + \frac{H(ct-|\mathbf{x}|)}{(1+t)^{\frac{3n-2}{4}}(ct-|\mathbf{x}|+\sqrt{t})^{\frac{1}{2}}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right),$$

$$|\mathbb{C}_n \underset{\mathbf{x}}{*} \mathbb{D}_2| \le O(1)\sum_{0\le|\alpha|\le n/2} t^{|\alpha|}\left|\int_{|\mathbf{y}|\le 1} D_{\mathbf{x}}^{\alpha}\mathbb{D}_2(\mathbf{x}+ct\mathbf{y})\mathbf{y}^{\alpha}dS_{\mathbf{y}}\right|$$

$$\le O(1)\sum_{0\le|\alpha|\le n/2}\frac{t^{|\alpha|}}{(1+t)^{(|\alpha|-1)/2+n}}\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(1+t)^{1/4}} + \frac{H(ct-|\mathbf{x}|)}{\sqrt{ct-|\mathbf{x}|+\sqrt{t}}}\right) + e^{-\frac{|\mathbf{x}|+t}{C}}$$

$$\le O(1)\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}} + \frac{H(ct-|\mathbf{x}|)}{(1+t)^{\frac{3n-2}{4}}(ct-|\mathbf{x}|+\sqrt{t})^{\frac{1}{2}}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right).$$

Hence for the even dimensional case, we get

$$\left|G_1^L(\mathbf{x},t)\right| \le O(1)\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{3n-1}{4}}} + \frac{H(ct-|\mathbf{x}|)}{(1+t)^{\frac{3n-2}{4}}(ct-|\mathbf{x}|+\sqrt{t})^{\frac{1}{2}}} + e^{-\frac{|\mathbf{x}|+t}{C}}\right). \quad (9)$$

For the short wave component, we adopt the local analysis method to give a description about all types of singular functions. When $|\boldsymbol{\xi}| \ge N$ for N sufficiently large, we have the following Taylor expansion for $\sigma_{\pm}$:

$$\begin{cases} \sigma_+ = -\frac{c^2}{\nu} + \sum_{j=1}^{n} a_j|\boldsymbol{\xi}|^{-2j} + O(|\boldsymbol{\xi}|^{-2(n+1)}), \\ \sigma_- = -\sigma_+ - \nu|\boldsymbol{\xi}|^2, \end{cases}$$

a_j are negative constants.

This non-decaying property results in the singularities of the fundamental solution G_i in spatial variable. To investigate the singularities, we approximate the spectra $\sigma_{\pm}$ by $\sigma_{\pm}^*$:

$$\begin{cases} \sigma_+^* = -\frac{c^2}{\nu} + \sum_{j=1}^{n} b_j(1+|\boldsymbol{\xi}|^2)^{-j} + J_0((1+|\boldsymbol{\xi}^2|)^{-(n+1)}), \\ \sigma_-^* = -\sigma_+^* - \nu|\boldsymbol{\xi}|^2, \end{cases}$$

b_j and J_0 are chosen suitably such that

$$\inf_{\boldsymbol{\xi}\in\mathcal{D}_{\kappa/C}}|\sigma_-^*(\boldsymbol{\xi}) - \sigma_+^*(\boldsymbol{\xi})| > 0, \sup_{\boldsymbol{\xi}\in\mathcal{D}_{\kappa/C}} Re(\sigma_{\pm}^*(\boldsymbol{\xi})) \le -J_0,$$

$$\sup_{\boldsymbol{\xi}\in\mathcal{D}_{\kappa/C}}|\boldsymbol{\xi}|^{2n+2}|\sigma_{\pm}(\boldsymbol{\xi}) - \sigma_{\pm}^*(\boldsymbol{\xi})| < \infty \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty.$$

Therefore, the approximated analytic spectra $\sigma_\pm^*$ given above satisfy

$$\left|\frac{\sigma_+ e^{\sigma_- t} - \sigma_- e^{\sigma_+ t}}{\sigma_+ - \sigma_-} - \frac{\sigma_+^* e^{\sigma_-^* t} - \sigma_-^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*}, \ \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} - \frac{e^{\sigma_+^* t} - e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*}\right|$$
$$\le \frac{O(1)}{(1+|\boldsymbol{\xi}|^2)^{(n+1)}}.$$

By Lemma 5.2 in the Appendix, we have

$$\left|\mathcal{F}^{-1}\left[\frac{\sigma_+ e^{\sigma_- t} - \sigma_- e^{\sigma_+ t}}{\sigma_+ - \sigma_-} - \frac{\sigma_+^* e^{\sigma_-^* t} - \sigma_-^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*}\right](\cdot, t)\right|_{L^\infty(\mathbb{R}^n)} = O(1),$$
$$\left|\mathcal{F}^{-1}\left[\frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} - \frac{e^{\sigma_+^* t} - e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*}\right](\cdot, t)\right|_{L^\infty(\mathbb{R}^n)} = O(1),$$

which asserts that all singularities are contained in $\frac{\sigma_+^* e^{\sigma_-^* t} - \sigma_-^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*}, \frac{e^{\sigma_+^* t} - e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*}$. Moreover, one can also prove that the errors of this approximation decay exponentially fast in the space-time domain, just like the proof in [7].

