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THE LAX-OLEINIK SEMIGROUP ON GRAPHS

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Abstract. We consider Tonelli Lagrangians on a graph, define weak KAM solutions, which happen to be the fixed points of the Lax-Oleinik semi-group, and identify their uniqueness set as the Aubry set, giving a representation formula. Our main result is the long time convergence of the Lax Oleinik semi-group. It follows that weak KAM solutions are viscosity solutions of the Hamilton-Jacobi equation [3, 4], and in the case of Hamiltonians called of eikonal type in [3], we prove that the converse holds.

1. **Introduction.** There has been an increasing interest in the study of dynamical systems and differential equation on networks. In the same vein, the study of control problems on networks has interesting applications in various fields. A typical optimal control problem is the minimum time problem, which consists of finding the shortest path between an initial position and a given target set. If the cost changes in a continuous way along the edges and the dynamics is continuous in time, the minimum time problem can be seen as a continuous-state continuous-time control problem where the admissible trajectories of the system are constrained to remain on the network. Control problems with state constrained in closures of open sets have been intensively studied but the there is much fewer literature on problems on networks.

Networks are the simplest examples of ramified spaces. Ramified spaces are the natural settings for problems of interaction between different media which are described by differential equations on the branches and transition conditions on the ramifications. Those interaction problems have different applications in physics, chemistry, and biology. Concerning the analysis of these models, the possibility of the application of several mathematical methods does not only depend on the structure of the differential equations on the branches, but also they depend strongly on the properties of the transition conditions. In fact, their effect on existence, uniqueness, and regularity of solutions is quite considerable. Many well-known results for elliptic and parabolic problems in the classical non-ramified situation have been extended to ramified spaces, providing results for boundary or initial

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value problems, now stablished for collections of media with transition conditions. In many cases, the core of the difficulties found on ramified spaces are already present in the simple setting of networks.

Viscosity solutions of the Hamilton Jacobi equation on networks have been studied for several authors, but the problem of the long time behaviour of the solutions to the Cauchy problem for the Hamilton Jacobi equation has not been considered. Upon addressing that problem, we found more convenient to follow the variational approach, considering the viscosity solution given by the Lax Oleinik semigroup.

In the first part of this article we study the Lax-Oleinik semi-group $\mathcal{L}_t$ defined by a Tonelli Lagrangian on a graph and prove that for any continuous function u, $\mathcal{L}_t u + ct$ converges as $t \to \infty$ where c is the critical value of the Lagrangian.

For Lagrangians on compact manifolds, Fathi [6, 7] proved the convergence using the Euler-Lagrange flow and conservation of energy. In our case we do not have those tools but we can follow ideas of Roquejoffre [11] and Davini-Siconolfi [5]. See also [10] for the convergence when the manifold is the whole euclidian space.

Camilli and collaborators [1, 4, 3] have studied viscosity solutions of the Hamilton-Jacobi equation, and given sufficient conditions for a set to be a uniqueness set and a representation formula. See also the recent papers [2, 9].

In the second part of this article we prove that, under the assumption that the Lagrangian is symmetric at the vertices, the sets of weak KAM and viscosity solutions of the Hamilton-Jacobi equation coincide.

We consider a graph G without boundary consisting of finite sets of unoriented edges $\mathcal{I} = \{I_j\}$ and vertices $\mathcal{V} = \{e_l\}$. The interior of I_j is $I_j - \mathcal{V}$. Parametrizing each edge by arc length $\sigma_j : I_j \to [0, s_j]$ we can write its tangent bundle as $TI_j = I_j \times \mathbb{R}$ and

$$TG = \bigcup_j \{j\} \times TI_j / \sim$$

where $(i, x, v) \sim (j, y, w) \iff (i, x, v) = (j, y, w)$ or $x = y \in I_i \cap I_j, v = w = 0$. Thus, a function $L : TG \to \mathbb{R}$ is given by a collection of functions $L_j : TI_j \to \mathbb{R}$ such that $L_i(e_l, 0) = L_j(e_l, 0)$ for $e_l \in I_i \cap I_j$. A Lagrangian in G is a function $L : TG \to \mathbb{R}$ such that each L_j is C^k, $k \geq 2$, and $L_j(x, \cdot)$ is strictly convex and super-linear for any $x \in I_j$. We will say that a Lagrangian is *symmetric at the vertices* if at each vertex e_l there is a function $\lambda_l : \{u \in \mathbb{R} : u \geq 0\} \to \mathbb{R}$ such that $L_j(e_l, z) = \lambda_l(|z|)$ if $e_l \in I_j$. As an example consider the mechanical Lagrangian given by $L_j(x, v) = \frac{1}{2}v^2 - U_j(x)$, with $U_j(e_l) = a_l$ if $e_l \in I_j$. For $x \in I_j \setminus \mathcal{V}$, we say that (x, v) points towards $\sigma^{-1}(s_j)$ if $v > 0$ and points towards $\sigma^{-1}(0)$ if $v < 0$.

We say that $(\sigma_j^{-1}(0), v)$ is an I_j-incoming or outgoing vector according to whether $v > 0$ or $v < 0$, and we say that $(\sigma_j^{-1}(s_j), v)$ is an I_j-incoming or outgoing vector according to wether $v < 0$ or $v > 0$. We let $T^+_{e_l} I_j$ ($T^-_{e_l} I_j$) to be the set of I_j-outgoing (incoming or zero) vectors in $T_{e_l} I_j$.

2. Basic properties of the action.

2.1. A distance on a graph.

We start defining a distance in the most natural way. We say a continuous path $\alpha : [a, b] \to G$ is a *unit speed geodesic* (u.s.g.) if there is a partition $a = t_0 < \ldots < t_m = b$ such that for each $1 \leq i \leq m$ there is $j(i)$ such that

$$\alpha([t_0, t_1]) \subset I_{j(1)}, \alpha([t_{i-1}, t_i]) = I_{j(i)}, \ldots, \alpha([t_{m-1}, t_m]) \subset I_{j(m)},$$

$\sigma_{j(i)} \circ \alpha|_{[t_{i-1},t_i]}$ is differentiable and either $(\sigma_{j(i)} \circ \alpha)' \equiv 1$ or $(\sigma_{j(i)} \circ \alpha)' \equiv -1$. We set the length of a u.s.g. to be

$$\ell(\alpha) = |\sigma_{j(1)}(\alpha(t_1)) - \sigma_{j(1)}(\alpha(a))| + \sum_{i=2}^{m-1} s_{j(i)} + |\sigma_{j(m)}(\alpha(b)) - \sigma_{j(m)}(\alpha(t_{m-1}))|$$

and define a distance on G by

$$d(x, y) = \min\{\ell(\alpha) : \alpha : [a, b] \to G \text{ is a u.s.g.}, \alpha(a) = x, \alpha(b) = y\}$$

2.2. **Absolute continuity.** We say a path $\gamma : [a, b] \to G$ is *absolutely continuous* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any finite collection of disjoint intervals $\{[c_i, d_i]\}$ with $\sum_i (d_i - c_i) < \delta$ we have $\sum_i d(\gamma(c_i), \gamma(d_i)) < \varepsilon$.

If $\gamma : [a, b] \to G$, $\gamma(t) \in \mathcal{V}$, we define $\dot\gamma(t) = 0$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $d(\gamma(s), \gamma(t)) < \varepsilon|s - t|$ when $|s - t| < \delta$.

Let $\gamma : [a, b] \to G$ be absolutely continuous and consider the closed set $V = \gamma^{-1}(\mathcal{V})$ so that $(a, b) \setminus V = \bigcup_i (a_i, b_i)$ where the intervals (a_i, b_i) are disjoint and $\gamma([a_i, b_i]) \subset I_{j(i)}$. It is clear that each $\sigma_{j(i)} \circ \gamma : [a_i, b_i] \to [0, s_{j(i)}]$ is absolutely continuous. We set $\dot\gamma(t) = (\sigma_{j(i)} \circ \gamma)'(t)$ whenever is defined.

Next Proposition will allow us to define the action of an absolutely continuous curve.

Proposition 1. *Let $\gamma : [a, b] \to G$ be absolutely continuous and $V = \gamma^{-1}(\mathcal{V})$*

(a) $\dot\gamma = 0$ Lebesgue almost everywhere in V.

(b) $\dot\gamma$ is integrable and for any $[c, d] \subset [a, b]$ we have $d(\gamma(c), \gamma(d)) \le \int_c^d |\dot\gamma|$.

Proof. Write $(a, b)\setminus V = \bigcup_i (a_i, b_i)$ as above with the intervals (a_i, b_i) disjoint. Since $\dot\gamma = 0$ on the interior of V and $\cup\{a_n, b_n\}$ is numerable to stablish item (a) it remains to prove that $\dot\gamma = 0$ Lebesgue almost everywhere in $\partial V \setminus \cup\{a_n, b_n\}$.

Let $\bar s = \min_j s_j$ and take $\delta > 0$ such that $d(\gamma(t_1), \gamma(t_2)) < \bar s$ if $|t_1 - t_2| < \delta$. There is N such that $b_i - a_i < \delta$ for $i > N$. Since $\gamma(a_i), \gamma(b_i) \in \mathcal{V}$ we have that $\gamma(a_i) = \gamma(b_i)$ for $i > N$. We change the labeling of the first N terms to have $a_1 < b_1 \le a_2 < \cdots < b_m$ and $\gamma(a_i) = \gamma(b_i)$ for $i > m$. Letting $J_0 = [a, a_1]$, $J_i = [b_i, a_{i+1}]$, $1 \le i < m$, $J_m = [b_m, b]$, and $V_i = V \cap J_i$ we have $\gamma(V_i) = e_{l_i}$, $0 \le i \le m$. We can forget about the cases $b_i = a_{i+1}$.

