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PERTURBATIONS OF MINIMIZING MOVEMENTS AND CURVES OF MAXIMAL SLOPE

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Abstract. We modify the De Giorgi's minimizing movements scheme for a functional ϕ, by perturbing the dissipation term, and find a condition on the perturbations which ensures the convergence of the scheme to an absolutely continuous perturbed minimizing movement. The perturbations produce a variation of the metric derivative of the minimizing movement. This process is formalized by the introduction of the notion of curve of maximal slope for ϕ with a given rate. We show that if we relax the condition on the perturbations we may have many different meaningful effects; in particular, some perturbed minimizing movements may explore different potential wells.

1. **Introduction.** The method of minimizing movements was introduced by De Giorgi to define a notion of evolution under very weak hypotheses. It consists in introducing a time-discretization scale and a corresponding time-discrete curve by solving an iterative Euler-type scheme. By refining the time scale we obtain a continuous curve.

Ambrosio, Gigli, and Savaré developed this method in [4] to formulate a notion of gradient flow in a complete metric space (S, d) for a given proper functional $\phi : S \to (-\infty, +\infty]$. They considered a time discretization $\tau = \{\tau_n\}$, of amplitude $|\tau|$ which tends to zero, and defined a time-discrete motion $U^\tau : [0, +\infty) \to S$ (called a discrete solution) by interpolating a sequence (U_n^τ) which solves the recursive scheme

$$U_n^\tau \in \underset{u\in S}{\operatorname{argmin}} \left\{ \phi(u) + \frac{d^2(u, U_{n-1}^\tau)}{2\tau_n} \right\}, \quad n \geq 1,$$

starting from a given initial datum U_0^τ. The minimization is localized, through the dissipation term $d^2(u, U_{n-1}^\tau)/2\tau_n$, in a neighborhood of the previous step of amplitude depending on τ. Under suitable assumptions on ϕ, when $|\tau| \to 0$, the discrete solutions converge to an absolutely continuous curve $U : [0, +\infty) \to S$ (a minimizing movement). Denoting with $|U'|$ the metric derivative of U, and with $|\partial^- \phi|$ the relaxed metric slope of ϕ (Definition 2.6 and 3.1), they also proved that these minimizing movements are curves of maximal slope for ϕ; *i.e.*, for every $t \geq s \geq 0$,

$$\phi(U(t)) - \phi(U(s)) \leq -\frac{1}{2}\int_s^t |U'|^2(\xi)d\xi - \frac{1}{2}\int_s^t |\partial^- \phi|^2(U(\xi))d\xi,$$

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provided that $|\partial^- \phi|$ is a strong upper gradient for ϕ; *i.e.*,

$$|\phi(v(t)) - \phi(v(s))| \leq \int_s^t |\partial^- \phi|(v(\xi))|v'|(\xi)d\xi$$

for any absolutely continuous curve v. Therefore minimizing movements are solutions of the metric gradient-flow

$$\begin{cases} |U'|(t) = |\partial^- \phi|(U(t)) \\ (\phi \circ U)'(t) = -|U'|(t)|\partial^- \phi|(U(t)), \end{cases} \qquad \text{for almost every } t \geq 0.$$

A few years earlier, Jordan, Kinderlehrer, and Otto had used minimizing movements in [12] to study Fokker-Plank equations in which the drift term was the gradient of a potential field; they proved that minimizing movements, obtained by using Wasserstein metric as the dissipation, were solutions of the Fokker-Plank equation. This work was generalized in [4] to build a theory of Wasserstein gradient-flows in the space of probability measures.

The De Giorgi's idea has been adopted also by Almgren, Taylor, and Wang to study motion by mean curvature of boundaries in $\mathbb{R}^n$ in their seminal work [3], which started a vast amount of literature.

We will introduce a variation of the method described above where the amplitude of the neighborhoods of minimization depends not only on τ, but also on a given sequence of positive coefficients (a_n^τ), which we call *perturbation*. In this paper, we will consider a uniform time-discretization for simplicity, but every regular partition of the positive half-line can be considered; with an abuse of notation, we will denote $\tau = |\tau|$. We will modify the scheme by multiplying these coefficients to the dissipation, mimicking the perturbation effect of a noise term. Hence, we will consider discrete solutions $u^\tau : [0, +\infty) \to S$ which interpolate sequences (u_n^τ) solving

$$u_n^\tau \in \operatorname*{argmin}_{u \in S} \left\{ \phi(u) + a_n^\tau \frac{d^2(u, u_{n-1}^\tau)}{2\tau} \right\}, \quad n \geq 1,$$

on a uniform time-partition of amplitude τ. The sequences (u_n^τ) are equal to those defined in [4], previously denoted as (U_n^τ), obtained considering $\tau_n = \tau / a_n^\tau$. Nevertheless, the interpolation curves u^τ and U^τ are made on different time-discretizations, therefore they converge to different motions. A limit of the discrete solutions of this scheme will be called *perturbed minimizing movement*. We will also note that, if the perturbations are regular enough, we can apply directly the classical method of Ambrosio, Gigli, and Savaré, and obtain the perturbed minimizing movements through a change of variable. However we will consider very general hypotheses on the perturbations, and this allows us to use an analogous method, slightly modifying the classical one. These perturbations may also be seen as a variation of the functional, considering $\phi(u)/a_n^\tau$ in the minimization problem. For the interested readers we suggest the work by Fleissner and Savaré [11].

Following the results of [4], we will prove that, under suitable hypotheses on ϕ, if the perturbations are such that the inverses are locally uniformly integrable, discrete solutions converge to an absolutely continuous perturbed minimizing movement (Theorem 2.4). We will show that these minimizing movements satisfy the energy estimate

$$\phi(u(t)) - \phi(u(s)) \leq -\frac{1}{2} \int_s^t a^*(\xi)|u'|^2(\xi)d\xi - \frac{1}{2} \int_s^t \frac{1}{a^*(\xi)} |\partial^- \phi|^2(u(\xi))d\xi$$

for every $t \geq s \geq 0$, where a^* is a function such that $1/a^*$ is a weak limit in L^1_{loc} of $\{1/a^\tau\}$. Therefore we will say that perturbed minimizing movements are curves of maximal slope for ϕ with rate $1/a^*$, provided that $|\partial^- \phi|$ is a strong upper gradient (Theorem 3.9).

By means of many different examples, we will see that, if some of the conditions on the perturbations are not satisfied, discrete solutions may diverge, or converge to a discontinuous curve. We will show that this condition can be relaxed, renouncing to the continuity of the perturbed minimizing movements, which in general may be assumed to be piecewise absolutely continuous. This leads us to observe that perturbed minimizing movements for multi-well energy functionals may explore local minima, while in the classical case the motion would be confined in a single potential well (Example 4.4 and 4.6).

Recently, the method of minimizing movements expounded in [4] has been applied to a family of functionals $\{\phi_\varepsilon\}$ instead of a single one, so that the discrete solutions $\{u^{\tau,\varepsilon}\}$ depend also on ε. Conditions which ensure the convergence of the discrete solutions to a curve of maximal slope for the Γ-limit of the energies, as τ and ε tend to zero, was exhibited in particular cases, for instance by Sandier and Serfaty in [13] and Colombo and Gobbino in [8]. While a wider treatment has been given by Braides, Colombo, Gobbino, and Solci in [7], or by Fleissner in [10]. The general case may present different limits, corresponding to the relation between the two small parameters ε and τ, as shown by Braides in [6], and precised for oscillating potentials by Ansini, Braides and Zimmer in [2] (see also [1]). The perturbation approach may be applied also in these cases, but it will not be treated in this paper.

2. **Perturbed minimizing movements.** Following the notation in [4], let (S, d) be a complete metric space, and let σ be a Hausdorff topology, weaker than the one induced by the metric, and such that d is σ-lower semicontinuous.

Fixed a positive constant τ^*, for every $\tau \in (0, \tau^*)$, which stands for the *time-discretization scale*, we consider a sequence $(a_n^\tau)_{n=1}^\infty$ of strictly positive real numbers. We call any such sequences (with τ fixed) a *perturbation*, and we will extend it to the function

$$a^\tau : (0, +\infty) \to (0, +\infty), \quad a^\tau(t) := a^\tau_{\lceil \frac{t}{\tau} \rceil}. \tag{1}$$

We consider a functional $\phi : S \to (-\infty, +\infty]$, sometimes called the energy functional, and we assume that it is proper; *i.e.*, it is not identically equal to $+\infty$. For $\tau \in (0, \tau^*)$, we denote as $(u_n^\tau)_{n=0}^\infty$ any sequence solving the recursive minimum problem

$$\begin{cases} u_0^\tau \in D(\phi) \\ u_n^\tau \in \underset{u \in S}{\operatorname{argmin}} \left\{ \phi(u) + a_n^\tau \frac{d^2(u, u_{n-1}^\tau)}{2\tau} \right\} \quad n \geq 1, \end{cases} \tag{2}$$

for a given initial datum u_0^τ. If such a sequence exists, it is called a *discrete solution* for the *implicit Euler-type scheme along ϕ at time-discretization scale τ with perturbation a^τ, and initial datum u_0^τ*. This sequence is identified with the corresponding interpolation curve

$$u^\tau : [0, +\infty) \to S, \quad u^\tau(t) := u_{\lceil \frac{t}{\tau} \rceil}.$$

The n-th element u_n^τ of a discrete solution is called a *discrete step*, or simply a *step*.

In the case that a^τ is the constant function 1, the scheme (2) is equal to the recursive scheme (2.0.4) presented in [4].

Definition 2.1. If there exist a sequence $\{\tau_k\} \subset (0, \tau^*)$ which tends to zero such that for every k there exists a discrete solution u^{τ_k} for the scheme (2), and a curve $u : [0, +\infty) \to S$ such that $u^{\tau_k}(t)$ σ-converges to $u(t)$ as $k \to +\infty$, for all $t \geq 0$, then u is called a (generalized) $\{a^\tau\}$-*perturbed minimizing movement for* ϕ.

We sometimes say that u is a (a^{τ_k})-*perturbed minimizing movement for* ϕ when we want to highlight the role of the sequence (τ_k).

2.1. **Basic assumptions.** We consider the following hypotheses for the energy functional ϕ. Lower semicontinuity and compactness hypotheses are the same as those considered in Section 2.1 [4]; whereas, we have to slightly modify the coercivity assumption, in order to consider the case in which a_n^τ/τ are not equibounded from below:

H1 (lower semicontinuity) ϕ is σ-lower semicontinuous;

H2 (coercivity) there exists $u^* \in S$ such that for any constant $c > 0$

$$\inf_{u \in S} \left\{ \phi(u) + c\, d^2(u, u^*) \right\} > -\infty;$$

H3 (compactness) for all $c > 0$ the set $\{u \in S \,|\, d(u, u^*) \leq c, \phi(u) \leq c\}$ is σ-compact.

Remark 1. The employ of the auxiliary topology σ allows us to use scheme (2) for a wide class of functionals. Indeed, if $S = X$ is a reflexive Banach space, every weakly lower-semicontinuous functional satisfies the compactness hypothesis for the weak topology, but not for the norm.

The previous hypotheses ensure the following result (a modification of Lemma 2.2.1 and Corollary 2.2.2 [4]).

Proposition 1 (Existence of discrete solutions). *Let* $\phi : S \to (-\infty, +\infty]$ *satisfy hypotheses* H1-H3. *Let* (a_n^τ) *be a sequence of strictly positive numbers. Then for all* τ *there exists a discrete solution* u^τ *for the scheme* (2).

Proof. Fixed τ and $n \geq 1$, we suppose that there exist the first $n-1$ steps $(u_i^\tau)_{i=0}^{n-1}$ of a discrete solution. For any $v \in S$, we consider the functional

$$\Phi_{\tau,n,v}(u) = \phi(u) + a_n^\tau \frac{d^2(u, v)}{2\tau}$$

whose minima are the n-th step of a discrete solution, when $v = u_{n-1}^\tau$. Since ϕ and d are σ-lower semicontinuous, then $\Phi_{\tau,n,v}$ satisfies the same property too. Let u^* be as in H2, by the triangular inequality and Young's inequality, we have $d^2(u, u^*) \leq 2d^2(u, v) + 2d^2(v, u^*)$ for any $u \in S$, which implies that $d^2(u, v) \geq d^2(u, u^*)/2 - d^2(v, u^*)$. Therefore, by the coercivity assumption H2, we get

$$\begin{aligned} \Phi_{\tau,n,v}(u) \geq & \left(\phi(u) + a_n^\tau \frac{d^2(u, u^*)}{8\tau} \right) - a_n^\tau \frac{d^2(v, u^*)}{4\tau} + a_n^\tau \frac{d^2(u, v)}{4\tau} \\ \geq & C - a_n^\tau \frac{d^2(v, u^*)}{4\tau} + a_n^\tau \frac{d^2(u, v)}{4\tau} \end{aligned}$$

where $C = C(\tau, n)$ is a constant. From the previous formula, we have that

$$d^2(u, v) \leq 4\tau \left(\frac{1}{a_n^\tau} \Phi_{\tau,n,v}(u) + \frac{d^2(v, u^*)}{4\tau} - \frac{C}{a_n^\tau} \right),$$

so that the sublevels $\{u \in S \,|\, \Phi_{\tau,n,v}(u) \leq c\} \subset \{u \in S \,|\, \phi(u) \leq c\}$ are bounded; hence σ-precompact by hypothesis H3. The result follows by applying the Weierstrass theorem. $\square$

If we consider perturbations $\{a^\tau\}$ as in (1) regular enough (*e.g.* bounded and $a^\tau \geq \alpha > 0$), we can directly apply the method of Ambrosio, Gigli, and Savaré to the scheme (2) to approach the problem (see Remark 4). Nevertheless, for more general perturbations, the application of the classical method is not immediate or not possible. Moreover we want to distinguish the role of the coefficients a_n^τ from the one of the time-discretization scale, in order to highlight the perturbation effect on the dissipation. Hence, in the following, we will recall the main results presented in [4].