Now we seek out all the singularities.

$$\frac{\sigma_+^* e^{\sigma_-^* t} - \sigma_-^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*} = e^{\sigma_+^* t} - \frac{\sigma_+^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*} + \frac{\sigma_+^* e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*}.$$

The first term is

$$e^{\sigma_+^* t} = e^{-\frac{c^2 t}{\nu}}(1 + \sum_{j=1}^{n} c_j (1+|\boldsymbol{\xi}|^2)^{-j} + O((1+|\boldsymbol{\xi}^2|)^{-(n+1)})),$$

where c_j are negative constants, it can be estimated as follows

$$\left|\mathcal{F}^{-1}[e^{\sigma_+^* t}] - e^{-c^2 t/\nu}\delta(\mathbf{x}) - e^{-t/C} Y_n(\mathbf{x})\right| \le e^{-\frac{|\mathbf{x}|+t}{C}}.$$

The second and third terms contain no singularities

$$\frac{\sigma_+^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*} = O(1) e^{-t/C}\left(\sum_{j=1}^{n} \frac{1}{(1+|\boldsymbol{\xi}|^2)^j} + O\left(\frac{1}{(1+|\boldsymbol{\xi}|^2)^{n+1}}\right)\right),$$
$$\frac{\sigma_+^* e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*} = O(1) e^{-t/C - Ct|\boldsymbol{\xi}|^2}\left(\sum_{j=1}^{n} \frac{1}{(1+|\boldsymbol{\xi}|^2)^j} + O\left(\frac{1}{(1+|\boldsymbol{\xi}|^2)^{n+1}}\right)\right),$$

and we have

$$\left|\mathcal{F}^{-1}\left[\frac{\sigma_+^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*}\right] - e^{-t/C} Y_n(\mathbf{x})\right| \le e^{-\frac{|\mathbf{x}|+t}{C}},$$
$$\left|\mathcal{F}^{-1}\left[\frac{\sigma_+^* e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*}\right] - L_n(\mathbf{x}) e^{-\frac{|\mathbf{x}|+t}{C}}\right| \le e^{-\frac{|\mathbf{x}|+t}{C}}.$$

Hence we get the estimates for the short wave component

$$|G_1^S(\mathbf{x}, t) - j_n(\mathbf{x}, t) - e^{-c^2 t/\nu}\delta_n(\mathbf{x}) - e^{-t/C} Y_n(\mathbf{x})| \le e^{-\frac{|\mathbf{x}|+t}{C}}. \tag{10}$$

Combing (9) and (10) together, we prove the estimate of the first fundamental solution $G_1(\mathbf{x}, t)$. The second one can be proved similarly and we omit it. $\square$

3.2. **Long time decaying rate estimates for the solution of the Cauchy problem.** The fundamental solutions $G_i (i = 1, 2)$ give the representation of solution $u(\mathbf{x}, t)$ for Cauchy problem:

$$\begin{aligned}\partial_{\mathbf{x}}^{\alpha} u(\mathbf{x}, t) = & \int_{\mathbb{R}^n} \partial_{\mathbf{x}}^{\alpha} \left(G_1(\mathbf{x}-\mathbf{y}, t)\, u_0(\mathbf{y}) + \partial_{\mathbf{x}}^{\alpha} G_2(\mathbf{x}-\mathbf{y}, t)\, u_1(\mathbf{y})\right) d\mathbf{y} \\ & + \int_0^t \int_{\mathbb{R}^n} \partial_{\mathbf{x}}^{\alpha} G_2(\mathbf{x}-\mathbf{y}, t-\tau)\, div f(u)(\mathbf{y}, \tau) d\mathbf{y} d\tau \\ \equiv & \ \partial_{\mathbf{x}}^{\alpha} I(\mathbf{x}, t) + \partial_{\mathbf{x}}^{\alpha} N(\mathbf{x}, t). \end{aligned} \tag{11}$$

If more assumptions are imposed on the initial data, one can get the following decaying rate estimates by using the integration by parts.

Theorem 3.2. [4] *Assume that the initial data $u_0(\mathbf{x})$ and $u_1(\mathbf{x})$ satisfy*

$$\|u_0\|_{H^l \cap W^{1,1}} \le O(1)\varepsilon, u_1(\mathbf{x}) = \partial_{x_1} h(\mathbf{x}), \|h\|_{H^{l+1} \cap L^1} \le O(1)\varepsilon$$

for ε sufficiently small, $l \ge [\frac{n}{2}] + 3$, there exists a unique global classical solution to the Cauchy problem of system (1) with $k \ge \max\{2, 1 + \frac{2}{n}\}$. The solution has the following optimal $L^p(\mathbb{R}^n)$ estimates for $|\alpha| \le l - [\frac{n}{2}] - 1$:

$$\|\partial_{\mathbf{x}}^{\alpha} u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \le O(1)\varepsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \quad p \in [2, \infty].$$

4. **Long time behavior of solution for the initial-boundary value problem.** To get the sharp estimates of the long time decaying rate for the solution of the initial boundary value problem, we take use of all the information that the solution of linearized problem has. Hence the Green's functions for the initial-boundary value problem are constructed. By comparing the symbols of the fundamental solutions and the Green's functions in the partial-Fourier and Laplace transformed space, we get the simplified Green's functions for the initial-boundary value problem.

4.1. **The Green's functions for the half space problem with the absorbing and radiative boundary condition.** It is sufficient to construct the Green's functions in the transformed space-time domain since the fundamental solutions and the Green's functions are closely related in the transformed variables.

