

INTERIOR REGULARITY TO THE STEADY INCOMPRESSIBLE SHEAR THINNING FLUIDS WITH NON-STANDARD GROWTH

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ABSTRACT. We consider weak solutions to the equations of stationary motion of a class of non-Newtonian fluids which includes the power law model. The power depends on the spatial variable, which is motivated by electrorheological fluids. We prove the existence of second order derivatives of weak solutions in the shear thinning cases.

1. Introduction. Traditionally, the Navier-Stokes equations have received quite a bit of attention. Recently, attention to the behavior of fluids with various viscosities has been increasing dramatically. It is because we can find such fluids everywhere. For example, water, yogurt, lubricants, sand in water, ink, gum solutions, nail polish, ketchup, molasses, ice, paint, custard, paper pulp, even blood in our body. The behavior of many of them can be described in the power law model. In that sense, we are interested in the power law model, which is a generalized Navier-Stokes system.

As mentioned in [16], electrorheological fluids are viscous liquids, that are characterized by their ability to undergo significant changes in their mechanical properties when an electric field is applied. The motion is governed by a system of partial differential equations consisting of electric field E , polarization, density ρ , velocity u , pressure π , and deviatoric stress tensor S . Refer to [11] for the detail description. In [11] the viscosity is the form of power law model, and the power p depends on the electric field $p(|E|^2)$.

In this article we consider the following stationary system related with non-Newtonian fluids:

$$\begin{cases} -\operatorname{div} \{S(x, D(u))\} + u \cdot \nabla u + \nabla \pi = g, \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } \Omega, \quad (1)$$

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where $g \in L^\infty(\Omega)$ is a given exterior force, and Ω is a bounded domain in \mathbb{R}^n with $n = 2, 3$. Here, D denotes the symmetric part of ∇ , given by

$$D^{ij}(u)(x) = \frac{1}{2} (\partial_i u^j + \partial_j u^i)(x) \in \mathbb{R}_{sym}^{n \times n}.$$

In addition, $S(x, D(u))$ denotes the deviatoric stress. Then $T := S - \pi I$ is called the full shear stress, such that $-\operatorname{div} T$ represents the sum of the internal forces due to friction, which depends mostly on the material of the fluid.

The continuous deviatoric stress tensor $S : \Omega \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ is assumed to be C^1 -regular in the gradient variable z for every $z \in \mathbb{R}^n \setminus \{0\}$, with $S_z(\cdot)$ being Carathéodory regular and satisfying the following nonstandard growth, monotonicity and continuity assumptions:

$$\begin{cases} |S(x, z)| + |S_z(x, z)|(|z|^2 + \mu^2)^{\frac{1}{2}} \leq L(|z|^2 + \mu^2)^{\frac{p(x)-1}{2}}, \\ \nu(|z_1|^2 + |z_2|^2 + \mu^2)^{\frac{p(x)-2}{2}} |z_1 - z_2|^2 \leq \langle S(x, z_1) - S(x, z_2), z_1 - z_2 \rangle, \\ |S(x_1, \xi) - S(x_2, \xi)| \leq L\omega(|x_1 - x_2|) [1 + |\log(|\xi|^2 + \mu^2)|] \\ \quad \times \left[(|\xi|^2 + \mu^2)^{\frac{p(x_1)-1}{2}} + (|\xi|^2 + \mu^2)^{\frac{p(x_2)-1}{2}} \right] \end{cases} \quad (2)$$

for every $z \in \mathbb{R}^n \setminus \{0\}$, $z_1, z_2, \xi \in \mathbb{R}^n$ and $x, x_1, x_2 \in \Omega$, where $0 < \nu \leq L$ and $\mu \geq 0$ are fixed numbers. The variable exponent function $p(\cdot) : \Omega \rightarrow [0, +\infty)$ is continuous with modulus of continuity $\omega : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\gamma_1 \leq p(x) \leq \gamma_2 \leq 2 \quad \text{and} \quad |p(x) - p(y)| \leq \omega(|x - y|) \quad \text{for } x, y \in \Omega, \quad (3)$$

where $\gamma_1 > 3/2$ if $n = 2$ and $\gamma_1 > 9/5$ if $n = 3$. We assume that the variable exponent $p(\cdot)$ is Lipschitz continuous, i.e.

$$\omega(r) \leq c_p r,$$

for some constant $c_p > 0$. For simplicity, we set

$$p_0 = p(x_0), \quad p_1 := \inf_{B_{4R}(x_0)} p(x) \quad \text{and} \quad p_2 := \sup_{B_{4R}(x_0)} p(x)$$

for some fixed $B_{4R}(x_0) \subset\subset \Omega$.

Here, we denote the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$, by the set of all measurable functions $f : \Omega \rightarrow \mathbb{R}^n$ satisfying

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|f|(x)}{\lambda} \right)^{p(x)} dx \leq 1 \right\} < \infty,$$

and the space $W_{0,\sigma}^{1,p(\cdot)}(\Omega)$ is the set of all divergence free $f \in W_0^{1,1}(\Omega)$ with $\|f\|_{L^{p(\cdot)}(\Omega)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega)} < \infty$. For more details, we refer to [6, 9]. We say that $u \in W_{loc}^{1,p(\cdot)}(\Omega)^n$ is a weak solution of (1), if u satisfies

$$\int_{\Omega} S(x, D(u)) : D(\varphi) dx - \int_{\Omega} u \otimes u : D(\varphi) dx = \int_{\Omega} g \cdot \varphi dx, \quad \forall \varphi \in W_{0,\sigma}^{1,p(\cdot)}(\Omega)^n,$$

where $u \otimes v := \{u^i v^j\}_{i,j}$ is a tensor product of u and v in \mathbb{R}^n .

