

SIR RUMOR SPREADING MODEL WITH TRUST RATE DISTRIBUTION

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ABSTRACT. In this paper, we study a rumor spreading model in which several types of ignorants exist with trust rate distributions λ_i , $1 \leq i \leq N$. We rigorously show the existence of a threshold on a momentum type initial quantity related to rumor outbreak occurrence regardless of the total initial population. We employ a steady state analysis to obtain the final size of the rumor. Using numerical simulations, we demonstrate the analytical result in which the threshold phenomenon exists for rumor size and discuss interaction between the ignorants of several types of trust rates.

1. Introduction. Currently it is the digital era where there is a steady flood of information. Such information inundation makes a variety of mass media more important, for example, newspapers, broadcast, social network system media, and public speaking. Before developing mass media, rumors propagated by word of mouth and played a crucial role in communication between people or groups. This process can be understood as a kind of homogenization of information system and social interaction [1]. With the emergence of multimedia and social media, rumors have spread faster and have wide transmissions [5]. However, some harmful and powerful rumor outbreaks arise from such wide transmission via these media [10, 14, 18]. Moreover, their influence causes multiple effects for a variety of situations rather than the mono effect for localized situations [12].

As a benefit in return for the homogenization, personality is more heavily emphasized and the diversity of people has garnered much attention in our social

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community. In the microscopic viewpoint of rumor spreading, this variety of characteristics is important. Many researchers already have investigated that the degree of belief is important in rumor spreading [8, 16]. From this perspective, we assume that there will be various groups that share the same trust rate. In this paper, we propose an SIR type rumor spreading model with given spreading rate distributions λ_i , $1 \leq i \leq N$. Each spreading rate distribution λ_i represents a different character of several classes of groups. With the distribution λ_i , we present a sufficient condition for the rumor outbreak. To obtain a rumor outbreak, we need a sufficiently large initial quantity that represents the momentum of rumor spreading. We also provide numerical simulations to verify our analysis with several combinations of different parameters. We notice that the authors in [19] studied an SIR type rumor spreading model with a reputation mechanism and considered a probability that represents the reputation of the opinion, where reputation is related to the trust rate. They used a single ignorant SIR model. On the other hand, we consider a model with several classes of ignorants and trust rate distributions λ_i , $1 \leq i \leq N$.

Next we provide a brief historical review of the rumor spreading model. Starting the pioneering studies by Daley and Kendall [3, 4], a lot of researchers have studied rumor spreading and tried to build mathematical models [11, 17]. Zanette [23, 24] numerically obtained the existence of a critical threshold for a rumor spreading model regarding small-world networks. In [13], the authors derived the mean-field equation of complex heterogeneous networks. For other topological settings, see [7, 15]. Most mathematical models for rumor spreading are based on the epidemic model. In [27], the authors considered an SIR type rumor spreading model with forgetting mechanism. See also [6, 28] for other models with forgetting mechanisms. In [26], the authors added a hibernator variable to the SIR type rumor spreading model. Similarly, in [20, 22], the authors adapted several new variables to construct a more realistic model for the rumor spreading phenomena. In [25], the authors employed the probability that ignorants directly become stiflers when they are aware of a rumor. We refer to papers [2, 9, 29] for other rumor spreading models.

The paper is organized as follows. In Section 2, we present the trust distribution and its mechanism in the SIR type model. In Section 3, we derive a single equation for the rumor size ϕ . In Section 4, we provide proof of the main theorem. In Section 5, we demonstrate our results by numerical simulations. Finally, we summarize our results in Section 6.

Notation: Throughout the paper, we use the following simplified notation:

$$I(t) = (I_1(t), \dots, I_N(t)), \quad \dot{I} = (\dot{I}_1, \dots, \dot{I}_N), \quad \dot{I}^n = (\dot{I}_1^n, \dots, \dot{I}_N^n), \quad n, N \in \mathbb{N}.$$

2. Spreading rate distribution model. Let V and E be sets of vertices and edges, respectively. Representing the individuals and contacts as the vertices and edges, a social network leads to an undirected graph $G = (V, E)$. A general social network is close to a small-world network [21]. In [13], the authors derived the mean-field rate equation based on the Poisson distribution of small-world network. As in [13], we consider a rumor spreading model based on the SIR model of a homogeneous network.

There are three groups of populations: ignorants (I), spreaders (S), and stiflers (R). At the first stage, ignorants contact a spreader, realize a rumor, and accept the hearsay. According to the acceptance with rate λ , some of them become spreaders and then spread the rumor to other ignorants. However, as spreaders lose interest in spreading the rumor, they become stiflers. Based on this mechanism, the SIR

rumor spreading model [3, 4] is given by

$$\begin{aligned}\dot{I} &= -k\lambda SI, \\ \dot{S} &= k\lambda SI - kS(\sigma_1 S + \sigma_2 R), \\ \dot{R} &= k\sigma S(S + R),\end{aligned}$$

where k is the average degree of the network, λ is the spreading rate of the system, σ_1 is the contact rate between spreaders, and σ_2 is the contact rate between spreaders and stiflers.

As in [25], we assume that spreaders lose their interest in rumors with probability δ and become stiflers. Moreover, stiflers interact with spreaders to create other stiflers from other spreaders. Spreaders also have a negative effect on other spreaders because they consider the rumor to be outdated if the spreader meets with other spreaders frequently. For simplicity, we assume that the contact rates σ_1 and σ_2 are the same, say $\sigma = \sigma_1 = \sigma_2$. As mentioned before, we consider several groups of ignorants with different spreading rates λ_i , $i = 1, \dots, N$, motivated by [8, 16]. The mean-field equation is then given by

$$\begin{aligned}\dot{I}_i &= -k\lambda_i SI_i, \quad i = 1, \dots, N, \\ \dot{S} &= \sum_{i=1}^N k\lambda_i SI_i - k\sigma S(S + R) - \delta S, \\ \dot{R} &= k\sigma S(S + R) + \delta S,\end{aligned}\tag{1}$$

subject to initial data $I_i(0) = \mathring{I}_i$, $i = 1, \dots, N$, $S(0) = \mathring{S}$, $R(0) = \mathring{R}$, where I_i is the population density of the i th group of ignorants with spreading (trust) rate λ_i . S and R are the density of spreaders and stiflers, respectively. Moreover, σ is the contact rate between spreaders and stiflers, and δ denotes the decay rate of spreaders to stiflers.

Throughout this paper, we assume that

- (1) k , σ and δ are fixed positive constants.
- (2) The spreading (trust) rate distribution λ_i is nonnegative constant for each $1 \leq i \leq N$.