Our aim is to use Proposition 3.3.1 [4], which is a generalization of the Ascoli-Arzelá theorem, to obtain the convergence of the discrete solutions; *i.e.*, the existence of a perturbed minimizing movement. For the reader's convenience, we recall the result below.

Lemma 2.2 (**Proposition 3.3.1** [4]). *Fixed $T > 0$, let $v^\tau : [0,T] \to S$ be a family of curves (indexed by τ) and let $\theta : [0,T] \times [0,T] \to [0,+\infty)$ be a function such that*

$$\lim_{(s,t)\to(r,r)} \theta(s,t) = 0, \quad \textit{for every } r \in [0,T]\backslash I,$$

where I is a discrete set. If $\{v^\tau\}$ are such that

(i) $\{v^\tau(t) \,|\, t \in [0,T], \tau \in (0,\tau^*)\}$ is σ-precompact
(ii) for every $t, s \in [0,T]$ $\limsup_{\tau\to 0} d(v^\tau(t), v^\tau(s)) \leq \theta(s,t)$,

then there exist a curve $v : [0,T] \to S$ continuous in $[0,T]\backslash I$, and a sequence $\tau_k \to 0$ such that $v^{\tau_k}(t)$ σ-converge to $v(t)$ as $k \to +\infty$, for all $t \in [0,T]$.

In order to apply Lemma 2.2 to the discrete solutions, they must satisfy assumptions (i) and (ii), which replace the usual equiboundedness and equicontinuity properties of the Ascoli-Arzelá theorem. Hence we add the following hypotheses:

H4 (control of initial data) there exists a constant C_0 such that $d(u_0^\tau, u^*) \leq C_0$ and $\phi(u_0^\tau) \leq C_0$;
H5 (local uniform integrability) the family $\{1/a^\tau\}$ is uniformly integrable in $[0,T]$ for all $T > 0$.

Remark 2. Assumption H5 implies that the family $\{1/a^\tau\}$ is weakly convergent, up to subsequences, in $L^1_{loc}(0,+\infty)$ by the Dunford-Pettis theorem.

We denote as $a^* : (0,+\infty) \to [0,+\infty]$ any measurable function such that $1/a^*$ is a weak limit for $\{1/a^\tau\}$, with the assumption that, if $1/a^*(t) = 0$ or $+\infty$ then $a^*(t) = +\infty$ or 0, respectively. This notation is inspired by the fact that periodic perturbations, which oscillate between two or more values, weakly converge to the function that constantly assume the value of the inverse of the harmonic mean, sometimes denoted by a^*.

Note that, by the local uniform integrability, $\{1/a^\tau\}$ is uniformly bounded in $L^1(0,T)$ for every $T > 0$; hence we may define

$$C_{0,T} := \sup_{\tau\in(0,\tau^*)} \left\| \frac{1}{a^\tau} \right\|_{L^1(0,T)}.$$

2.2. **Regularity of discrete solutions.** Assumptions H4 and H5 provide to the discrete solutions the regularity properties (i) and (ii) of Lemma 2.2. Before proving it, we note first that the energy functional ϕ has a decreasing behavior along any discrete solution (u_n^τ). In fact, setting $u = u_{n-1}^\tau$ in the n-th minimization problem of scheme (2), we have

$$\phi(u_n^\tau) + a_n^\tau \frac{d^2(u_n^\tau, u_{n-1}^\tau)}{2\tau} \le \phi(u_{n-1}^\tau). \tag{3}$$

This inequality also leads us to observe that the increments of the discrete solutions have an upper bound

$$d^2(u_n^\tau, u_{n-1}^\tau) \le \frac{2\tau}{a_n^\tau}\big(\phi(u_{n-1}^\tau) - \phi(u_n^\tau)\big). \tag{4}$$

First, we recall a useful discrete version of the Gronwall Lemma.

Lemma 2.3 (**Lemma 3.2.4** [4]). *Fixed an integer N, for any $1 \le n \le N$, let $b_n, \tau_n \in [0, +\infty)$, and let A and α be two positive constant such that $\alpha\tau_n < 1/2$, and $b_n \le A + \alpha \sum_{i=1}^n \tau_i b_i$, for every $1 \le n \le N$. Then*

$$b_n \le 2Ae^{2\alpha \sum_{i=1}^{n-1} \tau_i}, \quad \textit{for every } 1 \le n \le N.$$

Proposition 2 (**Equicompactness of discrete orbits**). *Let ϕ satisfy assumption* H2*, let $\{u_0^\tau\}$ be initial data satisfying* H4*. Let $\{a^\tau\}$ be a family of perturbations as in (1) such that $\{1/a^\tau\}$ is locally L^1-equibounded, and let $C_{0,T}$ defined as in Remark 2. If there exists a discrete solution u^τ, then for every $T > 0$ there exists a positive constant C_T such that*

$$d(u^\tau(t), u^*) \le C_T, \quad \phi(u^\tau(t)) \le C_0, \quad \textit{for every } t \in [0, T], \tag{5}$$

where C_0 is the same as in H4*. In addition, if hypothesis* H3 *is satisfied, the set of all discrete orbits $\{u^\tau(t) \,|\, t \in [0,T],\, \tau \in (0, \tau^*)\}$ is σ-precompact.*

Proof. We define $\alpha := (2C_{0,T})^{-1}$. Reasoning as in the proof of Lemma 3.2.2 [4], for any $1 \le n \le \lfloor T/\tau \rfloor$, by Young's inequality, the triangular inequality, and (4) we have that

$$\begin{aligned}
\frac{1}{2}d^2(u_n^\tau, u^*) - \frac{1}{2}d^2(u_0^\tau, u^*) &= \sum_{i=1}^n \frac{1}{2}d^2(u_i^\tau, u^*) - \frac{1}{2}d^2(u_{i-1}^\tau, u^*) \\
&= \sum_{i=1}^n d^2(u_i^\tau, u^*) - \frac{1}{2}d^2(u_i^\tau, u^*) - \frac{1}{2}d^2(u_{i-1}^\tau, u^*) \\
&\le \sum_{i=1}^n d^2(u_i^\tau, u^*) - d(u_i^\tau, u^*)d(u_{i-1}^\tau, u^*) \\
&\le \sum_{i=1}^n d(u_i^\tau, u_{i-1}^\tau)d(u_i^\tau, u^*) \\
&\le \frac{2}{\alpha}\sum_{i=1}^n a_i^\tau \frac{d^2(u_i^\tau, u_{i-1}^\tau)}{2\tau} + \frac{\alpha}{4}\sum_{i=1}^n \frac{\tau}{a_i^\tau} d^2(u_i^\tau, u^*) \\
&\le \frac{2}{\alpha}\big(\phi(u_0^\tau) - \phi(u_n^\tau)\big) + \frac{\alpha}{4}\sum_{i=1}^n \frac{\tau}{a_i^\tau} d^2(u_i^\tau, u^*).
\end{aligned}$$

By hypothesis H2, there exists a constant C (depending on $C_{0,T}$ defined in Remark 2) such that $-\phi(u_n^\tau) \le \alpha d^2(u_n^\tau, u^*)/8 - C$. This yields

$$d^2(u_n^\tau, u^*) \le \frac{8}{\alpha}\big(\phi(u_0^\tau) - C\big) + 2d^2(u_0^\tau, u^*) + \alpha \sum_{i=1}^n \frac{\tau}{a_i^\tau} d^2(u_i^\tau, u^*).$$

We define $A := 8\big(C_0 - C\big)/\alpha + 2C_0^2$. By assumption H4 we get

$$d^2(u_n^\tau, u^*) \le A + \alpha \sum_{i=1}^{n} \frac{\tau}{a_i^\tau} d^2(u_i^\tau, u^*).$$

Applying Lemma 2.3 with $b_n = d^2(u_n^\tau, u^*)$ and $\tau_n = \tau/a_n^\tau$ we prove the first inequality in (5);

$$d(u_n^\tau, u^*) \le \sqrt{2eA} =: C_T.$$

By formula (3) we have that $\phi(u_n^\tau)$ is a monotone sequence; therefore, $\phi(u_n^\tau) \le \phi(u_0^\tau) \le C_0$ for every $\tau > 0$, $n \ge 1$. Finally, (5) and assumption H3 imply the σ-precompactness of the discrete orbits $\{u^\tau(t)\}$. □

Proposition 3 (**Equicontinuity of discrete solutions**). *Let ϕ satisfy assumption* H2, *and let the initial data $\{u_0^\tau\}$ and the constant C_0 be as in* H4. *Let $\{a^\tau\}$ be a family of perturbations as in (1) such that $\{1/a^\tau\}$ is locally L^1-equibounded. If there exists a discrete solution u^τ, then for every $T > 0$ there exist a constant $C = C(C_0, T)$ and a function*

$$\theta_T : [0,T] \times [0,T] \to [0, +\infty), \quad \theta_T(t,s) := C \left(\sup_{\tau \in (0,\tau^*)} \int_s^t 1/a^\tau(\xi) d\xi \right)^{\frac{1}{2}}, \tag{6}$$

such that

$$d(u^\tau(t), u^\tau(s)) \le \theta_T(s, t+\tau), \quad \textit{for every } t, s \in [0,T].$$

Proof. We set $n = \lceil t/\tau \rceil$ and $m = \lceil s/\tau \rceil$ (for simplicity we consider $t > s$). Applying the triangular inequality and the discrete Holder's inequality to (4) we have that

$$\begin{aligned} d(u_n^\tau, u_m^\tau) &\le \sum_{i=m+1}^{n} \left(\frac{2\tau}{a_i^\tau} \big(\phi(u_{i-1}^\tau) - \phi(u_i^\tau)\big) \right)^{\frac{1}{2}} \\ &\le \Bigg(\sum_{i=m+1}^{n} \frac{\tau}{a_i^\tau} \Bigg)^{\frac{1}{2}} \Big(2\big(\phi(u_m^\tau) - \phi(u_n^\tau)\big) \Big)^{\frac{1}{2}}. \end{aligned}$$

By the coercivity condition H2, the energy is bounded from below on bounded sets, so by the first of (5) we get $\inf\{\phi(u^\tau(t)) \,|\, t \in [0,T],\, \tau > 0\} =: m_T > -\infty$. Hence

$$d(u^\tau(t), u^\tau(s)) \le \sqrt{2(C_0 - m_T)} \Bigg(\int_{m\tau}^{n\tau} \frac{1}{a^\tau(\xi)} d\xi \Bigg)^{\frac{1}{2}}. \tag{7}$$

Let $\tilde\theta_T : [0,T] \times [0,T] \to [0, +\infty)$ be the function $\tilde\theta_T(s,t) := \sup \big\{ \int_s^t 1/a^\tau(\xi) d\xi \,\big|\, \tau \in (0, \tau^*) \big\}$. We have that $\int_t^{n\tau} 1/a^\tau(\xi) d\xi \le \tilde\theta_T(s, t+\tau)$; thus

$$\Bigg(\int_{m\tau}^{n\tau} \frac{1}{a^\tau(\xi)} d\xi \Bigg)^{\frac{1}{2}} \le \Bigg(\int_s^t \frac{1}{a^\tau(\xi)} d\xi + \int_t^{n\tau} \frac{1}{a^\tau(\xi)} d\xi \Bigg)^{\frac{1}{2}} \le \big(2\tilde\theta_T(s, t+\tau) \big)^{\frac{1}{2}},$$

so that, by (7) we get

$$d(u^\tau(t), u^\tau(s)) \le 2\sqrt{C_0 - m_T} \big(\tilde\theta_T(s, t+\tau) \big)^{\frac{1}{2}}.$$

Denoting $C := 2\sqrt{C_0 - m_T}$ we obtain the thesis. □

2.3. **A convergence result.**

Theorem 2.4 (**Existence of perturbed minimizing movements**). *Let $\phi : S \to (-\infty, +\infty]$ satisfy assumptions* H1-H3, *let $\{u_0^\tau\}$ be a family of initial data satisfying* H4, *and let $\{a^\tau\}$ be perturbations as in* (1) *satisfying* H5. *Then there exists a continuous $\{a^\tau\}$-perturbed minimizing movement u for ϕ.*

Proof. By Proposition 1, there exist $\{u^\tau\}$ discrete solutions for every $\tau \in (0, \tau^*)$. We consider the restriction to $[0,1]$ of u^τ. As a consequence of H5, θ_T defined in (6) is a modulus of continuity. By Proposition 2 and 3 we can apply Lemma 2.2. Hence, there exists a sequence $(\tau_{1,k})$ such that $u^{\tau_{1,k}}(t)$ σ-converges to $u_1(t)$ for all $t \in [0,1]$. Note that, since θ_T is a modulus of continuity, the set I defined in Lemma 2.2 is empty, therefore the limit is continuous. Inductively, we can consider a subsequence $(\tau_{h,k})$ of $(\tau_{h-1,k})$ such that the restrictions to $[0,h]$ of $u^{\tau_{h,k}}(t)$ σ-converge to $u_h(t)$, for all $t \in [0,h]$, with $u_{h-1} \equiv u_h$ in $[0,h-1]$. Hence we can extract a subsequence $u^{\tau_k} := u^{\tau_{k,k}}$ σ-converging to a continuous perturbed minimizing movement $u : [0,+\infty) \to S$. □

Hypothesis H5 imposes an additional regularity to the perturbed minimizing movements, which actually are absolutely continuous. To obtain it, we have to recall the notion of discrete derivative for discrete piecewise-constant functions. We also recall the definition of absolutely continuous curve in complete metric spaces and of its metric derivative (see *e.g.* Definition 1.1.1 and Theorem 1.1.2 [4]).