In [13], the existence of weak solutions was provided for constant $p > \frac{2n}{n+2}$. In [14, 15], the existence of strong solution of (1) was proved when p is constant. In [14], the solution belongs to $C^{1,\alpha}$ for $p > \frac{3}{2}$ when $n = 2$ (up to boundary), and for $p > \frac{6}{5}$ when $n = 3$ (interior regularity). In [15], for $\frac{3n}{n+2} < p < 2$, it is shown that $(1 + |D(u)|)^{\frac{p-2}{2}} \nabla D(u) \in L_{loc}^2(\Omega)^{n \times n}$, and $u \in W_{loc}^{2,t}(\Omega)^n$ for all $1 \leq t < 2$ when

$n = 2$, and $u \in W_{loc}^{2, \frac{3p}{1+p}}(\Omega)^n$ when $n = 3$. In [4], for $S(z) = (1 + |z|^{p-2})z$ under slip or no-slip boundary conditions a regularity is provided for shear thickening fluid $p > 2$.

In case of anisotropic dissipative potential f , where $S = \nabla f$ and

$$\lambda(1 + |z|^2)^{\frac{p_1-2}{2}}|A|^2 \leq \nabla^2 f(z)(A, A) \leq \Lambda(1 + |z|^2)^{\frac{p_2-2}{2}}|A|^2$$

with exponents $1 < p_1 \leq p_2 < \infty$ and $2 \leq p_2 < p_1 \frac{n+2}{n}$, the existence of weak solutions is given in [3]. For the anisotropic fluid, it is shown in [5] that $u \in W_0^{1,p_1}(\Omega)^n \cap W_{loc}^{2,s}(\Omega)^n$ for some $s > 1$ for $p_1 > 6/5$ when $n = 2$, and $p_1 > 9/5$ when $n = 3$, and $p_2 < p_1 \frac{n+1}{n}$. Also in that article, there is a good introduction about isotropic and anisotropic cases, and the power depending on x , $p(x)$.

For a variable $p(x)$ depending on E , in [16] it is proved that a weak solution of (1) exists and it has the second weak derivative for $\gamma_1 > \frac{3n}{n+2}$ and $\mu > 0$. Nonetheless, it is not shown there that the second weak derivative of a weak solution belongs to Lebesgue space with variable exponent. It is just shown that a weak solution belongs to $W^{2,\gamma_1}(\Omega)^n$.

The lower bound of $p(\cdot)$, γ_1 , that a weak solution of (1) exists is decreased to $\frac{2n}{n+2}$ in [10]. But if $\gamma_1 > \frac{2n}{n+2}$, we cannot say that $u \otimes u : D(\varphi) \in L^1$ for all $u, \varphi \in W_0^{1,p(\cdot)}(\Omega)^n$. In this case, u is not able to be taken as a test function, hence we just consider $\gamma_1 > \frac{3n}{n+2}$.

On the other hand, in [7] it was proved that a solution of $p(x)$ -Laplace equation has weak second derivative in L^2 . In that paper, $\mu = 0$ is allowed, but the model is not related with fluid directly. The existence of strong solution of (1) for $S(x, D(u)) = (\mu + |D(u)|)^{p(x)-2}D(u)$ with $\mu > 0$ was proved in [8] for $\gamma_1 > 2$. And the same result was proved in [11] for $\gamma_1 > \frac{9}{5}$.

In this paper, we handle non-Newtonian problem (1) where μ is allowed to be 0 and $p(\cdot)$ is Lipschitz continuous. The result of this paper is the following theorem.

Theorem 1.1. *Let $\mu \in [0, \infty)$. We assume that (2) are satisfied and $p(\cdot)$ is Lipschitz continuous function with (3). Let u be a weak solution of (1) with (2). Then u has the following properties:*

$$(1 + |D(u)|^2)^{\frac{p(x)-2}{4}} \nabla D(u) \in L_{loc}^2(\Omega)^{n^3} \tag{4}$$

and for all $1 \leq t < 2$,

$$\begin{cases} u \in W_{loc}^{2,t}(\Omega)^n & \text{when } n = 2, \\ u \in W_{loc}^{2, \min\{\frac{3p(\cdot)}{1+p(\cdot)}, t\}}(\Omega)^n & \text{when } n = 3. \end{cases} \tag{5}$$

2. Proof of main theorem. To simplify the notation, the letter c will always denote any positive constant, which may vary throughout the paper. Moreover, we denote $p' = \frac{p}{p-1}$ as the Hölder conjugate exponent of p and $p^* = \frac{np}{n-p}$ as the Sobolev exponent of p for every $p \in (1, n)$.

We recall a useful lemma about higher integrability from [1].