We will consider a family of initial data $\mathring{I}_1, \dots, \mathring{I}_N$, \mathring{S} and \mathring{R} such that

$$\mathring{T} = \left(\sum_{j=1}^N \mathring{I}_j \right) + \mathring{S} + \mathring{R}$$

for a fixed total initial population \mathring{T} .

Next, we define the rumor size, a momentum type quantity of the initial data and rumor outbreak.

Definition 2.1. [19, 25, 26] For a solution (I, S, R) to system (1), we define the rumor size ϕ of (I, S, R) :

$$\phi(t) = \int_0^t S(\tau) d\tau.$$

Definition 2.2. Let (I, S, R) be a solution to system (1) with initial data \mathring{I} , \mathring{S} and \mathring{R} . We define initial reliability of ignorants $M_1(\mathring{I})$:

$$M_1(\mathring{I}) = \sum_{i=1}^N \lambda_i \mathring{I}_i$$

and total population with initial $T(0) = \mathring{T}$:

$$T(t) = \left(\sum_{i=1}^N I_i(t) \right) + S(t) + R(t).$$

Definition 2.3. Let (I, S, R) be solutions to system (1) subject to initial data \mathring{I} , \mathring{S} and $\mathring{R} = 0$. If $\phi(t)$ in Definition 2.1 converges as $t \rightarrow \infty$, we define ϕ^∞ as the final size of the rumor:

$$\phi^\infty(\mathring{I}, \mathring{S}) := \lim_{t \rightarrow \infty} \phi(t).$$

Definition 2.4. For a given initial data $\mathring{I}, \mathring{S}$ and \mathring{R} of system (1) with $\mathring{S} = 0$ and $\mathring{R} = 0$, let $\mathring{I}^n, \mathring{S}^n$ and \mathring{R}^n be sequences satisfying

$$\mathring{I}^n \rightarrow \mathring{I}, \quad \mathring{S}^n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$\mathring{S}^n > 0, \quad \mathring{R}^n = 0 \quad \text{for } n \in \mathbb{N}.$$

We additionally assume that the total populations are the same:

$$\mathring{T} = \left(\sum_{i=1}^N \mathring{I}_i \right) + \mathring{S} + \mathring{R} = \left(\sum_{i=1}^N \mathring{I}_i^n \right) + \mathring{S}^n + \mathring{R}^n = \mathring{T}^n.$$

We say that a rumor outbreak occurs if the following limit exists

$$\phi_e(\mathring{I}) = \lim_{n \rightarrow \infty} \phi^\infty(\mathring{I}^n, \mathring{S}^n)$$

and $\phi_e(\mathring{I})$ is positive, where $\phi^\infty(\mathring{I}^n, \mathring{S}^n)$ is the final size of the rumor for the initial data $\mathring{I}^n, \mathring{S}^n, \mathring{R}^n$.

Remark 1. (1) In [13, 15, 25], the authors define that the rumor outbreak occurs if

$$\lim_{\mathring{R} \rightarrow 0} R(\infty) > 0.$$

This is essentially equivalent to Definition 2.4. We use $\phi(\infty)$ instead of $R(\infty)$, since $R(t) = R(\phi(t))$. See the result in Lemma 3.3.

(2) The rumor spreading begins with one spreader. Therefore, $\mathring{S} = 1/N$, where N is the total population number. Generally, N is a large number and this implies that $\mathring{S} \approx 0$. However, if we assume that $\mathring{I} = \mathring{T}$ and $\mathring{S} = \mathring{R} = 0$, then the corresponding solution $(I(t), S(t), R(t))$ is the trivial stationary solution. Therefore, to get an intrinsic property of the system, we have to consider a limit of sequence with initial data $\mathring{S}^n \rightarrow 0$.

The following is the main theorem of this paper.

Theorem 2.5. Let k, σ, δ and \mathring{T} be positive constants and $\mathring{R} = \mathring{R}^n = 0$. Let $\{(I^n(t), S^n(t), R^n(t))\}$ be a sequence of solutions to system (1) subject to initial data $\mathring{I}^n, \mathring{S}^n$ and $\mathring{R}^n = 0$, respectively.

We assume that each \mathring{S}^n is positive, $\mathring{I}^n \rightarrow \mathring{I}$ and $\mathring{S}^n \rightarrow 0$ for an N -dimensional vector \mathring{I} , and

$$\mathring{T} = \left(\sum_{k=1}^N \mathring{I}_k^n \right) + \mathring{S}^n = \sum_{i=1}^N \mathring{I}_i.$$

Then, there exists the following limit of steady states:

$$\phi_e = \phi_e(\mathring{I}) = \lim_{\mathring{S}^n \rightarrow 0, \mathring{I}^n \rightarrow \mathring{I}} \phi^\infty(\mathring{I}^n, \mathring{S}^n),$$

where $\phi^\infty(\mathring{I}^n, \mathring{S}^n)$ is the final size of the rumor for the initial data \mathring{I}^n and \mathring{S}^n with $\mathring{R}^n = 0$.

Furthermore, if $kM_1(\mathring{I}) > \delta$, $\phi_e(\mathring{I})$ is positive and if $kM_1(\mathring{I}) \leq \delta$, $\phi_e(\mathring{I})$ is zero.

Remark 2. An equivalent condition of occurring a rumor outbreak is

$$kM_1(\mathring{I}) > \delta.$$

3. An equation for rumor size ϕ . In this section, we derive a single equation for ϕ and consider the steady state analysis for the rumor spreading model (1) via a derived single equation of ϕ from the next argument.

Lemma 3.1. Let (I, S, R) be a solution to system (1) with an initial data \mathring{I} , \mathring{S} and \mathring{R} . Then each $I_i(t)$ satisfies

$$I_i(t) = \mathring{I}_i e^{-k\lambda_i \phi(t)}, \quad i = 1, \dots, N, \tag{2}$$

where $\phi(t)$ is a function defined in Definition 2.1.

