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UNIFORM STABILITY AND MEAN-FIELD LIMIT FOR THE AUGMENTED KURAMOTO MODEL

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Abstract. We present two uniform estimates on stability and mean-field limit for the "augmented Kuramoto model (AKM)" arising from the second-order lifting of the first-order Kuramoto model (KM) for synchronization. In particular, we address three issues such as synchronization estimate, uniform stability and mean-field limit which are valid uniformly in time for the AKM. The derived mean-field equation for the AKM corresponds to the dissipative Vlasov-McKean type equation. The kinetic Kuramoto equation for distributed natural frequencies is not compatible with the frequency variance functional approach for the complete synchronization. In contrast, the kinetic equation for the AKM has a similar structural similarity with the kinetic Cucker-Smale equation which admits the Lyapunov functional approach for the variance. We present sufficient frameworks leading to the uniform stability and mean-field limit for the AKM.

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1. **Introduction.** Synchronization of weakly coupled oscillators is ubiquitous in our nature, e.g., rhythmic heart beatings of pacemaker cells, synchronous flashing of fireflies and collective hand clapping in a concert hall, etc [1, 6, 34, 35]. After Huygen's observation on two pendulum clocks hanging on the same bar, collective behaviors of weakly coupled oscillators have been reported from time to time in scientific literature (see [34]). However, major scientific progress on the collective dynamics of complex systems was initiated by Winfree and Kuramoto about a half century ago in [26, 27, 41]. Recently, research on the collective dynamics of complex systems has received lots of attention due to engineering applications in sensor network, mobile network and control of unmanned aerial vehicles (UAV) etc. In [22], the authors observed a formal analogy between the Cucker-Smale flocking model and the Kuramoto model for synchronization, and provide a quantitive estimate for the synchronization based on Lyapunov functional approach. In the paper, we further investigate this formal analogy and study dynamic asymptotic properties of the AKM.

Consider an ensemble of Kuramoto oscillators lying on the nodes of the complete graph with N-nodes, and assume that the state of an oscillator is described by a real-valued function "*phase*". Let θ_i be the phase of the i-th oscillator whose dynamics is given by the Kuramoto model:

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_i), \quad 1 \le i \le N, \tag{1}$$

where κ and ν_i denote the coupling strength, and the natural frequency of the i-th Kuramoto oscillator, respectively. On the other hand, it is well known in [28] that the dynamics of system (1) with $N \gg 1$ can be described by the corresponding mean-field equation, namely kinetic Kuramoto equation. To be specific, let $f = f(\theta, \nu, t)$ be the one-particle distribution function at phase θ, natural frequency ν at time t. Then, the kinetic Kuramoto equation reads as follows.

$$\begin{aligned} &\partial_t f + \partial_\theta(L[f]f) = 0, \quad (\theta,\nu) \in \mathbb{T}\times\mathbb{R},\ t>0, \\ &L[f](\theta,\nu,t) := \nu + \kappa\int_0^{2\pi}\int_{-\infty}^{\infty} \sin(\theta_* - \theta) f(\theta_*,\nu_*,t)d\nu_* d\theta_*, \end{aligned} \tag{2}$$

where $\mathbb{T}$ denotes the 1-dimensional torus. The Kuramoto model (1) and its corresponding mean-field kinetic equation (2) have been extensively studied in applied mathematics, control theory and statistical physics communities from various aspects, e.g., existence of partial and fully phase-locked states [2, 8, 29, 39], emergence of complete synchronization [4, 5, 7, 9, 10, 11, 14, 17, 18, 19, 22, 25, 37], stability of partial and fully phase-locked states and incoherent state [3, 30, 31, 32], slow-fast dynamics [23], existence of the critical coupling strength and its computing algorithm for phase-locked states [13, 38], phase-transition phenomena at critical coupling strength [1], relation with other models [22, 36] and rigorous mean-field limit [28] etc. For details, we refer to survey articles and a book [1, 12, 20, 34, 35]. Our main concern is to derive complete synchronization estimate for (2). For identical oscillators with $g(\nu) = \delta_0$, the kinetic equation (2) can be reduced to

$$\begin{aligned} &\partial_t f + \partial_\theta(L[f]f) = 0, \quad \theta \in \mathbb{T},\ t>0, \\ &L[f](\theta,t) := \kappa\int_0^{2\pi} \sin(\theta_* - \theta) f(\theta_*,t) d\theta_*. \end{aligned} \tag{3}$$

In this case, the variance function $\Lambda[f] := \int_{\mathbb{T}^2} |\theta - \theta_*|^2 f(\theta,t) f(\theta_*,t) d\theta_* d\theta$ can measure the emergence of complete (phase) synchronization. However, for distributed natural frequencies, the functional $\Lambda[f]$ cannot be used for the complete synchronization. Then, it is not clear how to derive a complete synchronization estimate directly for the kinetic equation (2) without lifting corresponding particle results as in [7]. This motivates the works in this paper.

Next, we briefly explain how to bypass the aforementioned difficulty for the complete synchronization of the kinetic equation (2) for distributed natural frequencies. Our idea is to lift the first-order model (1) into a second-order model by introducing an auxiliary frequency variable $\omega_i = \dot{\theta}_i$, i.e., we differentiate the equation (1) with respect to t and obtain an augmented Kuramoto model (AKM):

$$\begin{aligned} &\dot{\theta}_i = \omega_i, \quad t > 0,\ 1 \le i \le N, \\ &\dot{\omega}_i = \frac{\kappa}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta_i)(\omega_k - \omega_i). \end{aligned} \tag{4}$$

On the other hand, we consider a mean-field limit $N \to \infty$. In this case, we set $f = f(\theta, \omega, t)$ to be a probability density function corresponding to (4). Then, a formal BBGKY Hierarchy argument yields the kinetic equation:

$$\begin{aligned} &f_t + \omega \partial_\theta f + \kappa \partial_\omega (L[f] f) = 0, \quad (\theta, \omega) \in \mathbb{T} \times \mathbb{R},\ t > 0, \\ &L[f](\theta, \omega, t) := \int_0^{2\pi} \int_{-\infty}^{\infty} \cos(\theta_* - \theta)(\omega_* - \omega) f(\theta_*, \omega_*, t)\, d\theta_* d\omega_*. \end{aligned} \tag{5}$$

As an analogy with the kinetic Cucker-Smale model in [16], we can use the frequency variance functional

$$\Lambda_1[f] := \int_{\mathbb{T}^2 \times \mathbb{R}^2} |\omega - \omega_*|^2 f(\theta, \omega, t) f(\theta_*, \omega_*, t) d\omega_* d\omega d\theta_* d\theta$$

to measure the emergence of complete synchronization for (5).

In this paper, we are interested in the following questions for (4) and (5):

- (Q1): Under what conditions, can system (4) exhibit the complete synchronization?
- (Q2): Is the system (4) uniformly ℓ_p-stable with respect to initial data?
- (Q3): Can we derive the mean-field kinetic equation (5) from the particle model (4) as $N \to \infty$ uniformly in time?

The first two questions might be generalized to the locally coupled Kuramoto model on a general symmetric and connected networks. However, the last question, i.e., uniform mean-field limit can be treated only for mean-field couplings (e.g., BBGKY hierarchy arguments break down for the locally coupled case). As aforementioned, since our main motivation is to study the complete synchronization of the kinetic level in a direct manner, we consider only the complete network case.

The main results of this paper are four-fold: First, we provide a sufficient framework for the complete synchronization estimate. Our sufficient conditions are expressed in terms of the coupling strength κ, the diameter of the set of natural frequencies and initial data, and they are free of the number of oscillators (see Theorem 3.2). Second, we provide the uniform stability estimate of (4) with respect to initial data in a metric equivalent to the ℓ_p-distance in phase space. Our uniform stability roughly says that the ℓ_p-distance between two configurations at time t is uniformly bounded by the constant multiple of initial ℓ_p-distance between two initial data (see Theorem 4.3). Third, we present a uniform-in-time mean-field limit for (4) as a direct application of the exponential flocking estimate in Theorem 3.2 and uniform-in-time stability estimates in Theorem 4.3. Our last result is to derive the complete synchronization estimate for the mean-field kinetic equation (5) using a robust Lyapunov functional approach.

The rest of this paper is organized as follows. In Section 2, we briefly discuss theoretical minimum for the Kuramoto model and its augmented model. In Section 3, we present a synchronization estimate for the AKM (4). In Section 4, we present a uniform ℓ_p-stability estimate for the augmented model. In Section 5, we study the uniform mean-field limit from the particle model (4) to the corresponding kinetic equation uniformly in time, and we also study the complete frequency synchronization estimate for the kinetic equation. In Section 6, we provide a synchronization estimate for the kinetic Kuramoto equation (5) with distributed natural frequencies. Finally, Section 7 is devoted to a brief summary of our main results and discussion for future works.

Before we proceed to the next section, we introduce the notation which will be used in the rest of the paper.

Notation. When we discuss the distance in the spatial dimension $\mathbb{T}$, we use the orthodromic distance: let θ be a constant in $\mathbb{R}$, then we define

$$|\theta|_o := |\bar{\theta}| \quad \text{where} \quad \bar{\theta} \in (-\pi, \pi] \quad \text{and} \quad \theta \equiv \bar{\theta} \pmod{2\pi}.$$

In the following discussion, we only consider the case in which the oscillators are confined in a half circle. It is obvious that

$$|\theta|_o = |\theta| \quad \text{for} \quad \theta \in (-\pi, \pi).$$

For notational simplicity, we use $|\cdot|$ instead of $|\cdot|_o$ for the spatial distance. Throughout the paper, we use the following simplified notation: for $Z := (z_1, \cdots, z_N)$, we set

$$\begin{aligned} D(Z) &:= \max_{1\le i,j\le N} |z_i - z_j|, \\ \|Z\|_p &:= \Big(\sum_{i=1}^N |z_i|^p \Big)^{\frac{1}{p}}, \quad p \in [1, \infty), \\ \|Z\|_\infty &:= \max_{1\le i\le N} |z_i|, \end{aligned}$$

and

$$\Theta := (\theta_1, \cdots, \theta_N), \quad \Omega := (\omega_1, \cdots, \omega_N), \quad \mathcal{V} := (\nu_1, \cdots, \nu_N).$$

2. **Preliminaries.**

In this section, we briefly review a theoretical minimum for the Kuramoto model and the augmented Kuramoto model.