Definition 2.5. Let (S,d) be a metric space, let $\{t_n\} \subset \mathbb{R}$ be such that $t_{n-1} < t_n$, and let denote $J = \bigcup_n (t_{n-1}, t_n)$. Let $f : J \to S$ be such that $f(t) = f_n$, if $t \in (t_{n-1}, t_n)$. We define the *discrete derivative* of f as the function $|f'| : J \to \mathbb{R}$ such that

$$|f'|(t) = \frac{d(f_n, f_{n-1})}{t_n - t_{n-1}}, \quad \text{if } t \in (t_{n-1}, t_n].$$

Hence, for any discrete solution u^τ of the scheme (2), taking $t_n = n\tau$, we denote its discrete derivative as

$$\left|(u^\tau)'\right|(t) = \frac{d(u_n^\tau, u_{n-1}^\tau)}{\tau}, \quad \text{if } t \in ((n-1)\tau, n\tau]\,. \tag{8}$$

Definition 2.6. Let (S,d) be a complete metric space, and let $v : (a,b) \to S$ be an absolutely continuous curve; *i.e.* there exists $m \in L^1(a,b)$ such that

$$d(v(t), v(s)) \leq \int_s^t m(\xi) d\xi, \quad \text{for every } a < s \leq t < b.$$

Then, we define the metric derivative of v in $t \in (a,b)$ as

$$|v'|(t) = \lim_{s \to t} \frac{d(v(t), v(s))}{|t-s|}.$$

This limit is defined almost everywhere, and it coincides with the minimal $m \in L^1(a,b)$ satisfying the previous inequality.

Proposition 4. *Let ϕ satisfy* H2 *and* H3, *let $\{u_0^\tau\}$ be a family of initial data satisfying* H4, *and let $\{a^\tau\}$ be perturbations as in* (1) *satisfying* H5. *Suppose that there exists u a (a^{τ_k})-perturbed minimizing movement for ϕ. Then there exists a function $A \in L^1_{loc}(0,+\infty)$, and a subsequence $(\tau_{k'})$ of (τ_k) such that*

(i) $|(u^{\tau_{k'}})'|$ *weakly converge to A in $L^1_{loc}(0,+\infty)$,*
(ii) $u \in AC_{loc}(0,+\infty; S)$, *and $|u'|(t) \leq A(t)$ for almost every $t \in [0,+\infty)$.*

Proof. Let θ_T be defined as in (6), and $\theta_T^+(x) = \sup\{\theta_T(s,t) \,|\, t,s \in [0,T], |t-s| \leq x\}$. Integrating the discrete derivatives defined as in (8) in an interval (s,t), and reasoning as in proof of Proposition 3, by the uniform integrability of $\{1/a^\tau\}$ in $[0,T]$ we get

$$\begin{aligned}\int_s^t |(u^{\tau_k})'|(\xi)d\xi = &\sum_{i=m+1}^{n-1} d(u_i^{\tau_k}, u_{i-1}^{\tau_k}) + \frac{m\tau_k - s}{\tau_k} d(u_m^{\tau_k}, u_{m-1}^{\tau_k}) \\ &+ \frac{t-(n-1)\tau_k}{\tau_k} d(u_n^{\tau_k}, u_{n-1}^{\tau_k}) \\ \leq &\theta_T(s,t) + \frac{|m\tau_k - s| + |t-(n-1)\tau_k|}{\tau_k}\theta_T^+(\tau_k),\end{aligned}$$

for every $0 \leq s < t < T$, where $n = \lceil t/\tau_k \rceil$, $m = \lceil s/\tau_k \rceil$. This yields the uniform integrability of the discrete derivatives; *i.e.*, their weak compactness in $L^1(0,T)$ which proves (i).

By formula (8) we have that

$$d(u^{\tau_{k'}}(t), u^{\tau_{k'}}(s)) \leq \sum_{i=m}^{n} \int_{(i-1)\tau_{k'}}^{i\tau_{k'}} \frac{d(u_i^{\tau_{k'}}, u_{i-1}^{\tau_{k'}})}{\tau_{k'}} d\xi = \int_{(m-1)\tau_{k'}}^{n\tau_{k'}} |(u^{\tau_{k'}})'|(\xi)d\xi.$$

Taking the limit, by the σ-lower semicontinuity of d and the weakly convergence of the discrete derivatives proved at point (i), we get

$$d(u(t),u(s)) \leq \liminf_{k'\to+\infty} d(u^{\tau_{k'}}(s), u^{\tau_{k'}}(t)) \leq \lim_{k'\to+\infty} \int_{m\tau_{k'}}^{n\tau_{k'}} |(u^{\tau_{k'}})'|(\xi)d\xi = \int_s^t A(\xi)d\xi$$

so that $u \in AC_{loc}(0,+\infty;S)$. Therefore the metric derivative $|u'|$ exists almost everywhere, and for its minimality $|u'|(t) \leq A(t)$ for almost every $t \geq 0$. □

3. **Curves of maximal slope with a given rate.** This section is devoted to proving that, under suitable assumptions on ϕ, the perturbed minimizing movements as in Definition (2.1) are curves of steepest descend for the functional ϕ, in a sense that will be precised in the following. This is a generalization of Theorem 2.3.3 [4], but the presence of the perturbations yields a variation of the velocity of the curves.

First, we remind the crucial concept of strong upper gradient for a functional (see *e.g.* Definition 1.2.1 [4]).

Definition 3.1. Let $\phi : S \to (-\infty, +\infty]$ be a proper functional. A map $g : S \to [0,+\infty]$ is a *strong upper gradient* for ϕ if for every $u \in AC(a,b;S)$ the function $g \circ v$ is measurable and

$$|\phi(u(t)) - \phi(u(s))| \leq \int_s^t g(u(\xi))|u'|(\xi)d\xi, \quad \text{for all } a < s \leq t < b.$$

Slightly modifying Definition 1.3.2 [4], we introduce the following notion.

Definition 3.2 (Curve of maximal slope with a given rate). Let $\phi : S \to (-\infty, +\infty]$ be a proper functional, let $\lambda : (a,b) \to [0,+\infty]$ be a measurable function, and assume that $1/\lambda(t) = +\infty$ or 0 when $\lambda(t) = 0$ or $+\infty$ respectively. A curve $u \in AC(a,b;D(\phi))$ is a *curve of maximal slope for* ϕ *with respect to a strong upper gradient* g *with rate* λ if $\phi \circ u$ is equal almost everywhere to a nonincreasing function φ in (a,b), and for all $a < s \leq t < b$

$$\varphi(t)-\varphi(s)\le -\frac{1}{2}\int_{I_0\cap(s,t)}\frac{1}{\lambda(\xi)}|u'|^2(\xi)d\xi-\frac{1}{2}\int_{I_\infty\cap(s,t)}\lambda(\xi)g(u(\xi))^2d\xi$$
$$|u'|(\xi)=0,\quad \text{for almost every } \xi\in I_0$$
$$g(u(\xi))=0,\quad \text{for almost every } \xi\in I_\infty,$$

where $I_0=\{\xi\in(a,b)|\lambda(\xi)=0\}$, and $I_\infty=\{\xi\in(a,b)|\lambda(t)=+\infty\}$.

In order to simplify the notation, we assume that zero multiplied by $+\infty$ is null in the integral inequality, and that $\phi\circ u=\varphi$ in all (a,b); hence we will always write the following compact form

$$\phi(u(t))-\phi(u(s))\le -\frac{1}{2}\int_s^t\frac{1}{\lambda(\xi)}|u'|^2(\xi)d\xi-\frac{1}{2}\int_s^t\lambda(\xi)g(u(\xi))^2d\xi. \tag{9}$$

Note that, if $\lambda\equiv 1$, u is a curve of maximal slope for ϕ with respect to g, according to the classical definition given by Ambrosio, Gigli, and Savaré.

Applying Young's inequality to (9) and the definition of strong upper gradient we get

$$\phi(u(t))-\phi(u(s))=-\int_s^t\left(\frac{1}{2\lambda(\xi)}|u'|^2(\xi)+\frac{\lambda(\xi)}{2}g^2(u(\xi))\right)d\xi=-\int_s^t g(u(\xi))|u'|(\xi)d\xi$$

hence we have $(g\circ u)|u'|\in L^1_{loc}(a,b)$ and $\phi\circ u\in AC_{loc}(a,b)$. Furthermore, in Young's inequality the equal sign holds if and only if the terms are the same, so that every curve of maximal slope with rate λ satisfies the following metric gradient-flow;

$$\begin{cases}|u'|(t)=\lambda(t)g(u(t))\\ (\phi\circ u)'(t)=-|u'|(t)g(u(t)),\end{cases}\quad \text{for almost every } t\in(a,b).$$

We recall the definition of descending metric-slope (see *e.g.* Definition 1.2.4 [4]).

Definition 3.3. Let $\phi:S\to(-\infty,+\infty]$ be a proper functional. We define the *local slope* of ϕ at a point of its domain $u\in D(\phi)$ as

$$|\partial\phi|(u)=\limsup_{v\to u}\frac{\big(\phi(u)-\phi(v)\big)_+}{d(u,v)}.$$

We denote as $|\partial^-\phi|(u)$ the *relaxed slope* of ϕ (with respect to σ); *i.e.*, the lower σ-semicontinuous envelope of $|\partial\phi|$ in S.

As mentioned before, we will prove that perturbed minimizing movements are curves of maximal slope for ϕ with respect to $|\partial^-\phi|$ with a rate depending on the perturbations $\{a^\tau\}$.

3.1. **The Moreau-Yosida approximation scheme.** In order to obtain the energy estimate for a perturbed minimizing movement, we will prove that discrete solutions satisfy an energy estimate as well, and then taking the limit as $\tau\to 0$, obtain (9). In the following two sections, we introduce an approximation scheme, analogous to the one presented in Chapter 3 [4], to work with discrete solutions.

Definition 3.4. Let $\phi:S\to(-\infty,+\infty]$ be a proper functional. Fixed τ, for any $\delta\in(0,\tau]$ and $u\in S$, we define

$$\phi^\tau_{\delta,n}(u)=\inf_{v\in S}\left\{\phi(v)+a^\tau_n\frac{d^2(u,v)}{2\delta}\right\}$$

as the *n-th Moreau-Yosida approximation of* ϕ *at scale* τ for the scheme (2). Moreover, we denote $J^\tau_{\delta,n}(u)$ =argmin$_{v\in S}\{\phi(v)+a^\tau_n d^2(u,v)/2\delta\}$, and if $J^\tau_{\delta,n}(u)\neq\emptyset$, then we define

$$d^{\tau,+}_{\delta,n}(u)=\sup_{v\in J^\tau_{\delta,n}(u)} d(v,u),\quad d^{\tau,-}_{\delta,n}(u)=\inf_{v\in J^\tau_{\delta,n}(u)} d(v,u). \tag{10}$$

Remark 3. If ϕ satisfies assumptions H1-H3 then $\phi^\tau_{\delta,n}(u)>-\infty$ for any $u\in S$, therefore the set $J^\tau_{\delta,n}(u)$ is not empty.

It is know that Moreau-Yosida approximation has some monotonicity and continuity properties. Since $\phi^\tau_{\delta,n}(u)$ is equal to the classical one with $\tau=\delta/a^\tau_n$, we can apply Lemma 3.1.2 [4]. Hence, for all τ and $n\geq 1$, the map $(\delta,u)\mapsto\phi^\tau_{\delta,n}(u)$ is continuous, and for all $0<\delta_0<\delta_1\leq\tau$

$$\phi(u)\geq\phi^\tau_{\delta_0,n}(u)\geq\phi^\tau_{\delta_1,n}(u),\quad d(v_0,u)\leq d(v_1,u), \tag{11}$$

for all $v_0\in J^\tau_{\delta_0,n}(u)$ and $v_1\in J^\tau_{\delta_1,n}(u)$. In particular, if the functional ϕ satisfies hypotheses H1-H3, it holds

$$\begin{aligned}&\lim_{\delta\searrow 0}\phi^\tau_{\delta,n}(u)=\phi(u)\\ &\lim_{\delta\searrow 0}d^{\tau,+}_{\delta,n}(u)=0,\quad \text{if } u\in\overline{D(\phi)},\end{aligned} \tag{12}$$

and $d^\tau_{\delta,n}(u):=d^{\tau,+}_{\delta,n}(u)=d^{\tau,-}_{\delta,n}(u)$ for almost every $\delta\in(0,\tau]$, for all $n\geq 1$. Furthermore, as a direct application of Theorem 3.1.4 [4] to any Moreau-Yosida approximant, $\delta\mapsto\phi^\tau_{\delta,n}(u)$ are not only continuous but Lipschitz functions for $\delta\in(0,\tau)$, and

$$\frac{d\phi^\tau_{\delta,n}(u)}{d\delta}=-\frac{a^\tau_n}{2}\left(\frac{d^\tau_{\delta,n}(u)}{\delta}\right)^2,\quad\text{for almost every }\delta\in(0,\tau]. \tag{13}$$

We also have a slope estimate; applying Lemma 3.1.3 [4] with $\tau=\delta/a^\tau_n$ we get

$$|\partial\phi|(u^\tau_{\delta,n})\leq a^\tau_n\frac{d(u^\tau_{\delta,n},u)}{\delta}. \tag{14}$$

These properties will be very useful in the following.