Proof. From the first equation of system (1), we have

$$\frac{d}{dt} \log I_i(t) = -k\lambda_i S(t), \quad i = 1, \dots, N.$$

Integrating the above relation gives

$$\log I_i(t) = \log \mathring{I}_i - \int_0^t k\lambda_i S(\tau) d\tau, \quad i = 1, \dots, N.$$

For the population density of $S(t)$, we can obtain the following formula for I_i :

$$I_i(t) = \mathring{I}_i e^{-\int_0^t k\lambda_i S(\tau) d\tau}, \quad i = 1, \dots, N.$$

Clearly, we have $\frac{d}{dt} \phi(t) = S(t)$, which implies that $\phi(t) - \phi(0) = \int_0^t S(\tau) d\tau$. \square

Lemma 3.2. Let (I, S, R) be a solution to system (1) with initial data \mathring{I} , \mathring{S} and \mathring{R} . For the given initial data, the total population $T(t)$ of the solution to system (1) is conserved. Thus, we have

$$T(t) = \mathring{T}, \quad \text{for any } t > 0. \tag{3}$$

Proof. Note that the summation of all equations in system (1) yields

$$\left(\sum_{i=1}^N \dot{I}_i \right) + \dot{S} + \dot{R} = 0.$$

Integrating the above equation leads to

$$T(t) = \left(\sum_{i=1}^N I_i(t) \right) + S(t) + R(t) = \left(\sum_{i=1}^N \mathring{I}_i \right) + \mathring{S} + \mathring{R} = \mathring{T}.$$

\square

Remark 3. By the conservation property (3) in Lemma 3.2 and the formula in (2), we easily obtain the following formula for $S(t)$:

$$S(t) = \overset{\circ}{T} - \sum_{i=1}^N I_i(t) - R(t) = \overset{\circ}{T} - \sum_{i=1}^N \overset{\circ}{I}_i e^{-k\lambda_i \phi(t)} - R(t). \quad (4)$$

Lemma 3.3. Let (I, S, R) be a solution to system (1) with initial data $\overset{\circ}{I}$, $\overset{\circ}{S}$ and $\overset{\circ}{R}$. Then $R(t)$ is a function depending on ϕ :

$$R(t) = R(\phi(t)) = \overset{\circ}{R} + k\sigma \overset{\circ}{T} \phi(t) - k\sigma \sum_{\lambda_i \neq 0} \overset{\circ}{I}_i \frac{1 - e^{-k\lambda_i \phi(t)}}{k\lambda_i} - k\sigma \sum_{\lambda_i = 0} \overset{\circ}{I}_i \phi(t) + \delta\phi(t).$$

Proof. The third equation in system (1) gives us that

$$R(t) - \overset{\circ}{R} = \int_0^t \dot{R}(\tau) d\tau = \int_0^t [k\sigma S(\tau)(S(\tau) + R(\tau)) + \delta S(\tau)] d\tau.$$

From (4) and the definition of ϕ : $\phi(t) = \int_0^t S(\tau) d\tau$, we get

$$\begin{aligned} R(t) - \overset{\circ}{R} &= k\sigma \int_0^t S(\tau) [S(\tau) + R(\tau)] d\tau + \delta\phi(t) \\ &= k\sigma \int_0^t S(\tau) \left(\overset{\circ}{T} - \sum_{i=1}^N I_i(\tau) \right) d\tau + \delta\phi(t) \\ &= k\sigma \int_0^t \phi'(\tau) \left(\overset{\circ}{T} - \sum_{i=1}^N \overset{\circ}{I}_i e^{-k\lambda_i \phi(\tau)} \right) d\tau + \delta\phi(t) \\ &= k\sigma \overset{\circ}{T} \int_0^t \phi'(\tau) d\tau - \sum_{\lambda_i \neq 0} k\sigma \int_0^t \phi'(\tau) \overset{\circ}{I}_i e^{-k\lambda_i \phi(\tau)} d\tau \\ &\quad - \sum_{\lambda_i = 0} k\sigma \int_0^t \phi'(\tau) \overset{\circ}{I}_i e^{-k\lambda_i \phi(\tau)} d\tau + \delta\phi(t) \\ &= K_1 + K_2 + K_3 + K_4. \end{aligned}$$

We directly have K_1 and K_4 with

$$K_1 = k\sigma \overset{\circ}{T} \phi(t) \quad \text{and} \quad K_4 = \delta\phi(t).$$

For K_2 , we use $\lambda_i \neq 0$ for all $1 \leq i \leq N$. By the change of variables and the fact that $\phi(0) = 0$, we obtain

$$\begin{aligned} K_2 &= - \sum_{\lambda_i \neq 0} k\sigma \int_0^t \phi'(\tau) e^{-k\lambda_i \phi(\tau)} d\tau \\ &= - \sum_{\lambda_i \neq 0} k\sigma \int_{\phi(0)}^{\phi(t)} e^{-k\lambda_i \eta} d\eta \\ &= - \sum_{\lambda_i \neq 0} k\sigma \frac{e^{-k\lambda_i \phi(0)} - e^{-k\lambda_i \phi(t)}}{k\lambda_i} \\ &= - \sum_{\lambda_i \neq 0} k\sigma \frac{1 - e^{-k\lambda_i \phi(t)}}{k\lambda_i}. \end{aligned}$$

Similarly, we have K_3 with

$$\begin{aligned} K_3 &= - \sum_{\lambda_i=0} k\sigma \int_0^t \phi'(\tau) e^{-k\lambda_i\phi(\tau)} d\tau \\ &= - \sum_{\lambda_i=0} k\sigma \int_0^t \phi'(\tau) d\tau \\ &= - \sum_{\lambda_i=0} k\sigma(\phi(t) - \phi(0)) \\ &= - \sum_{\lambda_i=0} k\sigma\phi(t). \end{aligned}$$

Therefore, the above elementary calculations yield

$$R(t) - \dot{R} = k\sigma\dot{T}\phi(t) - k\sigma \sum_{\lambda_i \neq 0} \dot{I}_i \frac{1 - e^{-k\lambda_i\phi(t)}}{k\lambda_i} - k\sigma \sum_{\lambda_i=0} \dot{I}_i\phi(t) + \delta\phi(t).$$

□

The formula (4) of $S(t)$ implies that

$$\frac{d\phi(t)}{dt} = S(t) = \dot{T} - \sum_{i=1}^N \dot{I}_i e^{-k\lambda_i\phi(t)} - R(t).$$

By the result in Lemma 3.3, we derive a single decoupled equation for ϕ such that

$$\begin{aligned} \frac{d\phi(t)}{dt} &= \dot{T} - \sum_{i=1}^N \dot{I}_i e^{-k\lambda_i\phi(t)} - \dot{R} - k\sigma\dot{T}\phi(t) \\ &\quad + k\sigma \sum_{\lambda_i \neq 0} \dot{I}_i \frac{1 - e^{-k\lambda_i\phi(t)}}{k\lambda_i} + k\sigma \sum_{\lambda_i=0} \dot{I}_i\phi(t) - \delta\phi(t). \end{aligned}$$

