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PROPAGATION OF REGULARITY AND FINITE-TIME COLLISIONS FOR THE THERMOMECHANICAL CUCKER-SMALE MODEL WITH A SINGULAR COMMUNICATION

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Abstract. We study dynamical behaviors of the ensemble of *thermomechanical* Cucker-Smale (in short TCS) particles with singular power-law communication weights in velocity and temperatures. For the particle TCS model, we present several sufficient frameworks for the global regularity of solution and a finite-time breakdown depending on the blow-up exponents in the power-law communication weights at the origin where the relative spatial distances become zero. More precisely, when the blow-up exponent in velocity communication weight is greater than unity and the blow-up exponent in temperature communication weights is more than twice of blow-up exponent in velocity communication, we show that there will be no finite time collision between particles, unless there are collisions initially. In contrast, when the blow-up exponent of velocity communication weight is smaller than unity, we show that there can be a collision in finite time. For the kinetic TCS equation, we present a local-in-time existence of a unique weak solution using the suitable regularization and compactness arguments.

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1. **Introduction.** Collective behaviors of classical and quantum many-body systems are ubiquitous in our biological, chemical and physical complex systems in nature, e.g., flocking of birds, swarming of fish, aggregation of bacteria, or synchronization of neurons, synchronization of Josephson junction arrays, etc. [28, 29, 30]. In previous literature, many mathematical models were proposed to describe these collective dynamics. Among those models, our main interest in this paper lies on the thermomechanical Cucker-Smale (in short, TCS) model which is recently introduced in [19]. Consider an ensemble consisting of N TCS particles, i.e., the Cucker-Smale particles with internal temperature variables, and let $x_i(t), v_i(t)$ and $\theta_i(t)$ be the spatial position, velocity, and temperature of the i-th TCS particle at time t. Then, the ensemble of TCS particles is governed by the following first-order ordinary differential equations:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t>0, \quad i=1,2,\cdots,N, \\ \frac{dv_i}{dt} &= \frac{\kappa_1}{N}\sum_{j=1}^N \phi(r_{ij})\left(\frac{v_j}{\theta_j}-\frac{v_i}{\theta_i}\right), \\ \frac{d\theta_i}{dt} &= \frac{\kappa_2}{N}\sum_{j=1}^N \zeta(r_{ij})\left(\frac{1}{\theta_i}-\frac{1}{\theta_j}\right), \end{aligned} \tag{1}$$

subject to initial data:

$$(x_i(0), v_i(0), \theta_i(0)) =: (x_i^0, v_i^0, \theta_i^0), \qquad \sum_{i=1}^N v_i^0 = 0. \tag{2}$$

Here $r_{ij} := |x_i - x_j|$ denotes the Euclidean distance between i-th and j-th particles, and the positive constants κ_i, $i=1,2$ represent the coupling strengths for velocities and temperatures, and the interaction kernels $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ are generally non-increasing functions in their arguments. Note that the zero mean velocity condition $(2)_2$ will be propagated along the dynamics (1). To fix the idea, we consider the following specific forms of communication weights:

$$\phi(s) := \frac{1}{s^\alpha}, \qquad \zeta(s) = \frac{1}{s^\beta}, \quad \text{with} \quad \alpha, \beta > 0, \quad s > 0. \tag{3}$$

As its name suggests, TCS model is one of generalizations for the Cucker-Smale (in short, C-S) model introduced in [9]:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t>0, \quad i=1,2,\cdots,N, \\ \frac{dv_i}{dt} &= \frac{\kappa}{N}\sum_{j=1}^N \phi(r_{ij})(v_j - v_i). \end{aligned} \tag{4}$$

Note that for the same constant temperatures $\theta_i = \theta_\infty > 0$, system (1) is formally reduced to the C-S model with $\kappa = \frac{\kappa_1}{\theta_\infty}$ (see [5, 6, 8, 9, 10, 11, 15, 14, 16, 17, 18, 22, 23] and a recent survey article [4, 7]). Recently, the TCS model [19] has been studied in many different aspects, including the case of interaction kernels are spatially dependent, uniform stability issue, and kinetic and hydrodynamic limits [13, 12]. In those previous works, the interaction kernels are always assumed to be Lipschtiz continuous and regular. However, as in the case of the original C-S model [1, 3, 26, 25], we can also consider the case when the interaction kernels ϕ and ζ are singular to avoid collisions between particles. At least formally, as the number of particles tends to infinity, the dynamics of the whole ensemble $\{(x_i, v_i, \theta_i)\}$ can be effectively described by the Vlasov type equation which can be obtained via the BBGKY hierarchy in the formal level. More precisely, let $f = f(x, v, \theta, t)$ be the one-particle probability density function of the ensemble of TCS particle at position x, velocity v and temperature θ at time t. Then, the dynamics of the density function f is governed by the following Cauchy problem for the Vlasov type equation:

$$\begin{aligned}
&\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (\mathcal{F}[f]f) + \partial_\theta(\mathcal{G}[f]f) = 0, \quad x, v \in \mathbb{R}^d,\ \theta \in \mathbb{R}_+,\ t > 0,\\
&\mathcal{F}[f](z,t) := -\kappa_1 \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi(|x - x_*|)\Big(\frac{v}{\theta} - \frac{v_*}{\theta_*}\Big) f(z_*, t)\, dz_*,\\
&\mathcal{G}[f](x,\theta,t) := \kappa_2 \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \zeta(|x - x_*|)\Big(\frac{1}{\theta} - \frac{1}{\theta_*}\Big) f(z_*, t)\, dz_*,\\
&z = (x, v, \theta), \quad dz := dx\, dv\, d\theta,
\end{aligned} \tag{5}$$

subject to the initial condition:

$$f(z, 0) \ =: \ f^0(z), \quad z = (x, v, \theta) \in \mathbb{R}^{2d} \times \mathbb{R}_+. \tag{6}$$

Below, we briefly discuss our main issues to be explored throughout the paper. In this paper, we are interested in the following two questions for the Cauchy problems (1) - (2) and (5) - (6):

- (Q1) : Can the strong singularity in ϕ and ζ prevent the spatial collisions between TCS particles? If it is true, what will be the sufficient conditions leading to the absence of collisions?
- (Q2) : Can we establish the well-posedness of the kinetic TCS equation at least locally in time?

Next, we comment on the above questions. The collision avoidance between particles has been studied for the Cucker-Smale model and its variants in [1, 8, 3, 26, 25] due to the possible applications of the C-S model (4) to the traffic control of unmanned aerial vehicles and robot systems. More precisely, some special class of initial configurations are taken to guarantee collision avoidance for strong communication weight with $\alpha \geq 1$ in [1], and later those initial conditions are completely removed in [3]. On the other hand, due to the singularity of the interaction kernel, the system (1) may not be well-defined after collision of particles. If two particles collide, their relative position converges to 0, which makes singularity on the R.H.S. of (1). For weakly singular communication weights with $\alpha < 1$, the collisions can occur for some initial configurations. Moreover, the kinetic TCS equation (5) has a singular kernel. Thus, the well-posedness issue is not clear at all. In this paper, we consider the concept of weak solutions and study its local-in-time well-posedness.

The main novelty of this paper are three-fold. First, we show that the particle TCS model (1) - (3) with a strong communication weights $\alpha \geq 1,\ \beta \geq 2\alpha$ cannot have collisions unless they have them initially. The proof for the non-existence of collisions will be made via the contradiction argument. Suppose that there will be a finite time collision between some particles, say l and i at time t_0. Then, we consider the set of all particles to collide with l-th particle at time t_0, and we will denote it by $[l]$. Then, for this colliding particles at time t_0, we set

$$\|X\|_{[l]}(t) := \sqrt{\sum_{i,j\in[l]} |x_i(t) - x_j(t)|^2} \quad \text{and} \quad \|V\|_{[l]}(t) := \sqrt{\sum_{i,j\in[l]} |v_i(t) - v_j(t)|^2},$$

Then, we will show that $\|X\|_{[l]}$ and $\|V\|_{[l]}$ satisfy the following relation (see Section 4.1):

$$|\Phi(\|X\|_{[l]}(t))| \le \int_0^t \phi(\|X\|_{[l]}(\tau))\|V\|_{[l]}(\tau)d\tau + |\Phi(\|X\|_{[l]}(0))|. \tag{7}$$

where Φ is anti-derivative of ϕ, i.e., $\Phi(x) = \int^x \phi(t)\,dt$ which satisfies

$$\lim_{r\to 0+} \Phi(r) = \infty, \quad \text{for } \alpha \ge 1. \tag{8}$$

By laborious estimates, we can show that the R.H.S. of (7) is finite for all $t \ge 0$. However, at the first collision time t_0, $\lim_{t\to t_0-} X(t) = 0$. Then, this and (8) imply

$$\lim_{t\to t_0-} |\Phi(\|X\|_{[l]}(t))| = \infty$$

which gives a contradiction. As long as there are no collisions, the R.H.S. of (1) is still locally Lipschitz. Hence, the classical Cauchy-Lipschitz theory can be applied to (1) - (3), and this yields the global smooth solutions. Second, for weakly singular communication ϕ with $\alpha < 1$, we can show the existence of non-collisional initial configuration leading to the finite-time collisions (see Section 4.3). Third, we present a local existence and uniqueness of weak solutions to the corresponding kinetic equation (5), which can be formally derived by the standard BBGKY hierarchy from the particle system (1). The rigorous derivation of (5) through the mean-field limit is recently obtained in [12] when the interaction kernels are regular enough, e.g., ϕ and ζ are bounded Lipschitz functions. On the other hand, it is not known whether system (5) has a unique regular solution and exhibits an emergent behavior when at least one of the interaction kernels ϕ or ζ is singular. In order to show the local-in-time existence of weak solutions to the equation (5), we use the 1-Wasserstein metric which is defined by

$$W_1(\mu_1,\mu_2) := \inf_{\gamma\in\Gamma(\mu_1,\mu_2)} \int_{\mathbb{R}^{2d}} |x-y|d\gamma(x,y),$$

for two probability measures $\mu_1, \mu_2 \in \mathcal{P}_1(\mathbb{R}^d)$, where the $\Gamma(\mu_1,\mu_2)$ is defined as a set of all joint probability measures whose marginals are μ_1 and μ_2 respectively. Here $\mathcal{P}_1(\mathbb{R}^d)$ stands for the set of probability measures with bounded moments of order 1.

The rest of this paper is organized as follows. In Section 2, we provide a several preliminaries for the TCS model, which will be crucially used in the later sections. In Section 3, we briefly review some relevant results for the C-S model and present our main results on the global existence of solutions and local existence of weak solutions for the particle system (1)-(2). In Section 4, we provide the conditions on the communication weight ϕ and ζ to guarantee the global existence of the TCS system (1). We also give some examples of finite-time collision when the condition suggested in Section 3 is violated. In Section 5, we provide the existence and uniqueness of local-in-time weak solutions of kinetic TCS model. Finally, Section 6 is devoted to the summary of our main results and some discussion on the future works.

Notation: Throughout this paper, $|\cdot|$ denotes the standard Euclidean ℓ^2-norm in $\mathbb{R}^d$, and Ω can be either $\mathbb{R}^d$ or $\mathbb{R}^{2d}$ or $\mathbb{R}^{2d}\times\mathbb{R}_+$. For a function $f:\ \Omega\times[0,\infty)\to\mathbb{R}$, $\|f\|_{L^p}$ represents the usual $L^p(\Omega)$-norm and

$$\|f\|_{L^1\cap L^p} := \|f\|_{L^1} + \|f\|_{L^p}, \quad \text{and} \quad \|f\| := \|f\|_{L^\infty(0,\tau;L^1\cap L^p)}.$$

Moreover, we will use the notation of volume element $dz = dx\, dv\, d\theta$ in the extended phase space $\mathbb{R}^{2d} \times \mathbb{R}_+$.

2. **Preliminaries.** In this section, we briefly review theoretic minimum of the TCS model and kinetic TCS equation and Wasserstein distances.

2.1. **Basic a priori estimates.** First, we study a priori estimates for the particle TCS model (1) which will be crucially used in later sections. For position, velocity and temperature configurations X, V, Θ, respectively, we define their diameters as follows: For $t \geq 0$,

$$D(X(t)) := \max_{1\leq i,j\leq N} |x_i(t) - x_j(t)|, \quad D(V(t)) := \max_{1\leq i,j\leq N} |v_i(t) - v_j(t)|,$$

and

$$D(\Theta(t)) := \max_{1\leq i,j\leq N} |\theta_i(t) - \theta_j(t)|.$$

The most basic property of system (1) is the monotonicity and boundedness of temperature, position and velocity diameters.