Define the function $f_i : J_i \to \mathbb{R}$ by $f_i(t) = d(e_{l_i}, \gamma(t))$. For $t, s \in J_i$ we have $|f_i(t) - f_i(s)| \le d(\gamma(t), \gamma(s))$, so f_i is absolutely continuous and then f_i' exists Lebesgue almost everywhere in J_i. Let $t \in \partial V_i \setminus \bigcup_n \{a_n, b_n\}$ be a point where f_i' exists. There is a sequence $n_k \to \infty$ such that $a_{n_k} \to t$ and $\gamma(a_{n_k}) = e_{l_i}$. Thus $f_i'(t) = 0$, which means that $\dot\gamma(t) = 0$.

If $(a_j, b_j) \subset J_i$ then $|\dot\gamma| = |f_i'|$ Lebesgue almost everywhere in (a_j, b_j), so that

$$\int_{J_i} |\dot\gamma| = \int_{J_i} |f_i'|,$$

and then

$$\int_a^b |\dot\gamma| = \sum_{i=0}^m \int_{J_i} |\dot\gamma| + \sum_{i=1}^m \int_{a_i}^{b_i} |\dot\gamma| < \infty.$$

It is also easy to see that for $t, s \in J_i$ we have $d(\gamma(t),\gamma(s)) \le \int_s^t |f_i'|$, and using the partition $a \le a_1 < b_1 \le a_2 < \cdots < b_m \le b$ to make a partition of $[c,d]$ we get $d(\gamma(c),\gamma(d)) \le \int_c^d |\dot\gamma|$. □

2.3. **Lower semicontinuity and apriori bounds.** In this crucial part of the paper we prove that in the framework of graphs we have the lower semicontinuity of the action and apriori bounds for the Lipschitz norm of minimizers. The proofs have the same spirit as in euclidean space, paying attention to what happens at the vertices. Denote by $\mathcal{C}^{ac}([a,b])$ the set of absolutely continuous functions $\gamma : [a,b] \to G$ provided with the topology of uniform convergence. We define the action of $\gamma \in \mathcal{C}^{ac}([a,b])$ as

$$A(\gamma) = \int_a^b L(\gamma(t),\dot\gamma(t))dt$$

A minimizer is a $\gamma \in \mathcal{C}^{ac}([a,b])$ such that for any $\alpha \in \mathcal{C}^{ac}([a,b])$ with $\alpha(a) = \gamma(a)$, $\alpha(b) = \gamma(b)$ we have

$$A(\gamma) \le A(\alpha)$$

The following two properties of the Lagrangian are important to achieve our goal and follow from its strict convexity and super-linearity.

Proposition 2. *If $C \ge 0$, $\varepsilon > 0$, there is $\eta > 0$ such that for $x, y \in I_j$, $d(x,y) < \eta$ and $v, w \in \mathbb{R}$, $|v| \le C$, we have*

$$L(y,w) \ge L(x,v) + L_v(x,v)(w-v) - \varepsilon.$$

Proposition 3. *If $L_{vv} \ge \theta > 0$, $C \ge 0$, $\varepsilon > 0$, there is $\eta > 0$ such that for $x, y \in I_j$, $d(x,y) < \eta$ and $v, w \in \mathbb{R}$, $|v| \le C$, we have*

$$L(y,w) \ge L(x,v) + L_v(x,v)(w-v) + \frac{3\theta}{4}|w-v|^2 - \varepsilon.$$

Lemma 2.1. *Let L be a Lagrangian on G. If a sequence $\gamma_n \in \mathcal{C}^{ac}([a,b])$ converges uniformly to the curve $\gamma : [a,b] \to G$ and*

$$\liminf_{n\to\infty} A(\gamma_n) < \infty$$

then the curve γ is absolutely continuous and

$$A(\gamma) \le \liminf_{n\to\infty} A(\gamma_n).$$

Proof. By the super-linearity of L we may assume that $L \ge 0$. Let $c = \liminf_{n\to\infty} A(\gamma_n)$. Passing to a subsequence we can assume that

$$A(\gamma_n) < c+1, \quad \forall n \in \mathbb{N}$$

Fix $\varepsilon > 0$ and take $B > 2(c+1)/\varepsilon$. Again by super-linearity there is a positive number $C(B)$ such that

$$L(x,v) \ge B|v| - C(B), \quad x \in G \setminus \mathcal{V}, v \in \mathbb{R}$$

From Proposition 1 and $L \ge 0$, for $E \subset [a,b]$ measurable we have

$$-C(B)\text{Leb}(E) + B\int_E |\dot\gamma_n| \le \int_E L(\gamma_n,\dot\gamma_n) + \int_{[a,b]\setminus E} L(\gamma_n,\dot\gamma_n) \le c+1.$$

Thus

$$\int_E |\dot\gamma_n| \le \frac{1}{B}(c+1+C(B)\mathrm{Leb}(E)) \le \frac{\varepsilon}{2} + \frac{C(B)\mathrm{Leb}(E)}{B}.$$

Choosing $0<\delta<\dfrac{\varepsilon B}{2C(B)}$ we have that

$$\mathrm{Leb}(E)<\delta \Rightarrow \forall n\in\mathbb{N} \int_E |\dot\gamma_n| < \varepsilon.$$

Since the sequence $\dot\gamma_n$ is uniformly integrable, we have that γ is absolutely continuous and $\dot\gamma_n$ converges to $\dot\gamma$ in the $\sigma(L^1,L^\infty)$ weak topology.

Set $V=\gamma^{-1}(\mathcal{V})$. Let $\varepsilon>0$ and $E_k=\big\{t : |\dot\gamma(t)|\le k, d(t,V)\ge \frac{1}{k}\big\}$.

By Propositions 2, 1, for n large,

$$\int_{\gamma^{-1}(e_l)} \big[L(e_l,0)+L_v(e_l,0)\dot\gamma_n(t)-\varepsilon\big] \le \int_{\gamma^{-1}(e_l)} L(\gamma_n,\dot\gamma_n),$$
$$\int_{E_k} \big[L(\gamma,\dot\gamma)+L_v(\gamma,\dot\gamma)(\dot\gamma_n-\dot\gamma)-\varepsilon\big] \le \int_{E_k} L(\gamma_n,\dot\gamma_n),$$
$$\int_{E_k\cup V} \big[L(\gamma,\dot\gamma)+L_v(\gamma,\dot\gamma)(\dot\gamma_n-\dot\gamma)-\varepsilon\big] \le \int_{E_k\cup V} L(\gamma_n,\dot\gamma_n) \le A(\gamma_n).$$

Letting $n\to+\infty$ we have that

$$\int_{E_k\cup V} L(\gamma,\dot\gamma) \le c+\varepsilon\,(b-a)$$

Since $E_k\uparrow [a,b]\setminus V$ when $k\to+\infty$ and $L\ge 0$, we have

$$A(\gamma)=\lim_{k\to+\infty}\int_{E_k\cup V} L(\gamma,\dot\gamma)\le c+\varepsilon\,(b-a)$$

Now let $\varepsilon\to 0$. □

Lemma 2.1 implies

Theorem 2.2. *Let L be a Lagrangian on G. The action $A:\mathcal{C}^{ac}([a,b])\to\mathbb{R}\cup\{\infty\}$ is lower semicontinuous.*

Proposition 4. *Let L be a Lagrangian on G.*

The set $\{\gamma\in\mathcal{C}^{ab}([a,b]) : A(\gamma)\le K\}$ is compact with the topology of uniform convergence.

Let $C_t=\sup\{L(x,v) : x\in G, |v|\le \frac{\mathrm{diam}(G)}{t}\}$, then for any minimizer $\gamma:[a,b]\to G$ with $b-a\ge t$ we have

$$A(\gamma)\le C(b-a).$$

Proposition 5. *Suppose $\gamma_n\in\mathcal{C}^{ab}([a,b])$ converge uniformly to $\gamma:[a,b]\to G$ and $A(\gamma_n)$ converges to $A(\gamma)$, then $\dot\gamma_n$ converges to $\dot\gamma$ in $L^1[a,b]$*

Proof. Let $F\subset[a,b]$ be a finite union of intervals. From Lemma 2.1 we have

$$\int_F L(\gamma,\dot\gamma)\le\liminf_n\int_F L(\gamma_n,\dot\gamma_n) \text{ and } \int_{[a,b]\setminus F} L(\gamma,\dot\gamma)\le\liminf_n\int_{[a,b]\setminus F} L(\gamma_n,\dot\gamma_n)$$

Since

$$\lim_n \int_F L(\gamma_n,\dot\gamma_n)+\int_{[a,b]\setminus F} L(\gamma_n,\dot\gamma_n)=\lim_n A(\gamma_n)=A(\gamma)$$

we have

$$\lim_n \int_F L(\gamma_n, \dot\gamma_n) = \int_F L(\gamma, \dot\gamma). \tag{1}$$