For simplicity, we define

$$\begin{aligned} F(\phi) &= F(\phi, \dot{I}, \dot{S}, \dot{R}, \dot{T}) \\ &:= \dot{T} - \sum_{i=1}^N \dot{I}_i e^{-k\lambda_i\phi} - \dot{R} - k\sigma\dot{T}\phi \\ &\quad + k\sigma \sum_{\lambda_i \neq 0} \dot{I}_i \frac{1 - e^{-k\lambda_i\phi}}{k\lambda_i} + k\sigma \sum_{\lambda_i=0} \dot{I}_i\phi(t) - \delta\phi. \end{aligned} \tag{5}$$

Then, $\phi(t)$ is the solution to the following single equation subject to initial data $\phi(0) = 0$.

$$\dot{\phi}(t) = F(\phi(t)). \tag{6}$$

4. Steady state analysis. In this section, we use the steady state analysis to obtain the threshold phenomena for asymptotic behavior of the solution to system (1). To obtain the asymptotic behavior of solutions to (6), we first consider a steady state ϕ to (6). In other words, the steady state ϕ satisfies

$$F(\phi) = 0, \tag{7}$$

where $F(\phi) = F(\phi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T})$ is the function defined in (5). Therefore, ϕ is one of solutions to equation (7). Using a standard method from [19, 25, 26], we can obtain a sufficient condition for the existence of nontrivial zeros. We define G and H by

$$G(\phi) = G(\phi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T}) = \mathring{T} - \sum_{i=1}^N \mathring{I}_i e^{-k\lambda_i \phi} - \mathring{R} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i \frac{1 - e^{-k\lambda_i \phi}}{k\lambda_i} \quad (8)$$

and

$$H(\phi) = H(\phi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T}) = \left(\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i \right) \phi. \quad (9)$$

Then, we divide F into two parts G and H with $F(\phi) = G(\phi) - H(\phi)$. Note that F is the difference between the exponential part G and the linear part H . Hence equation (7) is equivalent to the following:

$$H(\phi) = G(\phi).$$

Lemma 4.1. *Let $G(\cdot)$ be a function defined in (8) for a given initial data $\mathring{I}, \mathring{S}, \mathring{R}$ and \mathring{T} . If $\mathring{T} = \sum_{i=1}^N \mathring{I}_i + \mathring{S} + \mathring{R}$, the following properties hold:*

$$G(0) = \mathring{S}, \quad \text{and} \quad \lim_{\phi \rightarrow \infty} G(\phi) < \infty, \quad \text{if } \mathring{R} = 0.$$

Proof. By the definition of G , we have $G(0) = \mathring{T} - \sum_{i=1}^N \mathring{I}_i - \mathring{R} = \mathring{S}$. Relation (4) yields the first result in this lemma.

For the next result, we take the limit such that

$$\begin{aligned} \lim_{\phi \rightarrow \infty} G(\phi) &= \lim_{\phi \rightarrow \infty} \left(\mathring{T} - \sum_{i=1}^N \mathring{I}_i e^{-k\lambda_i \phi} - \mathring{R} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i \frac{1 - e^{-k\lambda_i \phi}}{k\lambda_i} \right) \\ &= \lim_{\phi \rightarrow \infty} \left(\mathring{T} - \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} - \sum_{\lambda_i=0} \mathring{I}_i - \mathring{R} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i \frac{1 - e^{-k\lambda_i \phi}}{k\lambda_i} \right) \\ &= \mathring{T} - \sum_{\lambda_i=0} \mathring{I}_i - \mathring{R} + k\sigma \sum_{\lambda_i \neq 0} \frac{\mathring{I}_i}{k\lambda_i}. \end{aligned}$$

Therefore, we have $\lim_{\phi \rightarrow \infty} G(\phi) = \mathring{S} + k\sigma \sum_{\lambda_i \neq 0} \frac{\mathring{I}_i}{k\lambda_i}$. Notice that the second term on the right hand side of the above equation is finite.

$$k\sigma \sum_{\lambda_i \neq 0} \frac{\mathring{I}_i}{k\lambda_i} < \infty.$$

Therefore, we conclude that $\lim_{\phi \rightarrow \infty} G(\phi) < \infty$. \square

Lemma 4.2. *Assume that k and σ are positive constants, $\lambda_i \geq 0$ and $\mathring{I}_i \geq 0$ for all $i = 1, \dots, N$. Let $G(\cdot)$ be the function defined in (8). If $M_1(\mathring{I}) > 0$, then the derivative $G'(\cdot)$ is positive and*

$$\frac{dG}{d\phi} = \sum_{\lambda_i \neq 0} \left(k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \mathring{I}_i e^{-k\lambda_i \phi} \right) > 0.$$

Proof. Note that the derivative of G is

$$\frac{dG}{d\phi} = -\frac{d}{d\phi} \sum_{i=1}^N \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi}.$$

We can calculate the first term in the above as

$$\begin{aligned} \frac{d}{d\phi} \sum_{i=1}^N \mathring{I}_i e^{-k\lambda_i \phi} &= \frac{d}{d\phi} \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} + \frac{d}{d\phi} \sum_{\lambda_i = 0} \mathring{I}_i e^{-k\lambda_i \phi} \\ &= \frac{d}{d\phi} \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} + \frac{d}{d\phi} \sum_{\lambda_i = 0} \mathring{I}_i \\ &= -\sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi}. \end{aligned}$$

Thus, if at least one non-zero $\lambda_i \mathring{I}_i$ exists, we have

$$\frac{dG}{d\phi} = \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} > 0.$$

□

Lemma 4.3. *Let k and σ be positive constants and $\delta \geq 0$. Let $H(\cdot)$ be the function defined in (9). If $\mathring{T} = \sum_{i=1}^N \mathring{I}_i + \mathring{S} + \mathring{R}$, and initial data \mathring{I}_i , \mathring{S} and \mathring{R} are nonnegative for all $i = 1, \dots, N$, then $H(\cdot)$ is a linear function with nonnegative slope and $H(0) = 0$.*

Proof. Note that

$$\begin{aligned} \delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i = 0} \mathring{I}_i &= \delta + k\sigma \left(\sum_{i=1}^N \mathring{I}_i + \mathring{S} + \mathring{R} \right) - k\sigma \sum_{\lambda_i = 0} \mathring{I}_i \\ &= \delta + k\sigma \left(\sum_{\lambda_i \neq 0} \mathring{I}_i + \mathring{S} + \mathring{R} \right). \end{aligned}$$

Therefore, $\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i = 0} \mathring{I}_i$ is positive and this means that $\delta + k\sigma \mathring{T} > k\sigma \sum_{\lambda_i = 0} \mathring{I}_i$.