Lemma 2.1. *For a positive constant $T \in (0, \infty)$, let (X, V, Θ) be a solution of (1) with initial data (X^0, V^0, Θ^0) in the time-interval $[0, T)$. Then, the diameters $D(X)$, $D(V)$ and $D(\Theta)$ satisfy contraction property and boundedness, respectively.*

1. *The temperature diameter $D(\Theta)$ is monotonically decreasing in t:*

$$D(\Theta(t)) \leq D(\Theta(s)), \quad 0 \leq s \leq t.$$

2. *The velocity and position diameters are bounded:*

$$\sup_{0\leq t\leq T} D(V(t)) \leq C_T, \qquad \sup_{0\leq t\leq T} D(X(t)) \leq \tilde{C}_T$$

where C_T and $\tilde{C}_T$ are positive constants depending on initial configuration (X^0, V^0, Θ^0) and T.

Proof. (1) For a given $t \geq 0$, we set

$$\theta_M(t) := \max_{1\leq i\leq N} \theta_i(t), \quad \theta_m(t) := \min_{1\leq i\leq N} \theta_i(t).$$

Then, it follows from $(1)_3$ that we have

$$\frac{d\theta_M}{dt} = \frac{\kappa_2}{N} \sum_{j=1}^{N} \zeta(r_{Mj}) \left(\frac{1}{\theta_M} - \frac{1}{\theta_j} \right) \leq 0,$$

and

$$\frac{d\theta_m}{dt} = \frac{\kappa_2}{N} \sum_{j=1}^{N} \zeta(r_{mj}) \left(\frac{1}{\theta_m} - \frac{1}{\theta_j} \right) \geq 0.$$

These yield

$$\theta_m(s) \leq \theta_m(t) \leq \theta_i(t) \leq \theta_M(t) \leq \theta_M(s), \quad t \geq s \geq 0.$$

Then, we have

$$D(\Theta(t)) \leq D(\Theta(s)), \quad t \geq s \geq 0.$$

(2) It follows from the differential inequalities in [12, Lemma 3.2] that we have

$$\frac{d}{dt} D(V) \leq -\frac{\kappa_1 \phi(D_X)}{\theta_M^0} D_V + 2\kappa_1 \frac{D(\Theta)}{(\theta_m^0)^2} D(V) \leq 2\kappa_1 \frac{D(\Theta^0)}{(\theta_m^0)^2} D(V).$$

Then, Gronwall's lemma yields

$$D(V(t)) \le D(V^0)\exp\left(2\kappa_1 \frac{D(\Theta^0)}{(\theta_m^0)^2}T\right) =: C_T.$$

Finally, note that the following differential inequality in [12, Lemma 3.2] also holds:

$$\frac{d}{dt}D(X) \le D(V).$$

Then, we use the boundedness of velocity diameter to get

$$D(X(t)) \le D(X(0)) + C_T T,$$

which implies the boundedness of diameter of position. $\square$

Next, for position and temperature vectors $X = (x_1, \cdots, x_N)$, $\Theta = (\theta_1, \cdots, \theta_N)$, we set a production functional: For $t \ge 0$,

$$\mathcal{P}(t) := \mathcal{P}(X(t), \Theta(t)) = \sum_{i,j=1}^N \zeta(r_{ij}(t)) \frac{|\theta_i(t) - \theta_j(t)|^2}{\theta_i(t)\theta_j(t)}.$$

Lemma 2.2. *Suppose that the coupling strength κ_2 is positive and let (X, V, Θ) be a global solution of (1) with initial data (X^0, V^0, Θ^0). Then, the functional $\mathcal{P}(t)$ is integrable in t:*

$$\int_0^\infty \mathcal{P}(t)dt \le \frac{N}{\kappa_2}\sum_{i=1}^N |\theta_i^0|^2.$$

Proof. We multiply $(1)_3$ by $2\theta_i$ to obtain

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N |\theta_i|^2 &= \frac{2\kappa_2}{N}\sum_{i,j=1}^N \zeta(r_{ij})\left(\frac{1}{\theta_i} - \frac{1}{\theta_j}\right)\theta_i = \frac{2\kappa_2}{N}\sum_{i,j=1}^N \zeta(r_{ij})\left(\frac{\theta_j - \theta_i}{\theta_i\theta_j}\right)\theta_i \\ &= \frac{2\kappa_2}{N}\sum_{i,j=1}^N \zeta(r_{ij})\left(\frac{\theta_i - \theta_j}{\theta_j\theta_i}\right)\theta_j = -\frac{\kappa_2}{N}\sum_{i,j=1}^N \zeta(r_{ij})\frac{|\theta_i - \theta_j|^2}{\theta_i\theta_j} = -\frac{\kappa_2}{N}\mathcal{P}.\end{aligned}$$

We integrate the above relation in time to get

$$\sum_{i=1}^N |\theta_i(t)|^2 + \frac{\kappa_2}{N}\int_0^t \mathcal{P}(s)ds = \sum_{i=1}^N |\theta_i^0|^2,$$

which yields the desired estimate. $\square$

Next, we study the propagation of velocity and temperature moments along the kinetic TCS equation. For this, we set

$$\langle 1\rangle := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f dz, \quad \langle v\rangle := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} vf\, dz, \quad \langle v^2\rangle := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |v|^2 f dz,$$

$$\langle\theta\rangle := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \theta f\, dz, \quad \langle\theta^2\rangle := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \theta^2 f dz, \quad \text{and}\langle\log\theta\rangle := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} (\log\theta) f\, dz.$$

In next lemma, we study propagation of above moments along (5).

Lemma 2.3. *Let $f = f(z, t)$ be a solution of kinetic TCS equation (5) decaying sufficiently fast at infinity $|x| = \infty$, $|v| = \infty$ and $\theta = 0, \infty$. Then, we have*

$$\frac{d\langle 1\rangle}{dt} = 0, \quad \frac{d\langle v\rangle}{dt} = 0, \quad \frac{d\langle\theta\rangle}{dt} = 0, \quad \frac{d\langle\theta^2\rangle}{dt} \le 0, \quad \textit{and} \quad \frac{d\langle\log\theta\rangle}{dt} \ge 0.$$

Proof. (i) The conservation of mass follows from the divergence form of (5).
(ii) We multiply (5) by v and integrate the resulting relation over the extended phase space using the exchange symmetry $(x,v,\theta) \leftrightarrow (x_*,v_*,\theta_*)$ to get

$$\begin{aligned}\frac{d\langle v\rangle}{dt} &= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} v\partial_t f\,dz\\ &= -\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} v(v\cdot\nabla_x f) + v\nabla_v\cdot(\mathcal{F}[f]f) - v\partial_\theta(\mathcal{G}[f]f)\,dz\\ &= d\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \mathcal{F}[f]f\,dz\\ &= d\kappa_1\int_{\mathbb{R}^{4d}\times(\mathbb{R}_+)^2} \phi(|x-x_*|)\left(\frac{v_*}{\theta_*}-\frac{v}{\theta}\right) f(z,t)f(z_*,t)\,dz\,dz_*\\ &= 0.\end{aligned}$$

The case for $\langle\theta\rangle$ can be treated similarly.
(iii) We multiply (5) by θ^2 and integrate over the extended phase space to get the dissipativity for $\langle\theta^2\rangle$:

$$\begin{aligned}\frac{d\langle\theta^2\rangle}{dt} &= -\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \theta^2 v\cdot\nabla_x f + \theta^2\nabla_v\cdot(\mathcal{F}[f]ff) + \theta^2\partial_\theta(\mathcal{G}[f]f)\,dz\\ &= 2\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \theta\mathcal{G}[f]f\,dz\\ &= 2\int_{\mathbb{R}^{4d}\times(\mathbb{R}_+)^2} \zeta(|x-x_*|)\left(\frac{1}{\theta}-\frac{1}{\theta_*}\right)\theta f(z,t)f(z_*,t)\;dz\,dz_*\\ &= -\int_{\mathbb{R}^{4d}\times(\mathbb{R}_+)^2} \zeta(|x-x_*|)\frac{(\theta-\theta_*)^2}{\theta\theta_*} f(z,t)f(z_*,t)\,dz\,dz_* \le 0.\end{aligned}$$

(iv) Finally, we can estimate $\langle\log\theta\rangle$ in a similar way:

$$\begin{aligned}\frac{d\langle\log\theta\rangle}{dt} &= -\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} (\log\theta)v\cdot\nabla_x f + (\log\theta)\nabla_v\cdot(\mathcal{F}[f]f) + (\log\theta)\partial_\theta(\mathcal{G}[f]f)\,dz\\ &= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \frac{1}{\theta}\mathcal{G}[f]f\,dz\\ &= \int_{\mathbb{R}^{4d}\times(\mathbb{R}_+)^2} \zeta(|x-x_*|)\left(\frac{1}{\theta}-\frac{1}{\theta_*}\right)\frac{1}{\theta} f(z,t)f(z_*,t)\,dz\,dz_*\\ &= \frac{1}{2}\int_{\mathbb{R}^{4d}\times(\mathbb{R}_+)^2} \zeta(|x-x_*|)\left(\frac{1}{\theta}-\frac{1}{\theta_*}\right)^2 f(z,t)f(z_*,t)\,dz\,dz_*\\ &\ge 0.\end{aligned}$$

□

2.2. **Forward characteristic curves.** In this subsection, we introduce forward characteristic curves associated with the kinetic TCS equation (5). First, we define forward characteristics $(x(s),v(s),\theta(s)) := (x(s;0,x,v,\theta), v(s;0,x,v,\theta), \theta(s;0,x,v,\theta))$ as a solution to the following ODE system:

$$\begin{cases} \dfrac{dx(s)}{ds} = v(s), \quad s > 0, \\ \dfrac{dv(s)}{ds} = \mathcal{F}[f](x(s), v(s), \theta(s)), \\ \dfrac{d\theta(s)}{ds} = \mathcal{G}[f](x(s), \theta(s)), \\ (x(0), v(0), \theta(0)) = (x, v, \theta). \end{cases} \tag{9}$$

Moreover, we also define the section of support of f for each variables which can be obtained as projections of the support of f in x, v and θ-variables:

$$\Omega_x(t) := \overline{\left\{ x \in \mathbb{R}^d \mid f(x, v, \theta, t) \neq 0 \right\}},$$
$$\Omega_v(t) := \overline{\left\{ v \in \mathbb{R}^d \mid f(x, v, \theta, t) \neq 0 \right\}},$$
$$\Omega_\theta(t) := \overline{\left\{ \theta \in \mathbb{R}_+ \mid f(x, v, \theta, t) \neq 0 \right\}}.$$

Lemma 2.4. *Let $f = f(z, t)$ be a solution of kinetic TCS equation (5). Then the set Ω_θ is contractive along the dynamics (5):*

$$\Omega_\theta(s) \subseteq \Omega_\theta(t), \quad 0 \le t \le s.$$

Proof. In fact, the maximum value of Ω_θ decreases, whereas the minimum value of Ω_θ increases as in particle model (see Lemma 2.1). It suffices to show that θ is decreasing along the characteristic curve which gives the maximum value of Ω_θ. The increase of a lower bound can be shown similarly using the characteristic curve that gives the minimum value of Ω_θ. To see this, we multiply $(9)_3$ by $\theta(s)$ to get

$$\begin{aligned} \frac{1}{2}\frac{d\theta(s)^2}{ds} &= \theta(s)\frac{d\theta(s)}{ds} \\ &= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \zeta(|x(s) - x_*|) \left(\frac{1}{\theta(s)} - \frac{1}{\theta_*} \right) \theta(s) f(z_*, t)\, dz_* \\ &= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \zeta(|x(s) - x_*|) \left(\frac{\theta_* - \theta(s)}{\theta(s)\theta_*} \right) \theta(s) f(z_*, t)\, dz_* \le 0, \end{aligned}$$

where we use $(\theta_* - \theta(s))\theta(s) \le 0$. This implies that the maximum value of Ω_θ is non-increasing. □

2.3. **Wasserstein distances.** In this section, we provide some basic properties of Wasserstein metric between two measures for later use. First, we begin with definition of push-forward measure in the following definition.

Definition 2.5. Let μ_1 be a Borel measure on $\mathbb{R}^d$ and $f : (\mathbb{R}^d, \mu_1) \to \mathbb{R}^d$ be a measurable function. Then we define the push-forward measure of μ_1 by f, which will be denoted by $\mu_2 = f\#\mu_1$:

$$\mu_2(B) = \mu_1(f^{-1}(B)), \quad B \subset \mathbb{R}^d.$$

Next, we list several results about push-forward measure and Wasserstein metric without proofs.

Proposition 1. [31, 32] *The following assertions hold.*

(1) Let μ_1 and μ_2 be two measures on $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function. Then,

$$\mu_2 = f\#\mu_1 \quad \Longleftrightarrow \quad \int_{\mathbb{R}^d} \phi \, d\mu_2 = \int_{\mathbb{R}^d} \phi \circ f d\mu_1, \quad \forall \phi \in C_b(\mathbb{R}^d).$$

(2) Suppose that μ_0 is measure on $\mathbb{R}^d$, which has bounded p-th moment, and let f_1 and f_2 be two measurable functions from $\mathbb{R}^d$ to itself. Then, we have

$$W_p^p(f_1\#\mu_0, f_2\#\mu_0) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\gamma(x,y) = \int_{\mathbb{R}^d} |f_1(x) - f_2(x)|^p d\mu_0(x),$$

where γ is any measure with marginals $f_1\#\mu_0$ and $f_2\#\mu_0$.