Taking Laplace transform in t and Fourier transform in $\mathbf{x}$ to the equations in (7), denoting the transformed variables by s and $\boldsymbol{\xi}$ respectively, we have the transformed fundamental solutions in the whole space $\mathbb{R}^n$:

$$G_1(\boldsymbol{\xi}, s) = \frac{s + \nu|\boldsymbol{\xi}|^2}{s^2 + (c^2 + \nu s)|\boldsymbol{\xi}|^2}, \quad G_2(\boldsymbol{\xi}, s) = \frac{1}{s^2 + (c^2 + \nu s)|\boldsymbol{\xi}|^2}.$$

Taking inverse Fourier transform only to the first component ξ_1 and using

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi_1 x_1}}{s^2 + c^2|\boldsymbol{\xi}|^2 + \nu s|\boldsymbol{\xi}|^2} d\xi_1 = \frac{1}{\nu s + c^2} \frac{e^{-\lambda|x_1|}}{2\lambda},$$

we get

$$G_1(x_1, \boldsymbol{\xi}', s) = \frac{1}{\nu s + c^2}\left(\nu\delta(x_1) + \frac{c^2 s}{\nu s + c^2} \frac{e^{-\lambda|x_1|}}{2\lambda}\right), \quad G_2(x_1, \boldsymbol{\xi}', s) = \frac{e^{-\lambda|x_1|}}{2\lambda(\nu s + c^2)},$$

$\lambda = \lambda(\boldsymbol{\xi}', s) = \sqrt{|\boldsymbol{\xi}'|^2 + \frac{s^2}{\nu s + c^2}}$.

In particular, when $\bar{x}_1 > 0$,

$$G_1(-\bar{x}_1, \boldsymbol{\xi}', s) = \frac{c^2 s}{(\nu s + c^2)^2} \frac{e^{-\lambda \bar{x}_1}}{2\lambda}, \quad G_2(-\bar{x}_1, \boldsymbol{\xi}', s) = \frac{e^{-\lambda \bar{x}_1}}{2\lambda(\nu s + c^2)}.$$

We now construct the Green's functions in the half space $\mathbf{x} \in \mathbb{R}_+^n$. The first step is to make the initial value zero by considering the function $E_i(x_1, \mathbf{x}', t; y_1) = \mathbb{G}_i(x_1, \mathbf{x}', t; y_1) - G_i(x_1 - y_1, \mathbf{x}', t)$:

$$\begin{cases} \partial_t^2 E_i - c^2 \Delta E_i - \nu \partial_t \Delta E_i = 0, \mathbf{x} \in \mathbb{R}_+^n, t > 0, \\ E_i|_{t=0} = 0, E_{it}|_{t=0} = 0, \\ (a_1 \partial_{x_1} + a_2) E_i(0, \mathbf{x}', t; y_1) = -\left. (a_1 \partial_{x_1} + a_2) G_i (x_1 - y_1, \mathbf{x}', t) \right|_{x_1=0}. \end{cases}$$

Taking Fourier transform only with respect to the tangential spatial variable $\mathbf{x}'$, Laplace transform with respect to time variable t, one has the following ODE problem

$$\begin{cases} s^2 E_i - (c^2 + \nu s) E_{i x_1 x_1} + (c^2 + \nu s) |\boldsymbol{\xi}'|^2 E_i = 0, \\ (a_1 \partial_{x_1} + a_2) E_i(0, \boldsymbol{\xi}', s; y_1) = (a_1 \partial_{y_1} - a_2) G_i\big(-y_1, \boldsymbol{\xi}', s\big) = -(a_1 \lambda + a_2) G_i\big(-y_1, \boldsymbol{\xi}', s\big). \end{cases}$$

Solving it and dropping out the divergent mode as $x \to +\infty$, using the boundary relationship, we have

$$E_i\left(x_1, \boldsymbol{\xi}', s; y_1\right) = -\frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} e^{-\lambda x_1} G_i\left(-y_1, \boldsymbol{\xi}', s\right) = -\frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} G_i\left(x_1 + y_1, \boldsymbol{\xi}', s\right),$$

where λ is defined as before.

Therefore the transformed Green's functions $\mathbb{G}_i\left(x_1, \boldsymbol{\xi}', s; y_1\right)$ $(i = 1, 2)$ are

$$\begin{aligned} \mathbb{G}_i\left(x_1, \boldsymbol{\xi}', s; y_1\right) &= G_i\left(x_1 - y_1, \boldsymbol{\xi}', s\right) - \frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} G_i\left(x_1 + y_1, \boldsymbol{\xi}', s\right) \\ &= G_i\left(x_1 - y_1, \boldsymbol{\xi}', s\right) + G_i\left(x_1 + y_1, \boldsymbol{\xi}', s\right) - \frac{2a_2}{a_2 - a_1 \lambda} G_i\left(x_1 + y_1, \boldsymbol{\xi}', s\right), \end{aligned}$$

which reveal the connection between fundamental solutions and the Green's functions.

Hence,

$$\begin{aligned} \mathbb{G}_i(x_1, \mathbf{x}', t; y_1) &= G_i(x_1 - y_1, \mathbf{x}', t) + G_i(x_1 + y_1, \mathbf{x}', t) \\ &\quad - \mathcal{F}^{-1}_{\boldsymbol{\xi}' \to \mathbf{x}'} \mathcal{L}^{-1}_{s \to t} \left[\frac{2a_2}{a_2 - a_1 \lambda} \right] \underset{\mathbf{x}', t}{*} G_i(x_1 + y_1, \mathbf{x}', t). \end{aligned}$$

The study of the Green's functions is reduced to be the estimate of the boundary operator $\mathcal{F}^{-1}_{\boldsymbol{\xi}' \to \mathbf{x}'} \mathcal{L}^{-1}_{s \to t} \left[\frac{2a_2}{a_2 - a_1 \lambda} \right]$. The function $\frac{1}{a_2 - a_1 \sqrt{|\boldsymbol{\xi}'|^2 + \frac{s^2}{\nu s + c^2}}}$ has the poles in the right half time space if $a_1 a_2 > 0$, which suggests that the boundary term will grow exponentially in time. In the following we only consider the case $a_1 a_2 < 0$.