This implies that $H(\cdot)$ is monotone increasing with $H(0) = 0$. □

Proposition 1. *Let k , σ and δ be positive constants. Let (I, S, R) be a solution to system (1) with initial data \mathring{I} , \mathring{S} and \mathring{R} . We assume that \mathring{S} is positive, $\mathring{I} \geq 0$, $\mathring{R} = 0$ and $\mathring{T} = \sum_{i=1}^N \mathring{I}_i + \mathring{S} + \mathring{R}$.*

Then there is a final rumor size $\phi^\infty(\mathring{I}, \mathring{S})$ such that

$$\phi^\infty(\mathring{I}, \mathring{S}) := \lim_{t \rightarrow \infty} \phi(t),$$

where $\phi(t)$ is the rumor size satisfying (6). Moreover, $\phi^\infty(\mathring{I}, \mathring{S})$ is the smallest positive zero of $F(\cdot)$ such that

$$F(\phi^\infty(\mathring{I}, \mathring{S})) = 0.$$

Proof. This follows from an elementary result of ordinary differential equations. Note that for given initial data $\mathring{S} > 0$, $\mathring{I} \geq 0$ and $\mathring{R} = 0$, the corresponding equation $\dot{\phi} = F(\phi)$ is autonomous. Therefore, we can apply Lyapunov's stability theorem. From the properties of $G(\cdot)$ and $H(\cdot)$ in Lemmas 4.1 and 4.3, we have

$$F(0) = G(0) - H(0) = \mathring{S} > 0,$$

and

$$\lim_{\phi \rightarrow \infty} F(\phi) = G(\phi) - H(\phi) = -\infty.$$

By the intermediate value theorem, there is at least one positive solution $\psi > 0$ such that $0 = F(\psi) = F(\psi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T})$ for a given initial data \mathring{I} , \mathring{S} and \mathring{R} . Let ϕ^∞ be the smallest element of the set of positive solutions $\{\psi > 0 : F(\psi) = 0\}$.

For a fixed $\mathring{S} > 0$, $F(0) = \mathring{S} > 0$. This shows that $\phi(t)$ is increasing near $t = 0$ and $F(x) > 0$ on $\{0 < x < \phi^\infty\}$. Moreover, $F(x)$ is differentiable with respect to x , which implies that $\phi(t) \rightarrow \phi^\infty$ as $t \rightarrow \infty$ by an elementary result of ordinary differential equations. \square

Proposition 2. *Let k , σ and δ be positive constants and $\lambda_i \geq 0$ for all $i = 1, \dots, N$. Suppose that $\mathring{I}_i \geq 0$, $i = 1, \dots, N$, $\mathring{R} = \mathring{S} = 0$ and \mathring{T} is a fixed positive constant with $\mathring{T} = \sum_{i=1}^N \mathring{I}_i$. If we assume that $M_1(\mathring{I}) > 0$, then the following statement holds.*

The equation $F(\phi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T}) = 0$ has a positive solution $\phi > 0$ if and only if $kM_1 > \delta$. Moreover, if $kM_1 \leq \delta$, then the equation has no positive zero and $\phi = 0$ is a solution to the equation.

Proof. We have, by Lemma 4.1 and 4.3,

$$G(0) = \mathring{S} = 0, \quad \lim_{\phi \rightarrow \infty} G(\phi) < \infty$$

and

$$H(0) = 0, \quad \lim_{\phi \rightarrow \infty} H(\phi) = \infty.$$

It follows that $F(0) = 0$ and

$$\lim_{\phi \rightarrow \infty} F(\phi) = -\infty. \quad (10)$$

Lemma 4.2 implies that the derivative of $G(\phi)$ is positive and

$$\frac{dG}{d\phi} = \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} > 0.$$

Moreover, if $\mathring{S} = 0$, then we have

$$\frac{dH}{d\phi}(\phi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T}) = \delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i.$$

Therefore,

$$\begin{aligned} \frac{dF}{d\phi} &= \frac{dG}{d\phi} - \frac{dH}{d\phi} \\ &= \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} - \left(\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i \right). \end{aligned} \quad (11)$$

So,

$$\begin{aligned} \frac{dF}{d\phi}(0) &= \frac{dG}{d\phi}(0) - \frac{dH}{d\phi}(0) = \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i - \left(\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i \right) \\ &= \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i - \delta = kM_1 - \delta. \end{aligned}$$

This yields

$$kM_1 > \delta \quad \Rightarrow \quad \frac{dF}{d\phi}(0) > 0, \quad (12)$$

and the continuity of F implies that there is a positive real number $\phi_\epsilon > 0$ such that

$$F(\phi_\epsilon) > 0.$$

Thus, the intermediate value theorem and (10) show that $F(\phi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T}) = 0$ has a positive solution $\phi > 0$.

In order to complete the proof of this proposition, it suffices to verify that the equation has a unique solution $\phi = 0$ on $\{\phi \geq 0\}$ if $kM_1 \leq \delta$.

We now assume that $kM_1 \leq \delta$. We rewrite (11) by

$$\begin{aligned} \frac{dF}{d\phi} &= \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} - \left(\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i \right) \\ &= \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i e^{-k\lambda_i \phi} - \left(\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i e^{-k\lambda_i \phi} \right) \\ &= \sum_{i=1}^N k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{i=1}^N \mathring{I}_i e^{-k\lambda_i \phi} - \delta - k\sigma \mathring{T}. \end{aligned}$$

Since we assume that $\sum_{i=1}^N k\lambda_i \mathring{I}_i = kM_1 \leq \delta$ and $\mathring{T} = \sum_{i=1}^N \mathring{I}_i$,

$$\begin{aligned} \frac{dF}{d\phi} &\leq \sum_{i=1}^N k\lambda_i \mathring{I}_i e^{-k\lambda_i \phi} + k\sigma \sum_{i=1}^N \mathring{I}_i e^{-k\lambda_i \phi} - \sum_{i=1}^N k\lambda_i \mathring{I}_i - k\sigma \sum_{i=1}^N \mathring{I}_i \\ &= \sum_{i=1}^N k\lambda_i \mathring{I}_i (e^{-k\lambda_i \phi} - 1) + k\sigma \sum_{i=1}^N \mathring{I}_i (e^{-k\lambda_i \phi} - 1). \end{aligned}$$

This implies that

$$\frac{dF}{d\phi} < 0 \quad \text{if } \phi > 0, \quad \text{and} \quad \frac{dF}{d\phi} \leq 0 \quad \text{if } \phi = 0.$$

The fact in (10) with $F(0) = 0$ leads us to

$$F(\phi) < 0 \quad \text{for } \phi > 0.$$

Therefore, we complete the proof. □

We are now ready to prove the main theorem.