2.1. **The Kuramoto model.**

In this subsection, we briefly discuss an associated conservation law, and review the state-of-the-art results on the complete synchronization for the Kuramoto model. First, we introduce a time-dependent quantity $\mathcal{C}(\Theta, \mathcal{V}, t)$:

$$\mathcal{C}(\Theta, \mathcal{V}, t) := \sum_{i=1}^N \theta_i - t \sum_{i=1}^N \nu_i.$$

Lemma 2.1. *Let $\Theta = \Theta(t)$ be a phase vector whose dynamics is governed by* (1)*. Then, the quantity $\mathcal{C}(\Theta, \mathcal{V}, t)$ is a constant of motion.*

$$\frac{d}{dt}\mathcal{C}(\Theta(t), \mathcal{V}, t) = 0, \quad t > 0.$$

Proof. Let $\Theta = \Theta(t)$ be a Kuramoto flow. Then, we have

$$\frac{d}{dt}\mathcal{C}(\Theta(t), \mathcal{V}, t) = \frac{d}{dt}\Big(\sum_{i=1}^N \theta_i - t\sum_{i=1}^N \nu_i\Big) = \sum_{i=1}^N \dot{\theta}_i - \sum_{i=1}^N \nu_i = 0.$$

This yields the desired estimate. □

Remark 1. Note that unless $\sum_{i=1}^N \nu_i$ is zero, the total phase $\sum_{i=1}^N \theta_i$ is not conserved.

Next, we discuss the equilibrium for the Kuramoto model (1). Note that the equilibrium solution $\Theta = (\theta_1, \cdots, \theta_N)$ is a solution to the following equilibrium system:

$$\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \sin(\theta_k - \theta_i) = 0, \quad 1 \le i \le N. \tag{6}$$

We sum up the equation (6) with respect to i to obtain

$$0 = \sum_{i=1}^N \nu_i + \frac{\kappa}{N}\sum_{i,k=1}^N \sin(\theta_k - \theta_i) = \sum_{i=1}^N \nu_i.$$

Thus, if the system (6) does have a solution, then the total sum of natural frequencies is zero. Hence if $\sum_{i=1}^N \nu_i \neq 0$, then system (6) does not have a solution. This leads to the need of the relaxed equilibria in the following definition. We first recall several definitions for a phase-locked state and asymptotic phase-locking.

Definition 2.2. [1, 12, 19] Let $\Theta(t) = (\theta_1(t), \cdots, \theta_N(t))$ be a time-dependent phase vector.

1. Θ is a phase-locked state if all relative phase differences are constants:
$$\theta_i(t) - \theta_j(t) = \theta_i(0) - \theta_j(0), \quad t \ge 0, \qquad 1 \le i, j \le N.$$
2. Θ exhibits asymptotic phase-locking (complete synchronization) if the relative frequencies tend to zero asymptotically:
$$\lim_{t\to\infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0, \quad 1 \le i, j \le N.$$

Remark 2. Note that solutions to the following equilibrium system:

$$\nu_i + \frac{\kappa}{N}\sum_{j=1}^N \sin(\theta_j - \theta_i) = 0, \quad i = 1, \cdots, N, \qquad \sum_{i=1}^N \nu_i = 0$$

correspond to phase-locked states of (1).

We next briefly review the state-of-the-art result for the Kuramoto model (1). It is well known in [11, 24, 36] that system (1) can be lifted as a dynamical system on $\mathbb{R}^N$ and can also be written as a gradient flow with an analytical potential in $\mathbb{R}^N$:

$$\dot{\Theta}(t) = -\nabla_\Theta V(\Theta), \quad \text{where} \quad V(\Theta) := -\sum_{k=1}^N \nu_k\theta_k + \frac{\kappa}{2N}\sum_{1\le k,l\le N}(1 - \cos(\theta_k - \theta_l)).$$

For a gradient flow system with analytical potential, uniform boundedness is equivalent to the convergence of solution toward the phase-locked state. As a dynamical system in $\mathbb{R}^N$, the uniform boundedness of (1) in $\mathbb{R}^N$ in a rotating frame moving with the speed of average of natural frequencies is not obvious since nonidentical oscillators can cross each other.

In the following theorem, we summarize the state-of-the-art results for the emergence of phase-locked states from generic initial data in a large coupling strength regime with $\kappa \gg 1$.

Theorem 2.3. [9, 11, 13, 19] *The following assertions hold.*

1. *Suppose that the initial phase configuration Θ^0 is confined in a half circle and the coupling strength κ is positive such that*

$$D(\Theta^0) < \pi, \quad \kappa > 0, \quad D(\mathcal{V}) = 0.$$

Then, for any solution $\Theta = \Theta(t)$ to (1), we have

$$\lim_{t\to\infty} D(\dot{\Theta}(t)) = 0.$$

2. *Suppose that the initial phase configuration Θ^0 is confined in a half circle and the coupling strength κ is sufficient large such that*

$$D(\Theta^0) < \pi, \quad \kappa > D(\mathcal{V}) > 0.$$

Then, for any solution $\Theta = \Theta(t)$ to (1), there exists a positive constant λ such that

$$D(\dot{\Theta}(t)) \le Ce^{-\lambda t}, \quad \textit{as } t \to \infty.$$

3. *Suppose that natural frequencies are distributed and initial configurations satisfy*

$$D(\mathcal{V}) > 0, \quad R^0 = \frac{1}{N}\Big|\sum_{k=1}^N e^{\mathrm{i}\theta_k^0}\Big| > 0, \quad \theta_j^0 \neq \theta_k^0, \quad 1 \le j \neq k \le N.$$

Then there exists a large coupling strength $\kappa_\infty > 0$ such that if $\kappa \ge \kappa_\infty$ there exists a phase-locked state Θ^∞ such that the solution with initial data Θ_0 satisfies

$$\lim_{t\to\infty} \|\Theta(t) - \Theta^\infty\|_\infty = 0,$$

where the norm $\|\cdot\|_\infty$ is the standard ℓ^∞-norm in $\mathbb{R}^N$.

Remark 3. 1. In the course of the proof for the first statement, we can show that there exists a finite time t_0 and $D^\infty \in (0, \frac{\pi}{2})$ such that

$$D(\Theta(t)) \le D^\infty, \quad \text{for } t \ge t_0.$$

2. The result of [11] does not yield detailed asymptotic dynamics of identical Kuramoto oscillators. However, when the diameter of the emergent phase-locked state is less than π and the coupling strength is sufficiently large, then we can show that convergence speed is at least exponential. See [9, 12, 13] for detailed discussion.

2.2. **The augmented Kuramoto model.** In this subsection, we discuss basic structural properties of the AKM and relationship between the Kuramoto model and AKM. We set averaged quantities and fluctuations of phase and frequency around them:

$$\theta_c := \frac{1}{N}\sum_{k=1}^{N}\theta_k, \quad \omega_c := \frac{1}{N}\sum_{k=1}^{N}\omega_k,$$
$$\hat{\theta}_i := \theta_i - \theta_c, \quad \hat{\omega}_i := \omega_i - \omega_c.$$

Then, it is easy to see that the averaged quantities and fluctuations satisfy

$$\dot{\theta}_c = \omega_c, \qquad \dot{\hat{\omega}}_i = \frac{\kappa}{N}\sum_{k=1}^{N}\cos(\hat{\theta}_k - \hat{\theta}_i)(\hat{\omega}_k - \hat{\omega}_i). \tag{7}$$

Lemma 2.4. *Let $\{(\theta_i, \omega_i)\}_{i=1}^N$ be a solution to (4). Then, the averaged quantities (θ_c, ω_c) satisfy the following relations:*

$$\omega_c(t) = \omega_c(0), \qquad \theta_c(t) = \theta_c(0) + t\omega_c(0), \qquad t \geq 0.$$

Proof. We sum (4) with respect to i and use the skew symmetry of $\cos(\theta_j - \theta_i)(\omega_j - \omega_i)$ in the transformation of $(i, j) \iff (j, i)$ to get

$$\frac{d}{dt}\sum_{i=1}^{N}\omega_i = \frac{\kappa}{N}\sum_{i,k=1}^{N}\cos(\theta_k - \theta_i)(\omega_k - \omega_i) = -\frac{\kappa}{N}\sum_{i,k=1}^{N}\cos(\theta_k - \theta_i)(\omega_k - \omega_i) = 0.$$

The second relation follows from $(7)_1$. □

Next, we discuss the relation between the first-order model (1) and the second-order model (4) which is stated in the following theorem.

Theorem 2.5. *The Kuramoto model (1) is equivalent to the augmented Kuramoto model (4) in the following sense.*

1. *If $\{\theta_i\}$ is a solution to (1) with initial data $\{\theta_i^0\}$, then $\{(\theta_i, \omega_i := \dot{\theta}_i)\}$ is a solution to (4) with well-prepared initial data $\{(\theta_i^0, \omega_i^0)\}$:*

$$\omega_i^0 := \nu_i + \frac{\kappa}{N}\sum_{j=1}^{N}\sin(\theta_j^0 - \theta_i^0), \quad i = 1, \cdots, N.$$

2. *If $\{(\theta_i, \omega_i)\}$ is a solution to (4) with initial data $\{(\theta_i^0, \omega_i^0)\}$, then $\{\theta_i\}$ is a solution to (1) with natural frequencies:*

$$\nu_i := \omega_i^0 - \frac{\kappa}{N}\sum_{j=1}^{N}\sin(\theta_j^0 - \theta_i^0), \quad i = 1, \cdots, N.$$

Proof. (i) Let $\Theta = \Theta(t)$ be a solution to (1) with initial data Θ^0. Then, it satisfies

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i). \tag{8}$$

We set

$$\omega_i = \dot{\theta}_i \tag{9}$$

and differentiate the above equation to obtain

$$\dot{\omega}_i = \frac{\kappa}{N}\sum_{k=1}^{N}\cos(\theta_k - \theta_i)(\omega_k - \omega_i). \tag{10}$$

We use (8) to find

$$\omega_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i). \tag{11}$$

Letting $t \to 0+$ in (11), we obtain

$$\omega_i^0 = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k^0 - \theta_i^0). \tag{12}$$

Finally, we combine (9), (10) and (12) to see that (θ_i, ω_i) is a solution to (4) with initial data (θ_i^0, ω_i^0).
(ii) Let $\{(\theta_i, \omega_i)\}$ be a solution to (4) with initial data $\{(\theta_i^0, \omega_i^0)\}$, i.e., it satisfies

$$\dot{\omega}_i = \frac{\kappa}{N}\sum_{k=1}^{N}\cos(\theta_k - \theta_i)(\omega_k - \omega_i).$$

Then, we use the relations:

$$\omega_i = \dot{\theta}_i \quad \text{and} \quad \cos(\theta_k - \theta_i)(\omega_k - \omega_i) = \frac{d}{dt}\sin(\theta_k - \theta_i)$$

to integrate (10) to obtain

$$\dot{\theta}_i = \omega_i = \omega_i^0 - \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k^0 - \theta_i^0) + \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i). \tag{13}$$

Then, we set

$$\nu_i := \omega_i^0 - \frac{\kappa}{N}\sum_{k=1}^{N}\sin(\theta_k^0 - \theta_i^0). \tag{14}$$

Finally, we combine (13) and (14) to recover the Kuramoto model. □

Before we close this section, we quote the following lemma to be crucially used in later sections.