(3) Let $\{\mu_n\}_{k=1}^\infty$ be a sequence of measures in $\mathcal{P}_1(\mathbb{R}^d)$ and $\mu \in \mathcal{P}_1(\mathbb{R}^d)$. Then μ_n converges to μ in the sense of Wasserstein 1-distance if and only if μ_n converges to μ weakly, and the first moment of μ_n converges to the first moment of μ:

$$\int_{\mathbb{R}^d} |x| d\mu_n(x) \to \int_{\mathbb{R}^d} |x| d\mu(x), \quad \text{as} \quad n \to \infty.$$

Note that the space $\mathcal{P}_p(\mathbb{R}^d)$, the space of all probability measure with finite p-th moment, equipped with the Wasserstein p-distance is Polish space. For $p = 1$, the Wasserstein 1-distance has a dual representation which is given as

$$W_1(\mu_1,\mu_2) = \sup\Big\{ \int_{\mathbb{R}^d} \phi(x) d(\mu_1 - \mu_2)(x) \;\Big|\; \phi \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}),\ \|\phi\|_{Lip} \le 1 \Big\},$$

where $\|\phi\|_{Lip}$ denotes the Lipschitz constant of ϕ.

3. **Presentation of main results.**

In this section, we briefly present our main results on the collision avoidance and local well-posedness of the particle TCS model and kinetic TCS equation, respectively.

First, we briefly review the previous results [3, 17, 24, 25, 26] on the particle and kinetic C-S models with power-law communication weights from three perspectives: "*flocking dynamics, non-existence* and *local-in-time well-posedness*". As far as the authors know, the particle C-S model with a singular communication weight was first treated in [17] to study the flocking dynamics and then the global regularity and emergence of finite-time collisions has been established in a series of works by Peszek and his collaborators [3, 25, 26]. In a recent work [24], Mucha and Peszek studied the existence of measure-valued solutions and weak-atomic uniqueness for the kinetic C-S equation with singular communication weights. In the sequel, we explicitly state our main results for the particle and kinetic TCS models. The detailed proofs will be given in the later sections.

3.1. **The particle TCS model.**

In this subsection, we provide two results on the collision avoidance and asymptotic mono-cluster flocking of the particle TCS model.

Our first result deals with the collision avoidance and finite-time collisions of the particle model (1) depending on the blow-up exponents at the singular point of the communication weight ϕ and ζ in (1).

Theorem 3.1. (Collision avoidance and collisions) *The following assertions hold.*

1. *Suppose that the blow-up exponents α and β in (3) and initial data (X^0, V^0, Θ^0) satisfy*

$$1 \le \alpha \le \frac{\beta}{2} \quad \text{and} \quad x_i^0 \ne x_j^0 \quad \text{for all } 1 \le i \ne j \le N.$$

Then, there exists a unique global solution (X, V, Θ) *to* (1)-(2) *satisfying*

$$x_i(t) \neq x_j(t) \quad \text{for all } 1 \le i \neq j \le N \text{ and } t \ge 0.$$

2. *Suppose that the exponent* α *in* (3), *number of particles and dimension* d *satisfy*

$$0 < \alpha < 1 \quad \text{and} \quad N = 2, \quad d = 1.$$

Then, there exists initial data (X^0, V^0, Θ^0) *such that the local solution* (X, V, Θ) *to* (1)-(2) *with initial data* (X^0, V^0, Θ^0) *satisfying a finite-time collision:*

$$x_1(t_c) = x_2(t_c), \quad \text{for some time } t_c > 0.$$

Remark 1. For the second assertion, note that if all initial temperatures are equal with θ^∞, then the TCS model (1) reduces to the Cucker-Smale model (4) with $\kappa = \frac{\kappa_1}{\theta^\infty}$. Then we can directly use the results in [26], where the finite collision between Cucker-Smale particles with singular communication weights are studied. (See Section 4.3 for more details).

As a direct corollary of Theorem 3.1, we have the following equivalence relation.

Corollary 1. *Let* (X, V, Θ) *be a solution of* (1) - (2) *with initial data* (X^0, V^0, Θ^0). *If* $\beta \ge 2\alpha$, *then the following two statements are equivalent:*

(i) The exponents α *and* β *in the communication weights satisfy* $\alpha \ge 1$.
(ii) For any $d \ge 1$ *and* $N \ge 2$, *the particles will not collide with each other if they are initially non-collisional.*

Next, we introduce a concept of mono-cluster flocking for the TCS model (1) in the following definition.

Definition 3.2. [19] Let $\mathcal{Z} = \{(x_i, v_i, \theta_i)\}$ be a global solution to TCS model (1)-(2). Then, the solution $\mathcal{Z}$ exhibits an asymptotic mono-cluster (global) flocking if the following estimates hold: For $1 \le i, j \le N$,

$$\sup_{0 \le t < \infty} |x_i(t) - x_j(t)| < \infty, \quad \lim_{t \to \infty} (|v_i(t) - v_j(t)| + |\theta_i(t) - \theta_j(t)|) = 0.$$

Remark 2. The mono-cluster flocking dynamics for the particle, kinetic, and hydrodynamic C-S models have been studied in [9, 11, 15, 17, 18, 20, 21, 22, 23, 27].

Our second result is concerned about the emergent dynamics of the TCS model (1) - (2) with singular communications.

Theorem 3.3. (Emergence of mono-cluster flocking) *Suppose that the exponents* α *and* β *in* (3) *and initial data* (X^0, V^0, Θ^0) *satisfy the relations:*

$$\begin{aligned}
&1 \le \alpha \le \frac{\beta}{2}, \qquad x_i^0 \neq x_j^0 \quad \text{for all } 1 \le i \neq j \le N, \quad D(\Theta^0) \le \frac{(\theta_m^0)^2 \phi(D(X^0))}{2C\theta_M^0}, \\
&D(V^0) \le \frac{\kappa_1}{2\theta_M^0} \int_{D(X^0)}^{D_X^\infty} \phi(s)ds, \quad \frac{8}{3C}\phi(D(X^0)) \le \phi(D_X^\infty) < \phi(D(X^0)),
\end{aligned} \tag{10}$$

where $C > 0$ *and* D_X^∞ *are some positive constants. Then we have a mono-cluster flocking:*

$$\begin{aligned}
&\sup_{0 \le t < \infty} D(X(t)) \le D_X^\infty, \quad D(\Theta(t)) \le D(\Theta^0) e^{-\frac{\kappa_2}{(\theta_M^0)^2}\zeta(D_X^\infty)t}, \\
&D(V(t)) \le D(V^0) \exp\left(-\frac{\kappa_1 \phi(D_X^\infty)}{\theta_M^0} t + \frac{2\kappa_1 (\theta_M^0)^2 D(\Theta^0)}{\kappa_2 (\theta_m^0)^2 \zeta(D_X^\infty)}\right), \quad \forall\, t \ge 0.
\end{aligned}$$

Remark 3. Note that the existence of global solutions is guaranteed by Theorem 3.1 at least under the conditions in (10) in the interactions kernels ϕ and ζ.

3.2. **The kinetic TCS equation.** We first recall the definition of weak solutions to (5) in the following definition.

Definition 3.4 (Weak solution)**.** For a given finite $\tau \in (0,\infty)$, $f = f(z,t)$ is a weak solution of (5) in the finite time interval $[0,\tau)$ if and only if the following conditions are satisfied:

1. $f \in L^\infty(0,\tau;(L^1_+ \cap L^p)(\mathbb{R}^{2d}\times\mathbb{R}_+)) \cap C((0,\tau);\mathcal{P}_1(\mathbb{R}^{2d}\times\mathbb{R}_+))$.
2. For any $\Phi \in C^\infty_c(\mathbb{R}^{2d}\times\mathbb{R}_+\times[0,\tau))$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f(z,\tau)\Phi(z,\tau)\,dz - \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f^0(z)\Phi(z,0)\,dz \\
&\qquad = \int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f(\partial_t\Phi + v\cdot\nabla_x\Phi + \mathcal{F}[f]\cdot\nabla_v\Phi + \mathcal{G}[f]\partial_\theta\Phi)\,dz\,dt.
\end{aligned} \tag{11}$$

Then, our third result is concerned with local-in-time existence of weak solutions.

Theorem 3.5. *Suppose that p,α,β and initial data f^0 satisfy the following relations:*

$$\begin{aligned}
&1 \le p \le \infty, \quad 0 < \alpha,\beta < \frac{d}{p^*} - 1, \quad f^0 \in (L^1_+ \cap L^p \cap \mathcal{P}_1)(\mathbb{R}^{2d}\times\mathbb{R}_+),\\
&\mathrm{supp}_v f^0(x,\cdot,\theta) \subset \mathbb{R}^d, \quad \mathrm{supp}_\theta f^0(x,v,\cdot) \subset \mathbb{R}_+, \quad \text{for each } x,v \in \mathbb{R}^d,\ \theta \in \mathbb{R}_+,
\end{aligned}$$

where p^* is the Hölder conjugate of p defined by $1/p + 1/p^* = 1$. Then, there exists a unique weak solution f to (5) in the sense of Definition 3.4 in the time interval $[0,\tau)$ for some $\tau > 0$ satisfying uniform stability in Wasserstein-1 distance: For two local-in-time solutions $f_i, i = 1,2$ to (5), we have

$$\sup_{0\le t\le\tau} W_1(f_1(t), f_2(t)) \le G_0 W_1(f_1^0, f_2^0),$$

where G_0 is a positive constant independent of the time τ.

4. **A global regularity v.s. finite time collision.** In this section, we provide the detailed proof of Theorem 3.1. For this, we use a similar strategy to that in [3] which is based on the construction of a system of locally dissipative differential inequalities (SDDI) for quantifying collisions between particles. Note that this idea is proposed in [17] to show the flocking behavior of the original Cucker-Smale model.

4.1. **Global regularity.** For a fixed $T \in (0,\infty)$, assume that there is $t_0 \in (0,T]$ in which at least two particles are colliding with another, i.e., there is two indices i and j such that

$$\lim_{t\to t_0-}(x_i(t) - x_j(t)) = 0.$$

Next, we will consider colliding particles and non-colliding particles separately. For the colliding particles, we will use the standard index changing trick $(i,j) \leftrightarrow (j,i)$ to estimate, and for the non-colliding particles, we will use the Lipschtiz continuity of interaction kernel ϕ away from 0. More precisely, suppose that t_0 is the first collision time, and define the set of all indices $j \in \{1,\cdots,N\}$ that the j-th particle collides with the l-th particle by

$$[l] := \left\{ j \in \{1,\cdots,N\} \mid r_{jl} \to 0 \quad \text{as } t \to t_0^- \right\},$$

i.e., $[l]$ denotes the set of all particles which will collide with the l-th particle at time $t = t_0^-$. Let δ be a positive constant such that

$$r_{jl}(t) \geq \delta > 0 \quad \text{in } [0, t_0) \text{ for all } j \notin [l].$$

Without loss of generality, we may assume $r_{jl} < 1$. Now we define

$$\|X\|_{[l]}(t) := \sqrt{\sum_{i,j\in[l]} |x_i(t) - x_j(t)|^2} \quad \text{and} \quad \|V\|_{[l]}(t) := \sqrt{\sum_{i,j\in[l]} |v_i(t) - v_j(t)|^2}. \tag{12}$$

Note that $\|X\|_{[l]}(t) \to 0$ as $t \to t_0^-$.

Lemma 4.1. *Suppose that the exponents α and β in (3) satisfy the condition*

$$1 \leq \alpha \leq \frac{\beta}{2},$$

and let (x_i, v_i, θ_i) be the solution of (1) with initial condition $(x_i^0, v_i^0, \theta_i^0)$, and let $t_0 > 0$ is the first collision time. Then $\|X\|_{[l]}$ and $\|V\|_{[l]}$ in (12) satisfy

$$\begin{aligned} &\frac{d}{dt}\|X\|_{[l]} \leq \|V\|_{[l]}, \quad 0 < t < t_0, \\ &\frac{d}{dt}\|V\|_{[l]} \leq -C_1\phi(\|X\|_{[l]})\|V\|_{[l]} + C_2\sqrt{\mathcal{P}(X,\Theta)} + C_3, \end{aligned} \tag{13}$$

where $C_i, i = 1, 2, 3$ are positive constants.