Lemma 2.1. *Let $u \in W_{loc}^{1,p(\cdot)}(\Omega)^n$ be a weak solution of (1) and assume that S fulfills conditions (2) with Lipschitz continuous variable exponent, $p(\cdot)$. Then there*

exist constants $c, \sigma > 0$, both depending on $n, \gamma_1, \gamma_2, c_p$ such that if $B_{2R} \subset\subset \Omega$, then

$$\begin{aligned} & \left(\int_{B_R} (|D(u)| + \mu)^{(1+\sigma)p(x)} dx \right)^{\frac{1}{1+\sigma}} \\ & \leq c \int_{B_{2R}} (|D(u)| + \mu)^{p(x)} dx + c \int_{B_{2R}} (|\nabla u|^{\gamma_1} + |u|^{\gamma_1^*} + 1) dx. \end{aligned}$$

Here, $\int_{B_R} f dx$ means the average of $f \in L^1(B_R)$ over B_R . We take a constant $R_0 \leq 1$ such that

$$16\omega(4R_0) \leq \sigma \tag{6}$$

and assume $0 < R \leq R_0$, throughout this paper. This assumption will be frequently used in the proof, for instance (11) and (12).

Remark 1. Although the authors only considered the case $\mu > 0$ in [1], the statement is still valid for $\mu = 0$, since the proof is also available for $\mu = 0$. In this paper, we can remove the dependence of c on γ_2 since $\gamma_2 \leq 2$.

Since $\lim_{t \rightarrow 0^+} t^\epsilon \log t = 0$ and $\lim_{t \rightarrow +\infty} \frac{\log t}{t^\epsilon} = 0$ for any $\epsilon > 0$, we have the following useful estimate.

Lemma 2.2. *There exists a constant $c(\epsilon) > 0$ depending only on ϵ such that*

$$\log t \leq c(\epsilon) + t^\epsilon + t^{-\epsilon}$$

for all $t > 0$.

Our proof is mainly based on that of [15]. We divide our proof of main theorem in three steps. At first, we derive estimation related to finite difference of $D(u)$. After then, applying fractional Sobolev embedding theorem, we can show that u is locally bounded. Finally, we prove that u has second derivatives using difference quotient method.

Step 1. Estimation of $(|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)-2}{2}} |\Delta_\lambda D(u)|^2$.

From the modulus of continuity, there exists a radius $R_0 > 0$ such that $\omega(8R_0) \leq \frac{\sigma}{16}$. We use the following weak formulation

$$\int_{\Omega} S(x, D(u)) : D(\varphi) dx = - \int_{\Omega} u_i \frac{\partial u_j}{\partial x_i} \varphi_j dx + \int_{\Omega} \pi \operatorname{div} \varphi dx + \int_{\Omega} g \cdot \varphi dx \tag{7}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)^n$, where $\pi \in L^{p(\cdot)' }(\Omega)$. Let $\eta \in C_c^\infty(B_{2R}(x_0))$, where $B_{2R}(x_0) \subset \Omega$ such that $\eta \equiv 1$ in $B_R(x_0)$, $|\nabla \eta| \leq \frac{2}{R}$ and $|\nabla^2 \eta| \leq \frac{4}{R^2}$ for some $0 < R \leq R_0$. Now we choose a test function $\varphi = \Delta_{-\lambda,k}(\eta^2 \Delta_{\lambda,k} u)$, where $\Delta_{\lambda,k} f(x) = f(x + \lambda e_k) - f(x)$. For simplicity, we will denote $\Delta_\lambda := \Delta_{\lambda,k}$ for $\lambda \in [-R/4, R/4]$ and $B_{2R} := B_{2R}(x_0)$. By simple computation, we see

$$\begin{aligned} & \int_{B_{2R}} S(x, D(u)) : D(\Delta_{-\lambda}(\eta^2 \Delta_\lambda u)) dx \\ & = \int_{B_{2R}} \Delta_\lambda S(x, D(u)) : D(\eta^2 \Delta_\lambda u) dx \\ & = \int_{B_{2R}} \eta^2 \Delta_\lambda S(x, D(u)) : D(\Delta_\lambda u) dx \\ & \quad + \int_{B_{2R}} 2\eta \Delta_\lambda S(x, D(u)) : D(\eta) \otimes \Delta_\lambda u dx. \end{aligned} \tag{8}$$

The ellipticity condition in (2)₂ implies that

$$\begin{aligned}
& \int_{B_{2R}} \eta^2 \Delta_\lambda S(x, D(u)) : D(\Delta_\lambda u) dx \\
&= \int_{B_{2R}} \eta^2 [S(x + \lambda e_k, D(u)(x + \lambda e_k)) - S(x, D(u)(x + \lambda e_k))] \Delta_\lambda D(u) dx \\
&\quad + \int_{B_{2R}} \eta^2 [S(x, D(u)(x + \lambda e_k)) - S(x, D(u)(x))] \Delta_\lambda D(u) dx \tag{9} \\
&\geq \int_{B_{2R}} \eta^2 [S(x + \lambda e_k, D(u)(x + \lambda e_k)) - S(x, D(u)(x + \lambda e_k))] \Delta_\lambda D(u) dx \\
&\quad + \nu \int_{O_1^+} \eta^2 (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)-2}{2}} |\Delta_\lambda D(u)|^2 dx,
\end{aligned}$$

where $O_1^+ := \{x \in B_{2R} : |Du|(x + \lambda e_k) + |Du|(x) + \mu > 0\}$.

Note that the set $O_1^+ = B_{2R}$ whenever $\mu > 0$. Indeed, we here introduced the set O_1^+ to prove Theorem 1.1 for $\mu = 0$, in a simple and unified way with respect to $\mu \in [0, \infty)$.