Lemma 2.6. [21] *Suppose that two nonnegative Lipschitz functions X and V satisfy the system of differential inequalities:*

$$\left|\frac{dX}{dt}\right| \le V, \qquad \frac{dV}{dt} \le -\alpha V + \gamma e^{-\alpha t} X, \quad a.e.\ t > 0,$$

where α and γ are positive constants. Then, X and V satisfy the uniform bound and decay estimates:

$$X(t) \le \frac{2M}{\alpha}(X(0) + V(0)), \qquad V(t) \le M(X(0) + V(0))e^{-\frac{\alpha t}{2}}, \quad t \ge 0,$$

where M is a positive constant defined by,

$$M := \max\left\{1, \frac{2\gamma}{\alpha e}\right\} + \frac{8\gamma}{\alpha^3 e^3}.$$

3. **Emergence of the complete synchronization.** In this section, we present complete synchronization estimates for the AKM (4) in ℓ_∞ and ℓ_p, $p \in [1, \infty)$ frameworks by deriving a system of Grönwall's inequalities.

3.1. **ℓ_∞-framework.** In this subsection, we present a synchronization estimate in ℓ_∞-framework. In each case, the synchronization estimates are obtained in the following three steps:

- Step A (Existence of positively invariant set). We identify a positively invariant set which is translation invariant in phase space.
- Step B (Derivation of Grönwall's inequality). We introduce a Lyapunov type functional and derive a Grönwall type differential inequality.
- Step C (Complete synchronization estimate). Once we derive a Grönwall type inequality for a suitable Lyapunov functional, suitable Grönwall's lemma and continuity arguments yield the desired synchronization estimate.

As candidates for Lyapunov functionals and an invariant set, we introduce phase and frequency diameters:

$$D(\Theta) := \max_{1\le i,j\le N} |\theta_i - \theta_j|, \qquad D(\Omega) := \max_{1\le i,j\le N} |\omega_i - \omega_j|,$$

and for $D^\infty < \frac{\pi}{2}$, we set

$$\mathcal{S}(D^\infty) := \Big\{ \Theta = (\theta_1, \cdots, \theta_N) \ : \ D(\Theta) < D^\infty \Big\}.$$

Lemma 3.1. *Suppose that initial data (Θ^0, Ω^0) and the coupling strength κ satisfy*

$$\Theta^0 \in \mathcal{S}(D^\infty), \qquad \kappa > \frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \text{where } D^\infty < \frac{\pi}{2}.$$

Then, the set $\mathcal{S}$ is positively invariant under the flow (4), i.e., for any solution $\Theta = \Theta(t)$ with initial data $\Theta^0 \in \mathcal{S}$, we have

$$\Theta(t) \in \mathcal{S}(D^\infty), \quad t \ge 0.$$

Proof. Let Θ^0 be a initial data with $D(\Theta^0) < D^\infty$. Suppose that $\mathcal{S}$ is not positively invariant under flow. Then, there exists a finite time t^* such that

$$t^* = \sup\{ \ t : \ D(\Theta(s)) < D^\infty, \quad 0 \le s \le t \}.$$

By the continuity, we have

$$D(\Theta(t^*)) = D^\infty.$$

On the other hand, for any indices i and j, we integrate $(4)_2$ from 0 to t for $t < t^*$ to obtain

$$\begin{aligned} \omega_i(t) - \omega_j(t) = \omega_i^0 - \omega_j^0 + \frac{\kappa}{N}\sum_{k=1}^N \int_0^t \cos(\theta_k - \theta_i)(\omega_k(s) - \omega_i(s))\, ds \\ - \frac{\kappa}{N}\sum_{k=1}^N \int_0^t \cos(\theta_k - \theta_j)(\omega_k(s) - \omega_j(s))\, ds. \end{aligned}$$

Now we choose maximal indices M and m which might be dependent on t:

$$\omega_M(t) := \max_{1\le i\le N} \omega_i(t), \quad \omega_m(t) := \min_{1\le i\le N} \omega_i(t).$$

Then, for $t \in [0, t^*]$, we get

$$\begin{aligned} D(\dot\Theta(t)) &= \omega_M(t) - \omega_m(t) \\ &= \omega_M(0) - \omega_m(0) + \frac{\kappa}{N}\sum_{k=1}^N \int_0^t \cos(\theta_k(s) - \theta_M(s))(\omega_k(s) - \omega_M(s))\,ds \\ &\quad - \frac{\kappa}{N}\sum_{k=1}^N \int_0^t \cos(\theta_k(s) - \theta_m(s))(\omega_k(s) - \omega_m(s))\,ds. \end{aligned} \tag{15}$$

For $0 \le s \le t^*$, we have

$$|\theta_k(s) - \theta_M(s)| \le D^\infty, \quad |\theta_k(s) - \theta_m(s)| \le D^\infty. \tag{16}$$

Therefore, it follows from (15) and (16) that we have

$$\begin{aligned} D(\dot\Theta(t)) &\le D(\dot\Theta(0)) + \frac{\kappa\cos D^\infty}{N}\sum_{k=1}^N \int_0^t (\omega_k(s) - \omega_M(s))\,ds \\ &\quad - \frac{\kappa\cos D^\infty}{N}\sum_{k=1}^N \int_0^t (\omega_k(s) - \omega_m(s))\,ds \\ &= D(\dot\Theta(0)) - \kappa\cos D^\infty \int_0^t D(\dot\Theta(s))\,ds. \end{aligned} \tag{17}$$

This yields

$$D(\dot\Theta(t)) \le D(\dot\Theta(0)) - \kappa\cos D^\infty \int_0^t D(\dot\Theta(s))\,ds. \tag{18}$$

We set

$$u(t) := \int_0^t D(\dot\Theta(s))\,ds.$$

Then, it is easy to see that

$$\dot u(t) = D(\dot\Theta(t)), \quad u(0) = 0, \quad \dot u(0) = D(\dot\Theta(0)). \tag{19}$$

Then, the relation (18) is equivalent to

$$\dot u(t) + \kappa\cos D^\infty u(t) \le \dot u(0). \tag{20}$$

Then, (19) and (20) yield

$$u(t) \le \frac{\dot u(0)}{\kappa\cos D^\infty}\left(1 - e^{-\kappa(\cos D^\infty)t}\right) \le \frac{\dot u(0)}{\kappa\cos D^\infty}, \quad t \ge 0. \tag{21}$$

On the other hand, since $D(\Theta(t^*)) = D^\infty$, there exist indices i and j such that

$$\theta_i(t^*) - \theta_j(t^*) = D^\infty.$$

Then, it follows from $(4)_1$ that we have

$$\begin{aligned} D^\infty &= \theta_i(t^*) - \theta_j(t^*) \\ &= \theta_i^0 - \theta_j^0 + \int_0^{t^*} (\omega_i(s) - \omega_j(s))\,ds \\ &\le D(\Theta^0) + \int_0^{t^*} D(\dot\Theta(s))\,ds \le D(\Theta^0) + \frac{D(\dot\Theta(0))}{\kappa\cos(D^\infty)} < D^\infty, \end{aligned}$$

where we used the hypothesis on κ and (21), which yields contradiction. □

Theorem 3.2. *Suppose that initial data and coupling strength satisfy*

$$\Theta^0 \in \mathcal{S}(D^\infty), \qquad \sum_{i=1}^N \omega_i^0 = 0, \qquad \kappa > \frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \textit{where } D^\infty < \frac{\pi}{2}.$$

Then, we have an exponential synchronization:

$$D(\Omega(t)) \le D(\Omega^0) e^{-\kappa \cos(D^\infty) t}, \quad t \ge 0.$$

Proof. Due to the conservation law in Lemma 2.4, we have

$$\sum_{i=1}^N \omega_i(t) = 0, \quad t \ge 0.$$

We set extremal indices M and m such that

$$\omega_M := \max_{1 \le i \le N} \omega_i, \qquad \omega_m := \min_{1 \le i \le N} \omega_i.$$

Then, it follows from $(4)_2$ that we have

$$\dot{\omega}_M = \frac{\kappa}{N} \sum_{k=1}^N \cos(\theta_k - \theta_i)(\omega_k - \omega_M) \le -\kappa \cos D^\infty \omega_M. \tag{22}$$

Similarly, we have

$$\dot{\omega}_m \ge -\kappa \cos D^\infty \omega_m. \tag{23}$$

Then, it follows from (22) and (23) that we have

$$\frac{d}{dt} D(\dot{\Theta}(t)) \le -\kappa \cos D^\infty D(\dot{\Theta}), \quad t > 0.$$

This yields the desired exponential decay estimate. □

3.2. **ℓ_p-framework with $p \in [1, \infty)$.** In this subsection, we present ℓ_p-estimate for (4) for later use. For phase and frequency vectors

$$\Theta = (\theta_1, \cdots, \theta_N) \quad \text{and} \quad \Omega = (\omega_1, \cdots, \omega_N),$$

we set $\|\Theta\|_p$ and $\|\Omega\|_p$:

$$\|\Theta\|_p := \Big(\sum_{i=1}^N |\theta_i|^p \Big)^{\frac{1}{p}}, \qquad \|\Omega\|_p := \Big(\sum_{i=1}^N |\omega_i|^p \Big)^{\frac{1}{p}}, \qquad p \in [1, \infty).$$

Proposition 1. *Suppose that initial data and coupling strength satisfy*

$$\Theta^0 \in \mathcal{S}(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0, \quad \kappa > \frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \textit{where } D^\infty < \frac{\pi}{2}.$$

