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EXACT AND POSITIVE CONTROLLABILITY OF BOUNDARY CONTROL SYSTEMS

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Abstract. We characterize the space of all *exactly* reachable states of an abstract boundary control system using a semigroup approach. Moreover, we study the case when the controls of the system are constrained to be *positive*. The abstract results are then applied to study flows in networks with static as well as dynamic boundary conditions.

1. **Introduction.** This paper is a continuation of [16, 17] where we introduced a semigroup approach to boundary control problems and applied it to the control of flows in networks. While in these previous works we concentrated on maximal *approximate* controllability, we now focus on the *exact*- and *positive* controllability spaces.

There is a vast literature on abstract boundary control problems as well as on the application of the abstract theory to various concrete boundary control systems on Euclidean spaces. For an overview and related references on that topic we refer to [17, Sec. 1].

As a simple motivation for our study, we consider as in [16] a transport process along the edges of a finite network. This system is subject to some transmission conditions in the vertices of the network (imposing, for example, conservation of the mass) which span the "*boundary space*" for our problem. We then like to control the behavior of this system by acting upon a single vertex only. In this context it is reasonable to ask the following questions.

- Can we reach *all* possible states in finite time? The answer to this question is in general negative since we are limited by the network structure, see, e.g., [16, Sec. 5]. Therefore, we only ask: Can we describe the *maximal* possible set of reachable states in some *finite* time?

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- Does controllability depend on the particular choice of the control node? Here the answer is affirmative, which is again demonstrated by some examples in [16, Sec. 5]. However, to our knowledge there is no simple characterization for nodes yielding maximal exact control.
- Which states can be reached if only *positive* controls are allowed? This question is very important since in many applications only positive controls are meaningful and one expects that the state of the system remains positive for all times.

In recent decades, the study of different partial differential equations on networks and similar structures gained a lot of interest. Here we restrict ourself to first order equations that model transport problems (or flows) in networks and are motivated by many real-life applications. In fact, such systems can be used to model, to control, and to optimize road traffic [24, 23, 9, 21, 19], water supply [27, 28], gas flow [5, 22] or supply chains [11], to mention just the most frequent applications. Moreover, we refer to [8] for a survey of related results in the context of nonlinear hyperbolic systems.

In the present work we will, however, only consider *linear* models and make heavy use of the theory of semigroups of linear operators in the spirit of [18]. The application of semigroup theory to flows in networks was initiated in [26, 30], see also the survey paper [13] and the detailed accounts in the monographs [6, Ch. 18] and [29]. For an example of an application of this approach to population models in biology we refer to [4]. The stability and control problems of linear flows in networks using semigroup approach were investigated in [16, 17, 25, 7]. Our aim here is to further generalize and refine these latter results.

This paper is organized as follows. In Section 2 we first recall our abstract framework from [17] as well as some basic results concerning boundary control systems. In Section 3 we then characterize boundary admissible control operators and describe the corresponding exact reachability space. In Section 4 we turn our attention to positive boundary control systems on Banach lattices. Finally, in Section 5 we apply our abstract results and explicitly compute the exact (positive) reachability spaces in three different examples of transport equations controlled at the boundary: in $\mathbb{C}^m$, in a network, and in a network with dynamic boundary conditions. Moreover, we return to the motivating questions stated above.

2. **The abstract framework.**

We start by recalling our setting from [17].

Abstract Framework 2.1. We consider

(i) three Banach spaces X, ∂X and U, called *state*, *boundary* and *control space*, respectively;
(ii) a closed, densely defined *system operator* $A_m : D(A_m) \subseteq X \to X$;
(iii) a *boundary operator* $Q \in \mathcal{L}([D(A_m)], \partial X)$;
(iv) a *control operator* $B \in \mathcal{L}(U, \partial X)$.