The proof of the main theorem. Let $\mathring{R}^n = \mathring{R} = 0$ and let \mathring{I}^n and \mathring{S}^n be sequences such that

$$\mathring{I}^n \rightarrow \mathring{I}, \quad \mathring{S}^n \rightarrow 0. \tag{13}$$

We denote

$$F(\phi, \mathring{S}^n) = F(\phi, \mathring{I}^n, \mathring{S}^n, \mathring{R}^n, \mathring{T}) \quad \text{and} \quad F_\infty(\phi) = F(\phi, \mathring{I}, \mathring{S}, \mathring{R}, \mathring{T}).$$

First, assume that $kM_1(\mathring{I}) > \delta$. By (12), we have

$$F_\infty(0) = 0 \quad \text{and} \quad \left. \frac{d}{d\phi} F_\infty(\phi) \right|_{\phi=0} > 0.$$

Note that (11) implies that

$$\begin{aligned} &\frac{\partial F}{\partial \phi}(\phi, \mathring{I}^n, \mathring{S}^n, \mathring{R}^n, \mathring{T}) \\ &= \sum_{\lambda_i \neq 0} k\lambda_i \mathring{I}_i^n e^{-k\lambda_i \phi} + k\sigma \sum_{\lambda_i \neq 0} \mathring{I}_i^n e^{-k\lambda_i \phi} - \left(\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i^n \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{\lambda_i \neq 0 \\ N}} k\lambda_i \mathring{I}_i^n e^{-k\lambda_i \phi} + k\sigma \sum_{\substack{\lambda_i \neq 0 \\ N}} \mathring{I}_i^n e^{-k\lambda_i \phi} - \left(\delta + k\sigma \mathring{T} - k\sigma \sum_{\lambda_i=0} \mathring{I}_i^n e^{-k\lambda_i \phi} \right) \\
 &= \sum_{i=1}^N k\lambda_i \mathring{I}_i^n e^{-k\lambda_i \phi} + k\sigma \sum_{i=1}^N \mathring{I}_i^n e^{-k\lambda_i \phi} - \delta - k\sigma \mathring{T} \\
 &= \sum_{i=1}^N k\lambda_i \mathring{I}_i^n \left(e^{-k\lambda_i \phi} - 1 \right) + \sum_{i=1}^N k\lambda_i \mathring{I}_i^n \\
 &\quad + k\sigma \sum_{i=1}^N \mathring{I}_i^n e^{-k\lambda_i \phi} - \delta - k\sigma \left(\left(\sum_{i=1}^N \mathring{I}_i^n \right) + \mathring{S}^n \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{\partial F}{\partial \phi}(\phi, \mathring{I}^n, \mathring{S}^n, \mathring{R}^n, \mathring{T}) &= \sum_{i=1}^N k\lambda_i \mathring{I}_i^n \left(e^{-k\lambda_i \phi} - 1 \right) + k\sigma \sum_{i=1}^N \mathring{I}_i^n \left(e^{-k\lambda_i \phi} - 1 \right) \\
 &\quad - k\sigma \mathring{S}^n + kM_1(\mathring{I}^n) - \delta.
 \end{aligned}$$

By (13), there is $N_0 \in \mathbb{N}$ such that if $n > N_0$, then

$$kM_1(\mathring{I}^n) - \delta > \frac{1}{2}(kM_1(\mathring{I}) - \delta)$$

and

$$k\sigma \mathring{S}^n \leq \frac{1}{4}(kM_1(\mathring{I}) - \delta).$$

Since \mathring{I}_i^n is bounded, we can choose ϵ_ϕ such that if $0 \leq \phi \leq \epsilon_\phi$, then

$$\left| \sum_{i=1}^N k\lambda_i \mathring{I}_i^n \left(e^{-k\lambda_i \phi} - 1 \right) + k\sigma \sum_{i=1}^N \mathring{I}_i^n \left(e^{-k\lambda_i \phi} - 1 \right) \right| \leq \frac{1}{4}(kM_1(\mathring{I}) - \delta).$$

Thus, there are constants $\epsilon_\phi > 0, N_0 \in \mathbb{N}$ such that if $0 \leq \phi < \epsilon_\phi$ and $n > N_0$, then

$$\frac{\partial F}{\partial \phi}(\phi, \mathring{I}^n, \mathring{S}^n, \mathring{R}^n, \mathring{T}) > 0. \tag{14}$$

In order to prove convergence, we consider differentiable functions $I(s), S(s)$ and $R(s)$ with respect to $s \in \mathbb{R}$ such that

$$I(1/n) = \mathring{I}^n, \quad S(1/n) = \mathring{S}^n, \quad R(1/n) = \mathring{R}^n, \quad n \in \mathbb{N},$$

and

$$\mathring{T} = \left(\sum_{i=1}^N I_i(s) \right) + S(s) + R(s).$$

We denote $F_p(\phi, s) = F_p(\phi, I(s), S(s), R(s), \mathring{T})$. Since $\frac{\partial}{\partial \phi} F(\phi, s) \Big|_{\phi=0, s=0} > 0$, the implicit function theorem implies that $\phi(s)$ is differentiable near $s = 0$. Thus, $\phi^\infty(\mathring{I}^n, \mathring{S}^n) = \phi(1/n)$ converges.

We now prove that $\phi_n^\infty \rightarrow \phi_e > 0$. Assume not, that is $\phi_n^\infty \rightarrow 0$ as $n \rightarrow \infty$. Then there is a $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $\phi_n^\infty(\mathring{S}) < \epsilon_\phi$. We may take $N_1 \in \mathbb{N}$ such that $N_1 > N_0$.

For $n > N_1$, we have $F(0, \mathring{I}^n, \mathring{S}^n, \mathring{R}^n, \mathring{T}) = \mathring{S}^n > 0$ and $F(\phi_n^\infty, \mathring{I}^n, \mathring{S}^n, \mathring{R}^n, \mathring{T}) = 0$. However, by (14), $\frac{\partial F}{\partial \phi}(\phi, \mathring{I}^n, \mathring{S}^n, \mathring{R}^n, \mathring{T}) > 0$. This is a contradiction. Thus we obtain the desired result.