Then for any solution $\{(\theta_i, \omega_i)\}_{i=1}^N$ to (4), we have

$$\Big| \frac{d}{dt} \|\Theta\|_p \Big| \le \|\Omega\|_p, \qquad \frac{d}{dt} \|\Omega\|_p \le -\kappa \cos(D^\infty) \|\Omega\|_p, \quad \textit{a.e. } t > 0. \tag{24}$$

Proof. (i) Note that

$$\frac{d|\theta_i|}{dt} \le |\omega_i|.$$

We multiply by $p|\theta_i|^{p-1}$ to the above relation, take a sum the resulting relation, and use Hölder's inequality to get the following estimate:

$$\frac{d}{dt} \sum_{i=1}^N |\theta_i|^p \le p \sum_{i=1}^N |\theta_i|^{p-1} |\omega_i| \le p \Big(\sum_{i=1}^N |\theta_i|^p \Big)^{\frac{p-1}{p}} \Big(\sum_{i=1}^N |\omega_i|^p \Big)^{\frac{1}{p}} \le p \|\Theta\|_p^{p-1} \|\Omega\|_p.$$

This yields the desired first differential inequality.
(ii) It follows from $(4)_2$ that we have

$$|\omega_i|\frac{d|\omega_i|}{dt} = \frac{1}{2}\frac{d|\omega_i|^2}{dt} = \frac{1}{2}\frac{d\omega_i^2}{dt} = \omega_i\frac{d\omega_i}{dt} = \frac{\kappa}{N}\sum_{j=1}^{N}\cos(\theta_j-\theta_i)\omega_i(\omega_j-\omega_i). \tag{25}$$

We use (25) to obtain

$$\begin{aligned}\frac{d\|\Omega\|_p^p}{dt} &= \sum_{i=1}^{N}\frac{d}{dt}|\omega_i|^p = \sum_{i=1}^{N}p|\omega_i|^{p-2}|\omega_i|\frac{d}{dt}|\omega_i| \\ &= \sum_{i=1}^{N}p|\omega_i|^{p-2}\Big[\frac{\kappa}{N}\sum_{j=1}^{N}\cos(\theta_j-\theta_i)\omega_i(\omega_j-\omega_i)\Big] \\ &= \frac{\kappa p}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos(\theta_j-\theta_i)|\omega_i|^{p-2}\omega_i(\omega_j-\omega_i) \\ &= \frac{\kappa p}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos(\theta_j-\theta_i)(\omega_j-\omega_i)\Big(|\omega_i|^{p-2}\omega_i-|\omega_j|^{p-2}\omega_j\Big).\end{aligned} \tag{26}$$

We use the monotonicity of $f(x)=|x|^{p-2}x$ to see

$$(\omega_j-\omega_i)(|\omega_i|^{p-2}\omega_i-|\omega_j|^{p-2}\omega_j)\le 0. \tag{27}$$

Then, we use (26), (27), $\sum_{i=1}\omega_i=0$ and a priori condition:

$$\cos(\theta_j-\theta_i)\ge\cos D^\infty$$

to obtain

$$\begin{aligned}\frac{d\|\Omega\|_p^p}{dt} &\le \frac{\kappa p\cos D^\infty}{2N}\sum_{i,j=1}^{N}(\omega_j-\omega_i)(|\omega_i|^{p-2}\omega_i-|\omega_j|^{p-2}\omega_j) \\ &= -\kappa p\cos D^\infty\sum_{i=1}^{N}|\omega_i|^p = -\kappa p\cos D^\infty\|\Omega\|_p^p.\end{aligned}$$

This yields the desired second differential inequality. □

Finally, we combine Proposition 1 and Lemma 3.1 to derive the exponential synchronization.

Theorem 3.3. *Let $\{(\theta_i,\omega_i)\}$ be a solution to (4) with initial data and coupling strength:*

$$\Theta^0\in\mathcal{S}(D^\infty),\quad \sum_{i=1}^{N}\omega_i^0=0,\quad \kappa>\frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty-D(\Theta^0))},\quad \textit{where } D^\infty<\frac{\pi}{2}.$$

Then, there exists a positive constant θ_p^∞ such that for $p\in[1,\infty)$,

$$\|\Omega(t)\|_p\le\|\Omega^0\|_p e^{-\kappa\cos(D^\infty)t},\quad \|\Theta(t)\|_p\le\theta_p^\infty,\quad t\ge 0.$$

Proof. The exponential decay of Ω follows from the second equation of (24). On the other hand, it follows from $(24)_1$ that we have

$$\Big|\|\Theta(t)\|_p-\|\Theta(0)\|_p\Big|\le\int_0^t\|\Omega(s)\|_p\,ds\le\frac{\|\Omega^0\|_p}{\kappa\cos D^\infty}\Big(1-e^{-\kappa\cos(D^\infty)t}\Big)\le\frac{\|\Omega^0\|_p}{\kappa\cos D^\infty}.$$

Thus, we have

$$\|\Theta(t)\|_p \le \|\Theta^0\|_p + \frac{\|\Omega^0\|_p}{\kappa \cos D^\infty} =: \theta_p^\infty(D^\infty, \kappa, \|\Theta^0\|_p, \|\Omega^0\|_p).$$

□

Thanks to Theorem 3.3, we can conclude that there exists a unique phase-locked state Θ^∞. Moreover, $\Theta(t)$ will tend to Θ^∞ exponentially.

Corollary 1. *Let $\{(\theta_i, \omega_i)\}$ be a solution to (4) with initial data $\{(\theta_i^0, \omega_i^0)\}$ and coupling strength κ:*

$$\Theta^0 \in \mathcal{S}(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0, \quad \kappa > \frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \text{where } D^\infty < \frac{\pi}{2}.$$

Then, for any solution $\{(\theta_i, \omega_i)\}$, there exists a unique phase lock state $\Theta^\infty := (\theta_1^\infty, \cdots, \theta_N^\infty)$ such that

$$|\theta_i(t) - \theta_i^\infty| \le Ce^{-\kappa \cos(D^\infty)t}, \quad i = 1, \cdots, N.$$

Proof. Let $\Theta = \Theta(t)$ be a solution to system (4). Then, since κ is sufficiently large, we have

$$\sup_{0 \le t < \infty} D(\Theta(t)) \le D^\infty.$$

Then, we use Theorem 3.2 to obtain

$$\begin{aligned} |\theta_i(\tilde{t}) - \theta_i(t)| &= \Big| \int_t^{\tilde{t}} \omega_i(s)\, ds \Big| \le \int_t^{\tilde{t}} |\omega_i(s)| ds \le \int_t^{\tilde{t}} \Big(\sum_{i=1}^N |\omega_i(s)|^p \Big)^{1/p} ds \\ &\le \int_t^{\tilde{t}} \|\Omega(s)\|_p\, ds \le \|\Omega^0\|_p \int_t^{\tilde{t}} e^{-\kappa \cos(\mathcal{D}^\infty)s}\, ds \\ &\le \frac{\|\Omega^0\|_p}{\kappa \cos D^\infty} \Big(e^{-\kappa(\cos D^\infty)t} - e^{-\kappa(\cos D^\infty)\tilde{t}} \Big). \end{aligned} \tag{28}$$

Then for any $\varepsilon > 0$, we can find a positive number T such that if $\tilde{t} \ge T$ and $t \ge T$, then

$$|\theta_i(\tilde{t}) - \theta_i(t)| < \varepsilon.$$

This immediately implies that there exists a unique asymptotic limit θ_i^∞. Moreover, we combine (28) to show that

$$|\theta_i(t) - \theta_i^\infty| \le Ce^{-\kappa(\cos D^\infty)t}.$$

□

4. **Uniform ℓ_p-stability estimate.** In this section, we study the uniform ℓ_p-stability for the augmented system (4) with respect to initial data.

Let $Z := (\Theta, \Omega)$ and $\tilde{Z} := (\tilde{\Theta}, \tilde{\Omega})$ be two solutions to (4) corresponding to initial data (Θ^0, Ω^0) and $(\tilde{\Theta}^0, \tilde{\Omega}^0)$, respectively. For the uniform stability estimate, we introduce a metric which is equivalent to ℓ_p-distance: for $p \in [1, \infty)$ and two solutions $Z = (\Theta, \Omega)$ and $\tilde{Z} = (\tilde{\Theta}, \tilde{\Omega})$, we define the distance as

$$d_p(Z(t), \tilde{Z}(t)) := \|\Theta(t) - \tilde{\Theta}(t)\|_p + \|\Omega(t) - \tilde{\Omega}(t)\|_p. \tag{29}$$

Next, we present a uniform ℓ_p-stability of system (4) with respect to initial data as follows.

Definition 4.1. The system (4) is uniformly ℓ_p-stable with respect to initial data if the following relation holds: For two solutions Z and $\tilde{Z}$ to (4) with initial data Z^0 and $\tilde{Z}^0$, respectively, there exists a positive constant G independent of t such that

$$d_p(Z(t), \tilde{Z}(t)) \le G d_p(Z_0, \tilde{Z}_0), \quad t \ge 0.$$

In the following lemma, we will derive differential inequalities for two subfunctionals $\|\Theta(t) - \tilde{\Theta}(t)\|_p$ and $\|\Omega(t) - \tilde{\Omega}(t)\|_p$ for $p \in [1, \infty)$.

Lemma 4.2. *Let (Θ, Ω) and $(\tilde{\Theta}, \tilde{\Omega})$ be two solutions to (4) corresponding to initial data (Θ^0, Ω^0) and $(\tilde{\Theta}^0, \tilde{\Omega}^0)$, respectively. Suppose that initial data and coupling strength satisfies following conditions:*

$$\Theta^0 \in \mathcal{S}(D^\infty), \quad \tilde{\Theta}^0 \in \mathcal{S}(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0 \quad \text{and} \quad \sum_{i=1}^N \tilde{\omega}_i^0 = 0,$$
$$\kappa > \max\Big\{ \frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \frac{D(\tilde{\Omega}^0)}{\cos(D^\infty)(D^\infty - D(\tilde{\Theta}^0))} \Big\}.$$

Then, we have

$$\begin{aligned} &\frac{d}{dt}\|\Theta - \tilde{\Theta}\|_p \le \|\Omega - \tilde{\Omega}\|_p, \quad a.e.,\ t > 0, \\ &\frac{d}{dt}\|\Omega - \tilde{\Omega}\|_p \le -\kappa\cos(D^\infty)\|\Omega - \tilde{\Omega}\|_p + 2\kappa\|\tilde{\Omega}^0\|_p e^{-\kappa\cos(D^\infty)t}\|\Theta - \tilde{\Theta}\|_p. \end{aligned} \tag{30}$$

Proof. • Case A (Derivation of the first inequality $(30)_1$). Note that $\theta_i - \tilde{\theta}_i$ satisfies

$$\frac{d}{dt}(\theta_i - \tilde{\theta}_i) = \omega_i - \tilde{\omega}_i.$$

This yields

$$\frac{d}{dt}|\theta_i - \tilde{\theta}_i| \le |\omega_i - \tilde{\omega}_i|.$$

We multiply by $p|\theta_i - \tilde{\theta}_i|^{p-1}$ on both sides, sum up the resulting relations with respect to i and apply Hölder's inequality to obtain

$$\frac{d}{dt}\sum_{i=1}^N |\theta_i - \tilde{\theta}_i|^p \le p\|\Theta - \tilde{\Theta}\|_p^{p-1}\|\Omega - \tilde{\Omega}\|_p.$$

This implies the desired estimate.