If $V = \gamma^{-1}(\mathcal{V}) \subset F$ then

$$\limsup_n \int_V L(\gamma_n, \dot\gamma_n) \le \lim_n \int_F L(\gamma_n, \dot\gamma_n) = \int_F L(\gamma, \dot\gamma).$$

Thus

$$\limsup_n \int_V L(\gamma_n, \dot\gamma_n) \le \int_V L(\gamma, \dot\gamma). \tag{2}$$

As in Lemma 2.1, the $\dot\gamma_n$ are uniformly integrable, so they converge to $\dot\gamma$ in the $\sigma(L_1, L_\infty)$ weak topology and then, for any Borel set B where $\dot\gamma$ is bounded,

$$\lim_n \int_B L_v(\gamma, \dot\gamma)(\dot\gamma_n - \dot\gamma) = 0. \tag{3}$$

Given $\varepsilon > 0$, from Proposition 3 we have that for n large enough

$$\frac{3\theta}{4} \int_{\gamma^{-1}(e_l)} |\dot\gamma_n|^2 \le \int_{\gamma^{-1}(e_l)} \big[L(\gamma_n, \dot\gamma_n) - L(e_l, 0) - L_v(e_l, 0)\dot\gamma_n + \varepsilon\big].$$

Which together with Proposition 1 and equations (2), (3) give $\limsup\limits_n \int\limits_V |\dot\gamma_n|^2 \le \varepsilon$ for any $\varepsilon > 0$. So

$$\lim_n \int_V |\dot\gamma_n|^2 = 0. \tag{4}$$

For $k > 0$ let $D_k := \{t \in [a,b] : |\dot\gamma(t)| > k\}$, $B_k := \{t \in [a,b] : d(t, V) > \frac{1}{k}\}$. Then $\lim\limits_{k\to\infty} \mathrm{Leb}(D_k) = \lim\limits_{k\to\infty} \mathrm{Leb}([a,b] \setminus V \setminus B_k) = 0$. Let F_k be a finite union of intervals such that $D_k \cap B_k \subset F_k \subset B_k$ and $\mathrm{Leb}(F_k \setminus (D_k \cap B_k)) < \frac{1}{k}$. Then $\lim\limits_{k\to\infty} \mathrm{Leb}(F_k) = 0$.

Given $\varepsilon > 0$, from Proposition 3 we have that for n large enough

$$\frac{3\theta}{4} \int_{B_k \setminus F_k} |\dot\gamma_n - \dot\gamma|^2 \le \int_{B_k \setminus F_k} \big[L(\gamma_n, \dot\gamma_n) - L(\gamma, \dot\gamma) - L_v(\gamma, \dot\gamma)(\dot\gamma_n - \dot\gamma) + \varepsilon\big].$$

From (1), (3) we get that $\limsup\limits_n \int\limits_{B_k \setminus F_k} |\dot\gamma_n - \dot\gamma|^2 \le \varepsilon$ for any $\varepsilon > 0$. So

$$\lim_n \int_{B_k \setminus F_k} |\dot\gamma_n - \dot\gamma|^2 = 0. \tag{5}$$

Since $\{\dot\gamma_n\}$ is uniformly integrable, given $\varepsilon > 0$, for k sufficiently large we have

$$\int_{F_k \cup [a,b] \setminus V \setminus B_k} |\dot\gamma_n - \dot\gamma| \le \int_{F_k \cup [a,b] \setminus V \setminus B_k} |\dot\gamma_n| + |\dot\gamma| < \varepsilon, \tag{6}$$

From (4), (5), (6) and Cauchy-Schwartz inequality, we have that for any $\varepsilon > 0$

$$\limsup_n \int_a^b |\dot\gamma_n - \dot\gamma| \le \lim_n \int\limits_V |\dot\gamma_n| + \limsup_n \int\limits_{F_k \cup [a,b] \setminus V \setminus B_k} |\dot\gamma_n - \dot\gamma| \; + \lim_n \int\limits_{B_k \setminus F_k} |\dot\gamma_n - \dot\gamma| \le \varepsilon$$

□

Lemma 2.3. *Let L be a Lagrangian in G. For $\varepsilon > 0$ there exists K_ε that is a Lipschitz constant for any minimizer $\gamma : [a,b] \to G$ with $b - a \ge \varepsilon$.*

Proof. Note that if γ is a minimizer and $\gamma(c,d) \subset I_j$ then $\gamma|_{(c,d)}$ is a solution of the Euler Lagrange equation for L_j.

Suppose the Lemma is not true, then by Proposition 1, for any $i \in \mathbb{N}$ there are a minimizer $\gamma_i : [s_i, t_i] \to G$ with $t_i - s_i \geq \varepsilon$ and a set $E_i \subset [s_i, t_i] \setminus \gamma_i^{-1}(\mathcal{V})$ with $\mathrm{Leb}(E_i) > 0$ such that $|\dot{\gamma}| > i$ on E_i. Let $c_i \in E_i$. Translating $[s_i, t_i]$ we can assume that $c_i = c$ for all i and taking a subsequence that there is $a \in \mathbb{R}$ such that γ_i is defined in $[a, a + \frac{\varepsilon}{2}]$, $c \in [a, a + \frac{\varepsilon}{2}]$. As $A(\gamma_i|[a, a + \frac{\varepsilon}{2}])$ is bounded, by Proposition 4 there is subsequence $\gamma_i|[a, a + \frac{\varepsilon}{2}]$ which converges uniformly to $\gamma : [a, a + \frac{\varepsilon}{2}] \to G$. Since γ is limit of minimizers, it is a minimizer and $A(\gamma) \leq \liminf A(\gamma_i|[a, a + \frac{\varepsilon}{2}])$. We can not have that $A(\gamma) < \limsup A(\gamma_i|[a, a + \frac{\varepsilon}{2}])$ because that would contradict that the γ_i are minimizers. Thus $A(\gamma) = \lim A(\gamma_i|[a, a + \frac{\varepsilon}{2}])$.

If $\gamma(c) \in I_j \setminus \mathcal{V}$, there is $\delta > 0$ such that $\gamma([c - \delta, c + \delta]) \subset I_j$.

If $\gamma(c) = e_l$ we have 2 possibilities (not mutually exclusive)

a) There is an edge I_j with $e_l \in I_j$ and infinitely many i's such that $\gamma_i(c) \in I_j$ and $\dot{\gamma}_i(c)$ points towards e_l.

b) There is an edge I_j with $e_l \in I_j$ and infinitely many i's such that $\gamma_i(c) \in I_j$ and $\dot{\gamma}_i(c)$ points towards the other vertex.

In case a) there is $\delta > 0$ such that $\gamma([c - \delta, c]) \subset I_j$.

In case b) there is $\delta > 0$ such that $\gamma([c, c + \delta]) \subset I_j$.

We have that γ is a solution of the Euler-Lagrange equation for L_j either on $[c - \delta, c]$ or on $[c, c + \delta]$ and then $|\dot{\gamma}(t)| \leq K$ on $[c - \delta, c]$ or $[c, c + \delta]$. For some $0 < \delta_1 < \delta$ we have that γ_i are solutions of the Euler-Lagrange equation for L_j on $[c - \delta_1, c]$ or on $[c, c + \delta_1]$. For i suficiently large, we have that $|\dot{\gamma}_i| > 2K$ either on $[c - \delta_1, c]$ or on $[c, c + \delta_1]$. This would contradict Proposition 5. □

3. **Weak KAM theory on graphs.**

The content of this section is similar to that for Lagrangians on compact manifolds. We only give the proofs that are different from those in the compact manifold case, which can be found in [7], as well as extensions in [8].

3.1. **The Peierls barrier.**

Given $x, y \in G$ let $\mathcal{C}^{ac}(x, y, t)$ be the set of curves $\alpha \in \mathcal{C}^{ac}([0, t])$ such that $\alpha(0) = x$ and $\alpha(t) = y$. For a given real number k define

$$h_t(x, y) = \min_{\alpha \in \mathcal{C}^{ac}(x,y,t)} A(\alpha)$$

and

$$h^k(x, y) = \liminf_{t \to \infty} h_t(x, y) + kt$$

Lemma 3.1. *For $\varepsilon > 0$ the function $F : [\varepsilon, \infty) \times G \times G \to \mathbb{R}$ defined by $F(t, x, y) = h_t(x, y)$ is Lipschitz.*

Lemma 3.2. *There exists a real c independent of x and y such that*

1. *For all $k > c$ we have $h^k(x, y) = \infty$.*
2. *For all $k < c$ we have $h^k(x, y) = -\infty$*
3. *$h^c(x, y)$ is finite. The function $h := h^c$ is called the* Peierls barrier.

Lemma 3.3. *The value c is the infimum of k such that $\int_\gamma L + k \geq 0$ for all closed curves γ.*

Definition 3.4. The *Mañé potencial* $\Phi : G \times G \to \mathbb{R}$ is defined by

$$\Phi(x, y) = \inf_{t>0} h_t(x, y) + ct.$$

Clearly we have $\Phi(x,y) \le h(x,y)$ for any $x, y \in G$.