For the last part, we assume that $kM_1 \leq \delta$. Then, $\phi = 0$ is the unique nonnegative solution to $F(\phi, \dot{I}, \dot{S}, \dot{R}, \dot{T}) = 0$. Moreover $\phi^\infty(\dot{I}^n, \dot{S}^n) > 0$. Let $\{\phi_{n_n}\}$ be any convergent subsequence of $\{\phi^\infty(\dot{I}^n, \dot{S}^n)\}$.

Let $\phi_{n_\infty} = \lim_{n \rightarrow \infty} \phi_{n_n} \geq 0$. Since F is a continuous function,

$$\lim_{n \rightarrow \infty} F(\phi_{n_n}, \dot{I}_{n_n}, \dot{S}_{n_n}, \dot{R}_{n_n}, \dot{T}) = F(\phi_{n_\infty}, \dot{I}, 0, 0, \dot{T}).$$

Therefore, by Proposition 2, $\phi_{n_\infty} = 0$, i.e., $\lim_{n \rightarrow \infty} \phi_{n_n} = 0$. We proved that any convergent subsequence of $\{\phi^\infty(\dot{I}^n, \dot{S}^n)\}$ converges to zero. Thus, by an elementary theorem in analysis,

$$\lim_{n \rightarrow \infty} \phi_n = 0.$$

□

5. Numerical simulation. In this section, we numerically provide the solutions to system (1) with respect to several δ and compare them with the analytical results in Section 4. The main result in this paper suggests that the condition for rumor outbreak is that $M_1(I_0)$ is less than δ/k .

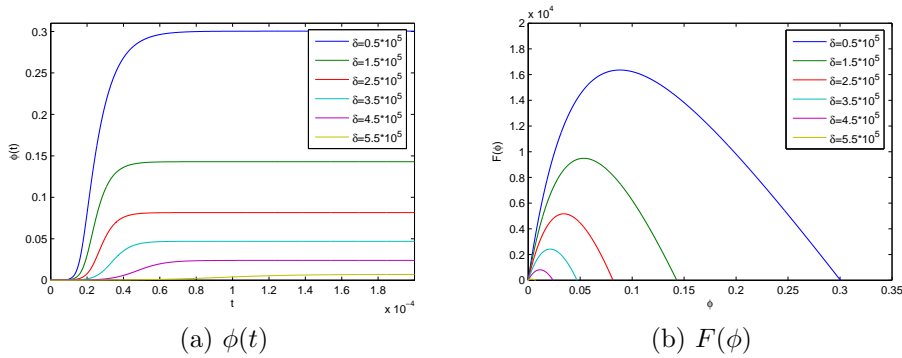


FIGURE 1. Temporal evolution of $\phi(t)$ with respect to δ and the plot of $F(\phi)$

For the numerical simulations, we used the fourth-order Runge-Kutta method with the following parameters:

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3, \quad \sigma = 0.2, \quad k = 10.$$

We take sufficiently large initial data $\dot{I}_i, i = 1, 2$ and 3, and sufficiently small \dot{S} such that

$$\dot{I}_1 = \dot{I}_2 = \dot{I}_3 = 9999, \quad \dot{S} = 3, \quad \dot{R} = 0.$$

To see the threshold phenomena for $M_1(I_0)$ and δ/k , we consider $\delta = 0.5 \times 10^5, 1.5 \times 10^5, 2.5 \times 10^5, 3.5 \times 10^5, 4.5 \times 10^5$ and 5.5×10^5 .

Note that in Proposition 1, we rigorously obtained that the final size of the rumor $\phi^\infty = \lim_{t \rightarrow \infty} \phi(t)$ is the smallest positive zero of $F(\phi)$. In Figure 1, we plotted the rumor size $\phi(t)$ and $F(\phi)$ with respect to several $\delta > 0$. Here, we can observe that the final size of the rumor ϕ^∞ is also the smallest positive zero of $F(\phi)$. Thus the numerical results are consistent with the analytical result.

In Figure 2, we plotted the densities of the spreaders (S) and the stiflers (R) with respect to several $\delta > 0$. Thus, we can see that the rumor size are proportional to $1/\delta$.

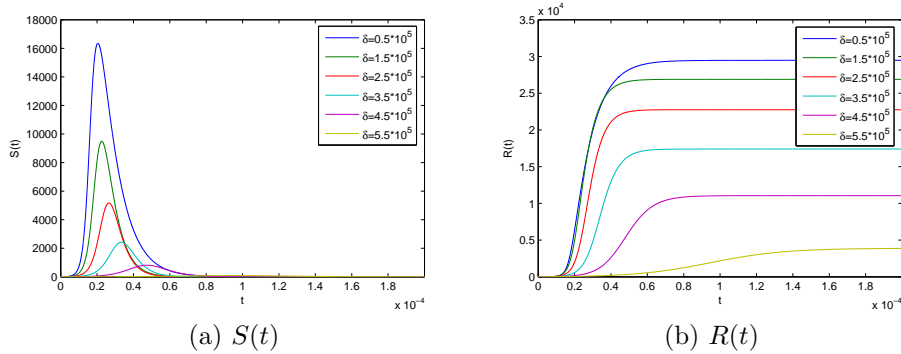


FIGURE 2. Temporal evolution of the densities $S(t)$ and $R(t)$ with respect to δ

In Figure 3, we plotted ϕ^∞ with respect to $\delta > 0$ to see the threshold phenomena for $M_1(I_0)$ and δ/k . Note that $\dot{S} \simeq 0$, $M_1(\dot{I}) = \lambda_1 \dot{I}_1 + \lambda_2 \dot{I}_2 + \lambda_2 \dot{I}_2 \simeq 6 \times 10^4$.

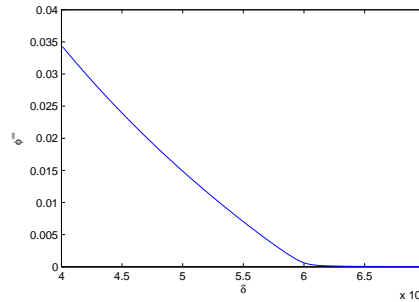


FIGURE 3. The final size of the rumor ϕ^∞ with respect to δ

From our analytic results, we can expect that the threshold occurs when $\delta = kM_1(\dot{I}) \simeq 6 \times 10^5$. In Figure 3, ϕ^∞ decreases almost linearly in δ , and the graph of $\phi^\infty(\delta)$ is very close to the line $\phi^\infty = 0$ as δ goes to 6×10^5 . Therefore, this numerical simulation coincides with our analytic result. By the way, a small error occurs between the analytical and the numerical results. We guess that this error may result from the nonzero assumption for $\dot{S} > 0$.