Instead of inversing the boundary symbol, we follow the differential equation method. Notice that

$$\begin{aligned} &\mathcal{F}^{-1}_{\boldsymbol{\xi}' \to \mathbf{x}'} \mathcal{L}^{-1}_{s \to t} \left[\frac{2a_2}{a_2 - a_1 \lambda} G_i\left(x_1 + y_1, \boldsymbol{\xi}', s\right) \right] \\ &= \left(-1 + 2 \frac{a_2}{a_1 \partial_{x_1} + a_2} \right) G_i(x_1 + y_1, \mathbf{x}', t), \end{aligned}$$

setting

$$g(x_1, \mathbf{x}', t) \equiv 2 \frac{a_2}{a_1 \partial_{x_1} + a_2} G_i(x_1, \mathbf{x}', t),$$

then the function $g(x_1, \mathbf{x}', t)$ satisfies

$$(a_2 + a_1 \partial_{x_1})\, g = 2a_2 G_i(x_1, \mathbf{x}', t).$$

Solving this ODE gives

$$g(x_1, \mathbf{x}', t) = 2a_2 \int_{x_1}^{\infty} e^{-\gamma(z-x_1)} G_i(z, \mathbf{x}', t)\, dz = 2a_2 \int_0^{\infty} e^{-\gamma z} G_i(x_1 + z, \mathbf{x}', t)\, dz,$$

where $\gamma \equiv -\frac{a_2}{a_1} > 0$.

Summarizing previous results we obtain

Lemma 4.1. *Assume that the spatial dimension $n \geq 2$, the Green's functions $\mathbb{G}_i(x_1, \mathbf{x}', t; y_1)$ $(i = 1, 2)$ of the linear initial-boundary value problem (3) and (4) can be represented as*

$$\mathbb{G}_i(x_1, \mathbf{x}', t; y_1) = G_i(x_1 - y_1, \mathbf{x}', t) + G_i(x_1 + y_1, \mathbf{x}', t) - g(x_1 + y_1, \mathbf{x}', t),$$

here $G_i(x_1, \mathbf{x}', t)$ are the fundamental solutions for (7).

Moreover, under the assumption of (6), we have the following L^p $(1 \leq p \leq \infty)$ estimates for $\mathbb{G}_i(x_1, \mathbf{x}', t; y_1)$:

$$\|\partial_{\mathbf{x}}^{\alpha} \mathbb{G}_1 * u_0\|_{L^p} = O(1)\varepsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\alpha|}{2}} + O(1)\varepsilon e^{-t/C},$$
$$\|\partial_{\mathbf{x}}^{\alpha}\partial_{x_i} \mathbb{G}_2 * u_1\|_{L^p} = O(1)\varepsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\alpha|}{2}} + O(1)\varepsilon e^{-t/C}.$$

Proof. Based on the longwave-shortwave decomposition, we can write the Green's functions $G_i(\mathbf{x}, t)$ as follows:

$$G_i(\mathbf{x}, t) = G_i^L(\mathbf{x}, t) + G_i^S(\mathbf{x}, t).$$

Setting

$$\mathbb{G}_i^L(x_1, \mathbf{x}', t; y_1) = G_i^L(x_1 - y_1, \mathbf{x}', t) + G_i^L(x_1 + y_1, \mathbf{x}', t) - g^L(x_1 + y_1, \mathbf{x}', t),$$
$$\mathbb{G}_i^S(x_1, \mathbf{x}', t; y_1) = G_i^S(x_1 - y_1, \mathbf{x}', t) + G_i^S(x_1 + y_1, \mathbf{x}', t) - g^S(x_1 + y_1, \mathbf{x}', t),$$

then

$$\mathbb{G}_i(x_1, \mathbf{x}', t; y_1) = \mathbb{G}_i^L(x_1, \mathbf{x}', t; y_1) + \mathbb{G}_i^S(x_1, \mathbf{x}', t; y_1).$$

By the explicit description of the fundamental solutions, one can easily get the following L^p $(1 \leq p \leq \infty)$ estimates for the case $n \geq 2$,

$$\|\partial_{\mathbf{x}}^{\alpha} \mathbb{G}_1^L(x_1, \mathbf{x}', t; y_1), \partial_{\mathbf{x}}^{\alpha}\partial_{x_i} \mathbb{G}_2^L(x_1, \mathbf{x}', t; y_1)\|_{L^p} \leq O(1) t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\alpha|}{2}}.$$

Applying Hausdorff-Young inequality Lemma 5.3, we have the L^p estimate

$$\|\partial_{\mathbf{x}}^{\alpha} \mathbb{G}_1^L(x_1, \mathbf{x}', t; y_1) * u_0(\mathbf{x})\|_{L^p} \leq O(1) t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\alpha|}{2}} \|u_0\|_{L^1},$$
$$\|\partial_{\mathbf{x}}^{\alpha}\partial_{x_i} \mathbb{G}_2^L(x_1, \mathbf{x}', t; y_1) * u_1(\mathbf{x})\|_{L^p} \leq O(1) t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\alpha|}{2}} \|u_1\|_{L^1}.$$

For the short wave component, when $|\alpha| = 0$ we have

$$\|\mathbb{G}_1^S(x_1, \mathbf{x}', t; y_1) * u_0(\mathbf{x})\|_{L^p} \leq O(1) e^{-t/C} \|u_0\|_{H^{k_p(0)}},$$
$$\|\mathbb{G}_2^S(x_1, \mathbf{x}', t; y_1) * u_1(\mathbf{x})\|_{L^p} \leq O(1) e^{-t/C} \|u_1\|_{H^{k_p(0)}}.$$