Proof. • **Step A** (Derivation of $(13)_1$): By definition in (12), we have

$$\frac{d}{dt}\|X\|_{[l]} \leq \|V\|_{[l]}.$$

• **Step B** (Derivation of $(13)_2$): We use $(1)_2$ to obtain

$$\begin{aligned} &\frac{d}{dt}\|v\|_{[l]}^2 \\ &= 2\sum_{i,j\in[l]} \left\langle v_i - v_j, \frac{\kappa_1}{N}\sum_{k=1}^N \phi(r_{ki})\left(\frac{v_k}{\theta_k} - \frac{v_i}{\theta_i}\right) - \frac{\kappa_1}{N}\sum_{k=1}^N \phi(r_{kij})\left(\frac{v_k}{\theta_k} - \frac{v_j}{\theta_j}\right)\right\rangle \\ &= \frac{2\kappa_1}{N}\left(\sum_{i,j,k\in[l]} + \sum_{\substack{i,j\in[l] \\ k\notin[l]}}\right)\left[\phi(r_{ki})\left\langle \frac{v_k}{\theta_k} - \frac{v_i}{\theta_i}, v_i - v_j\right\rangle - \phi(r_{kj})\left\langle \frac{v_k}{\theta_k} - \frac{v_j}{\theta_j}, v_i - v_j\right\rangle\right] \\ &=: \mathcal{I}_{11} + \mathcal{I}_{12}. \end{aligned}$$

$\diamond$ (Estimation for $\mathcal{I}_{11}$): We use the symmetry of the communication kernel ϕ and index exchange trick $(i,k) \longleftrightarrow (k,i)$ and Cauchy-Schwarz's inequality to get

$$\begin{aligned} \mathcal{I}_{11} &= \frac{2\kappa_1}{N}\sum_{i,j,k\in[l]} \phi(r_{ki})\left\langle \frac{v_k}{\theta_k} - \frac{v_i}{\theta_i}, v_i - v_j\right\rangle \\ &= -\frac{2\kappa_1}{N}\sum_{i,j,k\in[l]} \phi(r_{ki})\left\langle \frac{v_k}{\theta_k} - \frac{v_i}{\theta_i}, v_k - v_j\right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{\kappa_1}{N} \sum_{i,j,k\in[l]} \phi(r_{ki}) \left\langle \frac{v_k}{\theta_k} - \frac{v_i}{\theta_i}, v_i - v_k \right\rangle \\
&= -\frac{\kappa_1|[l]|}{N} \sum_{i,j\in[l]} \phi(r_{ij}) \left\langle \frac{v_i}{\theta_i} - \frac{v_j}{\theta_j}, v_i - v_j \right\rangle \\
&= -\frac{\kappa_1|[l]|}{N} \sum_{i,j\in[l]} \frac{\phi(r_{ij})}{\theta_i}|v_i - v_j|^2 + \frac{\kappa_1|[l]|}{N} \sum_{i,j\in[l]} \phi(r_{ij}) \frac{\theta_i-\theta_j}{\theta_i\theta_j} \langle v_j, v_i - v_j\rangle \\
&\le -\frac{\kappa_1|[l]|}{N} \sum_{i,j\in[l]} \frac{\phi(r_{ij})}{\theta_i}|v_i - v_j|^2 + \frac{\kappa_1|[l]|}{N} \left(\sum_{i,j\in[l]} \phi^2(r_{ij}) \frac{|\theta_i-\theta_j|^2}{\theta_i^2\theta_j^2} |v_j|^2 \right)^{\frac{1}{2}} \|V\|_{[l]},
\end{aligned}
$$

where we used the relation:

$$
\frac{v_i}{\theta_i} - \frac{v_j}{\theta_j} = \frac{v_i - v_j}{\theta_i} + \Big(\frac{1}{\theta_i} - \frac{1}{\theta_j}\Big) v_j = \frac{v_i - v_j}{\theta_i} + \frac{(\theta_j - \theta_i)}{\theta_i\theta_j} v_j.
$$

We use Lemma 2.1 and the assumption $\alpha \le \beta/2$ to obtain

$$
|v_j| = \frac{1}{N}\left|\sum_{k=1}^N (v_j - v_k)\right| \le \frac{1}{N}\sum_{k=1}^N |v_j - v_k| \le D(V) \le C_{t_0}, \quad \text{and} \quad \phi^2 \le \zeta \tag{14}
$$

for $0 \le t \le t_0$, where we used $\phi^2(s) \le \zeta(s)$ for $s < 1$. Then we find

$$
\mathcal{I}_{11} \le -\frac{\kappa_1|[l]|}{N} \sum_{i,j\in[l]} \frac{\phi(r_{ij})}{\theta_i}|v_i - v_j|^2 + \frac{\kappa_1 C_{t_0}^2|[l]|}{T_m^0} \left(\underbrace{\sum_{i,j=1}^N \zeta(r_{ij}) \frac{|\theta_i-\theta_j|^2}{\theta_i\theta_j}}_{=\mathcal{P}(X,\Theta)} \right)^{1/2} \|V\|_{[l]}.
$$

Now, we define positive constants C_1 and C_2 as

$$
C_1 := \frac{\kappa_1|[l]|}{2N\theta_M^0} \quad \text{and} \quad C_2 := \frac{\kappa_1 C_{t_0}^2|[l]|}{2\theta_m^0}.
$$

This gives

$$
\mathcal{I}_{11} \le -2C_1 \sum_{i,j\in[l]} \phi(r_{ij})|v_i - v_j|^2 + 2C_2\mathcal{P}^{1/2}\|V\|_{[l]}.
$$

• (Estimation for $\mathcal{I}_2$): We rearrange terms to get

$$
\begin{aligned}
\mathcal{I}_{12} &= \frac{2\kappa_1}{N} \sum_{\substack{i,j\in[l]\\ k\notin[l]}} \left[\phi(r_{ki}) \Big\langle \frac{v_k}{\theta_k} - \frac{v_i}{\theta_i}, v_i - v_j \Big\rangle - \phi(r_{kj}) \Big\langle \frac{v_k}{\theta_k} - \frac{v_j}{\theta_j}, v_i - v_j \Big\rangle \right] \\
&= \frac{2\kappa_1}{N} \sum_{\substack{i,j\in[l]\\ k\notin[l]}} \left[\phi(r_{ki}) \Big\langle \frac{v_j}{\theta_j} - \frac{v_i}{\theta_i}, v_i - v_j \Big\rangle + (\phi(r_{ki}) - \phi(r_{kj})) \Big\langle \frac{v_k}{\theta_k} - \frac{v_j}{\theta_j}, v_i - v_j \Big\rangle \right] \\
&=: \mathcal{I}_{121} + \mathcal{I}_{122}.
\end{aligned}
$$

$\diamond$ (Estimation for $\mathcal{I}_{121}$): We use Lemma 2.1 and (14) to obtain

$$\begin{aligned}
\mathcal{I}_{121} &= -\frac{2\kappa_1}{N}\sum_{\substack{i,j\in[l]\\ k\notin[l]}}\frac{\phi(r_{ki})}{\theta_i}|v_i-v_j|^2+\frac{2\kappa_1}{N}\sum_{\substack{i,j\in[l]\\ k\notin[l]}}\phi(r_{ki})\frac{\theta_i-\theta_j}{\theta_i\theta_j}\langle v_j, v_i-v_j\rangle\\
&\le \frac{2\kappa_1}{N}\sum_{\substack{i,j\in[l]\\ k\notin[l]}}\phi(r_{ki})\frac{\theta_i-\theta_j}{\theta_i\theta_j}\langle v_j, v_i-v_j\rangle\\
&\le \frac{2\kappa_1(N-|[l]|)}{N\delta^\alpha(\theta_m^0)^2}D(\Theta)D(V)\sum_{i,j\in[l]}|v_i-v_j|\\
&\le \frac{2\kappa_1(N-|[l]|)}{N\delta^\alpha(\theta_m^0)^2}D(\Theta^0)C_{t_0}|[l]|\|V\|_{[l]},
\end{aligned}\tag{15}$$

where we used Cauchy-Schwarz inequality in the last inequality.

$\diamond$ (Estimation for $\mathcal{I}_{122}$): We use the Lipschitz continuity of ϕ far from 0 and (14) to find

$$\begin{aligned}
\mathcal{I}_{122} &\le \frac{2\kappa_1}{N}\sum_{\substack{i,j\in[l]\\ k\notin[l]}}|\phi(r_{ki})-\phi(r_{kj})|\left|\left\langle\frac{v_k}{\theta_k}-\frac{v_j}{\theta_j}, v_i-v_j\right\rangle\right|\\
&\le \frac{2\kappa_1}{N}\sum_{\substack{i,j\in[l]\\ k\notin[l]}}|\phi(r_{ki})-\phi(r_{kj})|\left|\frac{v_k-v_j}{\theta_k}+v_j\left(\frac{1}{\theta_k}-\frac{1}{\theta_j}\right)\right||v_i-v_j|\\
&\le \frac{2\kappa_1}{N}\sum_{\substack{i,j\in[l]\\ k\notin[l]}}L_\delta|x_i-x_j|\left(\frac{D(V)}{\theta_m^0}\left(1+\frac{D(\Theta)}{\theta_m^0}\right)\right)|v_i-v_j|\\
&\le \frac{2\kappa_1L_\delta(N-|[l]|)}{N}\frac{C_{t_0}}{\theta_m^0}\left(1+\frac{D(\Theta^0)}{\theta_m^0}\right)\sum_{i,j\in[l]}|x_i-x_j||v_i-v_j|\\
&\le \frac{2\kappa_1L_\delta(N-|[l]|)}{N}\frac{C_{t_0}}{\theta_m^0}\left(1+\frac{D(\Theta^0)}{\theta_m^0}\right)\tilde{C}_{t_0}|[l]|\|V\|_{[l]},
\end{aligned}\tag{16}$$

where L_δ is a positive constant defined by the following relation:

$$L_\delta := ||\phi||_{\mathrm{Lip}(|r|\ge\delta)}.$$

Now we combine (15) and (16) to obtain

$$\begin{aligned}
\mathcal{I}_{12} &\le \frac{2\kappa_1(N-|[l]|)}{N\delta^\alpha(\theta_m^0)^2}D(\Theta^0)C_{t_0}|[l]|\|V\|_{[l]}\\
&\quad+\frac{2\kappa_1L_\delta(N-|[l]|)}{N}\frac{C_{t_0}}{\theta_m^0}\left(1+\frac{D(\Theta^0)}{\theta_m^0}\right)\tilde{C}_{t_0}|[l]|\|V\|_{[l]}\\
&=: 2C_3\|V\|_{[l]}.
\end{aligned}$$

Finally, we combine all the above estimates to get

$$\frac{d}{dt}\|V\|_{[l]}^2 \le \mathcal{I}_{11}+\mathcal{I}_{12} \le -2C_1\phi(\|x\|_{[l]})\|V\|_{[l]}^2+2C_2\mathcal{P}^{1/2}\|V\|_{[l]}+2C_3\|V\|_{[l]},$$

or equivalently,

$$\frac{d}{dt}\|V\|_{[l]} \le -C_1\phi(\|X\|_{[l]})\|V\|_{[l]}+C_2\sqrt{\mathcal{P}}+C_3.$$

□

4.2. **Proof of the first part in Theorem 3.1.** We are now ready to provide the proof of the first part of Theorem 3.1. We apply Grönwall's inequality for $(13)_2$ to yield

$$\begin{aligned}\|V\|_{[l]}(t) &\le \|V\|_{[l]}(s)e^{-C_1\int_s^t \phi(\|X\|_{[l]}(\tau))d\tau} \\ &\quad + \int_s^t \Big(C_2\sqrt{\mathcal{P}(\tau)}+C_3\Big)e^{-C_1\int_\tau^t \phi(\|X\|_{[l]}(\sigma)d\sigma)}d\tau.\end{aligned} \tag{17}$$

We set

$$\Phi(x) := \int^x \phi(y)\,dy. \tag{18}$$

On the other hand, we use $(13)_1$ and to obtain

$$|\Phi(\|X\|_{[l]}(t))| \le \underbrace{\int_s^t \phi(\|X\|_{[l]}(\tau))\|V\|_{[l]}(\tau)d\tau}_{=:\mathcal{J}(s,t)} + |\Phi(\|X\|_{[l]}(s))|. \tag{19}$$

Next, we claim:

$$|\mathcal{J}(t,s)| \le C_{\mathcal{J}} \quad \text{for } 0 \le s,t \le t_0.$$

Proof of claim: First, we set

$$\mathcal{B}(s,t) := e^{-C_1\int_s^t \phi(\|X\|_{[l]}(\sigma))d\sigma}.$$

Then, it is easy to see that

$$\partial_t\mathcal{B} = -C_1\phi(\|X\|_{[l]}(t))\mathcal{B}(s,t) \quad \text{and} \quad \mathcal{B}(\tau,t)\mathcal{B}(s,\tau) = \mathcal{B}(s,t), \quad \text{for } s\le\tau\le t. \tag{20}$$