Combining (7)-(9), we obtain

$$\begin{aligned}
I_0 &:= \nu \int_{O_1^+} \eta^2 (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)-2}{2}} |\Delta_\lambda D(u)|^2 dx \\
&\leq \int_{B_{2R}} \eta^2 [S(x, D(u)(x + \lambda e_k)) - S(x + \lambda e_k, D(u)(x + \lambda e_k))] D(\Delta_\lambda u) dx \\
&\quad - \int_{B_{2R}} 2\eta \Delta_\lambda S(x, D(u)) : D(\eta) \otimes \Delta_\lambda u dx \tag{10} \\
&\quad - \int_{B_{2R}} u_i \frac{\partial u_j}{\partial x_i} (\Delta_{-\lambda} (\eta^2 \Delta_\lambda u_j)) dx \\
&\quad + \int_{B_{2R}} \pi \operatorname{div} (\Delta_{-\lambda} (\eta^2 \Delta_\lambda u)) dx + \int_{B_{2R}} g \cdot \Delta_{-\lambda} (\eta^2 \Delta_\lambda u) dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

For simplicity, we set $\nu = 1$ since it does not make any trouble for the proof.

Estimation of I_1 . To estimate I_1 , we introduce a measurable set O_2^+ :

$$O_2^+ = \{x \in B_{2R} : |Du|(x + \lambda e_k) > 0\} \subset O_1^+.$$

We use the continuity assumption in (2)₃ to see

$$\begin{aligned}
|I_1| &\leq c\omega(\lambda) \int_{O_2^+} \eta^2 |D(\Delta_\lambda u)| [1 + |\log(|D(u)(x + \lambda e_k)|^2 + \mu^2)|] \\
&\quad \times \left[(|D(u)(x + \lambda e_k)|^2 + \mu^2)^{\frac{p(x+\lambda e_k)-1}{2}} \right. \\
&\quad \left. + (|D(u)(x + \lambda e_k)|^2 + \mu^2)^{\frac{p(x)-1}{2}} \right] dx \\
&\leq c|\lambda| \int_{O_2^+} \eta^2 |D(\Delta_\lambda u)| (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)-2}{4}} \\
&\quad \times [1 + |\log(|D(u)(x + \lambda e_k)|^2 + \mu^2)|]
\end{aligned}$$

$$\begin{aligned} & \times \left[(|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{2p(x+\lambda e_k)-p(x)}{4}} \right. \\ & \quad \left. + (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)}{4}} \right] dx \\ & \leq \epsilon \int_{O_1^+} \eta^2 (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)-2}{2}} |D(\Delta_\lambda u)|^2 dx \\ & \quad + c(\epsilon)|\lambda|^2 \int_{O_2^+} \eta^2 [1 + |\log(|D(u)(x + \lambda e_k)|^2 + \mu^2)|]^2 \\ & \quad \times \left[(|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{2p(x+\lambda e_k)-p(x)}{2}} \right. \\ & \quad \left. + (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)}{2}} \right] dx. \end{aligned}$$

Therefore, Lemma 2.2 and (6) reveals

$$\begin{aligned} |I_1| & \leq \epsilon I_0 \\ & \quad + c(\epsilon)|\lambda|^2 \int_{B_{2R}} \eta^2 \left[(|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{2p(x+\lambda e_k)-p(x)}{2}} \right. \\ & \quad \left. + (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)}{2}} \right] \\ & \quad \times \left[1 + (|D(u)(x + \lambda e_k)|^2 + \mu^2)^{\frac{\sigma\gamma_1}{16}} \right. \\ & \quad \left. + (|D(u)(x + \lambda e_k)|^2 + \mu^2)^{-\frac{\sigma\gamma_1}{16}} \right]^2 dx \\ & \leq \epsilon I_0 + c(\epsilon)|\lambda|^2 \int_{B_{2R}} \eta^2 (1 + |D(u)(x)|)^{(1+\sigma)p(x)} dx. \end{aligned} \tag{11}$$

Estimation of I_2 . Noting that (6) implies

$$p_2 \leq p_1 + \omega(4R_0) \leq p_1(1 + \sigma)$$

and

$$\frac{p_1(p_2 - 1)}{p_1 - 1} \leq p_1 + \frac{\omega(4R_0)p_1}{p_1 - 1} \leq p_1 + \frac{\omega(4R_0)\gamma_1}{\gamma_1 - 1} \leq p_1 + 8\omega(4R_0), \tag{12}$$

we discover

$$\begin{aligned} |I_2| & \leq 2 \left(\int_{B_{2R}} |S(x, D(u))|^{p_1'} dx \right)^{\frac{1}{p_1'}} \left(\int_{B_{2R}} |\Delta_{-\lambda}(\eta D(\eta) \otimes \Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ & \leq c|\lambda| \left(\int_{B_{2R}} (1 + |\nabla u|)^{p_1'(p(x)-1)} dx \right)^{\frac{1}{p_1'}} \\ & \quad \times \left(\int_{B_{2R}} |\nabla(\eta D(\eta) \otimes \Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ & \leq c|\lambda| \left(\int_{B_{2R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \right)^{\frac{1}{p_1'}} \\ & \quad \times \left(\frac{|\lambda|^{p_1}}{R^{2p_1}} \int_{B_{2R}} (1 + |\nabla u|^{p_1}) dx + \frac{1}{R^{p_1}} \int_{B_{2R}} |\eta \nabla(\Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}}. \end{aligned} \tag{13}$$