When $|\alpha| = 1$, using the integration by parts we have

$$\|\partial_{\mathbf{x}}^{\alpha} \mathbb{G}_1^S(x_1, \mathbf{x}', t; y_1) * u_0(\mathbf{x})\|_{L^p} \leq O(1) e^{-t/C} \|u_0\|_{H^{k_p(1)}},$$
$$\|\partial_{\mathbf{x}}^{\alpha} \mathbb{G}_2^S(x_1, \mathbf{x}', t; y_1) * u_1(\mathbf{x})\|_{L^p} \leq O(1) e^{-t/C} \|u_1\|_{H^{k_p(1)}}.$$

Therefore, we prove this lemma. ∎

4.2. **Existence for the nonlinear problem.** The study of boundary operator in the last section suggests that we can only consider the case $a_1 a_2 < 0$ for the nonlinear stability. The Green's functions $\mathbb{G}_i(x_1, \mathbf{x}', t; y_1)(i = 1, 2)$ give the representation of the solution $u(\mathbf{x}, t)$:

$$\begin{aligned}\partial_{\mathbf{x}}^{\alpha} u\left(\mathbf{x}, t\right) &= \partial_{\mathbf{x}}^{\alpha} \int_{\mathbb{R}_{+}^{n}} \left(\mathbb{G}_1\left(x_1, \mathbf{x}' - \mathbf{y}', t; y_1\right) u_0(\mathbf{y}) + \mathbb{G}_2\left(x_1, \mathbf{x}' - \mathbf{y}', t; y_1\right) u_1(\mathbf{y})\right) d\mathbf{y} \\ &\quad + \partial_{\mathbf{x}}^{\alpha} \int_0^t \int_{\mathbb{R}_{+}^{n}} \mathbb{G}_2\left(x_1, \mathbf{x}' - \mathbf{y}', t - \tau; y_1\right) div f(u)(\mathbf{y}, \tau) d\mathbf{y} d\tau \\ &\equiv \partial_{\mathbf{x}}^{\alpha} \mathcal{I}\left(\mathbf{x}, t\right) + \partial_{\mathbf{x}}^{\alpha} \mathcal{N}\left(\mathbf{x}, t\right). \end{aligned} \tag{12}$$

The unique existence of the initial-boundary value problem can be proved following the analogous procedure developed by [22], and one can also get the energy bound by using the weighted energy estimates [17]. Here due to the explicit expression of the Green's functions, we give the existence of the nonlinear problem following the way in [4].

Construct a sequence $\{u^m(\mathbf{x}, t)\}$ where $u^m(\mathbf{x}, t)$ is the solution of the following linear problem:

$$\begin{cases} \partial_t^2 u^m - c^2 \Delta u^m - \nu \partial_t \Delta u^m = div f(u^{m-1}), \mathbf{x} \in \mathbb{R}_{+}^{n}, t > 0, \\ u^m|_{t=0} = u_0(\mathbf{x}), \quad u_t^m|_{t=0} = u_1(\mathbf{x}), \\ \left(a_1 \partial_{x_1} u^m + a_2 u^m\right)\left(x_1 = 0, \mathbf{x}', t\right) = 0, \end{cases}$$

for $m \geq 1$ and $u^0(\mathbf{x}, t) = 0$.

Introduce a set of functions as follows:

$$M_{\varepsilon} = \{u(\mathbf{x}, t) | \sup_{t \geq 0} \{(1+t)^{\frac{n-1}{2}} \|u(\cdot, t)\|_{W^{1,\infty}} + (1+t)^{\frac{n-2}{4}} \|u(\cdot, t)\|_{H^1}\} \leq \varepsilon\}.$$

Then the M_{ε} is a non-empty and complete metric space under the measure $\rho(u, v) = \sup_{t \geq 0}\{(1+t)^{\frac{n-1}{2}} \|(u-v)(\cdot, t)\|_{W^{1,\infty}} + (1+t)^{\frac{n-2}{4}} \|(u-v)(\cdot, t)\|_{H^1}\}$. We have the following lemma

Lemma 4.2. *When the initial data u_0 and u_1 satisfy (6), we have $\{u^m(\mathbf{x}, t)\} \subseteq M_{\varepsilon}$ and it is a Cauchy sequence in M_{ε}. Thus one can find a limit $u(\mathbf{x}, t) \in M_{\varepsilon}$, which is a classical solution of the nonlinear problem (1) -(2).*

Proof. The basic idea is to design an iteration scheme which is based on the Green's functions for the half space $\mathbb{R}_{+}^{n}$. Then by using the decaying properties, the convergence sequence can be obtained and the global classical solution is proved.

Using the Green's functions to represent

$$\begin{aligned} u^m\left(\mathbf{x}, t\right) &= \int_{\mathbb{R}_{+}^{n}} \left(\mathbb{G}_1\left(x_1, \mathbf{x}' - \mathbf{y}', t; y_1\right) u_0\left(\mathbf{y}\right) + \mathbb{G}_2\left(x_1, \mathbf{x}' - \mathbf{y}', t; y_1\right) u_1(\mathbf{y})\right) d\mathbf{y} \\ &\quad + \int_0^t \int_{\mathbb{R}_{+}^{n}} \mathbb{G}_2\left(x_1, \mathbf{x}' - \mathbf{y}', t - \tau; y_1\right) div f(u^{m-1})(\mathbf{y}, \tau) d\mathbf{y} d\tau \end{aligned}$$

for $m \geq 2$.