3.2. **De Giorgi's interpolants.** To obtain the discrete energy estimate mentioned before, we use the De Giorgi's interpolation argument. Mimicking Definition 3.2.1 [4], we give the following notions.

Definition 3.5. Let u^τ be a discrete solution. Any curve $\tilde{u}^\tau:[0,+\infty)\to S$, which is an interpolation of the discrete values $\{u^\tau_n\}$ such that

$$\tilde{u}^\tau(t)=\tilde{u}((n-1)\tau+\delta)\in J^\tau_{\delta,n}(u^\tau_{n-1}),\quad\text{for } t=(n-1)\tau+\delta, \tag{15}$$

is called a De Giorgi's interpolant. We also define

$$G_\tau(t)=a^\tau_n\frac{d^\tau_{\delta,n}(u^\tau_{n-1})}{\tau},\quad\text{for } t=(n-1)\tau+\delta. \tag{16}$$

Proposition 5. *Let* ϕ *satisfy assumptions* H2 *and* H3*, let* $\{u^\tau_0\}$ *satisfy* H4*, and let* $\{a^\tau\}$ *be perturbations as in* (1) *satisfying* H5*. Suppose that there exist* u *a* (a^{τ_k})*-perturbed minimizing movement, and let* $(\tilde{u}^{\tau_k})$ *be a corresponding sequence of De Giorgi's interpolants. Then* $\tilde{u}^{\tau_k}$ *pointwise* σ*-converges to* u.

Proof. Fixed $t \in [0,T]$ with $t=(n-1)\tau+\delta$. Let θ_T be defined as in (6). By (15) and the second of formula (11) we get

$$d(\tilde{u}^\tau(t), u^\tau(t-\tau)) = d((u^\tau_{n-1})^\tau_{\delta,n}, u^\tau_{n-1}) \le d(u^\tau_n, u^\tau_{n-1}) \le \theta_T((n-1)\tau,(n+1)\tau) \quad (17)$$

hence Proposition 2 yields

$$d(\tilde{u}^\tau(t), u^*) \le d(\tilde{u}^\tau(t), u^\tau(t-\tau)) + d(u^\tau(t-\tau), u^*) \le 1 + C_T,$$

for τ small enough. Moreover, fixed $t,s \in [0,T]$, by Proposition 3 and (17) we obtain

$$\begin{aligned} d(\tilde{u}^\tau(t), \tilde{u}^\tau(s)) \le& d(\tilde{u}^\tau(t), u^\tau(t-\tau)) + d(u^\tau(t-\tau), u^\tau(s-\tau)) \\ &+ d(u^\tau(s-\tau), \tilde{u}^\tau(s)) \\ \le& \theta_T(t-\tau, t+\tau) + \theta_T(s-\tau, t) + \theta_T(s-\tau, s+\tau). \end{aligned}$$

Therefore we have proved that $(\tilde{u}^{\tau_k})$ satisfies hypotheses (i) and (ii) of Lemma 2.2. For any converging subsequence $(u^{\tau_{k'}})$, let v be its σ-pointwise limit, we get

$$\begin{aligned} d(v(t), u(t)) \le& \liminf_{k'\to+\infty} d(\tilde{u}^{\tau_{k'}}(t), u^{\tau_{k'}}(t)) \\ \le& \limsup_{k'\to+\infty} d(\tilde{u}^{\tau_{k'}}(t), u^{\tau_{k'}}(t-\tau_{k'})) + d(u^{\tau_{k'}}(t-\tau_{k'}), u^{\tau_{k'}}(t)) = 0, \end{aligned}$$

and the thesis follows. □

Lemma 3.6. *Let ϕ satisfy assumptions* H1-H3*, let $\{u^\tau_0\}$ satisfy* H4*, and let $\{a^\tau\}$ be perturbations as in* (1) *satisfying* H5*. Let u be a $\{a^\tau\}$-perturbed minimizing movement, and let G_τ be defined as in* (16)*. Then we have*

$$|\partial^-\phi|(u(t)) \le \liminf_{\tau\to 0} G_\tau(t), \quad \textit{for every } t \in [0,T].$$

Proof. Let $\{u^\tau\}$ be discrete solutions σ-converging (up to subsequences) to u, and let $\{\tilde{u}^\tau\}$ be the corresponding De Giorgi's interpolants. For any $t \ge 0$ we consider $n = \lceil t/\tau \rceil$. Taking $u = u^\tau_{n-1}$, recalling formula (15), by (14) we get

$$|\partial\phi|(\tilde{u}^\tau(t)) \le a^\tau_n \frac{d(\tilde{u}^\tau, u^\tau_{n-1})}{\delta} = G_\tau(t).$$

Applying Fatou's Lemma to the previous formula, by the σ-lower semicontinuity of $|\partial^-\phi|$ and Proposition (5) we get the thesis. □

3.3. **Perturbed minimizing movements are curves of maximal slope.** Using De Giorgi's interpolation scheme, we have the following a priori energy estimate for the discrete solutions.

Proposition 6. *Let ϕ satisfy assumptions* H1-H3*, let $\{a^\tau\}$ be perturbations as in* (1)*, and suppose that there exists $\{u^\tau\}$ a family of discrete solutions. For every $n \ge 1$ and τ we have*

$$\frac{1}{2}\int_0^{n\tau} a^\tau(\xi)|(u^\tau)'|^2(\xi)d\xi + \frac{1}{2}\int_0^{n\tau} \frac{1}{a^\tau(\xi)} G^2_\tau(\xi)d\xi = \phi(u^\tau_0) - \phi(u^\tau_n). \quad (18)$$

Proof. Integrating (13) on the interval (δ,τ) we get

$$\phi^\tau_{\delta,i}(u) - \phi^\tau_{\tau,i}(u) = -\int_\delta^\tau \frac{d\phi^\tau_{r,i}(u)}{dr}dr = \frac{1}{2}\int_\delta^\tau a^\tau_i \left(\frac{d^\tau_{r,i}(u)}{r}\right)^2 dr.$$

For all $v \in J^\tau_{\tau,i}(u)$, $\phi^\tau_{\tau,i}(u) = \phi(v) + a^\tau_i d^2(v,u)/2\tau$ which yields

$$\phi^\tau_{\delta,i}(u) - \phi(v) = \frac{1}{2}\int_\delta^\tau a^\tau_i \left(\frac{d^\tau_{r,i}(u)}{r}\right)^2 dr + a^\tau_i \frac{d^2(u,v)}{2\tau}. \quad (19)$$

Then, taking the limit for $\delta \searrow 0$ in (19), by the first of (12), for every $i \geq 1$ and $\tau \in (0, \tau^*)$, we have that

$$\begin{aligned}\phi(u) - \phi(v) =& a_i^\tau \frac{d^2(u,v)}{2\tau} + \frac{1}{2}\int_0^\tau a_i^\tau \left(\frac{d_{r,i}^\tau(u)}{r}\right)^2 dr\\ =& \frac{1}{2}\int_0^\tau a_i^\tau \left(\frac{d(u,v)}{\tau}\right)^2 dr + \frac{1}{2}\int_0^\tau \frac{1}{a_i^\tau}\left(a_i^\tau \frac{d_{r,i}^\tau(u)}{r}\right)^2 dr.\end{aligned}$$

Choosing $u = u_{i-1}^\tau$ and $v = u_i^\tau$ we get

$$\frac{1}{2}\int_0^\tau a_i^\tau |(u^\tau)'|^2((i-1)\tau + r)dr + \frac{1}{2}\int_0^\tau \frac{1}{a_i^\tau} G_\tau^2((i-1)\tau + r)dr = \phi(u_{i-1}^\tau) - \phi(u_i^\tau).$$

Taking the sum for all i from 1 to n we get the thesis. □

As in [4], the result that perturbed minimizing movements are curves of maximal slope with a given rate is obtained by taking the limit in the discrete energy estimate (18) as $\tau \to 0$. Nevertheless, the presence of the perturbation terms prevent from taking the limit directly. To work around this problem we need the two following results.

Lemma 3.7. *Let $\{a^\tau\}$ be a family of perturbations as in (1) satisfying hypothesis* H5*, and let $(1/a^{\tau_k})$ be a sequence L^1_{loc}-weakly converging to $1/a^*$. Then for every $t > 0$ we have*

$$\liminf_{k\to+\infty} \int_0^t \frac{1}{a^{\tau_k}(\xi)} G_{\tau_k}^2(\xi) d\xi \geq \int_0^t \frac{1}{a^*(\xi)} \liminf_{k\to+\infty} G_{\tau_k}^2(\xi) d\xi. \tag{20}$$

Proof. Given $\eta > 0$, let $g_\eta \in L^\infty(0,t)$ be such that $g_\eta(\xi) \leq \liminf_{k\to+\infty} G_{\tau_k}^2(\xi) - \eta$. We define $I_{n,\eta} := \{\xi \in [0,t] \,|\, g_\eta(\xi) \leq G_{\tau_k}^2(\xi), \forall k > n\}$. Then, we have that

$$\begin{aligned}\int_0^t \frac{1}{a^{\tau_k}(\xi)} G_{\tau_k}^2(\xi) d\xi =& \int_{I_{n,\eta}} \frac{1}{a^{\tau_k}(\xi)} G_{\tau_k}^2(\xi) + \int_{(0,t)\setminus I_{n,\eta}} \frac{1}{a^{\tau_k}(\xi)} G_{\tau_k}^2(\xi) d\xi\\ \geq& \int_{I_{n,\eta}} \frac{1}{a^{\tau_k}(\xi)} G_{\tau_k}^2(\xi) \geq \int_{I_{n,\eta}} \frac{1}{a^{\tau_k}(\xi)} g_\eta(\xi) d\xi,\end{aligned}$$

and taking the liminf we get

$$\liminf_{k\to+\infty} \int_0^t \frac{1}{a^{\tau_k}(\xi)} G_{\tau_k}^2(\xi) d\xi \geq \lim_{k\to+\infty} \int_{I_{n,\eta}} \frac{1}{a^{\tau_k}(\xi)} g_\eta(\xi) d\xi = \int_{I_{n,\eta}} \frac{1}{a^*(\xi)} g_\eta(\xi) d\xi.$$

Since $I_{n,\eta} \supset \{\xi \in [0,t] \,|\, \liminf_{h\to+\infty} G_{\tau_h}^2(\xi) - 1/n < G_{\tau_k}^2(\xi), k > n\}$ for n big enough, $\chi_{I_{n,\eta}}$ converges almost everywhere to $\chi_{[0,t]}$ as $n \to +\infty$, for any η; hence

$$\liminf_{k\to+\infty} \int_0^t \frac{1}{a^{\tau_k}(\xi)} G_{\tau_k}^2(\xi) d\xi \geq \lim_{n\to+\infty} \int_{I_{n,\eta}} \frac{1}{a^*(\xi)} g_\eta(\xi) d\xi = \int_0^t \frac{1}{a^*(\xi)} g_\eta(\xi) d\xi,$$

for every η and g_η as before. The result follows by monotone convergence. □

Lemma 3.8. *Let ϕ satisfy assumptions* H1-H3*, let $\{u_0^\tau\}$ satisfy* H4*, and let $\{a^\tau\}$ be perturbations as in (1) satisfying* H5*. Let (τ_k) be a sequence such that u is a (a^{τ_k})-perturbed minimizing movement, and $(1/a^{\tau_k})$ and $|(u^{\tau_k})'|$, defined as in (8),*

$L^1(0,t)$-weakly converge to $1/a^*$ and A respectively. Then there exists a subsequence $\{\tau_{k'}\} \subset \{\tau_k\}$ such that

$$\begin{aligned} &\liminf_{k'\to+\infty} \int_{E\cap(0,t)} a^{\tau_{k'}}(\xi)|(u^{\tau_{k'}})'|^2(\xi)d\xi \geq \int_{E\cap(0,t)} a^*(\xi)A^2(\xi)d\xi \\ &A(\xi)=0, \quad \textit{for almost every } \xi\in[0,t]\backslash E, \end{aligned} \tag{21}$$

where $E := \{t \in [0,+\infty) \,|\, a^*(t) < +\infty\}$.