Note that

$$\eta \nabla(\Delta_\lambda u) = \nabla(\eta \Delta_\lambda u) - (\nabla \eta) \cdot \Delta_\lambda u. \tag{14}$$

Applying (14) and Korn’s inequality, we have

$$\begin{aligned} & \int_{B_{2R}} |\eta \nabla(\Delta_\lambda u)|^{p_1} dx \\ & \leq c \int_{B_{2R}} |D(\eta \Delta_\lambda u)|^{p_1} dx + \frac{c}{R^{p_1}} \int_{B_{2R}} |\Delta_\lambda u|^{p_1} dx \\ & \leq c \int_{O_1^+} |\eta D(\Delta_\lambda u)|^{p_1} dx + \frac{c}{R^{p_1}} |\lambda|^{p_1} \int_{B_{3R}} |\nabla u|^{p_1} dx. \end{aligned} \tag{15}$$

And the first integral term in the last line is estimated as follows:

$$\begin{aligned} & \left(\int_{O_1^+} |\eta|^{p_1} |D(\Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ & \leq c \left(\int_{O_1^+} |\eta|^2 (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{(p(x)-2)}{2}} |D(\Delta_\lambda u)|^2 dx \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{B_{2R}} (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p_1(2-p(x))}{2(2-p_1)}} dx \right)^{\frac{2-p_1}{2p_1}} \\ & \leq c [I_0]^{\frac{1}{2}} \left(\int_{B_{3R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \right)^{\frac{2-p_1}{2p_1}}. \end{aligned} \tag{16}$$

Combining (13), (15) and (16), we obtain

$$\begin{aligned} \frac{|I_2|}{2} & \leq c \frac{|\lambda|^2}{R^2} \int_{B_{3R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \\ & \quad + \frac{|\lambda|}{R} \left(\int_{B_{3R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \right)^{\frac{1}{p_1}} \left(\int_{B_{2R}} |\eta|^{p_1} |D(\Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ & \leq c(\epsilon) \frac{|\lambda|^2}{R^2} \int_{B_{3R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx + \epsilon I_0. \end{aligned}$$

Estimation of I_3 will be postponed to Step 2.

Estimation of I_4 . Recalling Lemma 2.1 and the De Rham theory in [10], we see $\pi \in L^{q(\cdot)}(\Omega)$ for $q(x) = (1 + \sigma)p'(x)$. Observe that

$$\min_{B_{2R}} q(x) = \frac{p_2(1 + \sigma)}{p_2 - 1} > \frac{p_1}{p_1 - 1}$$

by the assumption $\omega(4R_0) < \frac{\sigma}{16}$. Via a similar estimating procedure of I_2 , we compute

$$\begin{aligned} |I_4| & = \left| \int_{B_{2R}} \pi \left(\Delta_{-\lambda, k} \left(\eta \frac{\partial \eta}{\partial x_i} \Delta_\lambda u_i \right) \right) dx \right| \\ & \leq c \frac{|\lambda|}{R} \|\pi\|_{L^{p_1'}} \left(\int_{B_{2R}} |\nabla(\eta \Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}}. \end{aligned} \tag{17}$$

In light of Korn's inequality, it holds that

$$\begin{aligned} & \left(\int_{B_{2R}} |\nabla(\eta\Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ & \leq c \frac{|\lambda|^2}{R^2} \left(\int_{B_{2R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \right)^{\frac{1}{p_1}} + c \left(\int_{O_1^+} |\eta|^{p_1} |D(\Delta_\lambda u)|^{p_1} dx \right)^{\frac{1}{p_1}}. \end{aligned} \quad (18)$$

Combining (16), (17) and (18), we obtain

$$\begin{aligned} |I_4| & \leq c \frac{|\lambda|^2}{R^2} \|\pi\|_{L^{p_1'}(B_{2R})} \left(\int_{B_{2R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \right)^{\frac{1}{p_1}} \\ & \quad + c(\epsilon) \frac{|\lambda|^2}{R^2} \|\pi\|_{L^{p_1'}(B_{2R})}^2 \left(\int_{B_{3R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \right)^{\frac{2-p_1}{p_1}} + \epsilon I_0. \end{aligned}$$

Estimation of I_5 . Korn's inequality yields

$$\begin{aligned} |I_5| & \leq c \|g\|_{L^\infty(B_{2R})} \int_{B_{2R}} |\Delta_{-\lambda}(\eta^2(\Delta_\lambda u))| dx \\ & \leq c |\lambda| \int_{O_1^+} |D(\eta^2 \Delta_\lambda u)| dx \\ & \leq c \frac{|\lambda|^2}{R} \int_{B_{3R}} |D(u)| dx + c |\lambda| \int_{O_1^+} |\eta|^2 |\Delta_\lambda D(u)| dx \\ & \leq c(\epsilon) \frac{|\lambda|^2}{R^2} \int_{B_{2R}} (1 + |D(u)|)^{(1+\sigma)p(x)} dx + \epsilon I_0. \end{aligned}$$

Consequently, it follows from combining the estimates of I_1 , I_2 , I_4 and I_5 , and (10) that

$$\begin{aligned} I_0 & \leq I_3 + c \frac{|\lambda|^2}{R^2} \int_{B_{2R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \\ & \quad + c \frac{|\lambda|^2}{R^2} \left(\int_{B_{3R}} (1 + |\nabla u|)^{(1+\sigma)p(x)} dx \right)^{\frac{2-p_1}{p_1}}. \end{aligned} \quad (19)$$

Step 2. Boundedness of u .