When $m = 1$, we define

$$u^1\left(\mathbf{x}, t\right) = \int_{\mathbb{R}_{+}^{n}} \left(\mathbb{G}_1\left(x_1, \mathbf{x}' - \mathbf{y}', t; y_1\right) u_0\left(\mathbf{y}\right) + \mathbb{G}_2\left(x_1, \mathbf{x}' - \mathbf{y}', t; y_1\right) u_1(\mathbf{y})\right) d\mathbf{y}.$$

Lemma 4.1 and Lemma 5.5 give

$$\|u^1(\cdot,t)\|_{L^2} \le O(1)\varepsilon(1+t)^{-\frac{n-2}{4}}, \quad \|u^1(\cdot,t)\|_{L^\infty} \le O(1)\varepsilon(1+t)^{-\frac{n-1}{2}}.$$

One can also get the estimates for the derivatives

$$\|\partial_{\mathbf{x}}^\alpha u^1(\cdot,t)\|_{L^2} \le O(1)\varepsilon(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}, \quad \|\partial_{\mathbf{x}}^\alpha u^1(\cdot,t)\|_{L^\infty} \le O(1)\varepsilon(1+t)^{-\frac{n-1}{2}-\frac{|\alpha|}{2}}.$$

Thus, we have $u^1(\mathbf{x},t) \in M_\varepsilon$. We assume then $j \le m-1$, $u^j(\mathbf{x},t) \in M_\varepsilon$.

Now considering $u^m(\mathbf{x},t)$, from the first inequality of Lemma 5.4, we have

$$\|f(u^{m-1})(\cdot,\tau)\|_{L^1} \le O(1)\|u^{m-1}\|_{L^2}^2 \|u^{m-1}\|_{L^\infty}^{k-2} \le O(1)\varepsilon^k(1+\tau)^{-\frac{(k-1)n-k}{2}},$$

$$\begin{aligned}\left\|divf(u^{m-1})(\cdot,\tau)\right\|_{L^1} &\le O(1)\left\|\partial_{x_i}u^{m-1}\right\|_{L^2}\left\|u^{m-1}\right\|_{L^2}\|u^{m-1}\|_{L^\infty}^{k-2}\\ &\le O(1)\varepsilon^k(1+\tau)^{-\frac{(k-1)n-k+1}{2}},\end{aligned}$$

and

$$\begin{aligned}\|u^m(\cdot,t)\|_{L^2} &\le O(1)\varepsilon(1+t)^{-\frac{n-2}{4}} + O(1)\varepsilon^k\int_0^t(1+t-\tau)^{-\frac{n-2}{4}}(1+\tau)^{-\frac{(k-1)n-k+1}{2}}d\tau\\ &\le O(1)\varepsilon(1+t)^{-\frac{n-2}{4}},\end{aligned}$$

$$\begin{aligned}\|\partial_{\mathbf{x}}^\alpha u^m(\cdot,t)\|_{L^2} &\le O(1)\varepsilon(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}\\ &\quad +O(1)\varepsilon^k\int_0^t(1+t-\tau)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}(1+\tau)^{-\frac{(k-1)n-k+1}{2}}d\tau\\ &\le O(1)\varepsilon(1+t)^{-\frac{n-2}{4}},\end{aligned}$$

since $n \ge 3$, $k > [\frac{n+1}{n-1}]$.

The L^∞ estimate is similar:

$$\|\partial_{\mathbf{x}}^\alpha u^m(\cdot,t)\|_{L^\infty} \le O(1)\varepsilon(1+t)^{-\frac{n-1}{2}} \quad \text{for} \quad |\alpha| \le 1,$$

These yield that $u^m(\mathbf{x},t) \in M_\varepsilon$.

Considering the equation of $v^m = u^m - u^{m-1}$, it satisfies

$$\begin{cases} \partial_t^2 v^m - c^2\Delta v^m - \nu\partial_t\Delta v^m = divf(u^{m-1}) - divf(u^{m-2}), \mathbf{x} \in \mathbb{R}^n_+, t > 0,\\ v^m|_{t=0} = 0, \quad v_t^m|_{t=0} = 0,\\ (a_1\partial_{x_1}v^m + a_2 v^m)(x_1 = 0, \mathbf{x}', t) = 0 \end{cases}$$

for $m \ge 2$ and $v^1(\mathbf{x},t) = u^1(\mathbf{x},t)$. Repeating the process above, with the help of the second inequality in Lemma 5.4, we get

$$\|v^m(\cdot,t)\|_{L^2} \le O(1)\int_0^t(1+t-\tau)^{-\frac{n-2}{4}}\|divf(u^{m-1}) - divf(u^{m-2})\|_{L^1}d\tau$$

$$\begin{aligned} &\le O(1)\int_0^t(1+t-\tau)^{-\frac{n-2}{4}}(\|u^{m-1}\|_{L^\infty} + \|u^{m-2}\|_{L^\infty})^{k-2}\\ &\qquad [\|v^{m-1}\|_{H^1}(\|u^{m-1}\|_{L^2} + \|u^{m-2}\|_{L^2}) + \|v^{m-1}\|_{L^2}(\|u^{m-1}\|_{H^1} + \|u^{m-2}\|_{H^1})]d\tau\\ &\le O(1)\|v^{m-1}\|_{H^1}\int_0^t(1+t-\tau)^{-\frac{n-2}{4}}(1+\tau)^{-\frac{(2k-3)(n-1)-1}{4}}d\tau\\ &\le O(1)\varepsilon^{k-1}\|v^{m-1}\|_{H^1},\end{aligned}$$

$$\begin{aligned}\|\partial_{\mathbf{x}}^{\alpha}v^{m}\left(\cdot,t\right)\|_{L^2} &\le O(1)\|v^{m-1}\|_{H^1}\int_0^t(1+t-\tau)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}(1+\tau)^{-\frac{(2k-3)(n-1)-1}{4}}d\tau\\ &\le O(1)\varepsilon^{k-1}\|v^{m-1}\|_{H^1}.\end{aligned}$$

The L^∞ estimate is similar, hence we have $\rho(u^m-u^{m-1})\le C\varepsilon^{k-1}\rho(u^{m-1}-u^{m-2})$. This concludes that $\{u^m\}$ is a Cauchy sequence in M_ε. Therefore, the limit $u(\mathbf{x},t)\in M_\varepsilon$, and it is the nonlinear solution of our problem. □