Proof. First, we restrict to the bounded case; $a^\tau(t) \leq M$ for all $\tau \in (0,\tau^*)$, $t > 0$, so that $E = [0,+\infty)$. Given $\eta > 0$, we define the truncation argument

$$a^\tau_\eta(t) := a^\tau(t) \vee \eta, \quad f_{\eta,\tau}(t) := |(u^\tau)'|(t)\chi_{\{t\,|\,a^\tau(t)\geq\eta\}}(t).$$

Since $1/a^\tau_\eta \leq 1/a^\tau$, the sequence $(1/a^{\tau_k}_\eta)$ is uniformly integrable in $[0,t]$. Let $(1/a^{\tau_{k'}}_\eta)$ be a subsequence $L^1(0,t)$-weakly converging to $1/a^*_\eta$. We get $a^*_\eta \geq a^*$ almost everywhere. Moreover we have

$$\int_0^t a^\tau_\eta(\xi)\big(f_{\eta,\tau}(\xi)\big)^2 d\xi \leq \int_0^t a^\tau(\xi)|(u^\tau)'|^2(\xi)d\xi. \tag{22}$$

By (22) and the discrete energy estimate (18) we have $\int_0^t a^\tau_\eta(\xi)\big(f_{\eta,\tau}(\xi)\big)^2 d\xi \leq \phi(u^\tau_0) - \phi(u^\tau(t))$. As in proof of Proposition 3, there exists a constant C such that $\phi(u^\tau_0) - \phi(u^\tau(t)) \leq C$, so that $\int_0^t \big(f_{\eta,\tau}(\xi)\big)^2 d\xi \leq C/\eta$. Therefore $\{f_{\eta,\tau_{k'}}\}$ is equibounded in $L^2(0,t)$ for η fixed. Unless extracting a subsequence, $(f_{\eta,\tau_{k'}})$ weakly converges, and let A_η denote the limit.

We define $I_{\eta,\tau} := \{\xi \in (0,t) \,|\, a^\tau(\xi) < \eta\}$ and $N_{\eta,\tau} := \{1 \leq n < t/\tau \,|\, a^\tau_n < \eta\}$. By definition of the truncated function $f_{\eta,\tau}$ we get

$$\big\||(u^\tau)'| - f_{\eta,\tau}\big\|_{L^1(0,t)} = \sum_{n\in N_{\eta,\tau}} d(u^\tau_n, u^\tau_{n-1}) \leq C \sum_{n\in N_{\eta,\tau}} \sqrt{\frac{\tau}{a^\tau_n}} \leq C\bigg(\int_{I_{\eta,\tau}} \frac{1}{a^\tau(\xi)} d\xi\bigg)^{\frac{1}{2}}.$$

Since $\|1/a^\tau\|_{L^1(0,t)} \geq \int_{I_{\eta,\tau}} 1/a^\tau(\xi)d\xi > |I_{\eta,\tau}|/\eta$, we have that $|I_{\eta,\tau}| < C_{0,t}\eta$, where $C_{0,t}$ is defined in Remark 2. Hence, by the uniform integrability of $\{1/a^\tau\}$, Proposition 4 and the Banach-Steinhaus theorem

$$\|A - A_\eta\|_{L^1(0,t)} \leq \liminf_{k'\to+\infty} \big\||(u^{\tau_{k'}})'| - f_{\eta,\tau_{k'}}\big\|_{L^1(0,t)} < o_\eta(1),$$

which yields that (A_η) strongly converges in $L^1(0,t)$ to A as $\eta \to 0$.

Now we define the functional $F^\tau_\eta(v) = \int_0^t a^\tau_\eta(\xi)|v'(\xi)|^2 d\xi$ in $H^1(0,t)$. For every η, we have that $\Gamma(L^2(0,t))$-$\lim_{k'\to+\infty} F^{\tau_{k'}}_\eta(v) = \int_0^t a^*_\eta(\xi)|v'(\xi)|^2 d\xi$ (see Theorem 2.35 and Example 2.36 [5]). Since $f_{\eta,\tau_{k'}} \rightharpoonup A_\eta$ in $L^2(0,t)$, by the liminf inequality of Γ-convergence

$$\liminf_{k'\to+\infty} F^{\tau_{k'}}_\eta(f_{\eta,\tau_{k'}}) \geq \int_0^t a^*_\eta(\xi)\big(A_\eta(\xi)\big)^2 d\xi. \tag{23}$$

Since $a^*_\eta \geq a^*$, gathering (22) and (23) we get

$$\liminf_{k'\to+\infty} \int_0^t a^{\tau_{k'}}(\xi)|(u^{\tau_{k'}})'|^2(\xi)d\xi \geq \liminf_{k'\to+\infty} F^\tau_\eta(f_{\eta,\tau_{k'}}) \geq \int_0^t a^*(\xi)\big(A_\eta(\xi)\big)^2 d\xi.$$

In particular, passing to the limit as $\eta \to 0$, by Lebesgue's theorem we get

$$\liminf_{k'\to+\infty} \int_0^t a^{\tau_{k'}}(\xi)|(u^{\tau_{k'}})'|^2(\xi)d\xi \geq \int_0^t a^*(\xi)A^2(\xi)d\xi.$$

We now remove the assumption that the perturbations are bounded using a truncation argument. For every $M > 1$, consider $a^\tau_M(t) = a^\tau(t) \wedge M$. The family $\{1/a^{\tau_k}_M\}$ is locally uniformly integrable. Let $(1/a^{\tau_{k'}}_M)$ be a weakly converging subsequence and $1/a^*_M$ its weak limit. On the truncated functions we can use the result obtained before so that, unless extracting a subsequence,

$$\begin{aligned}\liminf_{k'\to+\infty}\int_0^t a^{\tau_{k'}}(\xi)|(u^{\tau_{k'}})'|^2(\xi)d\xi &\geq \liminf_{k'\to+\infty}\int_0^t a^{\tau_{k'}}_M(\xi)|(u^{\tau_{k'}})'|^2(\xi)d\xi \\ &\geq \int_0^t a^*_M(\xi)A^2(\xi)d\xi.\end{aligned} \tag{24}$$

For every $T > 0$, we have

$$\begin{aligned}0 \leq \int_0^T \left(\frac{1}{a^\tau_M(\xi)} - \frac{1}{a^\tau(\xi)}\right)d\xi &= \int_{\{\xi | a^\tau(\xi)>M\}} \left(\frac{1}{a^\tau_M(\xi)} - \frac{1}{a^\tau(\xi)}\right)d\xi \\ &\leq \int_{\{\xi | a^\tau(\xi)>M\}} \frac{1}{M}d\xi \leq \frac{T}{M}.\end{aligned}$$

Hence the corresponding inequality holds for the weak limits as well;

$$0 \leq \int_0^T \left(\frac{1}{a^*_M(\xi)} - \frac{1}{a^*(\xi)}\right)d\xi \leq \frac{T}{M}.$$

Therefore $1/a^*_M$ converges strongly in $L^1(0,T)$ and almost everywhere to $1/a^*$ as $M \to +\infty$. Taking the limit as $M \to +\infty$ in (24)

$$\liminf_{k'\to+\infty}\int_{E\cap(0,t)} a^{\tau_{k'}}(\xi)|(u^{\tau_{k'}})'|^2(\xi)d\xi \geq \int_{E\cap(0,t)} a^*(\xi)A^2(\xi)d\xi.$$

Now we denote $\beta_{\alpha,M} := a^\tau_M \wedge \alpha$ with fixed $\alpha > 1$. Since $a^*_M(\xi)$ diverges for almost every $\xi \in [0,t]\backslash E$, we have that $\beta_{\alpha,M}$ converges almost everywhere to α on $[0,t]\backslash E$, as $M \to +\infty$. Then, for any $\alpha > 1$

$$\int_{[0,t]\backslash E} \beta_{\alpha,M}(\xi)A^2(\xi)d\xi \leq \int_{[0,t]\backslash E} a^*_M(\xi)A^2(\xi)d\xi \leq C,$$

and taking the limit as $M \to +\infty$, by Lebesgue's theorem we get

$$\int_{[0,t]\backslash E} A^2(\xi)d\xi \leq \frac{C}{\alpha}.$$

By the arbitrariness of α, $A = 0$ almost everywhere on $[0,t]\backslash E$. □

Theorem 3.9 (**Perturbed minimizing movements are curves of maximal slope**). *Let ϕ satisfy assumptions* H1-H3*, let $\{u^\tau_0\}$ be a family of initial data satisfying* H4*, let $\{a^\tau\}$ be perturbations as in* (1) *satisfying* H5*, and let $(1/a^{\tau_k})$ be a sequence L^1_{loc}-weakly converging to $1/a^*$. If the relaxed slope $|\partial^-\phi|$ is a strong upper gradient, and the following compatibility condition holds*

$$\lim_{k\to+\infty}\phi(u^{\tau_k}_0) = \phi(u(0)),$$

then every (a^{τ_k})-perturbed minimizing movement of the scheme (2) *is a curve of maximal slope for ϕ with respect to $|\partial^-\phi|$ with rate $1/a^*$.*

Proof. The existence of a (a^{τ_k})-perturbed minimizing movement is provided by Theorem 2.4. By the monotonicity of $\phi(u^\tau)$, the σ-lower semicontinuity of ϕ, and Helly's lemma we get $\lim_{k\to+\infty}\phi(u^{\tau_k}) \geq \phi(u(t))$. Let $(\tau_{k'})$ be a subsequence of (τ_k)

such that Proposition 4, Lemma 3.7 and 3.8 hold. Then by Lemma 3.7 and 3.8, we have that

$$\begin{aligned}\phi(u(0)) &= \lim_{k'\to+\infty}\phi(u_0^{\tau_{k'}})\\ &\geq \liminf_{k'\to+\infty}\frac{1}{2}\int_0^{\lfloor\frac{t}{\tau_{k'}}\rfloor}\left(a^{\tau_{k'}}(\xi)|(u^{\tau_{k'}})'|^2(\xi)+\frac{1}{a^{\tau_{k'}}(\xi)}G^2_{\tau_{k'}}(\xi)\right)d\xi+\phi(u^{\tau_{k'}}(t))\\ &\geq \frac{1}{2}\int_0^t\left(a^*(\xi)A^2(\xi)d\xi+\frac{1}{a^*(\xi)}\liminf_{k'\to+\infty}G^2_{\tau_{k'}}(\xi)\right)d\xi+\phi(u(t))\\ &\geq \frac{1}{2}\int_0^t\left(a^*(\xi)|u'|^2(\xi)+\frac{1}{a^*(\xi)}|\partial^-\phi|^2(u(\xi))\right)d\xi+\phi(u(t)).\end{aligned}$$

By Definition 3.2 we get the thesis. □

Remark 4. The case in which the perturbations $\{a^\tau\}$ defined in (1) have inverses that are globally uniformly integrable, and $\int_0^\infty 1/a^\tau(t)dt = +\infty$, can be studied directly applying the method of Ambrosio, Gigli, and Savaré.

In [4] a sequence of positive coefficients (τ_n), of amplitude $|\tau| := \sup_n \tau_n < +\infty$, is used as a time-discretization scale, provided that $\sum_n \tau_n = +\infty$. A sequence (U_n^τ) which solves

$$U_n^\tau \in \operatorname*{argmin}_{u\in S}\left\{\phi(u)+\frac{d^2(u,U_{n-1}^\tau)}{2\tau_n}\right\}$$

is called a discrete solution, starting from an initial datum $U_0^\tau \in D(\phi)$. We will refer to this as a *classical* discrete solution, to distinguish it from the perturbed one. If we consider $\tau_n = \tau/a_n^\tau$, the assumptions on (τ_n) are satisfied. By the change of parameter $\varphi_\tau(t) = \int_0^t 1/a^\tau(\xi)d\xi$ we can pass from a classical discrete solution to a discrete solution u^τ of the scheme (2) defined as $u^\tau(t) = U^\tau(\varphi_\tau(t))$, for every $t \geq 0$. In [4] it is proved that, if assumptions H1-H4 hold, discrete solutions pointwise σ-converge (up to subsequences) to a classical minimizing movement U as $|\tau| \to 0$. Moreover, it is absolutely continuous and satisfies the energy estimate

$$\frac{1}{2}\int_0^s |U'|^2(\xi)d\xi + \frac{1}{2}\int_0^s |\partial^-\phi|^2(U(\xi))d\xi = \phi(U(0)) - \phi(U(s))$$

for every $s \geq 0$, provided that the relaxed slope is a strong upper gradient; hence U is a curve of maximal slope for ϕ with respect to $|\partial^-\phi|$.

By the uniform integrability of the perturbations, the family $\{\varphi_\tau\}$ is equicontinuous, so (up to subsequences) it uniformly converges to a limit $\varphi(t) = \int_0^t 1/a^*(\xi)d\xi$. This proves the existence of an absolutely continuous perturbed minimizing movement $u = U\circ\varphi$, and its metric derivative satisfies $|u'|(t) = |U'|(\varphi(t))/a^*(t)$. Now, changing variable in the energy estimate with $\xi = \varphi(\zeta)$, we have that

$$\begin{aligned}\phi(U(0)) - \phi(U(\varphi(s))) =& \frac{1}{2}\int_0^{\varphi(s)} |U'|^2(\varphi(\zeta))\frac{1}{a^*(\zeta)}d\zeta\\ &+\frac{1}{2}\int_0^{\varphi(s)} |\partial^-\phi|^2(U(\varphi(\zeta)))\frac{1}{a^*(\zeta)}d\zeta\\ \phi(u(0)) - \phi(u(t)) =& \frac{1}{2}\int_0^t a^*(\zeta)|u'|^2(\zeta)d\zeta + \frac{1}{2}\int_0^t \frac{1}{a^*(\zeta)}|\partial^-\phi|^2(u(\zeta))d\zeta\end{aligned}$$

and obtain the result of Theorem 3.9; *i.e.*, perturbed minimizing movements are curves of maximal slope for ϕ with respect to $|\partial^-\phi|$ with rate $1/a^*$.

Now, we present three examples of perturbed minimizing movements in order to show the effects of the perturbations in well known frameworks.