Note that

$$\begin{aligned}\|V\|_{[l]}(t) &\le \|V\|_{[l]}(s)\mathcal{B}(s,t) + \int_s^t (C_2\sqrt{\mathcal{P}(\tau)}+C_3)\mathcal{B}(\tau,t)\,d\tau \\ &\le C_4\mathcal{B}(s,t) + \int_s^t (C_2\sqrt{\mathcal{P}(\tau)}+C_3)\mathcal{B}(\tau,t)\,d\tau.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathcal{J}(s,t) &\le \int_s^t \phi(\|X\|_{[l]}(\tau))\Big[C_4\mathcal{B}(s,\tau) + \int_s^\tau (C_2\sqrt{\mathcal{P}(\sigma)}+C_3)\mathcal{B}(\sigma,\tau)\,d\sigma\Big]\,d\tau \\ &=: \mathcal{I}_{31}+\mathcal{I}_{32}.\end{aligned} \tag{21}$$

Next, we estimate the terms $\mathcal{I}_{3i},\ i=1,2$ separately.
• (Estimation for $\mathcal{I}_{31}$): We use (20) together with the fact $\mathcal{B}(s,s)=1$ and $\mathcal{B}\ge 0$ to find

$$\mathcal{I}_{31} = C_4\int_s^t \phi(\|X\|_{[l]}(\tau))\mathcal{B}(\tau,s)\,d\tau = -\frac{C_4}{C_1}\int_s^t \partial_\tau(\mathcal{B}(\tau,s))\,d\tau \le \frac{C_4}{C_1}. \tag{22}$$

• (Estimation for $\mathcal{I}_{32}$): By direct calculation, we have

$$
\begin{aligned}
\mathcal{I}_{32} &= \int_s^t \phi(\|X\|_{[l]}(\tau))\Big[\int_s^\tau (C_2\sqrt{\mathcal{P}(\sigma)}+C_3)\frac{\mathcal{B}(\tau,s)}{\mathcal{B}(\sigma,s)}\,d\sigma\Big]d\tau \\
&= \int_s^t \phi(\|x\|_{[l]}(\tau))\mathcal{B}(\tau,s)\Big[\int_s^\tau (C_2\sqrt{\mathcal{P}(\sigma)}+C_3)\frac{1}{\mathcal{B}(\sigma,s)}\,d\sigma\Big]\,d\tau \\
&= -\frac{1}{C_1}\mathcal{B}(t,s)\int_s^t \Big(C_2\sqrt{\mathcal{P}(\sigma)}+C_3\Big)\frac{1}{\mathcal{B}(\sigma,s)}\,d\sigma \\
&+ \frac{1}{C_1}\int_s^t \Big[C_2\sqrt{\mathcal{P}(\tau)}+C_3\Big]\frac{\mathcal{B}(\tau,s)}{\mathcal{B}(\tau,s)}\,d\tau \\
&\le \frac{1}{C_1}\int_s^t \Big(C_2\sqrt{\mathcal{P}(\tau)}+C_3\Big)\,d\tau \\
&\le C_5,
\end{aligned} \tag{23}
$$

where we used integration by parts and $\int_0^t \mathcal{P}(s)ds < \infty$ in Lemma 2.2. In (21), we combine estimates (22) and (23) to obtain

$$\mathcal{J}(s,t) \le \frac{C_4}{C_1} + C_5.$$

However, if $\alpha \ge 1$,

$$\Phi(s) = \begin{cases} \dfrac{s^{1-\alpha}}{1-\alpha} & \text{if } \alpha > 1, \\ \log s & \text{if } \alpha = 1. \end{cases}$$

Hence we have

$$|\Phi(\|X\|_{[l]}(t))| \to \infty \quad \text{as } t \to t_0^-. \tag{24}$$

On the other hand in (19), we have

$$|\Phi(\|X\|_{[l]}(t))| < \infty,$$

which is contradictory to (24). □

4.3. **Proof of the second part in Theorem 3.1.** In this section, we provide some initial configurations leading to collision between TCS particles in finite time. Note that if all initial temperatures are the same, then the TCS system (1)-(2) can be reduced to the particle C-S model with singular communication weight which has been extensively studied in [1, 2, 3, 26, 25]. In particular, the authors in [26] showed that the finite time collision between the Cucker-Smale particles with singular weights under certain assumptions on the initial configurations. For the same initial temperatures, as a direct application of [26, Proposition 3.1], we have a counterexample leading to the finite time collision.

More precisely, consider a two-body system on the real line, and its initial configuration $(x_i^0, v_i^0, \theta_i^0)$, $i = 1, 2$ is given by the following conditions:

$$x_1^0 > x_2^0, \qquad v_2^0 - v_1^0 = \frac{\kappa_1}{\theta_0(1-\alpha)}(x_1^0 - x_2^0)^{1-\alpha}, \qquad \theta_1^0 = \theta_2^0 =: \theta^0 > 0, \quad 0 < \alpha < 1.$$

Then there exists a finite time $t_c < \infty$ such that $x_1(t_c) = x_2(t_c)$ (see [26] for detailed argument). Inspired by the above observation, we can construct the initial configurations leading to the finite time collision between TCS particles even for the case where the initial temperatures can be different from each other. For this, we again consider a two-body system on the real line:

$$
\begin{aligned}
&\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad t > 0,\\
&\dot{v}_1 = \frac{\kappa_1}{2}\phi(|x_1 - x_2|)\left(\frac{v_2}{\theta_2} - \frac{v_1}{\theta_1}\right), \quad \dot{v}_2 = \frac{\kappa_1}{2}\phi(|x_1 - x_2|)\left(\frac{v_1}{\theta_1} - \frac{v_2}{\theta_2}\right),\\
&\dot{\theta}_1 = \frac{\kappa_2}{2}\zeta(|x_1 - x_2|)\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right), \quad \dot{\theta}_2 = \frac{\kappa_2}{2}\zeta(|x_1 - x_2|)\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right),\\
&\phi(|x_1 - x_2|) = \frac{1}{|x_1 - x_2|^{\alpha}}, \quad \zeta(|x_1 - x_2|) = \frac{1}{|x_1 - x_2|^{\beta}}, \quad 0 < \alpha < 1, \quad \beta > 0.
\end{aligned} \tag{25}
$$

In the sequel, we will show that there exists an initial configuration leading to the finite time collision for the system (25).

Next, we define the difference of positions, velocities and temperatures of two particles as follows:

$$
x(t) := x_1(t) - x_2(t), \qquad v(t) := v_1(t) - v_2(t), \qquad \theta(t) := \theta_1(t) - \theta_2(t).
$$

Then the TCS dynamics can be rewritten in terms of (x, v, θ):

$$
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \kappa_1\phi(x)\left(\frac{v_2\theta}{\theta_1\theta_2} - \frac{v}{\theta_1}\right), \quad \frac{d\theta}{dt} = -\kappa_2\zeta(x)\frac{\theta}{\theta_1\theta_2}, \quad t > 0. \tag{26}
$$

To show the finite-time collisions in one-dimensional setting for the two-particle system, we consider the following initial configuration (see Figure 1):

$$
x_2^0 < x_1^0, \qquad v_2^0 > 0 > v_1^0, \qquad \theta_2^0 > \theta_1^0.
$$

FIGURE 1. Initial configuration

In the sequel, we will show that as long as there is no collisions between two particles, the ordering of velocity and temperatures will remain as it is, i.e., before the collision, we will have

$$
v_2(t) > 0, \quad \theta_2(t) > \theta_1(t), \quad \text{i.e.,} \quad \frac{v_2\theta}{\theta_1\theta_2} > 0.
$$

• **Step A:** First of all, we will prove $\theta_1(t) < \theta_2(t)$ until the collision happens if $0 < \theta_1^0 < \theta_2^0$. To show this, suppose that there exists $0 < t_* < \infty$ such that $\theta_1(t_*) = \theta_2(t_*)$ and $\theta_1(t) < \theta_2(t)$ for $0 \le t < t_*$. Moreover, suppose $|x(t)| > \delta$ for $0 \le t \le t_*$ to assure that there is no collision up to time t_*. Then from the equation (25), we get

$$
\frac{d\theta}{dt} = -\kappa_2\zeta(x)\frac{\theta}{\theta_1\theta_2} \le -\kappa_2\zeta(\delta)\frac{\theta}{(\theta_1^0)^2}, \quad 0 < t < t_*.
$$

Now, we use Grönwall's lemma to get

$$
\theta(t) \le \theta^0 \exp\left(-\frac{\kappa_2\zeta(\delta)t}{(\theta_1^0)^2}\right), \quad 0 \le t < t_*.
$$

This implies $\theta(t_*) \le \theta^0 \exp\left(-\frac{\kappa_2\zeta(\delta)t_*}{(\theta_1^0)^2}\right) < 0$, which is a contradiction to $\theta(t_*) = 0$. Thus we obtain that $\theta_1(t) < \theta_2(t)$ until the collision occurs.

• **Step B:** Next we will show that the velocity of each particle maintains its sign until the collision occurs. More precisely, we will show $v_2(t) > 0$. To show this, due to the zero-mean velocity condition, it suffices to show $v(t) < 0$ until the collision. Similar to Step A, we again assume that there exists $t_* < \infty$ such that

$$v(t_*) = 0, \quad \text{and} \quad v(t) < 0, \ |x(t)| > \delta, \quad \text{for} \quad 0 \le t < t_*.$$

Then it follows from (26) that

$$\frac{dv}{dt} = \kappa_1 \phi(x) \left(\frac{v_2 \theta}{\theta_1 \theta_2} - \frac{v}{\theta_1} \right) \le -\frac{\kappa_1 \phi(x) v}{\theta_1} \le -\frac{\kappa_1 \phi(\delta) v}{\theta_1^0}, \quad 0 < t < t_*.$$

Here, we use the fact that $v(t) < 0$ and $\theta(t) < 0$ for $0 \le t < t_*$. Again, it follows from Grönwall's lemma that we have

$$v(t) \le v^0 \exp\left(-\frac{\kappa_1 \phi(\delta) t}{\theta_1^0} \right), \quad 0 \le t < t_*,$$

which is a contradictory to $v(t_*) = 0$ since $v^0 < 0$. Hence, we have $v(t) < 0$ and consequently $v_2(t) > 0$ until the collision occurs.

• **Step C:** So far, we have obtained $v_2 > 0$ and $\theta < 0$ as long as there is no collisions between TCS particles. Thus, we have

$$\frac{v_2 \theta}{\theta_1 \theta_2} < 0,$$

This together with $(26)_2$ yields a differential inequality for v:

$$\frac{dv}{dt} \le -\frac{\kappa_1 \phi(x) v}{\theta_1} \le -\frac{\kappa_1 \phi(x) v}{\theta_1^0}.$$

Now, we set

$$\Phi(r) := \int^r \phi(s)\, ds = \frac{1}{1-\alpha} r^{1-\alpha},$$

i.e., Φ is a primitive of ϕ. Then the above differential inequality can be rewritten as

$$\frac{dv}{dt} \le -\frac{\kappa_1}{\theta_1^0} \frac{d}{dt} \left(\Phi(x) \right).$$

We integrate the above equation from 0 to $t > 0$ to obtain

$$v(t) - v^0 \le -\frac{\kappa_1}{\theta_1^0} (\Phi(x(t)) - \Phi(x^0)). \tag{27}$$

On the other hand, under our main assumptions in Theorem 3.1, we find

$$v^0 = -\frac{\kappa_1}{\theta_1^0 (1-\alpha)} (x_1^0 - x_2^0)^{1-\alpha} = -\frac{\kappa_1}{\theta_1^0} \Phi(x^0).$$

Thus, (27) is again reduced to the following sub-linear differential inequality:

$$\frac{dx}{dt} = v(t) \le -\frac{\kappa_1}{\theta_1^0} \Phi(x(t)) \le -\frac{\kappa_1}{\theta_1^0 (1-\alpha)} (x(t))^{1-\alpha},$$

which is equivalent to

$$\frac{d}{dt} \left(x(t)^\alpha \right) \le -\frac{\kappa_1 \alpha}{\theta_1^0 (1-\alpha)}.$$

This yields

$$x(t)^\alpha \le x_0^\alpha - \frac{\kappa_1 \alpha}{\theta_1^0 (1-\alpha)} t.$$

Hence, we have that the collision will occur at some time t_0 earlier than $(x_0^\alpha \theta_1^0(1-\alpha))/(\kappa_1\alpha)$. ☐

4.4. **Proof of Theorem 3.3.** In this subsection, we study a mono-clustering of the thermomechanical Cucker-Smale model (1)-(2). Note that the global existence of solutions is guaranteed by Theorem 3.1. The proof of Theorem 3.3 is exactly the same as in [12, Theorem 3.1]. More precisely, in [12], system (1)-(2) with regular weights are taken into account and the asymptotic emergent behavior is also obtained. However, the strategy used in [12] does not depend on the singularity of ϕ or ζ, thus we can directly apply it to our system with singular communications. Thus, we briefly sketch the details of the proof of Theorem 3.1 here. Below, we sketch the proof in two steps:

• **Step A** (Derivation of differential inequalities): We first derive a system of differential inequalities for the extreme values for positions, velocities, and temperatures as follows:

$$\begin{aligned}
&\left|\frac{dD(X)}{dt}\right| \le D(V), \quad t>0,\\
&\frac{dD(V)}{dt} \le -\frac{\kappa_1\phi(D(X))}{\theta_M^0}D(V) + 2\frac{\kappa_1}{(\theta_m^0)^2}D(\Theta)D(V),\\
&\frac{dD(\Theta)}{dt} \le -\frac{\kappa_2\zeta(D(X))}{(\theta_M^0)^2}D(\Theta).
\end{aligned} \tag{28}$$