For these operators and spaces and a *control function* $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, U)$ we then consider the *abstract Cauchy problem with boundary control*[1]

$$\begin{cases} \dot{x}(t) = A_m x(t), & t \geq 0, \\ Qx(t) = Bu(t), & t \geq 0, \\ x(0) = x_0 \in X. \end{cases} \tag{1}$$

[1]We denote by $\dot{x}(t)$ the derivative of x with respect to the "time" variable t.

A function $x(\cdot) = x(\cdot, x_0, u) \in \mathrm{C}^1(\mathbb{R}_+, X)$ with $x(t) \in D(A_m)$ for all $t \geq 0$ satisfying (1) is called a *classical solution*. Moreover, we denote the *abstract boundary control system* associated to (1) by $\Sigma_{\mathrm{BC}}(A_m, B, Q)$.

In order to investigate (1) we make the following standing assumptions which in particular ensure that the *un*controlled abstract Cauchy problem, i.e., (1) with $B = 0$, is well-posed.

Main Assumptions 2.2. *(i) The restriction $A \subset A_m$ with domain $D(A) := \ker Q$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X;*
(ii) the boundary operator $Q : D(A_m) \to \partial X$ is surjective.

Under these assumptions the following has been shown in [20, Lem. 1.2].

Lemma 2.3. *Let Assumptions 2.2 be satisfied. Then the following assertions are true for all $\lambda, \mu \in \rho(A)$.*
(i) $D(A_m) = D(A) \oplus \ker(\lambda - A_m)$;
(ii) $Q|_{\ker(\lambda - A_m)}$ is invertible and the operator

$$Q_\lambda := \left(Q|_{\ker(\lambda - A_m)}\right)^{-1} : \partial X \to \ker(\lambda - A_m) \subseteq X \tag{2}$$

is bounded;
(iii) $R(\mu, A)Q_\lambda = R(\lambda, A)Q_\mu$.

The following operators are essential to obtain explicit representations of the solutions of the boundary control problem (1).

Definition 2.4. For $\lambda \in \rho(A)$ we call the operator Q_λ introduced in (2) abstract *Dirichlet operator* and define

$$B_\lambda := Q_\lambda B \in \mathcal{L}\big(U, \ker(\lambda - A_m)\big) \subset \mathcal{L}(U, X).$$

By [17, Prop. 2.7] the solutions of (1) can be represented by the following extrapolated version of the variation of parameters formula. Here we use the standard notation for the extrapolated spaces and operators: X_{-1} denotes the completion of X with respect to the norm

$$\|x\|_{-1} := \|R(\lambda_0, A)x\|, \quad x \in X$$

for some fixed $\lambda_0 \in \rho(A)$, $T_{-1}(t) \in \mathcal{L}(X_{-1})$ is the unique bounded extension of the operator $T(t)$ to X_{-1}, and A_{-1} is the generator of the extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ with domain $D(A_{-1}) = X$, cf. [18, Sect. II.5.a].

Proposition 2.5. *Let $x_0 \in X$, $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, U)$ and $\lambda \in \rho(A)$. If $x(\cdot) = x(\cdot, x_0, u)$ is a classical solution of* (1)*, then it is given by the* variation of parameters formula

$$x(t) = T(t)x_0 + (\lambda - A_{-1}) \int_0^t T(t-s)B_\lambda u(s)\, ds, \quad t \geq 0. \tag{3}$$

Our aim in the sequel is to investigate which states in X can be *exactly* reached from $x_0 = 0$ by solutions of (1). To this end we have to impose an additional assumption which, by (3), ensures that solutions for L^p-controls remain in X.