• Case B (Derivation of the first inequality $(30)_2$). Note that $\omega_i - \tilde{\omega}_i$ satisfies

$$\begin{aligned} \frac{d}{dt}(\omega_i - \tilde{\omega}_i) &= \frac{\kappa}{N}\sum_{k=1}^N \Big[\cos(\theta_k - \theta_i)(\omega_k - \omega_i) - \cos(\tilde{\theta}_k - \tilde{\theta}_i)(\tilde{\omega}_k - \tilde{\omega}_i)\Big] \\ &= \frac{\kappa}{N}\sum_{k=1}^N \cos(\theta_k - \theta_i)\Big[(\omega_k - \omega_i) - (\tilde{\omega}_k - \tilde{\omega}_i)\Big] \\ &\quad + \frac{\kappa}{N}\sum_{k=1}^N \Big[\cos(\theta_k - \theta_i) - \cos(\tilde{\theta}_k - \tilde{\theta}_i)\Big](\tilde{\omega}_k - \tilde{\omega}_i) \\ &= \frac{\kappa}{N}\sum_{k=1}^N \cos(\theta_k - \theta_i)\Big[(\omega_k - \tilde{\omega}_k) - (\omega_i - \tilde{\omega}_i)\Big] \end{aligned}$$

$$
-\frac{\kappa}{N}\sum_{k=1}^{N}\sin\theta_{ik}^{*}\Big[(\theta_k-\theta_i)-(\tilde{\theta}_k-\tilde{\theta}_i)\Big](\tilde{\omega}_k-\tilde{\omega}_i). \tag{31}
$$

where θ_{ik}^{*} is located between $(\theta_k-\theta_i)$ and $(\tilde{\theta}_k-\tilde{\theta}_i)$ by mean value theorem. By multiplying $(\omega_i-\tilde{\omega}_i)$ on both sides of (31), we have

$$
\begin{aligned}
|\omega_i-\tilde{\omega}_i|\frac{d}{dt}|\omega_i-\tilde{\omega}_i| &= (\omega_i-\tilde{\omega}_i)\frac{d}{dt}(\omega_i-\tilde{\omega}_i)\\
&= \frac{\kappa}{N}\sum_{k=1}^{N}\cos(\theta_k-\theta_i)(\omega_i-\tilde{\omega}_i)\Big[(\omega_k-\tilde{\omega}_k)-(\omega_i-\tilde{\omega}_i)\Big]\\
&\quad -\frac{\kappa}{N}\sum_{k=1}^{N}\sin\theta_{ik}^{*}\Big[(\theta_k-\tilde{\theta}_k)-(\theta_i-\tilde{\theta}_i)\Big](\omega_i-\tilde{\omega}_i)(\tilde{\omega}_k-\tilde{\omega}_i).\\
&\le \frac{\kappa}{N}\sum_{k=1}^{N}\cos(\theta_k-\theta_i)(\omega_i-\tilde{\omega}_i)\Big[(\omega_k-\tilde{\omega}_k)-(\omega_i-\tilde{\omega}_i)\Big]\\
&\quad +\frac{\kappa}{N}\sum_{k=1}^{N}\Big[|\theta_k-\tilde{\theta}_k|+|\theta_i-\tilde{\theta}_i|\Big]|\omega_i-\tilde{\omega}_i||\tilde{\omega}_k-\tilde{\omega}_i|.
\end{aligned} \tag{32}
$$

We use (32) and similar argument used in Proposition 1 to obtain

$$
\begin{aligned}
\frac{d}{dt}\|\Omega-\tilde{\Omega}\|_p^p &= \sum_{i=1}^{N}\frac{d}{dt}|\omega_i-\tilde{\omega}_i|^p\\
&= \sum_{i=1}^{N}p|\omega_i-\tilde{\omega}_i|^{p-2}|\omega_i-\tilde{\omega}_i|\frac{d}{dt}|\omega_i-\tilde{\omega}_i|\\
&\le \frac{\kappa p}{N}\sum_{i,k}\cos(\theta_k-\theta_i)|\omega_i-\tilde{\omega}_i|^{p-2}(\omega_i-\tilde{\omega}_i)\Big[(\omega_k-\tilde{\omega}_k)-(\omega_i-\tilde{\omega}_i)\Big]\\
&\quad +\frac{\kappa p}{N}\sum_{i,k}\Big(|\theta_k-\tilde{\theta}_k|+|\theta_i-\tilde{\theta}_i|\Big)|\omega_i-\tilde{\omega}_i|^{p-1}|\tilde{\omega}_k-\tilde{\omega}_i|
\end{aligned} \tag{33}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
\sum_{i,k}|\theta_k-\tilde{\theta}_k||\omega_i-\tilde{\omega}_i|^{p-1}|\tilde{\omega}_k-\tilde{\omega}_i| &\le \Big(\sum_{i,k}|\omega_i-\tilde{\omega}_i|^p\Big)^{\frac{p-1}{p}}\Big(\sum_{i,k}|\theta_k-\tilde{\theta}_k|^p|\tilde{\omega}_k-\tilde{\omega}_i|^p\Big)^{\frac{1}{p}}\\
&\le ND(\tilde{\Omega})\|\Theta-\tilde{\Theta}\|_p\|\Omega-\tilde{\Omega}\|_p^{p-1}
\end{aligned}
$$

and similarly

$$
\sum_{i,k}|\omega_i-\tilde{\omega}_i|^{p-1}|\theta_i-\tilde{\theta}_i||\tilde{\omega}_k-\tilde{\omega}_i| \le ND(\tilde{\Omega})\|\Omega-\tilde{\Omega}\|_p^{p-1}\|\Theta-\tilde{\Theta}\|_p.
$$

Then, by using these estimation and relation (33), we obtain

$$
\frac{d}{dt}\|\Omega-\tilde{\Omega}\|_p^p \le -\kappa p\cos(D^\infty)\|\Omega-\tilde{\Omega}\|_p^p+2\kappa pD(\tilde{\Omega})\|\Theta-\tilde{\Theta}\|_p\|\Omega-\tilde{\Omega}\|_p^{p-1}.
$$

By applying the relation $D(\tilde{\Omega})\le\|\tilde{\Omega}\|_p$ and Theorem 3.3, we attain the desired result. □

We combine Lemma 2.6 and Lemma 4.2 to obtain the uniform ℓ_p-stability.

Theorem 4.3. *Suppose that initial data and coupling strength satisfy the following relations:*

$$\Theta^0 \in \mathcal{S}(D^\infty), \quad \tilde{\Theta}^0 \in \mathcal{S}(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0 \quad \textit{and} \quad \sum_{i=1}^N \tilde{\omega}_i^0 = 0,$$
$$\kappa > \max\Big\{\frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \frac{D(\tilde{\Omega}^0)}{\cos(D^\infty)(D^\infty - D(\tilde{\Theta}^0))}\Big\}.$$

Then, for any two solutions (Θ, Ω) and $(\tilde{\Theta}, \tilde{\Omega})$, we have uniform ℓ_p-stability estimate (29).

As a direct application of Theorem 4.3, we have the following corollary for the first-order Kuramoto model (1).

Corollary 2. *Suppose that initial data and coupling strength κ satisfy the following relations:*

$$\Theta^0 \in \mathcal{S}(D^\infty), \quad \tilde{\Theta}^0 \in \mathcal{S}(D^\infty), \quad \sum_{i=1}^N \nu_i = 0, \quad \sum_{i=1}^N \tilde{\nu}_i = 0,$$
$$\kappa > \max\Big\{\frac{D(\Omega^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \frac{D(\tilde{\Omega}^0)}{\cos(D^\infty)(D^\infty - D(\tilde{\Theta}^0))}\Big\}.$$

Then, for any two solutions Θ and $\tilde{\Theta}$ to (1) *with natural frequency $\mathcal{V} := (\nu_i)$ and $\tilde{\mathcal{V}} := (\tilde{\nu}_i)$ respectively, there exists a positive constant C independent of t such that*

$$\|\Theta(t) - \tilde{\Theta}(t)\|_p \le C\Big[\|\Theta^0 - \tilde{\Theta}^0\|_p + \|\mathcal{V} - \tilde{\mathcal{V}}\|_p\Big], \quad t \ge 0.$$

Proof. Let Θ and $\tilde{\Theta}$ be phase processes for (1) corresponding to the following initial data and natural frequencies, respectively:

$$(\theta_1^0, \cdots, \theta_N^0),\ (\nu_1, \cdots, \nu_N); \qquad (\tilde{\theta}_1^0, \cdots, \tilde{\theta}_N^0),\ (\tilde{\nu}_1, \cdots, \tilde{\nu}_N). \tag{34}$$

On the other hand, we also set initial frequencies:

$$\begin{aligned} \omega_i^0 &:= \nu_i + \frac{\kappa}{N}\sum_{j=1}^N \sin(\theta_j^0 - \theta_i^0), \\ \tilde{\omega}_i^0 &:= \tilde{\nu}_i + \frac{\kappa}{N}\sum_{j=1}^N \sin(\tilde{\theta}_j^0 - \tilde{\theta}_i^0). \end{aligned} \tag{35}$$