In this step, we shall prove that u is locally bounded. Note that Lemma 2.1 implies that u is locally Hölder continuous by Morrey embedding in case of $n = 2$ and $\gamma_1 = 2$. So it is not necessary to consider this case. We recall I_3 and apply integration by parts formula for finite difference to see

$$\begin{aligned} I_3 & = \int_{B_{2R}} \Delta_\lambda u_i \frac{\partial u_j}{\partial x_i}(x + \lambda e_k) \eta^2 \Delta_\lambda u_j dx + \int_{B_{2R}} u_i \Delta_\lambda \frac{\partial u_j}{\partial x_i} \eta^2 \Delta_\lambda u_j dx \\ & = : I_{3,1} + I_{3,2}. \end{aligned}$$

We now assume that

$$u \in W_{loc}^{1,s}(\Omega)^n \text{ for some } s \in [\gamma_1, n).$$

Next we estimate $I_{3,1}$

$$|I_{3,1}| \leq \int_{B_{2R}} |\Delta_\lambda u|^2 |\nabla u(x + \lambda e_k)| dx \leq \|\Delta_\lambda u\|_{L^{2s'}(B_{2R})}^2 \|u\|_{W^{1,s}(B_{2R})}^n.$$

Defining $\theta := \frac{s(n+2)-3n}{2n}$, we see $0 < \theta < 1$ and

$$\frac{1-\theta}{s^*} + \frac{\theta}{s} = \frac{1}{2s'},$$

where s' is the Hölder conjugate exponent of s and s^* is the Sobolev exponent of s . It then follows from interpolation that

$$\|\Delta_\lambda u\|_{L^{2s'}(B_{2R})}^2 \leq \|\Delta_\lambda u\|_{L^{s^*}(B_{2R})}^{2(1-\theta)} \|\Delta_\lambda u\|_{L^s(B_{2R})}^{2\theta} \leq c|\lambda|^{2\theta} \|u\|_{W^{1,s}(B_{3R})}^2$$

and

$$|I_{3,1}| \leq c|\lambda|^{2\theta} \|u\|_{W^{1,s}(B_{3R})}^3. \quad (20)$$

On the other hand, integration by parts formula for finite difference reveals

$$\begin{aligned} |I_{3,2}| &\leq \frac{1}{2} \left| \int_{B_{2R}} |u_i| (\Delta_\lambda u_j)^2 \frac{\partial \eta^2}{\partial x_i} dx \right| \\ &\leq \frac{c}{R} \|\Delta_\lambda u\|_{L^{2s'}(B_{2R})}^2 \|u\|_{W^{1,s}(B_{2R})} \\ &\leq \frac{c}{R} |\lambda|^{2\theta} \|u\|_{W^{1,s}(B_{3R})}^3. \end{aligned} \quad (21)$$

Merging up (20) and (21), we obtain

$$|I_3| \leq c\left(1 + \frac{1}{R}\right) |\lambda|^{2\theta} \|u\|_{W^{1,s}(B_{3R})}^2. \quad (22)$$

Finally, since $p(\cdot)$ is Lipschitz continuous, (19) and (22) give the boundedness of I_0 :

$$I_0 \leq c \frac{|\lambda|^{2\theta}}{R^2}, \quad (23)$$

where the constant c depends on $\|u\|_{W^{1,s}(B_{3R})}$ and $\|\pi\|_{L^{p'_1}(B_{2R})}$. Set

$$\hat{s} := \frac{2s}{s - \gamma_1 + 2}.$$

Then $\gamma_1 \leq \hat{s} < 2$ and $\frac{s(\hat{s}-2)}{\hat{s}} = \gamma_1 - 2$. Thus, Hölder's inequality and (23) reveal

$$\begin{aligned} &\int_{B_R} |\Delta_\lambda D(u)|^{\hat{s}} dx \\ &= \int_{O_1^+} (|D(u)(s + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)-2}{2} \frac{\hat{s}}{2}} |\Delta_\lambda D(u)(x)|^{\hat{s}} \eta^{\hat{s}} \\ &\quad \times (|D(u)(s + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{2-p(x)}{2} \frac{\hat{s}}{2}} dx \\ &\leq [I_0]^{\frac{\hat{s}}{2}} \left[\int_{B_{2R}} (|D(u)(s + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{2-p(x)}{2} \frac{\hat{s}}{2}} dx \right]^{\frac{2-\hat{s}}{2}} \\ &\leq c \frac{|\lambda|^{\theta \hat{s}}}{R^2} \left[\int_{B_{2R}} (1 + |D(u)(x)|)^s dx \right]^{\frac{2-\hat{s}}{2}}. \end{aligned} \quad (24)$$

This implies

$$D(u) \in W^{t, \hat{s}}(B_{R/2})^{n \times n} \quad \text{for any } t \in [0, \theta]$$

and then we obtain by Nikolskii's embedding theorem in [2] that

$$D(u) \in L^{\frac{n\hat{s}}{n-t\hat{s}}}(B_{R/2})^{n \times n} \quad \text{for any } t \in [0, \theta]. \quad (25)$$