6. Conclusion. In this paper, we consider the rumor spreading model with the trust rate distribution. The model consists of several ignorants with trust rates λ_i , $i = 1, \dots, N$. We provide the threshold phenomenon for this model by using classical steady state analysis. Fortunately, we can obtain the corresponding equation of rumor size for this model with several types of ignorants. We also present a rigorous proof for the convergence result. As seen from the steady state analysis, we obtain that the threshold phenomena depends on the decay rate δ of spreaders to stifiers

$$\text{and the momentum type quantity } M_1 \text{ defined in Definition 2.2: } M_1(\dot{I}) = \sum_{i=1}^N \lambda_i \dot{I}_i.$$

We conclude that a necessary and sufficient condition for rumor outbreak is that the trust momentum $M_1(\dot{I})$ is greater than some combination of parameters δ/k .

REFERENCES

- [1] P. Bordia and N. DiFonzo, [Problem solving in social interactions on the Internet: Rumor as social cognition](#), *Social Psychology Quarterly*, **67** (2004), 33–49.
- [2] G. Chen, H. Shen, T. Ye, G. Chen and N. Kerr, [A kinetic model for the spread of rumor in emergencies](#), *Discrete Dynamics in Nature and Society*, **2013** (2013), Art. ID 605854, 8 pp.
- [3] D. J. Daley and D. G. Kendall, [Epidemics and rumours](#), *Nature*, **204** (1964), 1118.
- [4] D. J. Daley and D. G. Kendall, [Stochastic rumours](#), *IMA Journal of Applied Mathematics*, **1** (1965), 42–55.
- [5] N. Fountoulakis, K. Panagiotou and T. Sauerwald, [Ultra-fast rumor spreading in social networks](#), *Proceedings of the Twenty-Third Annual ACM-SIAM symposium on Discrete Algorithms*, Society for Industrial and Applied Mathematics, 2012, 1642–1660.
- [6] J. Gu, W. Li and X. Cai, [The effect of the forget-remember mechanism on spreading](#), *European Physical Journal B*, **62** (2008), 247–255.
- [7] V. Isham, S. Harden and M. Nekovee, [Stochastic epidemics and rumours on finite random networks](#), *Physica A*, **389** (2010), 561–576.
- [8] M. E. Jaeger, S. Anthony and R. L. Rosnow, [Who hears what from whom and with what effect: A study of rumor](#), *Personality and Social Psychology Bulletin*, **6** (1980), 473–478.
- [9] J. Jiang, S. Gong and B. He, [Dynamical behavior of a rumor transmission model with Holling-type II functional response in emergency event](#), *Physica A*, **450** (2016), 228–240.
- [10] A. J. Kimmel and A.-F. Audrain-Pontevia, [Analysis of commercial rumors from the perspective of marketing managers: Rumor prevalence, effects, and control tactics](#), *Journal of Marketing Communications*, **16** (2010), 239–253.
- [11] D. Maki and M. Thomson, *Mathematical Models and Applications*, Prentice-Hall, Englewood Cliffs, 1973.
- [12] M. McDonald, O. Suleman, S. Williams, S. Howison and N. F. Johnson, [Impact of unexpected events, shocking news, and rumors on foreign exchange market dynamics](#), *Physical Review E*, **77** (2008), 046110.
- [13] Y. Moreno, M. Nekovee and A. Pacheco, [Dynamics of rumor spreading in complex networks](#), *Physical Review E*, **69** (2004), 066130.
- [14] M. Nagao, K. Suto and A. Ohuchi, [A media information analysis for implementing effective countermeasure against harmful rumor](#), *Journal of Physics, Conference Series*, **221** (2010), 012004.
- [15] M. Nekovee, Y. Moreno, G. Bianconi and M. Marsili, [Theory of rumour spreading in complex social networks](#), *Physica A*, **374** (2007), 457–470.
- [16] R. L. Rosnow, J. H. Yost and J. L. Esposito, [Belief in rumor and likelihood of rumor transmission](#), *Language & Communication*, **6** (1986), 189–194.
- [17] A. Sudbury, [The proportion of population never hearing a rumour](#), *Journal of Applied Probability*, **22** (1985), 443–446.
- [18] S. A. Thomas, [Lies, damn lies, and rumors: An analysis of collective efficacy, rumors, and fear in the wake of Katrina](#), *Sociological Spectrum*, **27** (2007), 679–703.
- [19] Y.-Q. Wang, X.-Y. Yang, Y.-L. Han and X.-A. Wang, [Rumor Spreading Model with Trust Mechanism in Complex Social Networks](#), *Communications in Theoretical Physics*, **59** (2013), 510–516.
- [20] J. Wang, L. Zhao and R. Huang, [2SI2R rumor spreading model in homogeneous networks](#), *Physica A*, **413** (2014), 153–161.
- [21] D. J. Watts and S. H. Strogatz, [Collective dynamics of small-world networks](#), *Nature*, **393** (1998), 440–442.
- [22] Y. Zan, J. Wua, P. Li and Q. Yua, [SICR rumor spreading model in complex networks: Counterattack and self-resistance](#), *Physica A*, **405** (2014), 159–170.
- [23] D. H. Zanette, [Critical behavior of propagation on small-world networks](#), *Physical Review E*, **64** (2001), 050901.
- [24] D. H. Zanette, [Dynamics of rumor propagation on small-world networks](#), *Physical Review E*, **65** (2002), 041908.
- [25] L. Zhao, H. Cui, X. Qiu, X. Wang and J. Wang, [SIR rumor spreading model in the new media age](#), *Physica A*, **392** (2013), 995–1003.
- [26] L. Zhao, J. Wang, Y. Chen, Q. Wang, J. Cheng and H. Cui, [SIHR rumor spreading model in social networks](#), *Physica A*, **391** (2012), 2444–2453.

- [27] L. Zhao, Q. Wang, J. Cheng, Y. Chen, J. Wang and W. Huang, [Rumor spreading model with consideration of forgetting mechanism: A case of online blogging LiveJournal](#), *Physica A*, **390** (2011), 2619–2625.
- [28] L. Zhao, W. Xie, H. O. Gao, X. Qiu, X. Wang and S. Zhang, [A rumor spreading model with variable forgetting rate](#), *Physica A*, **392** (2013), 6146–6154.
- [29] L. Zhu, H. Zhao and H. Wang, [Complex dynamic behavior of a rumor propagation model with spatial-temporal diffusion terms](#), *Information Sciences*, **349/350** (2016), 119–136.

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