Example 3.10. We consider $S = \mathbb{R}$. Let $\phi(t) = t^2$ be the energy functional, and let $\{u_0^\tau\}$ be a family of initial data converging to u_0 as $\tau \to 0$. We consider perturbations oscillating between two positive parameters $0 < \alpha \leq \beta$

$$a_n^\tau = \begin{cases} \alpha & n \text{ odd} \\ \beta & n \text{ even.} \end{cases}$$

The family $\{1/a^\tau\}$ weakly* converges in $L^\infty(0, +\infty)$ to its average; *i.e*, $1/a^* = (\alpha^{-1} + \beta^{-1})/2$, that is the inverse of the harmonic mean between α and β. All the hypotheses of Theorem 3.9 are satisfied, therefore there exists a gradient flow. It is the solution of $u' = -2u/a^*$ starting from u_0, that is $u(t) = u_0 e^{-2t/a^*}$. Discrete solutions u^τ are pictured in Figure 1.

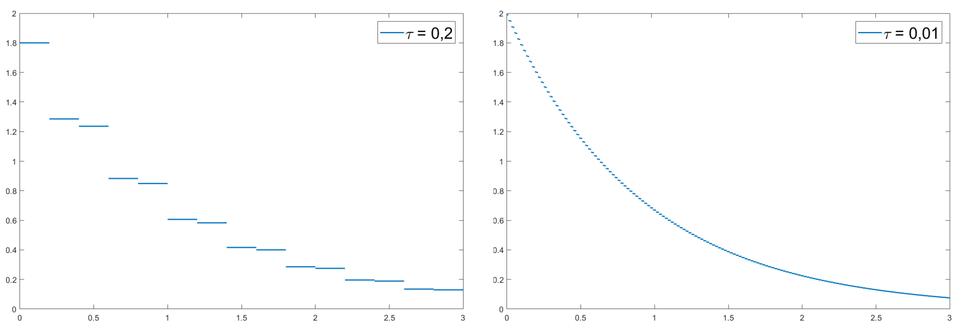

FIGURE 1. Graphs of the discrete solutions with different values of τ. The smaller jumps of u^τ correspond to the larger parameter β.

Note that we may consider divergent coefficients a_n^τ. They may produce a constant motion as in the following example.

Example 3.11 (**Pinning of the motion).** If we modify the previous perturbations considering $a_n^\tau = 1/\tau$, for every $\lceil 1/\tau \rceil < n \leq \lceil 2/\tau \rceil$, they produce a partially *pinned motion*; *i.e*, $u' \equiv 0$ (see Figure 2). In fact, Theorem 3.9 still holds, and according

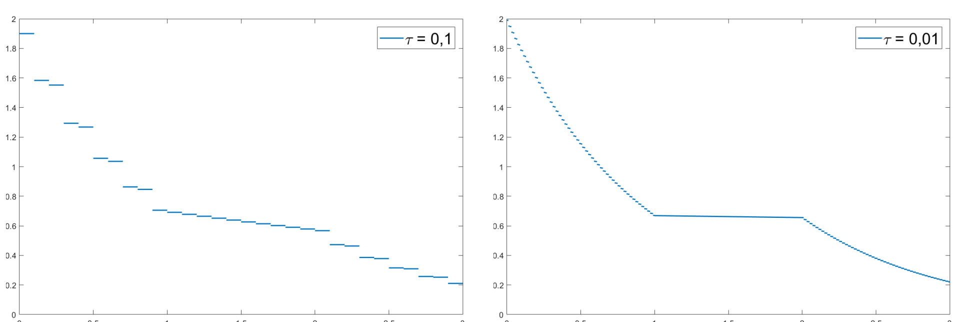

FIGURE 2. Pinned motion produced by perturbations diverging in the interval $(1, 2)$.

to Definition 3.2, the minimizing movement follows the perturbed gradient flow. In this case, in the interval $(1,2)$, the perturbations $1/a^\tau(t) = \tau$ converge to zero, or equivalently $1/a^*(t) = 0$.

Example 3.12 (**A perturbed heat equation**)**.** Given $\Omega \subset \mathbb{R}^d$ a regular domain, with $d \geq 2$, we consider $(S,d) = (L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$, and σ as the strong L^2-topology. We consider the Dirichlet energy functional

$$\phi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx & \text{if } u \in H^1_0(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

and an equibounded family of initial data $\{u^\tau_0\} \subset H^1_0(\Omega)$, so that H4 holds. The functional ϕ is strongly lower semicontinuous. Since $\phi > 0$, it satisfies H2. Moreover, by Sobolev's immersion, bounded sets of bounded energy are strongly precompact, which implies that hypothesis H3 holds.

It is known that the Fréchet subdifferential of the Dirichlet energy is minus the laplacian (see *e.g.* Theorem 4 in Subsection 9.6.3 [9]), and that, for convex and lower semicontinuous proper functionals on Hilbert spaces, the slope $|\partial\phi|(u) = \min\{\|\xi\| \,|\, \xi \in \partial\phi(u)\}$ is lower semicontinuous and a strong upper gradient (see Proposition 1.4.4 [4]). Hence $|\partial^-\phi|(u) = \|\Delta u\|_{L^2(\Omega)}$.

Let $\{a^\tau\}$ be any family of perturbations satisfying H5, and let $1/a^*$ be a weak limit. Recalling that, in Banach spaces, the metric derivative of an absolutely continuous curve is the norm of its derivative, then by Theorem 3.9 we get the perturbed gradient flow

$$\begin{cases} \|u'(t)\|_{L^2(\Omega)} = \frac{1}{a^*(t)}\|\Delta u(t)\|_{L^2(\Omega)} \\ (\phi \circ u)'(t) = u'(t)\Delta u(t), \end{cases} \qquad \text{for almost every } t \geq 0,$$

starting from $u_0 \in H^1_0(\Omega)$, which solves

$$\begin{cases} \partial_t u(t,x) = -\frac{1}{a^*(t)}\Delta u(t,x) & x \in \Omega \\ u_{|\partial\Omega} \equiv 0 \\ u(0) = u_0 \in H^1_0(\Omega), \end{cases} \qquad \text{for almost every } t > 0$$

in the distributional sense. In this case, the perturbation term $1/a^*$ takes the place of the thermal diffusivity coefficient in the classical heat equation. Nevertheless while the thermal diffusivity is a constant, $1/a^*$ changes in time.

4. **Relaxing the condition on the perturbations.** Hypothesis H5 plays a crucial role for the equicontinuity of the discrete solutions, and hence for their convergence to an absolutely continuous perturbed minimizing movement. Considering perturbations that do not satisfy it could create a lack of convergence or continuity, as it is shown by the next two examples in $\mathbb{R}$.

Example 4.1 (**Lack of convergence**)**.** We consider the functional $\phi(t) = -t$, and any bounded family of initial values. For the sake of simplicity we consider $u^\tau_0 \equiv 0$, so that assumptions H1-H4 hold.

We consider $a^\tau_n = 1/n$ for all $n \geq 1$. Hence, the family of $1/a^\tau(t) = \lceil t/\tau \rceil$ is not equibounded in $L^1(0,T)$, therefore it does not satisfy assumption H5. By the minimization scheme (2), we have that $u^\tau_n - u^\tau_{n-1} = n\tau$, and then

$$u^\tau_n = \tau \sum_{i=1}^n i = \tau \frac{n(n+1)}{2}.$$

The discrete solutions are $u^\tau(t) = \tau\lceil t/\tau\rceil(\lceil t/\tau\rceil + 1)/2$, and they diverge as $\tau \to 0$ for all $t > 0$.

Example 4.2 (**Lack of continuity**). We consider the functional $\phi(t) = t^2/2$, and initial data u_0^τ converging to $u_0 \neq 0$, otherwise we have a trivial motion because 0 is the global minimum of the energy. We consider the following perturbations

$$a^\tau(t) = \begin{cases} \tau & t \in I_\tau \\ 1 & \text{otherwise,} \end{cases} \qquad \text{where } I_\tau = \bigcup_{k\geq 1}\left(\left(\left\lceil \frac{k}{\tau}\right\rceil - 1\right)\tau, \left\lceil \frac{k}{\tau}\right\rceil \tau\right]. \tag{25}$$

For such perturbations, assumption H5 is not satisfied. In fact, taking $E_\tau = I_\tau \cap (0, 1]$, whose Lebesgue measures go to zero, we have that $\int_{E_\tau} 1/a^\tau(t)dt = 1$ for every τ, so that the uniform integrability is not satisfied.

In this case, the n-th step of discrete solutions of the scheme (2) is equal to

$$u_n^\tau = \frac{a_n^\tau}{\tau + a_n^\tau} u_{n-1}^\tau = \begin{cases} \frac{1}{2} u_{n-1}^\tau & \text{if } n = \lceil \frac{k}{\tau} \rceil, \text{ with } k \geq 1 \\ \frac{1}{1+\tau} u_{n-1}^\tau & \text{otherwise,} \end{cases}$$

and the discrete solutions are

$$u^\tau(t) = u_0^\tau 2^{-\lfloor t \rfloor}\left(\frac{1}{1+\tau}\right)^{\lceil \frac{t}{\tau}\rceil - \lfloor t \rfloor}.$$

Even if H5 does not hold, we still have the convergence of (u^τ). In this case we lose the continuity of the limit motion. In fact, taking the limit as $\tau \to 0$, we obtain the perturbed minimizing movement $u(t) = u_0 2^{-\lfloor t \rfloor} e^{-t}$, which is a piecewise absolutely continuous curve.

Remark 5. Note that Theorem 3.9 can be applied even if some coefficients a_n^τ tend to zero. As an example, slightly modifying the previous perturbations as

$$a^\tau(t) = \begin{cases} \tau^\alpha & t \in I_\tau \\ 1 & \text{otherwise,} \end{cases} \qquad \text{with } \alpha \in (0, 1), \tag{26}$$

assumption H5 is satisfied, because $\int_E 1/a^\tau(t)dt \leq |E|\tau^{1-\alpha}$ for every measurable set E. Hence, $(1/a^\tau)$ is locally uniformly integrable, and weakly converges to 1. Therefore, we can apply Theorem 3.9. Indeed

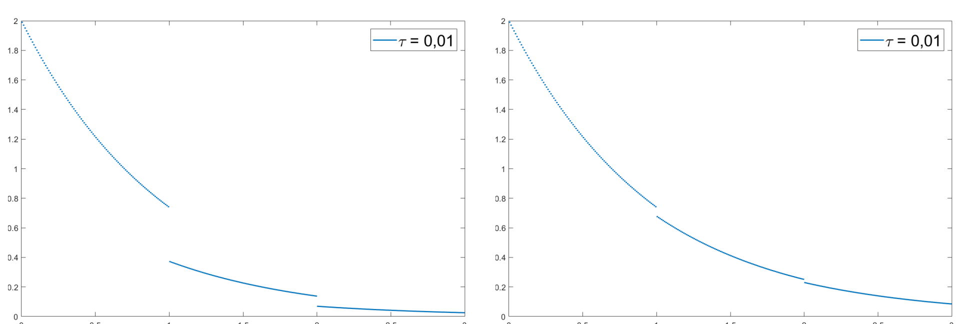

FIGURE 3. The graphs represent two discrete solutions for the same value of τ, corresponding respectively to perturbations as in (25) and (26). Note the discontinuous behavior on the left, while on the right jumps are going to disappear.

$$u_n^\tau = \begin{cases} \frac{1}{1+\tau^{1-\alpha}} u_{n-1}^\tau & \text{if } n = \lceil \frac{k}{\tau} \rceil, \text{ with } 1 \le k \\ \frac{1}{1+\tau} u_{n-1}^\tau & \text{otherwise,} \end{cases}$$

so the discrete solutions are

$$u^\tau(t) = u_0^\tau \left(\frac{1}{1+\tau^{1-\alpha}} \right)^{\lfloor t \rfloor} \left(\frac{1}{1+\tau} \right)^{\lceil \frac{t}{\tau} \rceil - \lfloor t \rfloor}.$$

Taking the limit as $\tau \to 0$ we get $u(t) = u_0 e^{-t}$.

The previous example suggests that, renouncing to the continuity of the perturbed minimizing movements, hypothesis H5 could be replaced by a relaxed one, which however ensures the convergence of the discrete solutions. Now, we consider the following assumption on $\{a^\tau\}$;

H5$'$ there exists a set of isolated points $I = \{t_j\} \subset [0, +\infty)$, and a family $\{I_\tau\}$ of sets unions of intervals with endpoints in $\tau\mathbb{Z}$, pointwise converging to I as $\tau \to 0$, such that $\{1/a^\tau \chi_{[0,+\infty)\setminus I_\tau}\}$ is locally uniformly integrable in $[0, +\infty)$, and $\{1/a^\tau\}$ is locally equibounded in the L^1-norm.

By substituting H5 with H5$'$, we are considering more general perturbations, which can violate the local uniform integrability in some intervals of the time discretization that accumulate around some isolated points, as for instance (a_n^τ) defined in (25).

Note that we cannot renounce to the local L^1-equiboundedness which is crucial in the proof of Proposition 2, as Example 4.1 shows.