We set

$$\sigma(s) := \frac{n\hat{s}}{n - \theta\hat{s}} = \frac{2ns}{(5 - \gamma_1)n - 2s},$$

then it follows that

$$\sigma(s) - s \geq \tau_0 := \frac{\gamma_1((n + 2)\gamma_1 - 3n)}{(5 - \gamma_1)n - 2\gamma_1} > 0.$$

Hence (25) can be written as

$$\nabla u \in L^\tau(B_{R/2})^{n \times n} \quad \text{for any } \tau \in [1, \sigma(s)].$$

Here, we have used Korn's inequality. According to Sobolev embedding theorem and the fact that $\sigma(s) \leq s^*$, we see

$$u \in W^{1,\tau}(B_{R/2})^n \quad \text{for any } \tau \in [1, \sigma(s)]$$

and so, by covering, we also have

$$u \in W_{loc}^{1,\tau}(\Omega)^n \quad \text{for any } \tau \in [1, \sigma(s)].$$

By bootstrap argument, we can conclude that

$$u \in W_{loc}^{1,\tau}(\Omega)^n \quad \text{for any } \tau \in [1, \sigma(n)],$$

where $n < \sigma(n) = \frac{2n}{3-\gamma_1}$. By Morrey embedding theorem, u is locally Hölder continuous, and so u is locally bounded.

Step 3. Completing the proof of Theorem 1.1.

For $1 < s < \frac{2n}{3-\gamma_1}$, we estimate I_3 again

$$\begin{aligned} |I_3| &\leq \left| \int_{B_{2R}} u_i \frac{\partial u_j}{\partial x_i} (\Delta_{-\lambda,k}(\eta^2 \Delta_\lambda u_j)) dx \right| \\ &\leq |\lambda| \|u\|_{L^\infty(B_{2R})^n} \|\nabla u\|_{L^s(B_{2R})^{n \times n}} \|D(\eta^2 \Delta_\lambda u)\|_{L^{s'}(B_{2R})^{n \times n}} \\ &\leq |\lambda| \|u\|_{L^\infty(B_{2R})^n} \|\nabla u\|_{L^s(B_{2R})^{n \times n}} \\ &\quad \times \left(\|\eta^2 D(\Delta_\lambda u)\|_{L^{s'}(B_{2R})^{n \times n}} + \frac{|\lambda|}{R} \|\nabla u\|_{L^{s'}(B_{2R})^{n \times n}} \right). \end{aligned} \tag{26}$$

We select $s = 4 - \gamma_1$. Then $2 \leq s < \frac{2n}{3-\gamma_1}$ and

$$\frac{(2 - p(x))s'}{2 - s'} = \frac{2 - p(x)}{2 - \gamma_1} (4 - \gamma_1) \leq (4 - \gamma_1) = s.$$

By following the calculations in (24), we also have

$$\begin{aligned} &\|\eta^2 D(\Delta_\lambda u)\|_{L^{s'}(B_{2R})^{n \times n}} \\ &\leq [I_0]^{\frac{1}{2}} \left(\int_{B_{2R}} (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{(2-p(x))s'}{2(2-s')}} dx \right)^{\frac{2-s'}{2s'}} \\ &\leq [I_0]^{\frac{1}{2}} \left(\int_{B_{2R}} (1 + |\nabla u|^s) dx \right)^{\frac{2-s'}{2s'}}. \end{aligned} \tag{27}$$

Applying (27) and Young's inequality to (26), we find

$$|I_3| \leq c(\epsilon) \frac{|\lambda|^2}{R} + \epsilon I_0. \tag{28}$$

Combining (19), (28) and the assumption $p(x) \leq 2$, we obtain

$$\begin{aligned} & \int_{B_R} (1 + |D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2)^{\frac{p(x)-2}{2}} \left| \frac{\Delta_\lambda D(u)}{\lambda} \right|^2 dx \\ &= \int_{O^+} (1 + |D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2)^{\frac{p(x)-2}{2}} \left| \frac{\Delta_\lambda D(u)}{\lambda} \right|^2 dx \\ &\leq \int_{O_1^+} (|D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2 + \mu^2)^{\frac{p(x)-2}{2}} \left| \frac{\Delta_\lambda D(u)}{\lambda} \right|^2 dx \leq c, \end{aligned} \quad (29)$$

where

$$O^+ := B_R \cap \{x \in \Omega : |Du|(x + \lambda e_k) + |Du|(x) + \mu > 0\} \subset O_2^+.$$

We now divide (27) by $|\lambda|$ to deduce

$$\frac{1}{|\lambda|} \|\Delta_\lambda \nabla u\|_{L^{s'}(B_R)^{n \times n}} \leq \frac{1}{|\lambda|} \|\eta^2 D(\Delta_\lambda u)\|_{L^{s'}(B_{2R})^{n \times n}} \leq c$$

for all $\lambda \in (-\frac{R}{4}, \frac{R}{4})$, where we have used Korn's inequality and (19).