Proposition 6. *Functions h and Φ have the following properties.*

1. $\Phi(x,z) \le \Phi(x,y) + \Phi(y,z)$.
2. $h(x,z) \le h(x,y) + \Phi(y,z)$, $h(x,z) \le \Phi(x,y) + h(y,z)$.
3. *h and Φ are Lipschitz*
4. *If $\gamma_n : [0, t_n] \to G$ is a sequence of absolutely continuous curves with $t_n \to \infty$ and $\gamma_n(0) \to x$, $\gamma_n(t_n) \to y$, then*

$$h(x,y) \le \liminf_{n\to\infty} A(\gamma_n) + ct_n. \tag{7}$$

Definition 3.5. A curve $\gamma : J \to G$ defined on an interval J is called

- *semi-static* if
$$\Phi(\gamma(t), \gamma(s)) = \int_t^s L(\gamma, \dot\gamma) + c(s-t)$$
for any $t, s \in J$, $t \le s$.
- *static* if
$$\int_t^s L(\gamma, \dot\gamma) + c(s-t) = -\Phi(\gamma(s), \gamma(t))$$
for any $t, s \in J$, $t \le s$.
- The *Aubry set* $\mathcal{A}$ is the set of points $x \in G$ such that $h(x,x) = 0$.

Notice that by item (2) in Proposition 6, $h(x,z) = \Phi(x,z)$ if $x \in \mathcal{A}$ or $z \in \mathcal{A}$.

Proposition 7. *If $\eta : \mathbb{R} \to G$ is static then $\eta(s) \in \mathcal{A}$ for any $s \in \mathbb{R}$.*

We do not have a Lagrangian flow and therefore we can not speak of conservation of energy. Nevertheless the following Proposition says that semi-static curves have energy $c(L)$.

Proposition 8. *Let $\eta : J \to G$ be semi-static. For almost every $t \in J$*

$$L_v(\eta(t), \dot\eta(t))\dot\eta(t) = L(\eta(t), \dot\eta(t)) + c$$

Proof. For $\lambda > 0$, let $\eta_\lambda(t) := \eta(\lambda t)$ so that $\dot\eta_\lambda(t) = \lambda\dot\eta(\lambda t)$ almost everywhere.
For $r, s \in J$ let

$$\mathcal{A}_{rs}(\lambda) := \int_{r/\lambda}^{s/\lambda} [L(\eta_\lambda(t), \dot\eta_\lambda(t)) + c]\, dt = \int_r^s [L(\eta(s), \lambda\dot\eta(s)) + c]\, \frac{ds}{\lambda}.$$

Since η is a free-time minimizer, differentiating $\mathcal{A}_{rs}(\lambda)$ at $\lambda = 1$, we have that

$$0 = \mathcal{A}'_{rs}(1) = \int_0^T [L_v(\eta(s), \dot\eta(s))\dot\eta(s) - L(\dot\eta(s), \dot\eta(s) - c]\, ds.$$

Since this holds for any $r, s \in J$ we have

$$L_v(\eta(t), \dot\eta(t))\dot\eta(t) = L(\eta(t), \dot\eta(t)) + c$$

for almost every $t \in J$. ◻

3.2. **Weak KAM solutions.** Following Fathi [7], we define weak KAM solutions and give some of their properties

Definition 3.6. Let c be given by Lemma 3.2.

- A function $u : G \to \mathbb{R}$ is *dominated* if for any $x, y \in G$, we have
$$u(y) - u(x) \leq h_t(x, y) + ct \quad \forall t > 0,$$
or equivalently
$$u(y) - u(x) \leq \Phi(x, y).$$
- $\gamma : I \to G$ *calibrates* a dominated function $u : G \to \mathbb{R}$ if
$$u(\gamma(s)) - u(\gamma(t)) = \int_t^s L(\gamma, \dot\gamma) + c(s - t) \quad \forall s, t \in I$$
- A continuous function $u : G \to \mathbb{R}$ is a *backward (forward) weak KAM solution* if it is dominated and for any $x \in G$ there is $\gamma : (-\infty, 0] \to G$ ($\gamma : [0, \infty) \to G$) that calibrates u and $\gamma(0) = x$

Corollary 1. *Any static curve $\gamma : J \to G$ calibrates any dominated function $u : G \to \mathbb{R}$*

Proposition 9. *For any $x \in G$, $h(x, \cdot)$ is a backward weak KAM solution and $-h(\cdot, x)$ is a forward weak KAM solution.*

Proof. By item (2) of Proposition 6, $h(x, \cdot)$ is dominated.

The standard construction of calibrating curves for compact manifolds involves the Euler Lagange flow that we do not have, so we use a diagonal trick. Let $\gamma_n : [-t_n, 0] \to G$ be a sequence of minimizing curves connecting x to y such that
$$h(x, y) = \lim_{n\to\infty} A(\gamma_n) + ct_n$$
By Lemma 2.3, $\{\gamma_n\}$ is uniformly Lipschitz and then equicontinuous. It follows from the Arzela Ascoli Theorem that there is a sequence $n^1_j \to \infty$ such that $\gamma_{n^1_j}$ converges uniformly on $[-1, 0]$. Again, by the Arzela Ascoli Theorem, there is a subsequence $(n^2_j)_j$ of the sequence $(n^1_j)_j$ such that $\gamma_{n^2_j}$ converges uniformly on $[-2, 0]$. By induction, this procedure gives for each $k \in \mathbb{N}$ a sequence $(n^k_j)_j$ that is a subsequence of the sequence $(n^{k-1}_j)_j$ and such that $\gamma_{n^k_j}$ converges uniformly on $[-k, 0]$ as $j \to \infty$. Letting $m_k = n^k_k$, the sequence γ_{m_k} converges uniformly on each $[-l, 0]$. For $s < 0$ define $\gamma(s) = \lim_{k\to\infty} \gamma_{m_k}(s)$. Fix $t < 0$, for k large $t + t_{m_k} \geq 0$ and
$$A(\gamma_{m_k}) + ct_{m_k} = \int_{-t_{m_k}}^{t} L(\gamma_{m_k}, \dot\gamma_{m_k}) + c(t + t_{m_k}) + \int_t^0 L(\gamma_{m_k}, \dot\gamma_{m_k}) - ct. \tag{8}$$
Since γ_{m_k} converges to γ uniformly on $[t, 0]$, we have
$$\liminf_{k\to\infty} \int_t^0 L(\gamma_{m_k}, \dot\gamma_{m_k}) \geq \int_t^0 L(\gamma, \dot\gamma).$$
From item (4) of Proposition 6 we have
$$h(x, \gamma(t)) \leq \liminf_{k\to\infty} \int_{-t_{m_k}}^{t} L(\gamma_{m_k}, \dot\gamma_{m_k}) + c(t + t_{m_k}).$$

Taking $\liminf_{k\to\infty}$ in (8) we get

$$h(x,y) \geq h(x,\gamma(t)) + \int_t^0 L(\gamma,\dot\gamma) - ct.$$

So γ calibrates $h(x,\cdot)$. □

From Proposition 9 we have

Corollary 2. *If $x \in \mathcal{A}$ there exists a curve $\gamma : \mathbb{R} \to G$ such that $\gamma(0) = x$ and for all $t \geq 0$*

$$h(\gamma(t),x) = -\int_0^t L(\gamma,\dot\gamma) - ct$$
$$h(x,\gamma(-t)) = -\int_{-t}^0 L(\gamma,\dot\gamma) - ct.$$

In particular the curve γ is static and calibrates any dominated function $u : G \to \mathbb{R}$.

Theorem 3.7. *The function $\Phi(x,\cdot)$ is a backward weak KAM solution if and only if $x \in \mathcal{A}$.*

Corollary 3. *Let $C \subset G$ and $w_0 : C \to \mathbb{R}$ be bounded from below. Let*

$$w(x) = \inf_{z\in C} w_0(z) + \Phi(z,x)$$

1. *w is the maximal dominated function not exceeding w_0 on C.*
2. *If $C \subset \mathcal{A}$, w is a backward weak KAM solution.*
3. *If for all $x,y \in C$*

$$w_0(y) - w_0(x) \leq \Phi(x,y),$$

then w coincides with w_0 on C.

For $u : G \to \mathbb{R}$ let $I(u)$ be the set of points $x \in G$ for which exists $\gamma : \mathbb{R} \to G$ such that $\gamma(0) = x$ and γ calibrates u.

Corollary 4.

$$\mathcal{A} = \bigcap_{u \text{ dominated}} I(u)$$

Proposition 10. *For each $x,y \in G$ with $x \neq y$ we can find $\varepsilon > 0$ and a curve $\gamma : [-\varepsilon,0] \to G$ such that $\gamma(0) = y$ and for all $t \in [0,\varepsilon]$*

$$\Phi(x,\gamma(0)) - \Phi(x,\gamma(-t)) = \int_{-t}^0 L(\gamma,\dot\gamma) + ct.$$

In particular, for each $x \in G$ the function $G \setminus \{x\} \to \mathbb{R}$; $y \mapsto \Phi(x,y)$ is a backward weak KAM solution.

Theorem 3.8. *$\mathcal{A}$ is nonempty and if $u : G \to \mathbb{R}$ is a backward weak KAM solution then*

$$u(x) = \min_{q\in\mathcal{A}} u(q) + h(q,x) \tag{9}$$

Corollary 5.

$$h(x,y) = \min_{q\in\mathcal{A}} h(x,q) + h(q,y) = \min_{q\in\mathcal{A}} \Phi(x,q) + \Phi(q,x)$$

4. The Lax semigroup and its convergence.

4.1. The Lax semigroup.

Let $\mathcal{F}$ be the set of real functions on G, bounded from below.