Theorem 4.3. *Let ϕ satisfy assumptions* H1-H3*, let $\{u_0^\tau\}$ satisfy* H4*, and let $\{a^\tau\}$ be perturbations as in (1) satisfying hypothesis* H5$'$*. Then there exists a piecewise absolutely continuous $\{a^\tau\}$-perturbed minimizing movement for the scheme (2).*

Moreover, let u be a (a^{τ_k})-perturbed minimizing movement, if $|\partial^-\phi|$ is a strong upper gradient and the following compatibility conditions hold

$$\lim_{k \to +\infty} \phi(u_0^{\tau_k}) = \phi(u_0), \quad \lim_{k \to +\infty} \phi(u^{\tau_k}) = \phi(u(t_j^+)), \quad \textit{for every } t_j \in I,$$

then u is a curve of maximal slope for ϕ with respect to $|\partial^-\phi|$ with rate $1/a^*$, in (t_j, t_{j+1}), starting from $u(t_j^+)$, for every $t_j \in I$, and in $(0, t_1)$ starting from u_0.

Proof. Fixed $T > 0$, by the L^1-equiboundedness of $\{1/a^\tau\}$, we can apply Proposition 2 and 3, for all $t \in [0, T]$. By H5$'$, θ_T defined in (6) is such that $\theta_T(s,t)$ tends to zero as $s, t \to r$ for any $r \in [0, T] \setminus I$. Hence, applying Lemma 2.2 to $\{u^\tau\}$, and a diagonal argument, as in proof of Theorem 2.4, there exists a limit curve u continuous in $[0, T] \setminus I$. Moreover

$$\begin{aligned} d(u(t_j^-), u(t_j^+)) &\le \liminf_{k \to +\infty} d(u^{\tau_{k,j}}(t_j - \tau_{k,j}), u^{\tau_{k,j}}(t_j + \tau_{k,j})) \\ &\le \liminf_{k \to +\infty} d(u^{\tau_{k,j}}(t_j - \tau_{k,j}), u^*) + d(u^*, u^{\tau_{k,j}}(t_j + \tau_{k,j})) \le 2C_T; \end{aligned}$$

hence the jumps are finite, and the result follows. Then let us define $v_j^\tau(t) := u^\tau(t - t_j)$ for every $t_j \in I$. For any v_j^τ we can apply Theorem 3.9 in $(0, t_{j+1} - t_j)$, and we get the thesis. ☐

Remark 6. In Theorem 4.3, it is not specified what happens in the points $t_j \in I$. This because the convergence of $u^\tau(t_j)$ depends on the convergence of I_τ to I. More precisely, in Example 4.2 the $\{a^\tau\}$-perturbed minimizing movement corresponding to the perturbations defined as in (25) is continuous from the right; *i.e.*,

$u^\tau(t_j) \to u(t_j^+)$. Nevertheless, if we consider $\{a^\tau\}$ corresponding to $I_\tau + \tau$, which still converges to I, the discrete solutions would be

$$u^\tau(t) = u_0^\tau 2^{\lfloor t-\tau \rfloor} \left(\frac{1}{1+\tau}\right)^{\lceil \frac{t-\tau}{\tau} \rceil - \lfloor t-\tau \rfloor},$$

so $u^\tau(t_j) \to u(t_j^-)$, and u is continuous from the left.

In the following, we will not be interested in the convergence on the points of I, so for the sake of simplicity we will deal with a specific kind of perturbations satisfying assumption H5', as follows.

Let δ be a positive real function defined on $(0, \tau^*)$. Let $\{a^\tau\}$ be a family of perturbations as in (1) such that

$$a^\tau(t) = \begin{cases} \delta(\tau) & \text{if } t \in I_\tau \\ 1 & \text{otherwise,} \end{cases} \qquad I_\tau := \bigcup_{t_j \in I} \left(\left(\left\lceil \frac{t_j}{\tau} \right\rceil - 1 \right) \tau, \left\lceil \frac{t_i}{\tau} \right\rceil \tau \right]. \tag{27}$$

In order to characterize δ such that $\{a^\tau\}$ satisfy H5', we first check the equiboundedness in L^1. We have that $\|1/a^\tau\|_{L^1(0,T)} = \#\{I \cap [0,T]\}\tau/\delta(\tau) + |[0,T] \setminus I_\tau|$. Therefore, to be bounded it should be $\delta(\tau) \geq O(\tau)$ as $\tau \to 0$. Since if $\delta(\tau) > O(\tau)$ the perturbations satisfy assumption H5, as can be seen in Remark 6, the interesting case is $\delta(\tau) = O(\tau)$. For simplicity, we will deal with $\delta(\tau) = \delta\tau$, with $\delta > 0$.

We can generalize these perturbations, considering a bounded sequence of positive coefficients (δ_j) such that $a^\tau(t_j) = \delta_j \tau$, for every $t_j \in I$, and perturbations $\{b^\tau\}$ satisfying H5 such that $a^\tau(t) = b^\tau(t)$, for every $t \notin I_\tau$.

Perturbations as in (27) could generate a discontinuous gradient flow as in Example 4.2, or non-trivial perturbed minimizing movements in cases in which classical minimizing movements are only the constant motion. Hence, perturbed minimizing movements can be used to obtain motion from a setting which does not allow it for classical minimizing movements, as we will see in the next section.

4.1. **Exploring lower energy states.** In the next three examples, we show how perturbed minimizing movements can be used to obtain a gradient-flow type motion for different multi-wells energy functionals. All the examples are set in the real line.

Example 4.4. We consider the functional

$$\phi(t) = \begin{cases} -t & t \in \mathbb{Z} \\ +\infty & \text{otherwise,} \end{cases}$$

and initial data $u_0^\tau \equiv 0$. Assumptions H1-H4 hold. If we consider perturbations satisfying assumption H5, by Theorem 2.4, we obtain continuous perturbed minimizing movements, but the only continuous curve on a discrete set (the domain of the energy is $\mathbb{Z}$) is the constant curve. Therefore every continuous perturbed minimizing movement for this problem, in particular the classical minimizing movement, is the trivial motion.

Given any $T, \delta > 0$, we consider a^τ as in (27) satisfying H5' with $I = T\mathbb{N}$ and $\delta(\tau) = \delta\tau$. First, note that the minimum of the function $t \mapsto \phi(t) + a_n^\tau(t - u_{n-1}^\tau)^2/2\tau$ is the integer nearest to the minimum of the continuous function $t \mapsto -t + a_n^\tau(t - u_{n-1}^\tau)^2/2\tau$. Let t_n^τ be such a minimum, we have that $t_n^\tau = u_{n-1}^\tau + \tau/a_n^\tau$. Hence the n-th step of the discrete solution of the scheme (2) is

$$u_n^\tau = \left\lfloor t_n^\tau + \frac{1}{2} \right\rfloor = u_{n-1}^\tau + \left\lfloor \frac{\tau}{a_n^\tau} + \frac{1}{2} \right\rfloor.$$

Considering $\tau < 1/2$, for any index n such that $a_n^\tau = 1$, we get $u_{n-1}^\tau = u_n^\tau$. Whereas, for indices n such that $a_n^\tau = \delta\tau$, we have that: if $1/\delta + 1/2 < 1$ then $u_n^\tau = u_{n-1}^\tau$; if $1/\delta + 1/2 > 1$ then $u_n^\tau > u_{n-1}^\tau$.

The case in which $1/\delta + 1/2 \in \mathbb{N}$ is a special case because t_n^τ is a half-integer, so the two closest integers are equidistant from t_n^τ. Hence $u_n^\tau \in u_{n-1}^\tau + 1/\delta + \{\pm 1/2\}$, and we say that there is a *bifurcation* of the motion.

For the sake of simplicity, we ignore the bifurcation phenomenon (which could be treated separately); hence we consider δ such that $1/\delta \notin \mathbb{N} + 1/2$. We define $N := \lfloor 1/\delta + 1/2 \rfloor$. If $n = \lceil Tk/\tau \rceil$ for some integer k, we get $u_n^\tau = u_{n-1}^\tau + N$, otherwise $u_n^\tau = u_{n-1}^\tau$. Hence the discrete solution is

$$u^\tau(t) = N \left\lfloor \left\lceil \frac{t}{\tau} \right\rceil \left\lceil \frac{T}{\tau} \right\rceil^{-1} \right\rfloor.$$

Taking the limit as $\tau \to 0$, we obtain the perturbed minimizing movement $u(t) = N\lfloor t/T \rfloor$, which is not the trivial motion when $\delta < 2$.

Example 4.5. Now, we consider the piecewise quadratic energy functional

$$\phi(t) = \begin{cases} t^2 & \text{if } t \le 0 \\ (t-1)^2 - 1 & \text{if } t > 0, \end{cases}$$

a family of initial data $\{u_0^\tau\}$ converging to $u_0 < 0$, and perturbations as in (27), satisfying assumption H5$'$, with $I = \mathbb{N}$ and $\delta(\tau) = \delta\tau$. If perturbations satisfy H5, any perturbed minimizing movement u will follow the perturbed gradient-flow of t^2, in particular $u(t) < 0$ for all $t \ge 0$; *i.e.*, it cannot exit the energy well. When H5$'$ holds, it is not immediately clear what happens. Therefore, we study the minimum of the function $t \mapsto \phi(t) + \bar{\delta}(t - \bar{u})^2/2$ depending on $\bar{\delta} > 0$ and $\bar{u}$. This minimum is the n-th step of the discrete solution, provided that $\bar{u} = u_{n-1}^\tau$ and $\bar{\delta} = \tau/a_n^\tau$. We study the minima separately in $t \le 0$ and $t > 0$, and then we compare them. We denote as $\phi_0(\bar{\delta}, \bar{u})$ and $\phi_1(\bar{\delta}, \bar{u})$ the minimum respectively in $t \le 0$ and $t > 0$, and $t_0(\bar{\delta}, \bar{u})$ and $t_1(\bar{\delta}, \bar{u})$ the corresponding minimizers. By computations we get

$$\phi_0(\bar{\delta}, \bar{u}) = \begin{cases} \frac{\bar{\delta}\bar{u}^2}{2+\bar{\delta}} & \text{if } \bar{u} < 0 \\ \frac{\bar{\delta}\bar{u}^2}{2} & \text{if } \bar{u} \ge 0, \end{cases} \qquad t_0(\bar{\delta}, \bar{u}) = \begin{cases} \frac{\bar{u}\bar{\delta}}{2+\bar{\delta}} & \text{if } \bar{u} < 0 \\ 0 & \text{if } \bar{u} \ge 0 \end{cases}$$

$$\phi_1(\bar{\delta}, \bar{u}) = \begin{cases} \frac{\bar{\delta}\bar{u}^2 - 2\bar{\delta}\bar{u} - 2}{2+\bar{\delta}} & \text{if } \bar{u} > -\frac{2}{\bar{\delta}} \\ \frac{\bar{\delta}\bar{u}^2}{2} & \text{if } \bar{u} \le -\frac{2}{\bar{\delta}}, \end{cases} \qquad t_1(\bar{\delta}, \bar{u}) = \begin{cases} \frac{2+\bar{u}\bar{\delta}}{2+\bar{\delta}} & \text{if } \bar{u} > -\frac{2}{\bar{\delta}} \\ 0 & \text{if } \bar{u} \le -\frac{2}{\bar{\delta}}. \end{cases}$$

If $\bar{u} \le -2/\bar{\delta}$, then $\phi_0(\bar{\delta}, \bar{u}) \le \phi_1(\bar{\delta}, \bar{u})$; if $\bar{u} \ge 0$, then $\phi_1(\bar{\delta}, \bar{u}) \le \phi_0(\bar{\delta}, \bar{u})$; while, if $-2\bar{\delta} < \bar{u} < 0$, we have that

$$\phi_0(\bar{\delta}, \bar{u}) \le \phi_1(\bar{\delta}, \bar{u}) \quad \iff \quad \frac{\bar{\delta}\bar{u}^2}{2+\bar{\delta}} \le \frac{\bar{\delta}\bar{u}^2 - 2\bar{\delta}\bar{u} - 2}{2+\bar{\delta}} \quad \iff \quad \bar{u} \le -\frac{1}{\bar{\delta}}.$$

Hence, if $\bar{u} \le -1/\bar{\delta}$, the minimum of the map $t \mapsto \phi(t) + \bar{\delta}(t - \bar{u})^2/2$ is $\bar{\delta}\bar{u}^2/(2 + \bar{\delta})$, is achieved in $\bar{\delta}\bar{u}/(2 + \bar{\delta})$. If $\bar{u} > -1/\bar{\delta}$, it is $(\bar{\delta}\bar{u}^2 - 2\bar{\delta}\bar{u} - 2)/(2 + \bar{\delta})$, is achieved in $(2 + \bar{u}\bar{\delta})/(2 + \bar{\delta})$.

Fixed $T > 0$ sufficiently large. Let u^τ be a discrete solution. By the previous analysis, the discrete solution passes to the lower energy state; *i.e.*, $u_n^\tau > 0$, if and only if $u_{n-1}^\tau > -\tau/a_n^\tau$. When $a_n^\tau = 1$, this condition is not satisfied for any n such that $\lceil n/\tau \rceil < T$ when τ is small enough, while when $a_n^\tau = \delta\tau$, it corresponds to the condition

$$u_{n-1}^\tau > -1/\delta. \tag{28}$$

As an example we may take $u_0^\tau \equiv -1$. If $t \in [0, (\lceil 1/\tau \rceil - 1)\tau]$, the motion follows the gradient flow of t^2, so $u^\tau(t) = -e^{-2t} + o_\tau(1)$. When $n = \lceil 1/\tau \rceil$, $a_n^\tau = \delta\tau$ and $u_{n-1}^\tau = -e^{-2} + o_\tau(1)$. Therefore, by condition (28), if $\delta < e^2$ the motion passes to the lower energy state; *i.e.*, $u_n^\tau > 0$ (see Figure 4). Otherwise, it follows the perturbed gradient-flow of t^2 remaining confined in the well; *i.e.*, $u_n^\tau < 0$.