Then, we solve the second-order system (4) with initial data (34) and (35). It follows from the equivalence relation between KM (1) and AKM (4) in Theorem 2.5 and Theorem 4.3 that we have

$$\|\Theta(t) - \tilde{\Theta}(t)\|_p \le C\Big[\|\Theta^0 - \tilde{\Theta}^0\|_p + \|\Omega^0 - \tilde{\Omega}^0\|_p\Big], \tag{36}$$

where $\Omega^0 = (\omega_1^0, \cdots, \omega_N^0)$. We again use the relations (35) to find

$$\begin{aligned} |\omega_i^0 - \tilde{\omega}_i^0| &\le |\nu_i - \tilde{\nu}_i| + \frac{\kappa}{N}\sum_{j=1}^N |\sin(\theta_j^0 - \theta_i^0) - \sin(\tilde{\theta}_j^0 - \tilde{\theta}_i^0)| \\ &\le |\nu_i - \tilde{\nu}_i| + \frac{\kappa}{N}\sum_{j=1}^N \Big(|\theta_j^0 - \tilde{\theta}_j^0| + |\theta_i^0 - \tilde{\theta}_i^0|\Big). \end{aligned}$$

This yields

$$\|\Omega^0 - \tilde{\Omega}^0\|_p \le C\Big[\|\Theta^0 - \tilde{\Theta}^0\|_p + \|\mathcal{V} - \tilde{\mathcal{V}}\|_p\Big]. \tag{37}$$

Finally, we combine (36) and (37) to obtain the desired stability estimate. □

5. **Uniform mean-field limit from the AKM to kinetic equation.** In this section, we present the uniform mean-field limit for the AKM in a measure theoretic framework. The limiting mean-field kinetic equation can be formally derived from the particle model (4) via the formal procedure of BBGKY hierarchy, and it can be rigorously justified using the standard empirical measure approximations and local-in-time stability estimates in Monge-Kantorovich distance which is equivalent to Wasserstein-1 distance in any finite time.

The formal BBGKY hierarchy procedure yields a formal mean-field limit of system (4) toward the mean-field kinetic equation as $N \to \infty$. More precisely, let $f = f(\theta, \omega, t)$ be the one-particle distribution function. Then, the kinetic equation reads as follows.

$$\begin{cases} f_t + \omega\partial_\theta f + \partial_\omega(L[f]f) = 0, \quad (\theta,\omega) \in \mathbb{T}\times\mathbb{R},\ t>0, \\ L[f](\theta,\omega,t) := \kappa \displaystyle\int_0^{2\pi}\int_{-\infty}^{\infty} \cos(\theta_* - \theta)(\omega_* - \omega) f(\theta_*, \omega_*, t)\, d\theta_* d\omega_*. \end{cases} \tag{38}$$

Recall that our main purpose of this section is to justify the rigorous transition from (4) to (38) in the mean-field limit ($N \to \infty$).

5.1. **A measure theoretic framework.** In this subsection, we briefly discuss some framework which embodies (4) and (38) in a common framework. For this, we first review concept of measure-valued solutions to (38).

Let $\mathcal{P}(\mathbb{T}\times\mathbb{R})$ be the set of all Radon probability measures with compact support on the phase space $\mathbb{T}\times\mathbb{R}$, which can be understood as normalized nonnegative bounded linear functionals on $C_0(\mathbb{T}\times\mathbb{R})$. For a probability measure $\mu \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$, we use a standard duality relation:

$$\langle \mu, f\rangle = \int_{\mathbb{T}\times\mathbb{R}} f(\theta,\omega)\, d\mu(\theta,\omega), \quad f \in C_0(\mathbb{T}\times\mathbb{R}).$$

Next, we recall several definitions to be used later.

Definition 5.1. [7] For $T \in [0,\infty)$, let $\mu_t \in L^\infty([0,T);\mathcal{P}(\mathbb{T}\times\mathbb{R}))$ be a measure-valued solution to (38) with initial data $\mu_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ if the following three assertions hold:

1. Total mass is normalized: $\langle \mu_t, 1\rangle = 1$.
2. μ is weakly continuous in t:
$$\langle \mu_t, f\rangle \text{ is continuous in } t \quad \forall\ f \in C_0^1(\mathbb{T}\times\mathbb{R}\times[0,T)).$$
3. μ satisfies the equation (38) in a weak sense: for $\forall\ \varphi \in C_0^1(\mathbb{T}\times\mathbb{R}\times[0,T))$,
$$\langle \mu_t, \varphi(\cdot,\cdot,t)\rangle - \langle \mu_0, \varphi(\cdot,\cdot,0)\rangle = \int_0^t \langle \mu_s, \partial_s\varphi + \omega\partial_\theta\varphi + L[\mu_s]\partial_\omega\varphi\rangle\, ds,$$

Remark 4. Note that for a solution $\{(\theta_i, \omega_i)\}$ to (4), the empirical measure

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{\theta_i} \otimes \delta_{\omega_i},$$

is a measure-valued solution in the sense of Definition 5.1 to (38). Thus, ODE solution to (4) can be understood as a measure-valued solution for (38). Likewise, the classical solution for the kinetic AKM model (38) is also a measure-valued solution as well. Thus, we can treat the particle and kinetic AKM models in the same framework.

We now discuss how to measure the distance between the solutions of (4) and (38) by equipping a metric to the probability measure space $\mathcal{P}(\mathbb{T}\times\mathbb{R})$, and the concept of local-in-time mean-field limit. In fact, we can endow Wasserstein-p distance W_p in the probability space $\mathcal{P}(\mathbb{T}\times\mathbb{R})$.

Definition 5.2. [33, 40]

1. For $p \in \mathbb{Z}_+$, let $\mathcal{P}_p(\mathbb{T}\times\mathbb{R})$ be a collection of all probability measures with finite p^{th} moment: for some $z_0 \in \mathbb{T}\times\mathbb{R}$
$$\langle \mu, \|z - z_0\|_p^p \rangle < +\infty.$$
Then, Wasserstein p-distance $W_p(\mu,\nu)$ is defined for any $\mu,\nu \in \mathcal{P}_p(\mathbb{T}\times\mathbb{R})$ as
$$W_p(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \Big(\int_{\mathbb{T}^2\times\mathbb{R}^2} \|z - z^*\|_p^p \, d\gamma(z,z^*) \Big)^{\frac{1}{p}},$$
where $\Gamma(\mu,\nu)$ denotes the collection of all probability measures on $\mathbb{T}^2\times\mathbb{R}^2$ with marginals μ and ν.
2. If $\lim_{p\to\infty} W_p$ exists, then we define W_∞ metric as the limit.
3. For any $T \in (0,\infty]$, the kinetic equation (38) is derivable from the particle model (4) in $[0,T)$, or equivalent to say the mean-field limit from the particle system (4) to the kinetic equation (38), which is valid in $[0,T)$, if for every solution μ_t of the kinetic equation (38) with initial data μ_0, the following condition holds: for some $p \in [1,\infty)$ and $t \in [0,T)$,
$$\lim_{N\to+\infty} W_p(\mu_0^N, \mu_0) = 0 \quad \Longleftrightarrow \quad \lim_{N\to+\infty} W_p(\mu_t^N, \mu_t) = 0,$$
where μ_t^N is a measure valued solution of the particle system (4) with initial data μ_0^N.

For later use, we quote two results on the approximation of a measure by empirical measures and mean-field limit in any finite time interval without proofs.

Proposition 2. [40] *For any given $p \in [1,\infty)$ and $\mu \in \mathcal{P}_p(\mathbb{T}\times\mathbb{R})$ with compact support, there exists a sequence of empirical measures $\mu^N \in \mathcal{P}_p(\mathbb{T}\times\mathbb{R})$ such that*
$$\mu^N \textit{ has a common compact support with } \mu \textit{ and } \quad \lim_{N\to+\infty} W_p(\mu^N,\mu) = 0.$$

Remark 5. The construction of the approximation can be followed by the method of Theorem 6.18 in the book [40] by finding a sequence of atomic measures $\sum_{j=1}^N a_j\delta_j$ with rational numbers a_j such that $\displaystyle\sum_{j=1}^N a_j = 1$.

5.2. **A uniform mean-field limit.** In this subsection, we present a uniform mean-field limit to the kinetic equation (38). We basically follow the approach given in [21], Corollary 1 and Lemma 3.2.

Theorem 5.3. *Suppose that the initial probability measure* $\mu_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$ *and coupling strength satisfy*

$$
\begin{aligned}
&D_\Theta^{\mu_0} \le D_\infty < \frac{\pi}{2}, \quad \int_{\mathbb{T}\times\mathbb{R}} \omega\,\mu_0(d\theta,d\omega) = 0, \quad \int_{\mathbb{T}\times\mathbb{R}} \mu_0\,(d\theta,d\omega) \le m_0,\\
&\int_{\mathbb{T}\times\mathbb{R}} (|\theta|^p + |\omega|^p)\,\mu_0(d\theta,d\omega) \le m_2, \quad \kappa > \frac{D_\omega^{\mu_0}}{\cos(D^\infty)(D^\infty - D_\Theta^{\mu_0})},
\end{aligned} \tag{39}
$$

where $D_\Theta^{\mu_0}$ *and* $D_\omega^{\mu_0}$ *are diameters of the projected supports of* μ_0 *in* θ *and* ω*-spaces. Then, the following assertions hold: for* $p \in [1,\infty)$,

1. *There exists a unique measure-valued solution* $\mu_t \in L^\infty([0,\infty);\mathcal{P}(\mathbb{T}\times\mathbb{R}))$ *to* (38) *with initial data* μ_0 *such that* μ_t *is approximated by empirical measure* μ_t^N *in Wasserstein-p distance uniformly in time:*

$$\overline{\lim_{N\to+\infty}} \sup_{t\in[0,+\infty)} W_p(\mu_t^N,\mu_t) = 0.$$

2. *Suppose that* ν_t *is the measure-valued solution to* (38) *with initial measure* ν_0 *which has the same property in* (39). *Then there exists nonnegative constant* G *independent of* t *such that*

$$W_p(\mu_t,\nu_t) \le G W_p(\mu_0,\nu_0), \quad t\in[0,\infty).$$

Proof. Since the overall proof of Theorem 5.3 is almost the same as that of Corollary 1.1 in [21], we will provide only sketch of the proof.