The backward Lax semigroup $\mathcal{L}_t : \mathcal{F} \to \mathcal{F}$, $t > 0$ is defined by

$$\mathcal{L}_t f(x) = \inf_{y \in G} f(y) + h_t(y, x).$$

It is clear that $f \in \mathcal{F}$ is dominated if and only if $f \le \mathcal{L}_t f + ct$ for any $t > 0$.

It follows at once that $\mathcal{L}_t \circ \mathcal{L}_s = \mathcal{L}_{t+s}$ and

$$\|\mathcal{L}_t f - \mathcal{L}_t g\|_\infty \le \|f - g\|_\infty \tag{10}$$

The proof of the following Lemma is the same as in the compact manifold case.

Lemma 4.1. *Given $\varepsilon > 0$ there is $K_\varepsilon > 0$ such that for each $u : G \to \mathbb{R}$ continuous, $t \ge \varepsilon$, we have $\mathcal{L}_t u : G \to \mathbb{R}$ is a Lipschitz with constant K_ε.*

Theorem 4.2. *A continuous function $u : G \to \mathbb{R}$ is a fixed point of the semigroup $\mathcal{L}_t + ct$ if and only if it is a backward weak KAM solution*

Proof. Suppose $u : G \to \mathbb{R}$ is a fixed point of the semigroup $\mathcal{L}_t + ct$. For each $T \ge 2$ there is a curve $\alpha_T : [-T, 0] \to G$ such that $\alpha_T(0) = x$ and

$$u(x) - u(\alpha_T(-T)) = A(\alpha_T) + cT.$$

By Lemma 2.3 $\{\alpha_T\}$ is uniformly Lipschitz. As in Propostion 9 one obtains a sequence $t_k \to \infty$ and $\gamma : (-\infty, 0] \to G$ such that α_{t_k} converges to γ, uniformly on each $[-n, 0]$.

By Lemma 2.2

$$\begin{aligned} \int_{-n}^0 L(\gamma, \dot\gamma) + nc &\le \liminf_{k\to\infty} \int_{-n}^0 L(\alpha_{t_k}, \dot\alpha_{t_k}) + nc \\ &= \liminf_{k\to\infty} u(x) - u(\alpha_{t_k}(-n)) \\ &= u(x) - u(\gamma(-n)) \end{aligned}$$

Suppose now that $u : G \to \mathbb{R}$ is a backward weak KAM solution. Since u is dominated, $u \le \mathcal{L}_t u + ct$. For $x \in G$ let $\gamma : (-\infty, 0] \to G$ be such that $\gamma(0) = x$ and for all $t > 0$

$$u(x) - u(\gamma(-t)) = \int_{-t}^0 L(\gamma, \dot\gamma) + ct.$$

Thus

$$u(x) \ge u(\gamma(-t)) + h_t(\gamma(t), x) + ct \ge \mathcal{L}_t u(x) + ct.$$

□

From Proposition 9 and Theorem 4.2 one obtains

Corollary 6. *The semigroup $\mathcal{L}_t + ct$ has fixed points.*

4.2. **Convergence of the Lax semigroup.** Without loss of generality assume $c = 0$. For $u \in C(G)$ define

$$v(x) := \min_{z \in G} u(z) + h(z, x). \tag{11}$$

Proposition 11. *Let $\psi = \lim_{n\to\infty} \mathcal{L}_{t_n} u$ for some $t_n \to \infty$, then*

$$\psi \ge v. \tag{12}$$

Proof. For $x \in G$ let $\gamma_n : [0, t_n] \to G$ be such that $\gamma_n(t_n) = x$ and

$$\mathcal{L}_{t_n} u(x) = u(\gamma_n(0)) + A(\gamma_n). \tag{13}$$

Passing to a subsequence if necessary we may assume that $\gamma_n(0)$ converges to $y \in G$. Taking $\liminf$ in (13), we have from item (4) of Proposition 6

$$\psi(x) = u(y) + \liminf_{n\to\infty} A(\gamma_n) \geq u(y) + h(y, x).$$

□

Proposition 12. *If $\mathcal{L}_t u$ converges as $t \to \infty$, then the limit is function v defined in* (11)*.*

Proof. For $x \in G$ let $z \in G$ be such that $v(z) = u(z) + h(z, x)$. Since $\mathcal{L}_t u(x) \leq u(z) + h_t(z, x)$, we have

$$\lim_{t\to\infty} \mathcal{L}_t u(x) \leq \liminf_{t\to\infty} u(z) + h_t(z, x) = v(z)$$

which together with Proposition (11) gives $\lim\limits_{t\to\infty} \mathcal{L}_t u = v$. □

Thus, given $u \in C(G)$ our goal is to prove that $\mathcal{L}_t u$ converges to v defined in (11).

Remark 1. Using Corollary 5 we can write (11) as

$$v(x) = \min_{y\in\mathcal{A}} \Phi(y, x) + w(y) \tag{14}$$

$$w(y) := \inf_{z\in G} u(z) + \Phi(z, y) \tag{15}$$

Item (1) of Corollary 3 states that w is the maximal dominated function not exceeding u. Items (2), (3) of the same Corollary imply that v is the unique backward weak KAM solution that coincides with w on $\mathcal{A}$.

Proposition 13. *Suppose that u is dominated, then $\mathcal{L}_t u$ converges uniformly as $t \to \infty$ to the function v given by* (11)*.*

Proof. Since u is dominated, the function $t \mapsto \mathcal{L}_t u$ is nondecreasing. As well, in this case, w given by (15) coincides with u. Items (1) and (3) of Corollary 3 imply that v is the maximal dominated function that coincides with u on $\mathcal{A}$ and then $u \leq v$ on G.

Since the semigroup $\mathcal{L}_t$ is monotone and v is a backward weak KAM solution

$$\mathcal{L}_t u \leq \mathcal{L}_t v = v \text{ for any } t > 0.$$

Thus the uniform limit $\lim\limits_{t\to\infty} u$ exists. □

We now address the convergence of $\mathcal{L}_t$ following the lines in [5] and [11].

For $u \in C(G)$ let

$$\omega_{\mathcal{L}}(u) := \{\psi \in C(G) : \exists t_n \to \infty \text{ such that } \psi = \lim_{n\to\infty} \mathcal{L}_{t_n} u\}.$$

$$\underline{u}(x) := \sup\{\psi(x) : \psi \in \omega_{\mathcal{L}}(u)\} \tag{16}$$

$$\overline{u}(x) := \inf\{\psi(x) : \psi \in \omega_{\mathcal{L}}(u)\} \tag{17}$$

From these and Proposition 11

Proposition 14. *Let $u \in C(G)$, v be the function given by* (11)*, $\underline{u}, \overline{u}$ defined in* (16) *and* (17)*. Then*

$$v \leq \overline{u} \leq \underline{u} \tag{18}$$

Proposition 15. *For $u \in C(G)$, function $\underline{u}$ given by (16) is dominated.*

Proof. Let $x, y \in G$. Given $\varepsilon > 0$ there is $\psi = \lim_{n\to\infty} \mathcal{L}_{t_n} u$ such that $\underline{u}(x) - \varepsilon < \psi(x)$. For $n > N(\varepsilon)$ and $a > 0$

$$\underline{u}(x) - 2\varepsilon < \psi(x) - \varepsilon \le \mathcal{L}_{t_n} u(x) = \mathcal{L}_a(\mathcal{L}_{t_n - a} u)(x) \le \mathcal{L}_{t_n - a} u(y) + h_a(y, x).$$

Choose a divergent sequence n_j such that $(\mathcal{L}_{t_{n_j} - a} u)_j$ converges uniformly. For $j > \bar{N}(\varepsilon)$, $\mathcal{L}_{t_{n_j} - a} u(y) < \underline{u}(y) + \varepsilon$, and then

$$\underline{u}(x) - 3\varepsilon < \mathcal{L}_{t_{n_j} - a} u(y) + h_a(y, x) - \varepsilon < \underline{u}(y) + h_a(y, x).$$

□

Denote by $\mathcal{K}$ the family of static curves $\eta : \mathbb{R} \to G$, and for $y \in \mathcal{A}$ denote by $\mathcal{K}(y)$ the set of curves $\eta \in \mathcal{K}$ with $\eta(0) = y$.