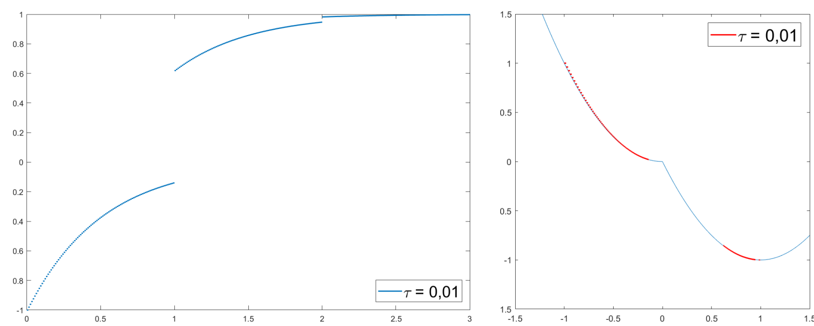

FIGURE 4. On the left the time chart of u^τ, on the right the plot of $\phi(u^\tau)$, corresponding to perturbations with $\delta = 1$. Note how the motion exits from the lower energy state when $t = 1$.

Note that the perturbed minimizing movement might not pass the well in 0 at its first discontinuity, but eventually it will (see Figure 5). In fact as in Example 4.2, for indices n such that $a_n^\tau = \delta\tau$, we get $u_n^\tau = u_{n-1}^\tau/(1+2\delta)$, and since the discrete solution decreases the energy we have $u_n^\tau > u_{n-1}^\tau$ for any index. Therefore $0 > u_n^\tau > u_0/(1+2\delta)^{\lceil n\tau \rceil} \to 0$ as $n \to +\infty$. Hence, for any δ, there exists an index n such that $a_n^\tau = \delta\tau$; *i.e.*, $n = \lceil k/\tau \rceil$ for some $k \in \mathbb{N}$, and $u_{n-1}^\tau > -1/\delta$, so that $u_n^\tau > 0$. Therefore $u(t) < 0$ for every $t < k$ and $u(t) > 0$ when $t > k$.

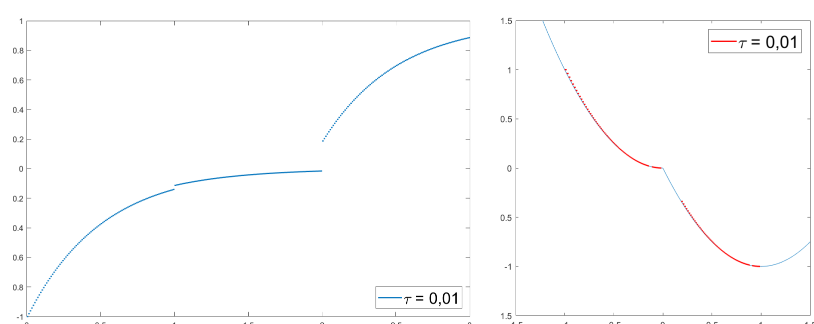

FIGURE 5. Graphs as in Figure 4 with $\delta = 8$. It is grater than the critical value e^2, indeed when $t = 1$ the motion does not exit the first well.

The argument of the previous example can be iterated when a multi-well quadratic energy functional is considered. While perturbed minimizing movements corresponding to perturbations satisfying assumption H5, in particular classical minimizing movements, would stagnate in the initial potential well, perturbed minimizing movements with perturbations as in H5' may explore different local minima.

Example 4.6. We consider the functional $\phi(t) = \min_{k\geq 0}\{(t-k)^2 - k\}$, initial data $\{u_0^\tau\}$ converging to $u_0^\tau < 0$ (so that the motion starts in the first potential well), and perturbations, satisfying H5', as in (27) with $I = \mathbb{N}$ and $\delta(\tau) = \delta\tau$.

We study the minimum of the function $t \mapsto \phi(t) + \bar{\delta}(t-\bar{u})^2/2$ using calculations done in Example 4.5 by changing variable t in $t-k$ or $t-(k-1)$, and adding k or $k-1$ respectively which does not affect the minimization. With same notation we get

$$\phi_k(\bar{\delta}, \bar{u}) = \frac{\bar{\delta}\bar{u}^2 - 2k\bar{\delta}\bar{u} + k(\bar{\delta}(k-1) - 2)}{2+\bar{\delta}}, \quad \text{if } (k-1) - \frac{2}{\bar{\delta}} < \bar{u} \leq k$$

$$t_k(\bar{\delta}, \bar{u}) = \frac{2k + \bar{\delta}\bar{u}}{2+\bar{\delta}}, \quad \text{if } (k-1) - \frac{2}{\bar{\delta}} < \bar{u} \leq k.$$

Hence, by comparing the minima, for any $k > 0$ we obtain that

$$\phi_{k-1}(\bar{\delta}, \bar{u}) \leq \phi_k(\bar{\delta}, \bar{u}) \leq \phi_{k+1}(\bar{\delta}, \bar{u}) \quad \Longleftrightarrow \quad (k-1) - \frac{1}{\bar{\delta}} \leq \bar{u} \leq k - \frac{1}{\bar{\delta}}.$$

Note that this is not in contrast with the previous condition on $\bar{u}$. From the previous computation, we have that the minimizer of the map $t \mapsto \phi(t) + \bar{\delta}(t-\bar{u})^2/2$ is $t_k(\bar{\delta}, \bar{u})$ whenever we have $\bar{u} \in (k-1-1/\bar{\delta}, k-1/\bar{\delta}]$.

The behavior of the motion depends on δ. Fixed $T > 0$ sufficiently large, we consider discrete steps u_n^τ such that $\lceil n/\tau \rceil < T$. First, we assume that $\delta \geq 1$. Let the discrete solution u^τ be in the interval $(h-1, h]$, that is the h-th energy well. As in condition (28) we have that, if $u_{n-1}^\tau \in (h-1, h-\frac{1}{\delta}]$, the discrete solution will not pass to another potential well, when τ is small enough; whereas, if it is in $(h-\frac{1}{\delta}, h]$, it will pass to the next well; *i.e*, $u_n^\tau \in (h, h+1]$, provided that n is such that $a_n^\tau = \delta\tau$.

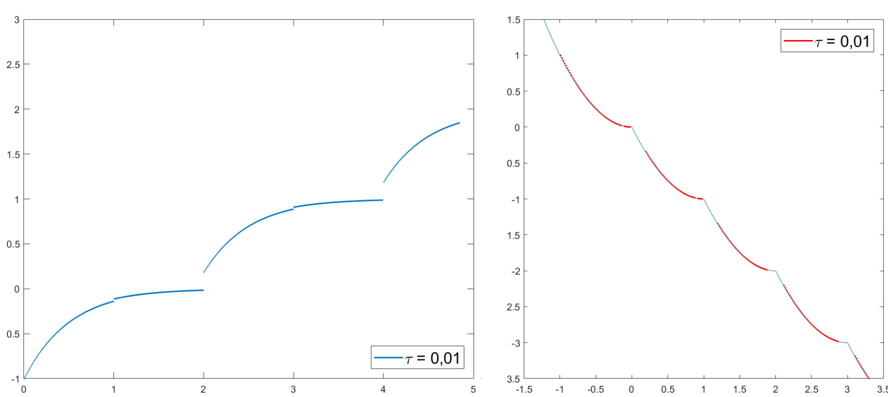

FIGURE 6. Graphs for $\delta = 8$. As in Figure 5, the motion does not exit the well when $t = 1$, but when $t = 2$ it does.

As pictured in Figure 7, if $\delta = 1$ the motion will always exit any potential well as soon as $a_n^\tau = \tau$, because the range of the positions of u_{n-1}^τ, in respect of which u_n^τ pass to the next well, is the whole interval $(h-1, h]$; hence the perturbed minimizing movement $u(t) \in (h-1, h)$ whenever $t \in (h, h+1)$, for every positive integer h.

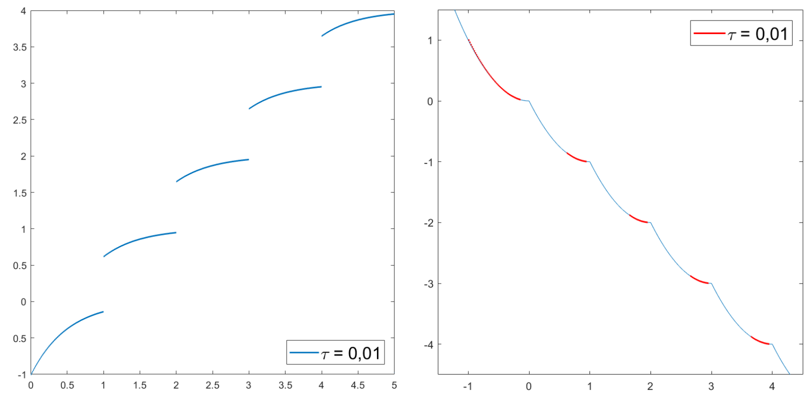

FIGURE 7. Graph for $\delta = 1$. The motion always passes to the very next potential well.

If $0 < \delta < 1$, the discrete solution will exit more than one well. If $u^\tau_{n-1} \in (h-1, h-1/\delta]$, it will pass through $\lfloor 1/\delta \rfloor$ wells, if $(h-1/\delta, h]$ through $\lceil 1/\delta \rceil$; *i.e.*, $u^\tau_n \in (h-1+\lfloor 1/\delta \rfloor, h+\lfloor 1/\delta \rfloor]$, or $u^\tau_n \in (h-1+\lceil 1/\delta \rceil, h+\lceil 1/\delta \rceil]$ respectively.

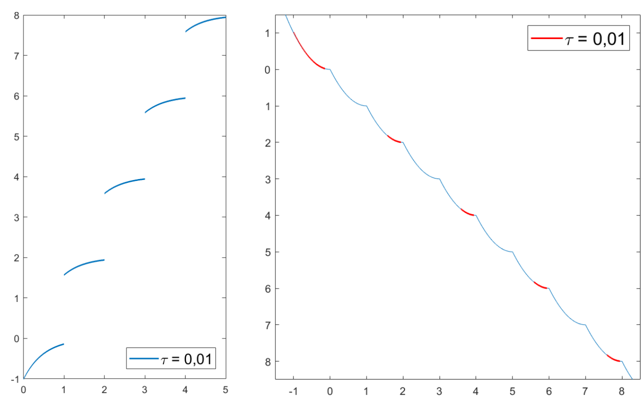

FIGURE 8. Graphs of a discrete solution passing through two potential wells at every jump discontinuity.

We conclude this paper studying a particular case in which the perturbations do not even satisfy the relaxed assumption H5'. The following result is stated in a restrictive situation and could be generalized, but the aim is to show that, even for uncontrolled perturbations, discrete solutions may converge to a perturbed minimizing movement.

Proposition 7. *Let* ϕ *satisfy assumptions* H1-H3, *and assume that it admits a unique global minimum* u_*. *Let* u^τ_0 *satisfy* H4, *and let* $\{a^\tau\}$ *be a family of perturbations as in* (1). *If there exists a time parameter* $t_0 > 0$ *such that*

(i) $a^\tau(t)$ *satisfies assumption* H5' *for every* $t \in [0, \lceil t_0/\tau \rceil \tau]$

(ii) $\lim_{\tau \to 0} a^\tau_{\lceil t_0/\tau \rceil}/\tau = 0$,

then there exists a $\{a^\tau\}$*-perturbed minimizing movement* u *such that* $u(t) = u_*$ *for every* $t > t_0$.

Proof. Let u^τ be a discrete solution. We denote $S_0 = \{u \in S \,|\, \phi(u) \le C_0\}$. By (i) we can apply Proposition 2 in $[0, t_0]$, and we get $\mathrm{diam}(S_0) \le 2C_{t_0}$. We write $N(\tau) = \lceil t_0/\tau \rceil$. The $N(\tau)$-th step of the discrete solution is

$$u^\tau_{N(\tau)} \in \operatorname*{argmin}_{u \in S_0} \left\{ \phi(u) + a^\tau_{N(\tau)} \frac{d^2(u^\tau_{N(\tau)-1}, u)}{2\tau} \right\}.$$

By (ii), for any $\varepsilon > 0$ there exists τ_ε such that $a^\tau_{N(\tau)}/\tau < \varepsilon / C_{t_0}$, and we get $\phi(u^\tau_{N(\tau)}) < \phi(u) + \varepsilon$ for every $u \in S_0$. By the monotonicity of $\phi(u^\tau)$, we have that

$$u^\tau_n \in \{u \in S \,|\, \phi(u) \le \phi(u_*) + \varepsilon\}, \quad \text{for every } n \ge N(\tau). \tag{29}$$

By a contradiction argument, suppose that there exists a constant η_0, an index $m > N(\tau)$, and $\tau_0 < \tau_\varepsilon$ such that $d(u^{\tau_0}_m, u_*) \ge \eta_0$. By the uniqueness of the minimum, $\inf_{S \setminus B_{\eta_0}} \phi > \phi(u_*)$. Then, we have that $\phi(u^\tau_m) - \phi(u_*) \ge \inf_{S\setminus B_{\eta_0}} \phi - \phi(u_*) > 0$, which is in contrast with (29). Applying Theorem 4.3 in $[0, t_0)$, and by (29), we get the thesis. □

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