• **Step B** (Exponential flocking from the SDDI) : The next step is showing the exponential flocking from the SDDI (28). To do this, we first assume that the following conditions for initial configuration hold: Suppose that there exist $X^\infty \ge 0$ such that

$$bD(\Theta(0)) \le \frac{a\phi(D(X(0)))}{4}, \quad D(V(0)) \le \frac{a}{2}\int_{D(X(0))}^{X^\infty}\phi(s)\,ds,$$

and

$$\frac{2}{3}\phi(D(X(0))) \le \phi(X^\infty) < \phi(DX(0)).$$

Then, we use the bootstrapping argument to conclude the following flocking estimation:

$$\sup_{0\le t<\infty} D(X(t)) \le X^\infty, \quad D(V(t)) \le D(V(0))e^{-a\phi(X^\infty)+\frac{bD(\Theta(0))}{c\zeta(X^\infty)}},$$

and

$$D(\Theta(t)) \le D(\Theta(0))e^{-c\zeta(X^\infty)t}, \quad t\ge 0.$$

☐

5. **Local well-posedness of the kinetic TCS equation.** In this section, we provide a local-in-time well-posedness of weak solutions (see Definition 3.4) to the kinetic TCS equation:

$$\begin{aligned}
&\partial_t f + \nabla_x\cdot(vf) + \nabla_v\cdot(\mathcal{F}[f]f) + \partial_\theta(\mathcal{G}[f]f) = 0, \quad x,v\in\mathbb{R}^d,\ \theta\in\mathbb{R}_+,\ t>0,\\
&\mathcal{F}[f](z,t) := -\kappa_1\int_{\mathbb{R}^{2d}\times\mathbb{R}_+}\phi(x-x_*)\Big(\frac{v}{\theta}-\frac{v_*}{\theta_*}\Big)f(z_*,t)\,dz_*,\\
&\mathcal{G}[f](x,\theta,t) := \kappa_2\int_{\mathbb{R}^{2d}\times\mathbb{R}_+}\zeta(x-x_*)\Big(\frac{1}{\theta}-\frac{1}{\theta_*}\Big)f(z_*,t)\,dz_*,
\end{aligned} \tag{29}$$

where the interaction kernels are given as follows:

$$\phi(s) := \frac{1}{s^\alpha}, \quad \zeta(s) = \frac{1}{s^\beta}, \quad \text{with} \quad \alpha, \beta > 0.$$

The existence of weak solutions to (29) will be obtained via a suitable weak limit of the regularized system for (29). To do so, we introduce a radially symmetric standard mollifiers $\eta \in C_c^\infty(\mathbb{R}^d)$ and its scaled family:

$$\eta(x) = \tilde{\eta}(|x|) \geq 0, \quad \operatorname{supp} \eta \subset B_1(0), \quad \int_{\mathbb{R}^d} \eta(x)\,dx = 1, \quad \eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\Big(\frac{x}{\varepsilon}\Big).$$

Now, we use this family of mollifier to mollify the communication kernels:

$$\phi_\varepsilon := \phi * \eta_\varepsilon \quad \text{and} \quad \zeta_\varepsilon = \zeta * \eta_\varepsilon \quad \text{for each } \varepsilon > 0.$$

With the regularized weights ϕ_ε and ζ_ε, we have the regularized system:

$$\begin{aligned}
&\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \nabla_v\cdot[\mathcal{F}^\varepsilon(f_\varepsilon)f_\varepsilon] + \partial_T[\mathcal{G}^\varepsilon(f_\varepsilon)f_\varepsilon] = 0, \quad x, v \in \mathbb{R}^d,\ \theta\in\mathbb{R}_+,\ t>0,\\
&\mathcal{F}^\varepsilon(f_\varepsilon)(z,t) := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_\varepsilon(|x-x_*|)\Big(\frac{v_*}{\theta_*} - \frac{v}{\theta}\Big) f_\varepsilon(z_*,t)\,dz_*,\\
&\mathcal{G}^\varepsilon(f_\varepsilon)(x,\theta,t) := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \zeta_\varepsilon(|x-x_*|)\Big(\frac{1}{\theta} - \frac{1}{\theta_*}\Big) f_\varepsilon(z_*,t)\,dz_*,\\
&f_\varepsilon(z,0) =: f^0(z).
\end{aligned} \tag{30}$$

Note that the global-in-time existence of solution to (30) can be proved by using standard method of characteristics since all of the kernels are regular enough.

5.1. **A priori estimates.** We next provide uniform estimates for f in L^p-norm and the velocity support of f with respect to the regularization parameter ε. For this, we present several lemmas.

Lemma 5.1. *For $p \in [1,\infty)$, let X and Y be two positive differentiable functions satisfying*

$$\frac{dX(t)}{dt} \leq CX^2(t)(Y^{\frac{d}{p'}}(t)+1), \quad \frac{dY(t)}{dt} \leq CX(t)Y(t)(Y^{\frac{d}{p'}}(t)+1), \quad t>0,$$

where C is a positive constant and p' is Hölder conjugate of p. Then, there exist $\tau < \infty$ and positive constant C such that

$$\sup_{t\in[0,\tau]} (X(t)+Y(t)) \leq C.$$

Proof. We set $\tilde{X}$ and $\tilde{Y}$:

$$\tilde{X} := X+1, \quad \tilde{Y} := Y+1.$$

Then, $\tilde{X}$ and $\tilde{Y}$ satisfy the following differential inequalities:

$$\begin{aligned}
\frac{d\tilde{X}}{dt} &= \frac{dX}{dt} \leq CX^2(Y^{\frac{d}{p'}}+1) \leq C\tilde{X}^2\tilde{Y}^{\frac{d}{p'}}, \quad t>0,\\
\frac{d\tilde{Y}}{dt} &= \frac{dY}{dt} \leq CXY(Y^{\frac{d}{p'}}+1) \leq C\tilde{X}\tilde{Y}^{\left(1+\frac{d}{p'}\right)}.
\end{aligned}$$

This yields

$$\frac{d}{dt}(\tilde{X}+\tilde{Y}) \leq C\tilde{X}\tilde{Y}^{\frac{d}{p'}}(\tilde{X}+\tilde{Y}) \quad \text{for} \quad t \geq 0. \tag{31}$$

On the other hands, we use Young's inequality to get

$$\tilde{X}\tilde{Y}^{\frac{d}{p'}} \leq C(\tilde{X}^{\frac{d}{p'+d}} + \tilde{Y}^{\frac{d}{p'+d}}) \leq C(\tilde{X}+\tilde{Y})^{\frac{d}{p'+d}}.$$

This together with (32) gives

$$\frac{d}{dt}(\tilde{X}+\tilde{Y}) \leq C(\tilde{X}+\tilde{Y})^{1+\frac{d}{p'+d}}.$$

Therefore, although the value $\tilde{X}+\tilde{Y}$ may blow up at the finite time $\tilde{t}$, we can choose smaller time $\tau < \tilde{t}$ so that we have the following uniform bound for $\tilde{X}+\tilde{Y}$ in time interval $[0,\tau]$

$$\sup_{0\leq t\leq\tau}(X(t)+Y(t)) \leq \sup_{0\leq t\leq\tau}(\tilde{X}(t)+\tilde{Y}(t)) \leq C,$$

which yields the desired estimate. □

In next lemma, we show that $\|f_\varepsilon\|_{L^1\cap L^p}$ and the velocity support of f satisfy the system of differential inequalities in Lemma 5.1. For $t\geq 0$, we set the velocity support R_v^ε:

$$R_v^\varepsilon := \max\left\{|v_0| : v_0 \in \overline{\{v\in\mathbb{R}^d : \exists z\in\mathbb{R}^{2d}\times\mathbb{R}_+ \text{ such that } f_\varepsilon(z,t)\neq 0\}}\right\}.$$

Lemma 5.2. *Let $f_\varepsilon = f_\varepsilon(z,t)$ be a solution for the regularized system (30). Then, there exists a positive constant $C>0$ independent of ε such that*

$$\begin{aligned}
&\frac{d}{dt}\|f_\varepsilon\|_{L^1\cap L^p} \leq C\left[(R_v^\varepsilon)^{\frac{d}{p'}}+1\right]\|f_\varepsilon\|^2_{L^1\cap L^p}, \quad t>0,\\
&\frac{dR_v^\varepsilon}{ds} \leq CR_v^\varepsilon\left[(R_v^\varepsilon)^{\frac{d}{p'}}+1\right]\|f_\varepsilon\|_{L^1\cap L^p}.
\end{aligned} \tag{32}$$

Proof. Below, we will derive the differential inequalities one by one.
- (Derivation of $(32)_1$): We use (3) to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon^p\,dz = -(p-1)\int_{\mathbb{R}^{2d}\times\mathbb{R}_+}(\nabla_v\cdot(\mathcal{F}^\varepsilon[f_\varepsilon])+\partial_\theta\mathcal{G}^\varepsilon[f_\varepsilon])f_\varepsilon^p\,dz. \tag{33}$$

To estimate the R.H.S. of (33), we use the standard cutoff function $\chi_1\in C_c^\infty(\mathbb{R}^d)$:

$$\chi_1(x) := \begin{cases} 1 & |x|\leq 1,\\ 0 & |x|>2. \end{cases}$$

We use a similar strategy in [2] to estimate $\|\nabla_v\cdot(\mathcal{F}^\varepsilon[f_\varepsilon])\|_\infty$. For the convenience of reader, we provide the detailed calculation. We separate $\phi_\varepsilon = \phi*\eta_\epsilon$ as

$$\phi*\eta_\epsilon = (\phi\chi_1)*\eta_\varepsilon + (\phi(1-\chi_1))*\eta_\varepsilon$$

and use Young's convolution inequality to get following inequalities:

$$\begin{aligned}
&\|(\phi\chi_1)*\eta_\varepsilon\|_{L^{p'}} \leq \|\phi\chi_1\|_{L^{p'}}\|\eta_\varepsilon\|_{L^1} = \|\phi\chi_1\|_{L^{p'}} < C,\\
&\|(\phi(1-\chi_1))*\eta_\varepsilon\|_{L^\infty} \leq \|\phi(1-\chi_1)\|_{L^\infty} \leq 1.
\end{aligned} \tag{34}$$

Now, thanks to the boundedness of velocity support and (34), we have

$$
\begin{aligned}
&|\nabla_v \cdot (\mathcal{F}^\varepsilon[f_\varepsilon])| \\
&\quad = \frac{d}{\theta} \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_\varepsilon(|x-x_*|) f_\varepsilon(z_*,t)\, dz_* \\
&\quad \le \frac{d}{\theta_m}\Big(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |(\phi\chi_1)*\eta_\varepsilon||f_\varepsilon|\, dz_* + \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |(\phi(1-\chi_1))*\eta_\varepsilon||f_\varepsilon|\, dz_* \Big) \\
&\quad \le C\Big(\|(\phi\chi_1)*\eta_\varepsilon(x) 1_{B(0,R_v^\varepsilon)}(v)\|_{L^{p'}} \|f\|_{L^p} + \|(\phi(1-\chi_1))*\eta_\varepsilon\|_{L^\infty}\|f_\varepsilon\|_1 \Big) \\
&\quad \le C(R_v)^{\frac{d}{p'}} \|\phi\chi_1\|_{L^{p'}}\|f_\varepsilon\|_{L^p} + \|\phi(1-\chi_1)\|_{L^\infty}\|f_\varepsilon\|_1 \\
&\quad = C\Big((R_v)^{\frac{d}{p'}} + 1 \Big) \|f_\varepsilon\|_{L^1\cap L^p},
\end{aligned}
$$

for some $C>0$ which is independent of ε. Similarly, we also estimate $\|\partial_\theta(\mathcal{G}^\varepsilon[f_\varepsilon])\|_\infty$ as

$$
\begin{aligned}
|\partial_\theta \mathcal{G}^\varepsilon[f_\varepsilon]| &\le \frac{1}{\theta_m^2}\Big(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |(\zeta\chi_1)*\eta_\varepsilon||f_\varepsilon|\, dz_* + \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |(\zeta(1-\chi_1))*\eta_\varepsilon||f_\varepsilon|\, dz_* \Big) \\
&\le C(R_v)^{\frac{d}{p'}} \|\zeta\chi_1\|_{L^{p'}}\|f_\varepsilon\|_{L^p} + \|\zeta(1-\chi_1)\|_{L^\infty}\|f_\varepsilon\|_1 \\
&= C\Big((R_v)^{\frac{d}{p'}} + 1 \Big) \|f_\varepsilon\|_{L^1\cap L^p}.
\end{aligned}
$$

Thus, we have

$$
\frac{d}{dt}\|f_\varepsilon\|_{L^1\cap L^p} \le C\Big((R_v)^{\frac{d}{p'}} + 1 \Big) \|f_\varepsilon\|_{L^1\cap L^p}^2 \quad \text{for} \quad t>0,
$$

where $C>0$ is independent of ε.