By difference quotient method (See [12, Section 5.8.2]), there exists the weak derivative of ∇u such that

$$\frac{1}{\lambda} \Delta_\lambda \nabla u \rightharpoonup \frac{\partial}{\partial x_k} \nabla u \text{ in } L^{s'}(B_R)^{n \times n}. \quad (30)$$

For the time being, we suppose that

$$\begin{aligned} & (1 + |D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2)^{\frac{p(x)-2}{4}} \\ & \rightarrow (1 + 2|D(u)(x)|^2)^{\frac{p(x)-2}{4}} \end{aligned} \quad \text{in } L^q(B_R) \quad (31)$$

for all $q \in [1, \infty)$. Then we see that (30) deduce

$$\begin{aligned} & (1 + |D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2)^{\frac{p(x)-2}{4}} \left(\frac{\Delta_\lambda D(u)}{\lambda} \right) \\ & \rightarrow (1 + 2|D(u)(x)|^2)^{\frac{p(x)-2}{4}} \left(\frac{\partial}{\partial x_k} D(u) \right) \end{aligned} \quad \text{in } L^1(B_R)^{n \times n}.$$

Since there exists a function $w_k \in L^2(B_R)$ by (29) such that

$$(1 + |D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2)^{\frac{p(x)-2}{4}} \left(\frac{\Delta_\lambda D(u)}{\lambda} \right) \rightharpoonup w_k \text{ in } L^2(B_R)^{n \times n},$$

we find $(1 + 2|D(u)(x)|^2)^{\frac{p(x)-2}{4}} \left| \frac{\partial}{\partial x_k} \nabla u \right| = w_k$.

Verification of (31). To do this, we write $h(x) = \frac{1}{2} + |D(u)(x)|^2$ and $h_\lambda(x) = \frac{1}{2} + |D(u)(x + \lambda e_k)|^2$, and compute

$$\begin{aligned}
& \int_{B_R} \left| \frac{1}{(h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}}} - \frac{1}{(2h(x))^{\frac{2-p(x)}{4}}} \right|^q dx \\
&= \int_{B_R} \left| \frac{(2h(x))^{\frac{2-p(x)}{4}} - (h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}}}{(h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}} (2h(x))^{\frac{2-p(x)}{4}}} \right|^q dx \\
&\leq c \int_{B_R} \frac{\left((2h(x))^{\frac{2-p(x)}{4}} + (h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}} \right)^{q-1}}{\left((h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}} (2h(x))^{\frac{2-p(x)}{4}} \right)^q} \\
&\quad \times \left((2h(x))^{\frac{2-p(x)}{4}} - (h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}} \right) dx \\
&\leq c \int_{B_R} \left((2h(x))^{\frac{2-p(x)}{4}} - (h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}} \right) dx.
\end{aligned}$$

And we estimate

$$\begin{aligned}
& (2h(x))^{\frac{2-p(x)}{4}} - (h(x) + h_\lambda(x))^{\frac{2-p(x)}{4}} \\
&= \int_0^1 \frac{d}{dt} (h(x) + h_\lambda(x) + t(h(x) - h_\lambda(x)))^{\frac{2-p(x)}{4}} dt \\
&\leq \frac{2-p(x)}{4} |h(x) - h_\lambda(x)| \int_0^1 (h(x) + h_\lambda(x) + t(h(x) - h_\lambda(x)))^{\frac{-2-p(x)}{4}} dt \\
&\leq \frac{2-p(x)}{2} |h(x) - h_\lambda(x)| \\
&= \frac{2-p(x)}{2} \left| |D(u)(x)|^2 - |D(u)(x + \lambda e_k)|^2 \right|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \int_{B_R} \left| (1 + |D(u)(x + \lambda e_k)|^2 + |D(u)(x)|^2)^{\frac{p(x)-2}{4}} - (1 + 2|D(u)(x)|^2)^{\frac{p(x)-2}{4}} \right|^q dx \\
&\leq c \int_{B_R} \left| |D(u)(x)|^2 - |D(u)(x + \lambda e_k)|^2 \right| dx.
\end{aligned}$$

Since the integral on the right-hand side above inequality converges to 0 as $\lambda \rightarrow 0$, (31) is valid, and (4) is proved.

To verify (5), we put $V = (1 + |D(u)|^2)^{\frac{p(x)}{4}}$. Then

$$\left| \frac{\partial V}{\partial x_k} \right| \leq c \left((1 + |D(u)|^2)^{\frac{p(x)}{4}} \log(1 + |D(u)|^2) + (1 + |D(u)|^2)^{\frac{p(x)-2}{4}} \left| \frac{\partial}{\partial x_k} D(u) \right| \right).$$

Using (4), the higher integrability of $D(u)$ and Lemma 2.2, we know $V \in W_{loc}^{1,2}(\Omega)$. Therefore,

$$V \in L_{loc}^s(\Omega) \text{ for all } s \in [1, \infty) \text{ if } n = 2, \quad V \in L_{loc}^6(\Omega) \text{ if } n = 3. \quad (32)$$

Note that following inequality holds by Young's inequality:

$$|a|^t = |b|^{\frac{t(p-2)}{2}} |a|^t |b|^{\frac{t(2-p)}{2}} < |b|^{p-2} |a|^2 + |b|^{\frac{t(2-p)}{2-t}}$$

for any $p \in [1, 2]$, $t \in [1, 2)$ and $|b| > 0$. For the case $n = 2$ with $t \in [1, 2)$, we estimate

$$\int_{B_R} \left| \frac{\partial D(u)}{\partial x_k} \right|^t dx \leq \int_{B_R} (1 + |D(u)|^2)^{\frac{p(x)-2}{2}} \left| \frac{\partial D(u)}{\partial x_k} \right|^2 dx \\ + \int_{B_R} (1 + |D(u)|^2)^{\frac{t(2-p(x))}{2(2-t)}} dx.$$

Then, we immediately deduce (5)₁ from (32) and Korn's inequality. One can prove (5)₂ in a similar way. This completes our proof.

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