• Step A (Extraction of Cauchy approximation for μ_0 in W_p). We take a sequence of empirical measures μ_0^N that approximate μ_0 satisfying

$$\lim_{N\to+\infty} W_p(\mu_0^N,\mu_0) = 0. \tag{40}$$

The existence of such approximation is guaranteed by [40]. Then, owing to (40), for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$W_p(\mu_0^n,\mu_0^m) < \varepsilon, \quad \textit{for } n,m > N(\varepsilon).$$

• Step B (Approximation of $W_p(\mu_0^n,\mu_0^m)$). Using the argument used in the proof of the Corollary 1.1 in [21], we can find a natural number M_{mn} such that

$$\left| W_p^p(\mu_0^n,\mu_0^m) - \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} \|z_{k0} - \bar{z}_{k0}\|_p^p \right| \le \varepsilon^p, \tag{41}$$

where, $z_{k0} := (\theta_{k0},\omega_{k0})$ and $\bar{z}_{k0} := (\bar{\theta}_{k0},\bar{\omega}_{k0})$ are support of initial approximated empirical measures μ_0^n and μ_0^m respectively.

• Step C (Lifting the information at time $s = 0$ to $s = t > 0$). Now, using (41) and the previous ℓ_p-stability in particle level, Theorem 4.3, we can directly estimate $W_p(\mu_t^n,\mu_t^m)$ as

$$W_p^p(\mu_t^n,\mu_t^m) \le 2^{p-1}G^p(W_p^p(\mu_0^n,\mu_0^m) + \varepsilon^p) \le 2^p G^p \varepsilon^p. \tag{42}$$

which implies that the sequence μ_t^n is Cauchy in W_p-metric. Thus, we can find a limit measure μ_t. We next apply similar arguments in [15] and show that the limit measure μ_t is the unique measure-valued solution of the kinetic equation (38) with initial data μ_0. Moreover, because of the estimate (42), we can conclude that for any ε, there exists a positive constant L, such that

$$\sup_{t\in[0,+\infty)} W_p(\mu_t^n,\mu_t) \le 4G\varepsilon, \quad \text{for} \quad n > L.$$

This yields

$$\overline{\lim_{N\to+\infty}} \sup_{t\in[0,+\infty)} W_p(\mu_t^N,\mu_t) = 0. \tag{43}$$

The uniform compact support of μ_t follows this uniform convergence.

• Step D (Uniform stability of kinetic equation). For measures μ_0 and ν_0 in $\mathcal{P}(\mathbb{T}\times\mathbb{R})$, let μ and ν be measure-valued solutions to (38). Then, it follows from (43) that for any $\varepsilon \ll 1$, there exists $N_0(\varepsilon) \in \mathbb{N}$ such that

$$W_p(\mu,\mu^n) < \frac{\varepsilon}{2}, \qquad W_p(\nu^n,\nu) < \frac{\varepsilon}{2} \quad \text{and} \quad n \ge N_0(\varepsilon).$$

Then, we use the above estimates and (42) to obtain

$$\begin{aligned} W_p^p(\mu_t,\nu_t) &\le \Big(W_p(\mu_t,\mu_t^n) + W_p(\mu_t^n,\nu_t^n) + W_p(\nu_t^n,\nu_t)\Big)^p \\ &\le \Big(\varepsilon + W_p(\mu_t^n,\nu_t^n)\Big)^p \\ &\le 2^{p-1}\Big(\varepsilon^p + W_p^p(\mu_t^n,\nu_t^n)\Big) \\ &\le 2^{p-1}\Big(2\varepsilon^p + G^p W_p^p(\mu_0^n,\nu_0^n)\Big). \end{aligned}$$

Letting $n\to\infty$, we have

$$W_p^p(\mu_t,\nu_t) \le 2^p\varepsilon^p + 2^{p-1}G^pW_p^p(\mu_0,\nu_0).$$

Since ε was arbitrary, we have the uniform W_p-stability:

$$W_p(\mu_t,\nu_t) \le 2^{\frac{p-1}{p}} G W_p(\mu_0,\nu_0), \quad t \ge 0.$$

□

Remark 6. The same arguments can be applied for the mean-field limit for the Kuramoto model to the corresponding kinetic equation uniform in time in the class of synchronizing solutions in the next section.

As a direct application of Theorem 5.3, we have the following synchronization estimate for the measure-valued solutions to (38).

Corollary 3. *Suppose that the assumptions (39) hold, and let μ_t be a measure-valued solution to (38) whose existence is guaranteed by Theorem 5.3. Then, we have the complete frequency synchronization:*

$$\Big(\int_{\mathbb{T}\times\mathbb{R}} |\omega|^p\, d\mu_t\Big)^{\frac{1}{p}} \le Ce^{-\kappa(\cos D^\infty)t}\Big(\int_{\mathbb{T}\times\mathbb{R}} |\omega|^p\, d\mu_0\Big)^{\frac{1}{p}}.$$

Proof. Let μ_t^N be a sequence of empirical measures appearing in the course of the proof of Theorem 3.2. Then, it follows from Theorem 3.2 that we have

$$\Big(\int_{\mathbb{R}^{2d}} |\omega|^p\, d\mu_t^N\Big)^{\frac{1}{p}} \le e^{-\kappa(\cos D^\infty)t}\Big(\int_{\mathbb{R}^{2d}} |\omega|^p\, d\mu_0^N\Big)^{\frac{1}{p}}. \tag{44}$$

On the other hand, since μ_t^N has a common compact support, we can view $\|\mathcal{V}\|_p^p$ as a test function. Then, due to Theorem 5.3, we have,

$$\lim_{N\to 0} W_p(\mu_t^N, \mu_t) = 0.$$

This implies the weak convergence of μ_t^N to μ_t. Thus, we can pass to the limit $N \to \infty$ to (44) to obtain

$$\Big(\int_{\mathbb{R}^{2d}} |\omega|^p \, d\mu_t\Big)^{\frac{1}{p}} \le e^{-\kappa(\cos D^\infty)t}\Big(\int_{\mathbb{R}^{2d}} |\omega|^p \, d\mu_0\Big)^{\frac{1}{p}}.$$

□

5.3. **Complete synchronization estimate.** In this subsection, we present an alternative approach for the complete synchronization estimate for (38) introduced in previous subsection. As mentioned in abstract, the kinetic equation (38) for the AKM is more suitable for the Lyapunov functional approach, compared to the kinetic Kuramoto equation for the KM with distributed natural frequencies. For simplicity of presentation, we suppress t-dependence in f:

$$f(\theta,\omega) := f(\theta,\omega,t), \quad \theta \in [0,2\pi], \ \omega \in \mathbb{R}.$$

Lemma 5.4. *Let f be a classical solution of (38) whose support is compact. Then, we have*

$$\frac{d}{dt}\int_0^{2\pi}\int_{-\infty}^{\infty} f \, d\omega d\theta = 0, \qquad \frac{d}{dt}\int_0^{2\pi}\int_{-\infty}^{\infty} \omega f \, d\omega d\theta = 0, \quad t > 0.$$

Proof. It directly comes from multiplying by 1 and ω to (38) and integrating the resulting relation over the phase and frequency space, hence we omit the detailed calculation. □

Next, we discuss the derivation of the complete (frequency) synchronization estimate. For this, we use the Lyapunov functional defined as follows.

$$\Lambda[f(t)] := \int_0^{2\pi}\int_{-\infty}^{\infty} |\omega-\omega_c|^2 f(\theta,\omega)\, d\omega d\theta, \qquad \omega_c := \frac{\int_0^{2\pi}\int_{\mathbb{R}} \omega f \, d\omega d\theta}{\int_0^{2\pi}\int_{\mathbb{R}} f \, d\omega d\theta}, \tag{45}$$

where ω_c is the mean frequency which is constant due to Lemma 5.4.

If complete synchronization occurs, it is natural to expect that frequency will converge to ω_c, i.e., the Lyapunov functional $\Lambda(f)$ converges to 0. To show this, we will use the standard Lyapunov functional estimate on $\Lambda(f)$.

Theorem 5.5. *Let f be a classical solution of (38) whose support is compact and initial datum f^0 satisfying*

$$D_\Theta^0 \le D^\infty < \frac{\pi}{2},$$

where D_Θ^0 is the diameter of support of f_0 projected to θ-space. Then, if the coupling strength κ is large enough, the Lyapunov functional $\Lambda[f]$ decays exponentially:

$$\Lambda[f(t)] \le \Lambda[f^0] e^{-2\kappa(\cos D^\infty)\|f_0\|_{L^1} t}, \quad \textit{as } t \to \infty.$$

Proof. It follows from Lemma 3.1 and condition for support of initial data that we have

$$D_\Theta(t) \le D^\infty < \frac{\pi}{2},$$

where $D_\Theta(t)$ is the diameter of support of $f(\cdot,\cdot,t)$. Now we use (45) and use the periodicity of f in θ-variable to obtain

$$\begin{aligned}
\frac{d}{dt}\Lambda[f] &= \int_0^{2\pi}\int_{\mathbb{R}} (\omega-\omega_c)^2 \partial_t f \, d\omega d\theta \\
&= -\int_0^{2\pi}\int_{\mathbb{R}} (\omega-\omega_c)^2 \omega \partial_\theta f \, d\omega d\theta - \int_0^{2\pi}\int_{\mathbb{R}} (\omega-\omega_c)^2 \partial_\omega (L[f]f) \, d\omega d\theta \\
&= \int_0^{2\pi}\int_{\mathbb{R}} 2(\omega-\omega_c)(L[f]f) \, d\omega d\theta \\
&= 2\kappa \int_{[0,2\pi]^2\times\mathbb{R}^2} \cos(\theta_*-\theta)(\omega-\omega_c)(\omega_*-\omega) f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&= 2\kappa \int_{[0,2\pi]^2\times\mathbb{R}^2} \cos(\theta_*-\theta)\omega(\omega_*-\omega) f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&\quad - 2\kappa \int_{[0,2\pi]^2\times\mathbb{R}^2} \cos(\theta_*-\theta)\omega_c(\omega_*-\omega) f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

Next, we estimate the terms $\mathcal{I}_i$ separately.