Proposition 16. *$\mathcal{K}$ is a compact metric space with respect to the uniform convergence on compact intervals.*

Proof. Let $\{\eta_n\}$ be a sequence in $\mathcal{K}$. By Lemma 2.3, $\{\eta_n\}$ is uniformly Lipschitz. As in Proposition 9 we obtain a sequence $n_k \to \infty$ such that η_{n_k} converges to $\eta : \mathbb{R} \to G$ uniformly on each $[a, b]$ and then η is static. □

Proposition 17. *Two dominated functions that coincide on $\mathcal{M} = \bigcup_{\eta\in\mathcal{K}} \omega(\eta)$ also coincide on $\mathcal{A}$.*

Proof. Let φ_1, φ_2 be two dominated functions coinciding on $\mathcal{M}$. Let $y \in \mathcal{A}$ and $\eta \in \mathcal{K}(y)$. Let $(t_n)_n$ be a diverging sequence such that $\lim_n \eta(t_n) = x \in \mathcal{M}$. By Corollary 1

$$\varphi_i(y) = \varphi_i(\eta(0)) - \Phi(y, \eta(0)) = \varphi_i(\eta(t_n)) - \Phi(y, \eta(t_n))$$

for every $n \in N, i = 1, 2$. Sending n to ∞, we get

$$\begin{aligned}\varphi_1(y) &= \lim_{n\to\infty} \varphi_1(\eta(t_n)) - \Phi(y, \eta(t_n)) = \varphi_1(x) - \Phi(y, x) = \varphi_2(x) - \Phi(y, x) \\ &= \lim_{n\to\infty} \varphi_2(\eta(t_n)) - \Phi(y, \eta(t_n)) = \varphi_2(y).\end{aligned}$$

□

Proposition 18. *Let $\eta \in \mathcal{K}$, $\psi \in C(G)$ and φ be a dominated function. Then the function $t \mapsto (\mathcal{L}_t\psi)(\eta(t)) - \varphi(\eta(t))$ is nonincreasing on $\mathbb{R}_+$.*

Proof. From Corollary 1, for $t < s$ we have

$$(\mathcal{L}_s\psi)(\eta(s)) - (\mathcal{L}_t\psi)(\eta(t)) \le \int_t^s L(\eta(\tau), \dot{\eta}(\tau))d\tau = \varphi(\eta(s)) - \varphi(\eta(t))$$

□

Lemma 4.3. *There is a $M > 0$ such that, if η is any curve in $\mathcal{K}$ and λ is sufficiently close to 1, we have*

$$\int_{t_1}^{t_2} L(\eta_\lambda, \dot{\eta}_\lambda) \le \Phi(\eta_\lambda(t_1), \eta_\lambda(t_2)) + M(t_2 - t_1)(\lambda - 1)^2 \tag{19}$$

for any $t_2 > t_1$, where $\eta_\lambda(t) = \eta(\lambda t)$.

Proof. Let $K > 0$ be a Lipschitz constant for any minimizer $\gamma : [a, b] \to G$ with $b - a > 1$, $2R = \sup\{|L_{vv}(x, v)| : |v| \le K\}$. For $\lambda \in (1 - \delta, 1 + \delta)$ fixed, using Proposition 8

$$\begin{aligned}\int_{t_1}^{t_2} L(\eta_\lambda(t), \dot\eta_\lambda(t))dt &= \int_{t_1}^{t_2} [L(\eta(\lambda t), \dot\eta(\lambda t)) + (\lambda - 1)L_v(\eta(\lambda t), \dot\eta(\lambda t))\dot\eta(\lambda t) \\ &\quad + \frac{1}{2}(\lambda - 1)^2 L_{vv}(\eta(\lambda t), \mu\dot\eta(\lambda t))(\dot\eta(\lambda t))^2]\, dt \\ &\le \lambda \int_{t_1}^{t_2} L(\eta(\lambda t), \dot\eta(\lambda t))\, dt + (t_2 - t_1)RK^2(\lambda - 1)^2 \\ &= \Phi(\eta(\lambda t_1), \eta(\lambda t_2)) + (t_2 - t_1)RK^2(\lambda - 1)^2\end{aligned}$$

□

Proposition 19. *Let $\eta \in \mathcal{K}$, $\psi \in C(G)$ and φ be a dominated function. Assume that $D^+((\psi - \varphi) \circ \eta)(0) \setminus \{0\} \ne \emptyset$ where D^+ denote the super-differential. Then for all $t > 0$ we have*

$$(\mathcal{L}_t\psi)(\eta(t)) - \varphi(\eta(t)) < \psi(\eta(0)) - \varphi(\eta(0)) \tag{20}$$

Proof. Fix $t > 0$. By Corollary 1 it is enough to prove (20) for $\varphi = -\Phi(\cdot, \eta(t))$. Since $\mathcal{L}_t(\psi + a) = \mathcal{L}_t\psi + a$ we can assume that $\psi(\eta(0)) = \varphi(\eta(0))$.

$$(\mathcal{L}_t\psi)(\eta(t)) - \varphi(\eta(t)) = (\mathcal{L}_t\psi)(\eta(t)) \le \int_{(1/\lambda - 1)t}^{t/\lambda} L(\eta_\lambda, \dot\eta_\lambda) + \psi(\eta((1 - \lambda)t)),$$

thus, by Lemma 4.3

$$(\mathcal{L}_t\psi)(\eta(t)) - \varphi(\eta(t)) \le \psi(\eta((1 - \lambda)t)) - \varphi(\eta((1 - \lambda)t)) + Mt(\lambda - 1)^2.$$

If $m \in D^+((\psi - \varphi) \circ \eta)(0) \setminus \{0\}$, we have

$$(\mathcal{L}_t\psi)(\eta(t)) - \varphi(\eta(t)) \le m((1 - \lambda)t) + o((1 - \lambda)t)) + Mt(\lambda - 1)^2,$$

where $\lim\limits_{\lambda \to 1} \dfrac{o((1 - \lambda)t)}{1 - \lambda} = 0$. Choosing appropriately λ close to 1, we get

$$(\mathcal{L}_t\psi)(\eta(t)) - \varphi(\eta(t)) < 0.$$

□

Proposition 20. *Suppose φ is dominated and $\psi \in \omega_{\mathcal{L}}(u)$. For any $y \in \mathcal{M}$ there exists $\gamma \in \mathcal{K}(y)$ such that the function $t \mapsto \psi(\gamma(t)) - \varphi(\gamma(t))$ is constant.*

Proof. Let $(s_k)_k$ and $(t_k)_k$ be diverging sequences, η be a curve in $\mathcal{K}$ such that $y = \lim\limits_k \eta(s_k)$, and ψ is the uniform limit of $\mathcal{L}_{t_k}u$. As in Proposition 9, we can assume that the sequence of functions $t \mapsto \eta(s_k + t)$ converges uniformly on compact intervals to $\gamma : \mathbb{R} \to G$, and so $\gamma \in \mathcal{K}$. We may assume moreover that $t_k - s_k \to \infty$, as $k \to \infty$, and that $\mathcal{L}_{t_k - s_k}u$ converges uniformly to $\psi_1 \in \omega_{\mathcal{L}}(u)$. By the semi-group property and (10)

$$\|\mathcal{L}_{t_k}u - \mathcal{L}_{s_k}\psi_1\|_\infty \le \|\mathcal{L}_{t_k - s_k}u - \psi_1\|_\infty$$

which implies that $\mathcal{L}_{s_k}\psi_1$ converges uniformly to ψ. From Proposition 18, we have that for any $\tau \in \mathbb{R}$ $s \mapsto (\mathcal{L}_s\psi_1)(\eta(\tau + s)) - \varphi(\eta(\tau + s))$ is a nonincreasing function in $\mathbb{R}^+$, and hence it has a limit $l(\tau)$ as $s \to \infty$, which is finite since $l(\tau) \ge -\|\overline{u} - \varphi\|_\infty$. Given $t > 0$, we have

$$l(\tau) = \lim_{k \to \infty} (\mathcal{L}_{s_k + t}\psi_1)(\eta(s_k + \tau + t)) - \varphi(\eta(s_k + \tau + t)) = (\mathcal{L}_t\psi)(\gamma(\tau + t)) - \varphi(\gamma(\tau + t))$$

The function $t \mapsto (\mathcal{L}_t\psi)(\gamma(\tau+t)) - \varphi(\gamma(\tau+t))$ is therefore constant on $\mathbb{R}^+$. Applying Proposition 19 to the curve $\gamma(\tau + \cdot) \in \mathcal{K}$, we have $D^+((\psi - \varphi) \circ \gamma)(\tau) \setminus \{0\} = \emptyset$ for any $\tau \in \mathbb{R}$. This implies that $\psi - \varphi$ is constant on γ. □

Proposition 21. *Let $\eta \in \mathcal{K}$, $\psi \in \omega_{\mathcal{L}}(u)$ and v be defined by (11). For any $\varepsilon > 0$ there exists $\tau \in \mathbb{R}$ such that*

$$\psi(\eta(\tau)) - v(\eta(\tau)) < \varepsilon.$$

Proof. Since the curve η is contained in $\mathcal{A}$, we have

$$v(\eta(0)) = \min_{z \in G} u(z) + \Phi(z, \eta(0)),$$

and hence $v(\eta(0)) = u(z_0) + \Phi(z_0, \eta(0))$, for some $z_0 \in G$. Take a curve $\gamma : [0, T] \to G$ such that

$$v(\eta(0)) + \frac{\varepsilon}{2} = u(z_0) + \Phi(z_0, \eta(0)) + \frac{\varepsilon}{2} > u(z_0) + \int_0^T L(\gamma, \dot{\gamma}) \geq \mathcal{L}_T u(\eta(0)).$$

Choosing a divergent sequence $(t_n)_n$ such that $\mathcal{L}_{t_n} u$ converges uniformly to ψ we have for n sufficiently large

$$\|\mathcal{L}_{t_n} u - \psi\|_\infty < \frac{\varepsilon}{2}, \quad t_n - T > 0.$$

Take $\tau = t_n - T$

$$\begin{aligned}
\psi(\eta(\tau)) - \frac{\varepsilon}{2} &< \mathcal{L}_{t_n} u(\eta(\tau)) = \mathcal{L}_\tau \mathcal{L}_T u(\eta(\tau)) = \mathcal{L}_T u(\eta(0)) + \int_0^\tau L(\eta, \dot{\eta}) \\
&< \frac{\varepsilon}{2} + v(\eta(0)) + \int_0^\tau L(\eta, \dot{\eta}) = \frac{\varepsilon}{2} + v(\eta(\tau))
\end{aligned}$$

□

From Propositions 20 and 21 we obtain

Theorem 4.4. *Let $\psi \in \omega_{\mathcal{L}}(u)$ and v be defined by (11). Then $\psi = v$ on $\mathcal{M}$.*

Theorem 4.5. *Let $u \in C(G)$, then $\mathcal{L}_t u$ converges uniformly as $t \to \infty$ to v given by (11).*

Proof. The function $\underline{u}$ is dominated and coincides with v on $\mathcal{M}$ by Theorem 4.4. Proposition 17 implies that $\underline{u}$ coincide with v on $\mathcal{A}$ and so does with w. By item (1) of Corollary 3 we have $\underline{u} \leq v$. □

5. Viscosity solutions of the Hamilton - Jacobi equation.

In this section we compare weak KAM and viscosity solutions.