4.3. **Long time decaying rate for the initial-boundary value problem.** Applying Lemma 4.1, the first part of (12) has the following estimate for $|\alpha|\le 1$,

$$\|\partial_{\mathbf{x}}^{\alpha}\mathcal{I}\left(\cdot,t\right)\|_{L^p}=O(1)\varepsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}.$$

Now we consider the decaying rates of the derivatives by making the ansatz for $|\alpha|\le 1$:

$$\|\partial_{\mathbf{x}}^{\alpha}u(\cdot,t)\|_{L^2}=A\varepsilon(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}},\quad \|\partial_{\mathbf{x}}^{\alpha}u(\cdot,t)\|_{L^\infty}=A\varepsilon(1+t)^{-\frac{n-1}{2}-\frac{|\alpha|}{2}},\tag{13}$$

for some positive constant A.

We first prove the L^2 estimate in (13). Straightforward computation shows that

$$\begin{aligned}&\|f(u)(\cdot,\tau)\|_{L^1}=O(1)\left\|u\right\|_{L^2}^2\left\|u\right\|_{L^\infty}^{k-2}\le O(1)A^k\varepsilon^k(1+\tau)^{-\frac{(k-1)n-k}{2}},\\ &|div f(u)(\cdot,\tau)\|_{L^1}=O(1)\left\|\partial_{x_i}u\right\|_{L^2}\left\|u\right\|_{L^2}\left\|u\right\|_{L^\infty}^{k-2}\le O(1)A^k\varepsilon^k(1+\tau)^{-\frac{(k-1)n-k+1}{2}}.\end{aligned}$$

Hence for $\partial_{\mathbf{x}}^{\alpha}\mathcal{N}(\mathbf{x},t)$

$$\begin{aligned}\|\partial_{\mathbf{x}}^{\alpha}\mathcal{N}(\cdot,t)\|_{L^2}&\le\left\|\int_0^t\int_{\mathbb{R}_+^n}\partial_{\mathbf{x}}^{\alpha}\mathbb{G}_2^S(x_1,\mathbf{x}',t;y_1)div f(u)\left(\mathbf{y},\tau\right)d\mathbf{y}d\tau\right\|_{L^2}\\ &\quad+\left\|\int_0^t\int_{\mathbb{R}_+^n}\partial_{\mathbf{x}}^{\alpha}\mathbb{G}_2^L(x_1,\mathbf{x}',t;y_1)div f(u)\left(\mathbf{y},\tau\right)d\mathbf{y}d\tau\right\|_{L^2}\\ &=\|\partial_{\mathbf{x}}^{\alpha}\mathcal{N}_S\|_{L^2}+\|\partial_{\mathbf{x}}^{\alpha}\mathcal{N}_L\|_{L^2}.\end{aligned}$$

Using Lemma 5.5, we have

$$\|\mathcal{N}_S\|_{L^2}\le O(1)A\varepsilon(1+t)^{-\frac{n-2}{4}},$$

$$\begin{aligned}\|\mathcal{N}_L\|_{L^2}&\le\left\|\int_0^t\int_{\mathbb{R}_+^n}\mathbb{G}_2^L(x_1,\mathbf{x}',t;y_1)\cdot div f(u)(\mathbf{y},\tau)d\mathbf{y}d\tau\right\|_{L^2}\\ &\le O(1)\int_0^t(1+t-\tau)^{-\frac{n-2}{4}}\left\|div f(u)(\cdot,\tau)\right\|_{L^1}d\tau\\ &\le O(1)A^k\varepsilon^k\int_0^t(1+t-\tau)^{-\frac{n-2}{4}}(1+\tau)^{-\frac{(k-1)n-k+1}{2}}d\tau\\ &\le O(1)A^k\varepsilon^k(1+t)^{-\frac{n-2}{4}}.\end{aligned}$$

For $|\alpha|=1$, using the integration by parts, we have the following estimate for the short wave part

$$\|\partial_{\mathbf{x}}^{\alpha}\mathcal{N}_S\|_{L^2}\ \le\ \left\|\int_0^t\int_{\mathbb{R}_+^n}\mathbb{G}_2^S(x_1,\mathbf{x}',t;y_1)\partial_{\mathbf{x}}^{\alpha}div f(u)\left(\mathbf{y},\tau\right)d\mathbf{y}d\tau\right\|_{L^2}$$

$$
\begin{aligned}
&+1_{\{\partial_{\mathbf{x}}^\alpha=\partial_{x_1}\}}\left\|\int_0^t\int_{\mathbb{R}^{n-1}}\mathbb{G}_2^S(x_1,\mathbf{x}',t;y_1)divf(u)\left(\mathbf{y},\tau\right)|_{y_1=0}^{\infty}d\mathbf{y}'d\tau\right\|_{L^2}\\
\le\;& O(1)A^k\varepsilon^k(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}},
\end{aligned}
$$

where

$$
1_{\{\partial_{\mathbf{x}}^\alpha=\partial_{x_1}\}}=\begin{cases}1,\text{if } \ \partial_{\mathbf{x}}^\alpha=\partial_{x_1},\\ 0,\quad\text{otherwise.}\end{cases}
$$