• (Derivation of $(32)_2$): Consider a characteristic curve similarly defined as in (9), generated by approximated solutions f_ε and g_ε. Then along that specific characteristic curve which gives the maximum modulus of velocity, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{ds}(R_v^\varepsilon)^2 &\le \frac{(R_v^\varepsilon)^2(s)}{\theta_m} \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi(|x(s)-x_*|) f(y,v_*,\theta_*,s)\, dz_* \\
&\le C(R_v^\varepsilon)^2 \left((R_v^\varepsilon)^{\frac{d}{p'}} + 1 \right) \|f_\varepsilon\|_{L^1\cap L^p}.
\end{aligned}
$$

Note that in the last inequality, we used similar estimate as in the previous step. Thus we have

$$
\frac{dR_v^\varepsilon}{ds} \le CR_v^\varepsilon \left((R_v^\varepsilon)^{\frac{d}{p'}} + 1 \right) \|f_\varepsilon\|_{L^1\cap L^p},
$$

where $C>0$ is independent of ε. □

By direct applications of Lemma 5.1 and Lemma 5.2, we have the following uniform bound estimates and stability estimate.

Proposition 2. *The following assertions hold.*

1. *(Uniform boundedness): Let f_ε be a solution of the regularized system (30). Then there exist a positive constant τ such that uniform $L^1\cap L^p$-estimate of f_ε and the boundedness of velocity support hold:*

$$
\sup_{t\in[0,\tau]} \|f_\varepsilon\|_{L^1\cap L^p} \le C, \quad R_v(t) := \sup_{t\in[0,\tau]} |\Omega_v(t)| \le C,
$$

where C is a positive constant independent of ε.

2. *Let f_ε and $f_{\varepsilon'}$ be two solutions of the system* (29). *Then there exists C independent of ε and ε' such that*

$$\frac{d}{dt}W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le C(W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) + \varepsilon + \varepsilon'), \quad \forall\ 0 \le t < \tau.$$

Proof. (1) The uniform boundedness follow from Lemma 5.1 and Lemma 5.2.
(2) The stability estimate can be done as for the regular case. First, we define the family of characteristic curves $Z_\varepsilon(s) := (x_\varepsilon(s), v_\varepsilon(s), \theta_\varepsilon(s))$ as a solution to the following ODEs:

$$\begin{aligned}
&\frac{d}{dt}x_\varepsilon(t; s, x, v, \theta) = v_\varepsilon(t; s, x, v, \theta), \quad 0 \le s \le t\\
&\frac{d}{dt}v_\varepsilon(t; s, x, v, \theta) = \mathcal{F}^\varepsilon[f_\varepsilon](Z_\varepsilon(t; s, x, v, \theta), t),\\
&\frac{d}{dt}\theta_\varepsilon(t; s, x, v, \theta) = \mathcal{G}^\varepsilon[f_\varepsilon](x_\varepsilon(t; s, x, v, \theta), \theta_\varepsilon(t; s, x, v, \theta), t),
\end{aligned} \tag{35}$$

and define $Z_{\varepsilon'}$ in similar way. Note that Z_ε is well-defined since ϕ_ε and ζ_ε are regular kernels. Now, we define the optimal transport map

$$\mathcal{T}^0(x, v, \theta) = (\mathcal{T}_1^0(x, v, \theta), \mathcal{T}_2^0(x, v, \theta), \mathcal{T}_3^0(x, v, \theta))$$

between $f_\varepsilon(t_0)$ and $f_{\varepsilon'}(t_0)$, i.e., $f_{\varepsilon'}(t_0) = \mathcal{T}^0 \# f_\varepsilon(t_0)$. Moreover, as in [17], we can obtain $f_\varepsilon(t) = Z_\varepsilon(t; t_0, \cdot, \cdot, \cdot)\# f_\varepsilon(t_0)$. We combine two observations to get

$$\mathcal{T}^t \# f_\varepsilon(t) = f_{\varepsilon'}(t), \quad \text{where} \quad \mathcal{T}' := Z_{\varepsilon'}(t; t_0, \cdot, \cdot, \cdot) \circ \mathcal{T}^0 \circ Z_\varepsilon(t_0; t, \cdot, \cdot, \cdot)$$

Now, it follows from the Proposition 1 that we have

$$W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |Z_\varepsilon(t; t_0, z) - Z_{\varepsilon'}(t; t_0, \mathcal{T}^0(z))| f_\varepsilon(z, t_0)\, dz =: Q_{\varepsilon,\varepsilon'}(t).$$

Then, it follows from (35) that we have

$$\begin{aligned}
&\frac{d}{dt}Q_{\varepsilon,\varepsilon'}(t)\Big|_{t=t_0+}\\
&\le \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \Big| v_\varepsilon(t; t_0, z) - v_{\varepsilon'}(t; t_0, \mathcal{T}^0(z))\Big| f_\varepsilon(z, t_0)\, dz\Big|_{t=t_0+}\\
&\quad + \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \Big| \mathcal{F}^\varepsilon[f_\varepsilon](Z_\varepsilon(t; t_0, z), t) - \mathcal{F}^{\varepsilon'}[f_{\varepsilon'}](Z_{\varepsilon'}(t; t_0, \mathcal{T}^0(z)), t)\Big| f_\varepsilon(z, t_0)\, dz\Big|_{t=t_0+}\\
&\quad + \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \Big| \mathcal{G}^\varepsilon[f_\varepsilon](x_\varepsilon(t; t_0, z), \theta_\varepsilon(t; t_0, z), t)\\
&\qquad\qquad - \mathcal{G}^{\varepsilon'}[f_{\varepsilon'}](x_{\varepsilon'}(t; t_0, \mathcal{T}^0(z)), \theta_{\varepsilon'}(t; t_0, \mathcal{T}^0(z)), t)\Big| f_\varepsilon(z, t_0)\, dz\Big|_{t=t_0+}\\
&=: \mathcal{I}_{41} + \mathcal{I}_{42} + \mathcal{I}_{43}.
\end{aligned}$$

Below, we estimate the terms $\mathcal{I}_{4i}$ separately.
• (Estimate of $\mathcal{I}_{41}$): By direct estimate, we have

$$\mathcal{I}_{41} = \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |v - \mathcal{T}_2^0(z)| f_\varepsilon(z, t_0)\, dz \le C W_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)).$$

• (Estimate of $\mathcal{I}_{42}$): We separate $\mathcal{I}_2$ in two parts to get

$$\mathcal{I}_{42} = \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \Big| \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_\varepsilon(x-y)\Big(\frac{v_*}{\theta_*} - \frac{v}{\theta}\Big) f_\varepsilon(z_*, t_0)\, dz_*$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_{\varepsilon'}(\mathcal{T}_1^0(z)-y)\left(\frac{v_*}{\theta_*}-\frac{\mathcal{T}_2^0(z)}{\mathcal{T}_3^0(z)}\right) f_{\varepsilon'}(z_*,t_0)\,dz_*\Big| f_\varepsilon(z,t_0)\,dz \\
&= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \Big| \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_\varepsilon(x-y)\left(\frac{v_*}{\theta_*}-\frac{v}{\theta}\right) f_\varepsilon(z_*,t_0)\,dz_* \\
&- \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_{\varepsilon'}(\mathcal{T}_1^0(z)-\mathcal{T}_1^0(z_*))\left(\frac{\mathcal{T}_2^0(z_*)}{\mathcal{T}_3^0(z_*)}-\frac{\mathcal{T}_2^0(z)}{\mathcal{T}_3^0(z)}\right) f_{\varepsilon'}(z_*,t_0)\,dz_*\Big| f_\varepsilon(z,t_0)\,dz.
\end{aligned}
$$

Now, we define the further subterms of $\mathcal{I}_{42}$ as

$$
\mathcal{I}_{42} =: \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |\mathcal{I}_{421}+\mathcal{I}_{422}| f_\varepsilon(z,t_0)\,dz \le \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |\mathcal{I}_{421}| + |\mathcal{I}_{422}| f_\varepsilon(z,t_0)\,dz,
$$

where

$$
\begin{aligned}
\mathcal{I}_{421} &:= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \left(\phi_\varepsilon(x-y)-\phi_{\varepsilon'}(\mathcal{T}_1^0(z)-\mathcal{T}_1^0(z_*))\right)\left(\frac{v_*}{\theta_*}-\frac{v}{\theta}\right) f_\varepsilon(z_*,t_0)\,dz_*, \\
\mathcal{I}_{422} &:= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_{\varepsilon'}(\mathcal{T}_1^0(z)-\mathcal{T}_1^0(z_*))\Big(\frac{v_*}{\theta_*}-\frac{v}{\theta}\Big) f_\varepsilon(z_*,t_0)\,dz_* \\
&\quad - \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi_{\varepsilon'}(\mathcal{T}_1^0(z)-\mathcal{T}_1^0(z_*))\left(\frac{\mathcal{T}_2^0(z_*)}{\mathcal{T}_3^0(z_*)}-\frac{\mathcal{T}_2^0(z)}{\mathcal{T}_3^0(z)}\right) f_\varepsilon(z_*,t_0)\,dz_*.
\end{aligned}
$$

$\diamond$ (Estimate of $\mathcal{I}_{421}$): Again, we add and subtract terms to find

$$
\begin{aligned}
\mathcal{I}_{421} &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |(\phi_\varepsilon-\phi_{\varepsilon'})(x-y)| \left|\frac{v_*}{\theta_*}-\frac{v}{\theta}\right| f_\varepsilon(z_*,t_0)\,dz_* \\
&\quad + \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \left|\phi_{\varepsilon'}(x-y)-\phi_{\varepsilon'}(\mathcal{T}_1^0(z)-\mathcal{T}_1^0(z_*))\right| \left|\frac{v_*}{\theta_*}-\frac{v}{\theta}\right| f_\varepsilon(z_*,t_0)\,dz_*.
\end{aligned}
$$

Now we estimate $|\phi_\varepsilon(x)-\phi(x)|$ as

$$
\begin{aligned}
&|\phi_\varepsilon(x)-\phi(x)| \\
&\quad\le \int_{\mathbb{R}^d} |\phi(x-y)-\phi(x)|\theta_\varepsilon(y)\,dy \le 2\int_{\mathbb{R}^d}\Big(\frac{1}{|x|^{1+\alpha}}+\frac{1}{|x-y|^{1+\alpha}}\Big)|y|\theta_\varepsilon(y)dy \\
&\quad\le 2\varepsilon\int_{\{y:|y|\le\varepsilon\}}\Big(\frac{1}{|x|^{1+\alpha}}+\frac{1}{|x-y|^{1+\alpha}}\Big)\theta_\varepsilon(y)\,dy \le \frac{C\varepsilon}{|x|^{1+\alpha}}.
\end{aligned}
\tag{36}
$$

Recall that the velocity support and temperature support have finite diameters at any finite time. Then, we use this fact together with the estimate (36) to obtain

$$
\begin{aligned}
&\int_{(\mathbb{R}^{2d}\times\mathbb{R}_+)^2} |(\phi_\varepsilon-\phi)(x-y)|\left|\frac{v_*}{\theta_*}-\frac{v}{\theta}\right| f_\varepsilon(z_*,t_0)f_\varepsilon(z,t_0)\,dzdz_* \\
&\quad\le C\varepsilon\int_{(\mathbb{R}^d\times\Omega_v(\tau)\times\Omega_T(\tau))^2} \frac{1}{|x-y|^{1+\alpha}} f_\varepsilon(z_*,t_0)f_\varepsilon(z,t_0)\,dzdz_*
\end{aligned}
$$

$$
\begin{aligned}
&\le C\varepsilon \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \Big(\int_{\{y:|x-y|<1\}\times\Omega_v(\tau)\times\Omega_T(\tau)} + \int_{\{y:|x-y|\ge 1\}\times\Omega_v(\tau)\times\Omega_T(\tau)} \\
&\qquad \frac{1}{|x-y|^{1+\alpha}} f_\varepsilon(z_*,t_0)\,dz_* \Big) f_\varepsilon(z,t_0)\,dz \\
&\le C\varepsilon \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \Big[\Big(\int_{\{y:|x-y|<1\}} \frac{1}{|x-y|^{(1+\alpha)p'}}\,dy \Big)^{\frac{1}{p'}} \|f_\varepsilon\|_{L^p} + \|f_\varepsilon\|_{L^1} \Big] f_\varepsilon(z,t_0)\,dz \\
&\le C\varepsilon \|f_\varepsilon\|^2 \\
&\le C\varepsilon.
\end{aligned}
$$

This yields

$$
\int_{(\mathbb{R}^{2d}\times\mathbb{R}_+)^2} |(\phi_\varepsilon - \phi_{\varepsilon'})(x-y)| \left| \frac{v_*}{\theta_*} - \frac{v}{\theta} \right| f_\varepsilon(z_*)\,dz_* \le C(\varepsilon+\varepsilon').
$$