Definition 5.1.
- A continuous real function φ defined on the neighborhood of e_l is C^1 if for every j with $e_l \in I_j$, $\varphi|I_j$ is C^1.
- A continuous real function φ defined on the neighborhood of (e_l, t) is C^1 if for every j with $e_l \in I_j$, $\varphi|I_j \times (t - \delta, t + \delta)$ is C^1.

Note that if $\alpha : [0, \delta] \to I_j$ is differentiable and $\alpha(0) = e_l$, then $\alpha'_+(0) \in T^-_{e_l} I_j$ and we have

$$D^j \varphi(e_l) z = (\varphi \circ \alpha)'_+(0).$$

We consider the Hamiltonian consisting in functions $H_j : I_j \times \mathbb{R} \to \mathbb{R}$ given by

$$H_j(x, p) = \max \left\{ -pz - L_j(x, z) : \begin{array}{ll} z \in T^-_x I_j, & x \in \mathcal{V} \\ z \in T_x I_j, & x \in I_j \setminus \mathcal{V} \end{array} \right\}$$

and the Hamilton Jacobi equations

$$H(x, Du(x)) = c, \tag{21}$$

$$u_t(x,t) + H(x, D_x u(x,t)) = 0. \tag{22}$$

Note that if L is symmetric at the vertices, then for any vertex e_l there is a function h_a such that $H_j(e_l, p) = h_a(|p|)$ for any j with $e_l \in I_j$. This kind of Hamiltonians are called of eikonal type [3].

The following definition appeared in [3] and [4].

Definition 5.2. A function $u : G \to \mathbb{R}$ is a

- *viscosity subsolution* of (21) if satisfies the usual definition in $G \setminus \mathcal{V}$ and for any C^1 function φ on the neighborhood of any e_l s.t. $u - \varphi$ has a maximum at e_l we have
$$\max\{H_j(e_l, D^j\varphi(e_l)) : e_l \in I_j\} \le c.$$
- *viscosity supersolution* of (21) if satisfies the usual definition in $G \setminus \mathcal{V}$ and for any C^1 function φ on the neighborhood of any e_l s.t. $u - \varphi$ has a minimum at e_l we have
$$\max\{H_j(e_l, D^j\varphi(e_l)) : e_l \in I_j\} \ge c$$
- *viscosity solution* if it is both, a subsolution and a supersolution.

A function $u : G \times [0,\infty) \to \mathbb{R}$ is a

- *viscosity subsolution* of (22) if satisfies the usual definition in $G \setminus \mathcal{V} \times [0,\infty)$ and for any C^1 function φ on the neighborhood of any (e_l, t) s.t. $u - \varphi$ has a maximum at (e_l, t) we have
$$\varphi_t(e_l, t) + \max\{H_j(e_l, D^j\varphi(e_l, t)) : e_l \in I_j\} \le c.$$
- *viscosity supersolution* of (21) if satisfies the usual definition in $G \setminus \mathcal{V} \times [0,\infty)$ and for any C^1 function φ on the neighborhood of any (e_l, t) s.t. $u - \varphi$ has a minimum at (e_l, t) we have
$$\varphi_t(e_l, t) + \max\{H_j(e_l, D^j\varphi(e_l, t)) : e_l \in I_j\} \ge c$$
- *viscosity solution* if it is both, a subsolution and a supersolution.

Proposition 22. *If $u : G \to \mathbb{R}$ is dominated then then it is a viscosity subsolution of* (21)*. If u is a backward weak KAM solution then it is a viscosity solution.*

Proof. Suppose $u : G \to \mathbb{R}$ is dominated. Let φ be a C^1 function on the neighborhood of e_l s.t. $u - \varphi$ has a maximum at e_l, j s.t. $e_l \in I_j$, $\alpha : [0, \delta] \to I_j$ differentiable with $\alpha(0) = e_l$, $z = \alpha'(0)$. Define $\gamma : [-\delta, 0] \to I_j$ by $\gamma(s) = \alpha(-s)$.

$$\varphi(e_l) - \varphi(\gamma(s)) \le u(e_l) - u(\gamma(s)) \le \int_s^0 L_j(\gamma, \dot\gamma) - cs$$
$$\frac{\varphi(e_l) - \varphi(\alpha(t)))}{t} \le \frac{1}{t}\int_{-t}^0 L_j(\gamma, \dot\gamma) + c$$
$$-D^j\varphi(e_l)z \le L_j(e_l, z) + c.$$

So u is a subsolution.

Let φ be a C^1 function on the neighborhood of e_l s.t. $u - \varphi$ has a minimum at e_l. Let $\gamma : (-\infty, 0] \to G$ be such that $\gamma(0) = e_l$ and for $t < 0$

$$u(e_l) - u(\gamma(t)) = \int_t^0 L_j(\gamma, \dot\gamma) - ct$$

Let $\delta > 0, j$ be such that $\gamma([-\delta, 0]) \subset I_j$.

$$\varphi(e_l) - \varphi(\gamma(s)) \geq \int_s^0 L_j(\gamma, \dot{\gamma}) - cs$$

Define $\alpha : [0, \delta] \to I_j$ by $\alpha(t) = \gamma(-t)$, $z = \alpha'(0)$,

$$\frac{\varphi(e_l) - \varphi(\alpha(t))}{t} \geq \frac{1}{t} \int_{-t}^0 L_j(\gamma, \dot{\gamma}) + c$$

$$-D^j \varphi(e_l) z \geq L_j(e_l, z) + c.$$

So u is a supersolution. □

Our approach to get a converse to Proposition 22 is to prove uniqueness of solutions to the Cauchy problem for (22), using a comparison principle. For that purpose the symmetry Lagrangian is a sufficient condition. It may be possible that other assumptions imply the required uniqueness or that a different approach gives a converse to Proposition 22.

Proposition 23. *Let $f : G \to \mathbb{R}$ be continuous and define $u : G \times [0, \infty) \to \mathbb{R}$ by $u(x, t) = \mathcal{L}_t f(x)$, then u is a viscosity solution of* (22)

Proof. Since $\mathcal{L}_t f = \mathcal{L}_{t-s}(\mathcal{L}_s f)$ if $0 \leq s < t$, for any $\gamma : [s, t] \to G$

$$u(\gamma(t), t) - u(\gamma(s), s) \leq \int_s^t L(\gamma, \dot{\gamma}) \tag{23}$$

and for any $x \in G$ there is $\gamma : [s, t] \to G$ with $\gamma(t) = x$ such that equality in (23) holds.

Let φ be a C^1 function on the neighborhood of (e_l, t) s.t. $u - \varphi$ has a maximum at (e_l, t), j s.t. $e_l \in I_j$, $\alpha : [0, \delta] \to I_j$ differentiable with $\alpha(0) = e_l$, $z = \alpha'(0)$. Define $\gamma : [t - \delta, t] \to I_j$ by $\gamma(s) = \alpha(t - s)$.

$$\varphi(e_l, t) - \varphi(\gamma(s), s) \leq u(e_l, t) - u(\gamma(s), s) \leq \int_s^t L_j(\gamma, \dot{\gamma})$$

$$\frac{\varphi(e_l, t) - \varphi(\alpha(t - s), s))}{t - s} \leq \frac{1}{t - s} \int_s^t L_j(\gamma, \dot{\gamma})$$

$$\varphi_t(e_l, t) - D_x^j \varphi(e_l, t) z \leq L_j(e_l, z).$$

So u is subsolution.