The long wave part gives

$$
\begin{aligned}
\|\partial_{\mathbf{x}}^\alpha\mathcal{N}_L\|_{L^2}\;\le\;&\left\|\int_0^t\int_{\mathbb{R}^n_+}\partial_{\mathbf{x}}^\alpha\mathbb{G}_2^L(x_1,\mathbf{x}',t;y_1)\cdot divf(u)(\mathbf{y},\tau)d\mathbf{y}d\tau\right\|_{L^2}\\
\le\;& O(1)\int_0^t(1+t-\tau)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}\left\|divf(u)(\cdot,\tau)\right\|_{L^1}d\tau\\
\le\;& O(1)A^k\varepsilon^k\int_0^t(1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{(k-1)n-k+1}{2}}d\tau\\
\le\;& O(1)A^k\varepsilon^k(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}.
\end{aligned}
$$

Hence we have for $|\alpha|\le 1$,

$$
\|\partial_{\mathbf{x}}^\alpha u(\cdot,t)\|_{L^2}=O(1)\varepsilon(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}.
$$

Similarly, we will have

$$
\|\partial_{\mathbf{x}}^\alpha u(\cdot,t)\|_{L^\infty}=O(1)\varepsilon(1+t)^{-\frac{n-1}{2}-\frac{|\alpha|}{2}}.
$$

Thus we verify our assumption (13).

Using the L^2-L^∞ interpolation lemma 5.6, we get

$$
\|u\|_{L^p}\le O(1)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}},2\le p\le\infty,
$$

and finish the proof of the main theorem.

5. **Appendix.**

Lemma 5.1. [11] *Let* $E(\mathbf{x},t)=e^{-\frac{|\mathbf{x}|^2}{C(t+1)}}$ *,we have the following estimates*

$$
\left|\int_{|\mathbf{y}|\le1}\frac{E(\mathbf{x}+ct\mathbf{y},t)\mathbf{y}^\alpha}{\sqrt{1-|\mathbf{y}|^2}}d\mathbf{y}\right|\le O(1)\left(\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(1+t)^{(2n-1)/4}}+\frac{H(ct-|\mathbf{x}|)}{(1+t)^{\frac{n-1}{2}}(ct-|\mathbf{x}|+\sqrt{t})^{\frac{1}{2}}}\right),
$$

$$
\left|\int_{|\mathbf{y}|=1}E(\mathbf{x}+ct\mathbf{y},t)\mathbf{y}^\alpha dS_{\mathbf{y}}\right|\le O(1)\frac{e^{-\frac{(|\mathbf{x}|-ct)^2}{C(t+1)}}}{(1+t)^{(n-1)/2}}.
$$

Lemma 5.2. [7] *Suppose a function* $f\in L^1(\mathbb{R}^n)$ *and its Fourier transform* $\mathcal{F}[f](\boldsymbol{\xi})$, $\boldsymbol{\xi}$ *is analytic in* $\mathcal{D}_\delta$ *and satisfies*

$$
|\mathcal{F}[f](\boldsymbol{\xi})|\le\frac{E}{(1+|\boldsymbol{\xi}|)^{n+1}},\quad for\ \ |Im(\xi_i)|\le\delta,\quad and\ \ i=1,2,\cdots,n.
$$

Then, the function $f(\mathbf{x})$ *satisfies*

$$
|f(\mathbf{x})|\le Ee^{-\delta|\mathbf{x}|/C},
$$

for any positive constant $C>1$.

Lemma 5.3. *{Hausdorff-Young inequality} For $1 \le p \le r \le q \le \infty$ satisfying $\frac{1}{q} - \frac{1}{p} = 1 - \frac{1}{r}$, the inequality*

$$\|f * g\|_{L^p} \le C\|f\|_{L^r}\|g\|_{L^q}.$$

holds, where $*$ denote the convolution.

The following lemma is used to deal with the nonlinear term, which is similar to that in [4]. It can be proved by Gagliardo-Nirenberg inequality.

Lemma 5.4. *Suppose $f(u)$ is smooth, where u is a vector function. Assume $f(u) = O(|u|^{1+m})$ ($m \ge 1$ is an integer) when $|u| \le 1$. Then for any integer $l \ge 0$, if $u, v \in W^{l,q}(\mathbb{R}^n_+) \cap L^p(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ and $\|u, v\|_{L^\infty} \le 1$, we have $f(u) - f(v) \in W^{l,r}(\mathbb{R}^n_+)$. Furthermore, the following inequalities holds:*

$$\begin{aligned}
&\|f(u)\|_{W^{l,r}} \le O(1)\|u\|_{W^{l,q}}\|u\|_{L^p}\|u\|^{m-1}_{L_\infty},\\
&\|f(u)-f(v)\|_{W^{l,r}} \le O(1)(\|u\|_{L_\infty} + \|v\|_{L_\infty})^{m-1}\\
&\qquad\qquad [\|u-v\|_{W^{l,q}}(\|u\|_{L^p} + \|v\|_{L^p}) + \|u-v\|_{L^p}(\|u\|_{W^{l,q}} + \|v\|_{W^{l,q}})].
\end{aligned}$$

Here $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 \le p, q, r \le \infty$.

Lemma 5.5. [25] *Let $r_1, r_2, r_3 > 0$, we have*

$$\int_0^t (1+t-\tau)^{-r_1}(1+\tau)^{-r_2} d\tau \le C(r_1, r_2)(1+t)^{-min\{r_1, r_2, r_1+r_2-1-\eta\}}$$

for an arbitrarily small $\eta > 0$, and

$$\int_0^t e^{-r_3(t-\tau)}(1+\tau)^{-r_1} d\tau \le (1+t)^{-r_1}.$$

Lemma 5.6. *Assume $1 \le p \le r \le q \le \infty$ and*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, 0 \le \theta \le 1.$$

If $u \in L^p(\mathbb{R}^n_+) \cap L^q(\mathbb{R}^n_+)$, one has $u \in L^r(\mathbb{R}^n_+)$, and $\|u\|_{L^r} \le \|u\|^\theta_{L^p}\|u\|^{1-\theta}_{L^q}$.

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