For the second term of $\mathcal{I}_{421}$, we again use the boundedness of velocity and temperature support and change of variables to get

$$
\begin{aligned}
&\int_{(\mathbb{R}^d\times\Omega_v(\tau)\times\Omega_T(\tau))^2} \left|\phi_{\varepsilon'}(x-y) - \phi_{\varepsilon'}(\mathcal{T}_1^0(z) - \mathcal{T}_1^0(z_*))\right| \left| \frac{v_*}{\theta_*} - \frac{v}{\theta} \right| f_\varepsilon(z,t_0) f_\varepsilon(z_*,t_0)\,dz dz_* \\
&\le C \int_{(\mathbb{R}^d\times\Omega_v(\tau)\times\Omega_T(\tau))^2} \frac{|\mathcal{T}_1^0(z) - x|}{|\mathcal{T}_1^0(z) - \mathcal{T}_1^0(z_*)|^{1+\alpha}} f_\varepsilon(z,t_0) f_\varepsilon(z_*,t_0)\,dz dz_* \\
&\quad + C \int_{(\mathbb{R}^d\times\Omega_v(\tau)\times\Omega_T(\tau))^2} \frac{|\mathcal{T}_1^0(z) - x|}{|x-y|^{1+\alpha}} f_\varepsilon(z,t_0) f_\varepsilon(z_*,t_0)\,dz dz_* \\
&\le C \max(\|f_\varepsilon\|, \|f_{\varepsilon'}\|) W_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)).
\end{aligned}
$$

Thus, we combine these estimation to get

$$
\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |\mathcal{I}_{421}| f_\varepsilon(z,t_0)\,dz \le C(W_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)) + \varepsilon + \varepsilon'). \tag{37}
$$

$\diamond$ (Estimate of $\mathcal{I}_{422}$) : Similar to $\mathcal{I}_{421}$, we divide the term into two parts as

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |\mathcal{I}_{422}| f_\varepsilon(z,t_0)\,dz \\
&\quad \le \int_{(\mathbb{R}^{2d}\times\mathbb{R}_+)^2} |\phi_{\varepsilon'}(\mathcal{T}_1^0(z) - \mathcal{T}_1^0(z_*))| \left| \frac{v_*}{\theta_*} - \frac{\mathcal{T}_2^0(z_*)}{\mathcal{T}_3^0(z_*)} \right| f_\varepsilon(z,t_0) f_\varepsilon(z_*,t_0)\,dz dz_* \\
&\qquad + \int_{(\mathbb{R}^{2d}\times\mathbb{R}_+)^2} |\phi_{\varepsilon'}(\mathcal{T}_1^0(z) - \mathcal{T}_1^0(z_*))| \left| \frac{v}{\theta} - \frac{\mathcal{T}_2^0(z)}{\mathcal{T}_3^0(z)} \right| f_\varepsilon(z,t_0) f_\varepsilon(z_*,t_0)\,dz dz_* \\
&\quad =: \mathcal{I}_{4221} + \mathcal{I}_{4222}
\end{aligned}
$$

However, it is easy to see that

$$
\begin{aligned}
\mathcal{I}_{4221} &= \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \left(\int_{\mathbb{R}^d\times\Omega_v(\tau)\times\Omega_T(\tau)} |\phi_{\varepsilon'}(\mathcal{T}_1^0(z) - \mathcal{T}_1^0(z_*))| f_\varepsilon(z,t_0)\,dz \right) \\
&\quad \times \left| \frac{v_*}{\theta_*} - \frac{\mathcal{T}_2^0(z_*)}{\mathcal{T}_3^0(z_*)} \right| f_\varepsilon(z_*,t_0)\,dz_* \\
&\le C\|f_{\varepsilon'}\| \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \left| \frac{v_*}{\theta_*} - \frac{\mathcal{T}_2^0(z_*)}{\mathcal{T}_3^0(z_*)} \right| f_\varepsilon(z_*,t_0)\,dz_* \\
&\le C W_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)).
\end{aligned}
$$

Similarly, we also can estimate $\mathcal{I}_{4222}$ to obtain

$$\mathcal{I}_{4222} \leq CW_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)).$$

Hence, we have

$$\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} |\mathcal{I}_{422}| f_\varepsilon(z,t_0)\,dz \leq \mathcal{I}_{4221} + \mathcal{I}_{4222} \leq CW_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)). \tag{38}$$

We combine (37)-(38) to get

$$\mathcal{I}_{42} \leq C(W_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)) + \varepsilon + \varepsilon').$$

• (Estimate of $\mathcal{I}_{43}$): We can estimate $\mathcal{I}_3$ in a same way with estimate of $\mathcal{I}_{42}$ and get

$$\mathcal{I}_{43} \leq C(W_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)) + \varepsilon + \varepsilon').$$

Now, we combine the estimation of $\mathcal{I}_{4i}$ for $i = 1, 2, 3$ to find

$$\frac{d}{dt} Q_{\varepsilon,\varepsilon'}(t)\Big|_{t=t_0+} \leq C(W_1(f_\varepsilon(t_0), f_{\varepsilon'}(t_0)) + \varepsilon + \varepsilon').$$

This implies that for arbitrary $t \in [0,\tau)$, we have

$$\frac{d}{dt} W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \leq C(W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) + \varepsilon + \varepsilon'),$$

where C is a positive constant which does not depend on ε or ε'. □

5.2. **Proof of Theorem 3.5.** Now, we are ready to prove the local-in-time existence and uniqueness of weak solution in the sense of Definition 3.4. Note that Proposition 2 implies that the family of regularized solution $\{f_\varepsilon\}_{\varepsilon\geq 0}$ is a Cauchy sequence in $C([0,\tau]; \mathcal{P}_1(\mathbb{R}^{2d}\times\mathbb{R}_+))$ and hence there exists a limit function f. Then, the remaining thing is to show that f is indeed the unique weak solution, and this completes the proof of Theorem 3.5.

• (Existence part): Fix any test function $\Phi \in C_c^\infty(\mathbb{R}^{2d}\times\mathbb{R}_+\times[0,\tau))$. Then the approximate solution f_ε satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon(z,\tau)\Phi(z,\tau)\,dz - \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon(z,0)\Phi(z,0)\,dz \\
&\qquad = \int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon(\partial_t\Phi + v\cdot\nabla_x\Phi + \mathcal{F}^\varepsilon[f_\varepsilon]\cdot\nabla_v\Phi + \mathcal{G}^\varepsilon[f_\varepsilon]\partial_\theta\Phi)\,dz\,dt.
\end{aligned} \tag{39}$$

Note that we can pass the limit $\varepsilon \to 0$ easily for the linear terms: As $\varepsilon \to 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon(z,\tau)\Phi(z,\tau)\,dz - \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon(z,0)\Phi(z,0)\,dz \\
&\qquad \longrightarrow \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f(z,\tau)\Phi(z,\tau)\,dz - \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f(z,0)\Phi(z,0)\,dz, \\
&\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon(\partial_t\Phi + v\cdot\nabla_x\Phi)\,dz\,dt \quad\longrightarrow\quad \int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f(\partial_t\Phi + v\cdot\nabla_x\Phi)\,dz\,dt.
\end{aligned} \tag{40}$$

Thus, we only need to show that as $\varepsilon \to 0$,

$$
\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon \mathcal{F}^\varepsilon[f_\varepsilon]\cdot\nabla_v\Phi\,dz\,dt \to \int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f\mathcal{F}[f]\cdot\nabla_v\Phi\,dz\,dt,\\
&\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon \mathcal{G}^\varepsilon[f_\varepsilon]\partial_\theta\Phi\,dz\,dt \to \int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f\mathcal{G}[f]\partial_\theta\Phi\,dz\,dt.
\end{aligned}
\tag{41}
$$

Since proofs of two limiting processes are almost same, we only focus on first limit in (41). Note that

$$
\begin{aligned}
&\left|\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} (f_\varepsilon \mathcal{F}^\varepsilon[f_\varepsilon] - f\mathcal{F}[f])\cdot\nabla_v\Phi\,dz\,dt\right|\\
&\quad\le \left|\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon\left(\mathcal{F}^\varepsilon[f_\varepsilon] - \mathcal{F}[f_\varepsilon]\right)\cdot\nabla_v\Phi\,dz\,dt\right|\\
&\qquad+ \left|\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon\left(\mathcal{F}[f_\varepsilon] - \mathcal{F}[f]\right)\cdot\nabla_v\Phi\,dz\,dt\right|\\
&\qquad+ \left|\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} (f_\varepsilon - f)\mathcal{F}[f]\cdot\nabla_v\Phi\,dz\,dt\right|\\
&\quad=: \mathcal{I}_{51} + \mathcal{I}_{52} + \mathcal{I}_{53}.
\end{aligned}
$$

Next, we estimate the terms $\mathcal{I}_{5i},\ i = 1,2,3$ separately.

$\diamond$ (Estimate of $\mathcal{I}_{51}$): As in [2, Section 3.2], we can estimate $\mathcal{I}_{51}$ as follows.

$$
\begin{aligned}
\mathcal{I}_{51} &= \left|\int_0^\tau \int_{(\mathbb{R}^{2d}\times\mathbb{R}_+)^2} (\phi_\varepsilon - \phi)(|x-x_*|)(\nabla_v\Phi)\cdot\left(\frac{v_*}{\theta_*} - \frac{v}{\theta}\right) f_\varepsilon(z,t) f_\varepsilon(z_*,t)\,dz\,dz_*\,dt\right|\\
&\le C\varepsilon \int_0^\tau \int_{(\mathbb{R}^d\times\Omega_v(\tau)\times\Omega_T(\tau))^2} \frac{1}{|x-x_*|^{1+\alpha}} f_\varepsilon(z,t) f_\varepsilon(z_*,t)\,dz\,dz_*\,dt\\
&\le C\varepsilon\|f_\varepsilon\|^2 \le C\varepsilon \to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned}
\tag{42}
$$

$\diamond$ (Estimate of $\mathcal{I}_{52}$): We set

$$
\mathcal{H}[f_\varepsilon](z,t) := \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} f_\varepsilon(z_*,t)\phi(|x_*-x|)\left(\frac{v}{\theta} - \frac{v_*}{\theta_*}\right)\cdot\nabla_{v_*}\Phi(z_*)\,dz_*.
$$

Then, we have

$$
\mathcal{I}_{52} = \left|\int_0^\tau \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \mathcal{H}[f_\varepsilon](z_*,t)(f_\varepsilon(z_*,t) - f(z_*,t))\,dz_*\,dt\right|.
$$

On the other hand, we employ similar arguments as in the proof of Lemma 5.2 and the estimate of $\mathcal{I}_{52}$ in the proof of Proposition 2, we find that the function $\mathcal{H}[f_\varepsilon]$ is locally Lipschitz and bounded uniformly in ε. Thus, by definition, we obtain

$$
\mathcal{I}_{52} \le C \sup_{0\le t\le\tau} W_1(f^\varepsilon(t), f(t)) \to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{43}
$$

$\diamond$ (Estimate of $\mathcal{I}_{53}$): Similarly to previous case, we can also show that $F[f]\cdot\nabla_v\Phi$ is locally Lipschitz and bounded, and consequently, we have

$$
\mathcal{I}_{53} \le C \sup_{0\le t\le\tau} W_1(f^\varepsilon(t), f(t)) \to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{44}
$$

In (41), we collect all estimates (42), (43) and (44) to obtain the limiting convergence $(41)_1$. One can prove $(41)_2$ in a similar way, as in the proof of $(41)_1$. Thus, the detailed proof of $(41)_2$ is omitted. Finally, in (39), we combine (40) and (41) to show that the limit function f satisfies the weak formulation (11).

• (Uniqueness and stability) : Let f_1 and f_2 be two weak solutions of equation (29) with same initial data f^0. Then, it follows from Proposition 2 that we have

$$\frac{d}{dt}W_1(f_1(t), f_2(t)) \leq CW_1(f_1(t), f_2(t)), \quad t \in [0, \tau).$$

Then, the Grönwall lemma yields the uniqueness of the solution.

6. **Conclusion.** In this paper, we have studied the dynamic features of the TCS model with singular power-law kernels in their velocity and temperature dynamics. For strong singularities in communication weights, collisions cannot occur in any finite time. Thus, the classical Cauchy-Lipschitz theory can be applied to yield the global existence of smooth solutions. In contrast, when the singularity is mildly weak, finite-time collisions can still occur from some prepared initial configurations. Hence, the global smooth solutions cannot be guaranteed in general. As far as the authors know, after a finite-time collision occur, there is no existence theory after collision time. Formal BBGKY hierarchy argument yields the kinetic TCS equation with singular kernel. For this kinetic equation with singular kernel, we also provide a local existence of weak solutions. At present, we do not have a global existence theory for weak or strong solutions for the kinetic TCS equation. This will be an interesting future work to be explored.

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