Let φ be a C^1 function on the neighborhood of (e_l, t) s.t. $u - \varphi$ has a minimum at (e_l, t). Let $\gamma : [t - 1, t] \to G$ be such that $\gamma(t) = e_l$ and

$$u(e_l, t) - u(\gamma(t - 1), t - 1) = \int_{t-1}^t L(\gamma, \dot{\gamma})$$

Let $\delta > 0, j$ be such that $\gamma([t - \delta, t]) \subset I_j$. For $s \in [t - \delta, t]$

$$\varphi(e_l, t) - \varphi(\gamma(s), s) \geq \int_s^t L_j(\gamma, \dot{\gamma})$$

Define $\alpha : [0, \delta] \to I_j$ by $\alpha(s) = \gamma(t - s)$, $z = \alpha'(0)$,

$$\frac{\varphi(e_l, t) - \varphi(\alpha(t - s), s)}{t - s} \geq \frac{1}{t - s} \int_s^t L_j(\gamma, \dot{\gamma})$$

$$\varphi_t(e_l, t) - D_x^j \varphi(e_l, t) z \geq L_j(e_l, z).$$

So u is supersolution. □

Proposition 24. *Suppose the Lagrangian is symmetric at the vertices. Let* $u, v : G \times [0,T] \to \mathbb{R}$ *be respectively a Lipschitz viscosity sub, supersolution of* (22) *such that* $u(x,0) \le v(x,0)$*, for any* $x \in G$*. Then* $u \le v$*.*

Proof. Suppose that there are x^*, t^* such that $\delta = u(x^*,t^*) - v(x^*,t^*) > 0$. Let $0 < \rho \le \dfrac{\delta}{4t^*}$ and define $\Phi : G^2 \times [0,T]^2$ by

$$\Phi(x,y,t,s) = u(x,t) - v(y,s) - \frac{d(x,y)^2 + |t-s|^2}{2\varepsilon} - \rho(t+s).$$

From the previous definitions we have

$$\frac{\delta}{2} \le \delta - 2\rho t^* = \Phi(x^*,x^*,t^*,t^*) \le \sup_{G^2\times[0,T]^2} \Phi = \Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon). \tag{24}$$

It follows from $\Phi(x_\varepsilon,x_\varepsilon,t_\varepsilon,t_\varepsilon) + \Phi(y_\varepsilon,y_\varepsilon,s_\varepsilon,s_\varepsilon) \le 2\Phi(x_\varepsilon,y_\varepsilon,t_\varepsilon,s_\varepsilon)$ that

$$\begin{aligned}\frac{d(x_\varepsilon,y_\varepsilon)^2 + |t_\varepsilon - s_\varepsilon|^2}{2\varepsilon} &\le u(x_\varepsilon,t_\varepsilon) - u(y_\varepsilon,s_\varepsilon) + v(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) \\ &\le C(d(x_\varepsilon,y_\varepsilon)^2 + |t_\varepsilon - s_\varepsilon|^2)^{1/2}\end{aligned}$$

Thus, there is a sequence $\varepsilon \to 0$ such that $x_\varepsilon, y_\varepsilon$ converge to $\bar{x} \in G$ and $t_\varepsilon, s_\varepsilon$ converge to $\bar{t} \in [0,T]$ and (24) gives

$$\frac{\delta}{2} \le \Phi(\bar{x},\bar{x},\bar{t},\bar{t}) \le u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}),$$

and so $\bar{t} \ne 0$. Define the test functions

$$\begin{aligned}\varphi(x,t) &= v(y_\varepsilon,s_\varepsilon) + \frac{d(x,y_\varepsilon)^2 + |t-s_\varepsilon|^2}{2\varepsilon} + \rho(t+s_\varepsilon) \\ \psi(y,s) &= u(x_\varepsilon,t_\varepsilon) - \frac{d(x_\varepsilon,y)^2 + |t_\varepsilon - s|^2}{2\varepsilon} - \rho(t_\varepsilon + s).\end{aligned}$$

$$\varphi_t(x_\varepsilon,t_\varepsilon) = \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \rho, \quad \psi_s(y_\varepsilon,s_\varepsilon) = \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} - \rho$$

Since $u-\varphi$ has maximum at $(x_\varepsilon,t_\varepsilon)$, $v-\psi$ has minimum at $(y_\varepsilon,s_\varepsilon)$, u is subsolution and v is supersolution,

$$\begin{aligned}2\rho = \varphi_t(x_\varepsilon,t_\varepsilon) - \psi_s(y_\varepsilon,s_\varepsilon) &\le \max\{H_j\big(y_\varepsilon, -D_y^j\Big(\frac{d(x_\varepsilon,y)^2}{2\varepsilon}\Big)(y_\varepsilon)\big) : y_\varepsilon \in I_j\} \\ &\quad - \max\{H_j\big(x_\varepsilon, D_x^j\Big(\frac{d(x,y_\varepsilon)^2}{2\varepsilon}\Big)(x_\varepsilon)\big) : x_\varepsilon \in I_j\}\end{aligned} \tag{25}$$

Since $\rho > 0$ we can not have $x_\varepsilon = y_\varepsilon$.

If $\bar{x}$ is not a vertex, $\bar{x} \in I_j$, for $\varepsilon > 0$ small we have

$$D_x^j\Big(\frac{d(x,y_\varepsilon)^2}{2\varepsilon}\Big)(x_\varepsilon) = \pm\frac{d(x_\varepsilon,y_\varepsilon)}{\varepsilon} = -D_y^j\Big(\frac{d(x_\varepsilon,y)^2}{2\varepsilon}\Big)(y_\varepsilon).$$

If we denote by $a(x_\varepsilon,y_\varepsilon)$ this common value, then (25) becomes

$$2\rho \le H_j(y_\varepsilon, a(x_\varepsilon,y_\varepsilon)) - H_j\big(x_\varepsilon, a(x_\varepsilon,y_\varepsilon)\big)$$

with $a(x_\varepsilon,y_\varepsilon)$ bounded as $\varepsilon \to 0$, giving a contradiction.

Suppose now that $\bar{x} = e_l$. For $\varepsilon > 0$ small we distinguish the following cases

1. Neither x_ε nor y_ε is a vertex. If $x_\varepsilon, y_\varepsilon \in I_j$, $d(x_\varepsilon, y_\varepsilon) = |\sigma_j(x_\varepsilon) - \sigma_j(y_\varepsilon)|$. If $x_\varepsilon \in I_i$, $y_\varepsilon \in I_j$, and $e_l \in I_i \cap I_j$, then $d(x_\varepsilon, y_\varepsilon) = d(x_\varepsilon, e_l) + d(e_l, y_\varepsilon)$. In both subcases

$$|D^i_x\Big(\frac{d(x,y_\varepsilon)^2}{2\varepsilon}\Big)(x_\varepsilon)| = \frac{d(x_\varepsilon,y_\varepsilon)}{\varepsilon}$$
$$|D^j_y\Big(\frac{d(x_\varepsilon,y)^2}{2\varepsilon}\Big)(y_\varepsilon)| = \frac{d(x_\varepsilon,y_\varepsilon)}{\varepsilon}$$

Then (25) becomes

$$2\rho \le H_j\big(y_\varepsilon, \pm\frac{d(x_\varepsilon,y_\varepsilon)}{\varepsilon}\big) - H_i\big(x_\varepsilon, \pm\frac{d(x_\varepsilon,y_\varepsilon)}{\varepsilon}\big).$$

2. Suppose $x_\varepsilon = e_l$, $y_\varepsilon \in I_j \setminus \mathcal{V}$.

$$|D^j_x\Big(\frac{d(x,y_\varepsilon)^2}{2\varepsilon}\Big)(e_l)| = \frac{d(e_l,y_\varepsilon)}{\varepsilon}$$
$$|D^j_y\Big(\frac{d(e_l,y)^2}{2\varepsilon}\Big)(y_\varepsilon)| = \frac{d(e_l,y_\varepsilon)}{\varepsilon}$$

Since

$$H_j\big(e_l, \pm\frac{d(e_l,y_\varepsilon)}{\varepsilon}\big) = h_a\big(e_l, \frac{d(e_l,y_\varepsilon)}{\varepsilon}\big),$$

we have that (25) becomes

$$2\rho \le H_j\big(y_\varepsilon, \pm\frac{d(e_l,y_\varepsilon)}{\varepsilon}\big) = h_a\big(e_l, \frac{d(e_l,y_\varepsilon)}{\varepsilon}\big).$$

3. If $y_\varepsilon = e_l$, $x_\varepsilon \in I_j \setminus \mathcal{V}$ we get in the same way that (25) becomes

$$2\rho \le h_a\big(e_l, \frac{d(x_\varepsilon,e_l)}{\varepsilon}\big) - H_j\big(x_\varepsilon, \pm\frac{d(x_\varepsilon,e_l)}{\varepsilon}\big).$$

Since $\dfrac{d(x_\varepsilon,y_\varepsilon)}{\varepsilon}$ remains bounded as $\varepsilon \to 0$, we get a contradiction. □

Corollary 7. *Suppose the Lagrangian is symmetric at the vertices. Let $u, v : G \times [0,T] \to \mathbb{R}$ be viscosity solutions of (22) such that $u(x,0) = v(x,0)$ for any $x \in G$. Then $u = v$.*

Corollary 8. *Suppose the Lagrangian is symmetric at the vertices. Let $f : G \to \mathbb{R}$ be a viscosity solution of (21), then f is a fixed point of the Lax semigroup $\mathcal{L}_t + ct$.*

Proof. We next show that $u(x,t) = f(x) - ct$ is a viscosity solution of (22). Proposition 23 and Corollary 7 then imply that $f - ct = \mathcal{L}_t f$.

Let φ be a C^1 function on the neighborhood of (e_l, t) s.t. $u - \varphi$ has a maximum at (e_l, t). Then $s \to -cs - \varphi(e_l, s)$ has a maximum at t and so $\varphi_t(e_l, t) = -c$. Since $f - \varphi(\cdot, t)$ has a maximum at e_l we have

$$\max\{H_j(e_l, D^j\varphi(e_l,t)) : x \in I_j\} \le c = -\varphi_t(e_l,t),$$

so u is a subsolution of (22). Similarly u is a supersolution of (22). □

Corollary 9. *Suppose the Lagrangian is symmetric at the vertices. Let $u : G \to \mathbb{R}$ be a viscosity solution of (21) then the representation formula (9) holds.*

Proof. By Proposition 22 and Corollary 8, u is a backward weak KAM solution and by Theorem (3.8), formula (9) holds. □

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