- (estimate of $\mathcal{I}_1$). By interchanging ω and ω_*, we obtain

$$\begin{aligned}
\mathcal{I}_1 &= 2\kappa \int_{[0,2\pi]^2\times\mathbb{R}^2} \cos(\theta_*-\theta)\omega(\omega_*-\omega) f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&= -\kappa \int_{[0,2\pi]^2\times\mathbb{R}^2} \cos(\theta_*-\theta)(\omega_*-\omega)^2 f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&\le -\kappa \cos D^\infty \int_{[0,2\pi]^2\times\mathbb{R}^2} ((\omega_*-\omega_c)-(\omega-\omega_c))^2 f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&= -\kappa \cos D_\infty \int_{[0,2\pi]^2\times\mathbb{R}^2} (\omega_*-\omega_c)^2 f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&\quad + 2\kappa \cos D^\infty \int_{[0,2\pi]^2\times\mathbb{R}^2} (\omega_*-\omega_c)(\omega-\omega_c) f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&\quad - \kappa \cos D^\infty \int_{[0,2\pi]^2\times\mathbb{R}^2} (\omega-\omega_c)^2 f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&= -2\kappa \cos D^\infty \int_{[0,2\pi]^2\times\mathbb{R}^2} (\omega-\omega_c)^2 f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* \\
&= -2\kappa \cos D^\infty \|f_0\|_{L^1} \Lambda[f],
\end{aligned}$$

where we use the condition $|\theta_*-\theta| \le D^\infty$ when θ and θ_* are contained in support of $f(\cdot,\cdot,t)$, which is guaranteed by condition on support of initial data.

- (estimate of $\mathcal{I}_2$). From the anti-symmetry of integrand, it is easy to see

$$\mathcal{I}_2 = -2\kappa\omega_c \int_{[0,2\pi]^2\times\mathbb{R}^2} \cos(\theta_*-\theta)(\omega_*-\omega) f(\theta,\omega) f(\theta_*,\omega_*) \, d\theta d\theta_* d\omega d\omega_* = 0.$$

From the estimation of $\mathcal{I}_i$, $i=1,2$, we derive following differential inequality

$$\frac{d}{dt}\Lambda[f] \le -2\kappa \cos D^\infty \|f_0\|_{L^1} \Lambda[f].$$

By using Grönwall's lemma, we can obtain the desired exponential decay. □

6. **Applications to the kinetic Kuramoto Model.** In this section, we study the uniform mean-field limit of the Kuramoto model and as a direct application of previous results, we also show the existence of phase-locked states for the kinetic Kuramoto equation via uniform mean-field limit by lifting particle results to the kinetic regime. For the local-in-time stability and mean-field limit of the kinetic Kuramoto equation, we refer to [28].

Let $f = f(\theta, \nu, t)$ be a one-oscillator probability density function for the ensemble of Kuramoto oscillators. Then, the dynamics of f is governed by the kinetic Kuramoto equation:

$$\begin{cases} \partial_t f + \partial_\theta (v[f] f) = 0, \quad (\theta, \nu) \in \mathbb{T} \times \mathbb{R},\ t > 0, \\ v[f](\theta, \nu, t) = \nu + \kappa \displaystyle\int_0^{2\pi} \sin(\theta_* - \theta) f(\theta_*, \nu_*, t)\, d\nu_* d\theta_*. \end{cases} \tag{46}$$

Note that the probability density function $g = g(\nu)$ for natural frequencies appears as a ν-marginal density function of f:

$$\int_0^{2\pi} f(\theta, \nu, t) d\theta = g(\nu).$$

Unlike to (38), it is not clear how to show the emergence of the complete frequency synchronization for (46) using the nonlinear functional approach as in Section 5.3. This is why we introduce a second order model (4) and its mean-field limit (38). Similar to Definition 5.2, we can define the measure valued solution of the kinetic Kuramoto equation (46).

Definition 6.1. [7] For $T \in [0, \infty)$, $\mu_t \in L^\infty([0, T); \mathcal{P}(\mathbb{T} \times \mathbb{R}))$ is a measure valued solution to (46) with initial data $\mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ if the following three assertions hold:

1. Total mass is normalized: $\langle \mu_t, 1 \rangle = 1$.
2. μ is weakly continuous in t:
$$\langle \mu_t, f \rangle \text{ is continuous in } t \quad \forall\ f(\theta, \nu) \in C_0^1(\mathbb{T} \times \mathbb{R} \times [0, T)).$$
3. μ satisfies the equation (46) in a weak sense: for $\forall\ \varphi \in C_0^1(\mathbb{T} \times \mathbb{R} \times [0, T))$,
$$\langle \mu_t, \varphi(\cdot, \cdot, t) \rangle - \langle \mu_0, \varphi(\cdot, \cdot, 0) \rangle = \int_0^t \langle \mu_s, \partial_s \varphi + v[\mu] \partial_\theta \varphi \rangle\, ds. \tag{47}$$

Remark 7. As mentioned in Remark 4, both the solution of the original Kuramoto model (1) and the solution of the kinetic equation (46) can be viewed as a measure valued solution in the sense of Definition 6.1. Thus, we can apply the Wasserstein metric in Definition 5.2 to measure the distance between two measure valued solutions.

According to Proposition 2 and Remark 5, we have the following result.

Theorem 6.2. *Suppose that initial probability measure $\mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ and coupling strength satisfy*

$$\begin{aligned} & D_\Theta^{\mu_0} \le D^\infty < \frac{\pi}{2}, \quad \int_0^{2\pi} \nu\, \mu_0(d\theta, d\nu) = 0, \quad \int_{\mathbb{T} \times \mathbb{R}} \mu_0(d\theta, d\nu) \le m_0, \\ & \int_{\mathbb{T} \times \mathbb{R}} (|\theta|^p + |\nu|^p) \mu_0(d\theta, d\nu) \le m_2, \quad \kappa > \frac{D_\omega^{\mu_0}}{\cos(D^\infty)(D^\infty - D_\Theta^{\mu_0})}. \end{aligned} \tag{48}$$

Then, the following assertions hold. For $p \in [1, \infty)$,

1. *There exists a unique measure-valued solution* $\mu_t \in L^\infty([0,\infty);\mathcal{P}(\mathbb{T}\times\mathbb{R}))$ *to* (46) *with initial data* μ_0 *such that* μ_t *is approximated by empirical measure* μ_t^N *in Wasserstein-p distance uniformly in time:*

$$\overline{\lim_{N\to+\infty}} \sup_{t\in[0,+\infty)} W_p(\mu_t^N, \mu_t) = 0.$$

2. *Moreover, if* $\tilde{\mu}_t$ *is the measure-valued solution to* (46) *with initial measure* $\tilde{\mu}_0$ *with compact support and finite moments* (48)*, then there exists nonnegative constant* G *independent of* t *such that*

$$W_p(\mu_t, \tilde{\mu}_t) \le G W_p(\mu_0, \tilde{\mu}_0), \quad t \in [0,\infty).$$

Proof. The construction of the proof is similar to Theorem 5.3. In fact, as the distribution of natural frequency ν does not have its own dynamics, i.e., it does not change in time, thus the variance of ν between μ and $\tilde{\mu}$ will be a constant, i.e.,

$$\inf_{\gamma\in\Gamma(\mu_t,\tilde{\mu}_t)} \int_{\mathbb{T}^2\times\mathbb{R}^2} |\nu-\bar{\nu}|^p \,\gamma(\nu,\bar{\nu}) = \inf_{\gamma\in\Gamma(\mu_0,\tilde{\mu}_0)} \int_{\mathbb{T}^2\times\mathbb{R}^2} |\nu-\bar{\nu}|^p \,\gamma(\nu,\bar{\nu}).$$

Therefore, we only need to control the variance on θ. Applying the uniform stability in Corollary 2, we can construct the uniform mean-field limit and stability for kinetic Kuramoto model (46) as same as Theorem 5.3. □

Now, for large time behavior, we can apply Corollary 1 to the approximate solution μ_t^N. Notice that the decay rate in Corollary 1 is independent of N. Hence, the mean-field limit preserves the decay rate, when N tends to infinity.

Corollary 4. (Emergence of a phase-locked state) *Suppose that the initial data* μ_0 *and coupling strength satisfy the assumptions* (39)*. Then, there exists a phase-locked state* μ_∞ *such that*

$$W_p(\mu_t, \mu_\infty) \le Ce^{-KD^\infty t}, \quad \text{as } t\to\infty.$$

Proof. It follows from Corollary 1 that for each μ_t^N, we have a unique asymptotic equilibrium μ_∞^N. Then from the uniform stability in Corollary 2, we can obtain the sequence $\{\mu_\infty^N\}$ is Cauchy, and thus generates a unique limit measure μ_∞. Moreover, it follows from Corollary 1 that we have

$$W_p(\mu_t^N, \mu_\infty^N) \le Ce^{-\kappa D^\infty t}.$$

Notice here the p-th moment of ν would be cancelled because μ_t^N and μ_∞^N has the same natural frequency distribution $\int \nu \mu_0^N(d\theta, d\Omega)$. Now for any $\varepsilon > 0$, we can find N_0 large enough such that, for $N \ge N_0$ we have

$$W_p(\mu_t,\mu_\infty) \le W_p(\mu_t,\mu_t^N) + W_p(\mu_t^N,\mu_\infty^N) + W_p(\mu_\infty^N,\mu_\infty) \le 2\varepsilon + Ce^{-\kappa D^\infty t}.$$

Thus, we have

$$W_p(\mu_t,\mu_\infty) \le Ce^{-\kappa D^\infty t}.$$

□

7. **Conclusion.** We presented the dynamic properties of the augmented Kuramoto model which is a second-order lifting of the Kuramoto model for synchronization. For the particle Kuramoto model with distributed natural frequencies, the complete (frequency) synchronization can be studied by analyzing the temporal evolution of $D(\dot{\Theta})$. However, it is not clear how to verify the complete synchronization for the corresponding mean-field kinetic equation directly. This is why we introduced a

second-order lifting of the Kuramoto model. For the corresponding kinetic equation, the complete frequency synchronization can be obtained via the Lyapunov functional measuring the dispersion of the frequency variations. Our proposed second-order model has a formal similarity to the Cucker-Smale flocking model. As long as the phase diameter is confined in a quarter arc, the flocking estimate, uniform ℓ_p-stability and mean-field limit for the extended Kuramoto model can be analyzed using the similar techniques done for the Cucker-Smale model. As aforementioned in Introduction, the reason that we focus on the mean-field case (all-to-all couplings) is that we are interested in the complete synchronization estimate for the kinetic Kuramoto equation at the kinetic level without lifting particle results to the kinetic level. Other than this, some of the estimates studied in this paper can be extended to a more general setting. For example, we considered synchronization dynamics of Kuramoto oscillators on the complete network, but some estimates such as the uniform stability and synchronization estimates at the particle level can be certainly extended to the locally coupled Kuramoto oscillators over the general networks, say symmetric and connected networks. Moreover, nonlinear stability and instability of the incoherent state can be studied for the proposed kinetic equation. We leave these interesting issues as a future work.

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