Definition 2.6. Let $1 \leq p \leq +\infty$. Then the control operator $B \in \mathcal{L}(U, \partial X)$ is called *p-boundary admissible* if there exist $t > 0$ and $\lambda \in \rho(A)$ such that

$$\int_0^t T(t-s)B_\lambda u(s)\, ds \in D(A) \quad \text{for all } u \in \mathrm{L}^p\big([0,t], U\big). \tag{4}$$

Remark 2.7. From Lemma 2.3.(iii) it follows that $(\lambda - A_{-1})Q_\lambda \in \mathcal{L}(\partial X, X_{-1})$, hence also

$$B_A := (\lambda - A_{-1})B_\lambda \in \mathcal{L}(U, X_{-1}),$$

are independent of $\lambda \in \rho(A)$. Then $B \in \mathcal{L}(U, \partial X)$ is p-boundary admissible if and only if B_A is p-admissible in the usual sense, cf. [31, Def. 4.1]. This implies that if (4) is satisfied for some $t > 0$ then it is satisfied for every $t > 0$. Moreover, we note that by [17, Lem. A.3], B is 1-boundary admissible if $\ker(\lambda - A_m) \subset F_1^A$, the Favard class of A (see [17, Def. A.1] and references there). Finally, since $\mathrm{L}^p\big([0,t], U\big) \subset \mathrm{L}^1([0,t], U)$ it follows that 1-boundary admissibility implies p-boundary admissibility for all $p > 1$.

Now assume that $B \in \mathcal{L}(U, \partial X)$ is p-boundary admissible. Then for fixed $\lambda \in \rho(A)$ and $t > 0$ the operators $\mathcal{B}_t^{\mathrm{BC}} : \mathrm{L}^p\big([0,t], U\big) \to X$ given by

$$\mathcal{B}_t^{\mathrm{BC}} u := (\lambda - A) \int_0^t T(t-s) B_\lambda u(s)\, ds = \int_0^t T_{-1}(t-s) B_A u(s)\, ds \tag{5}$$

are called the *controllability maps* of the system $\Sigma_{\mathrm{BC}}(A_m, B, Q)$, where the second integral initially is taken in the extrapolation space X_{-1}. Note that by the closed graph theorem $\mathcal{B}_t^{\mathrm{BC}} \in \mathcal{L}(\mathrm{L}^p\big([0,t], U\big), X)$. Hence, this definition is independent of the particular choice of $\lambda \in \rho(A)$ and gives the (unique) classical solution of (1) for given $u \in \mathrm{W}^{2,1}([0,t], U)$ and $x_0 = 0$. This motivates the following definition.

Definition 2.8. (a) The *exact reachability space in time* $t \geq 0$ of $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ is defined by[2]

$$e\mathcal{R}_t^{\mathrm{BC}} := \mathrm{rg}(\mathcal{B}_t^{\mathrm{BC}}).$$

Moreover, we define the *exact reachability space* (in arbitrary time) by

$$e\mathcal{R}^{\mathrm{BC}} := \bigcup_{t \geq 0} \mathrm{rg}(\mathcal{B}_t^{\mathrm{BC}})$$

and call $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ *exactly controllable* (in arbitrary time) if $e\mathcal{R}^{\mathrm{BC}} = X$.

(b) The *approximate reachability space in time* $t \geq 0$ of $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ is defined by

$$a\mathcal{R}_t^{\mathrm{BC}} := \overline{e\mathcal{R}_t^{\mathrm{BC}}}.$$

Moreover, we define the *approximate reachability space* (in arbitrary time) by

$$a\mathcal{R}^{\mathrm{BC}} := \overline{\bigcup_{t \geq 0} a\mathcal{R}_t^{\mathrm{BC}}}$$

and call $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ *approximately controllable* if $a\mathcal{R}^{\mathrm{BC}} = X$.

From [17, Thm. 2.12 & Cor. 2.13] we obtain the following properties and representations of the approximate reachability space.

Proposition 2.9. *Assume that $B \in \mathcal{L}(U, \partial X)$ is p-boundary admissible. Then the following holds.*

(i) $a\mathcal{R}^{BC}$ is a closed linear subspace of X which is invariant under $(T(t))_{t \geq 0}$ and $R(\lambda, A)$ for $\lambda > \omega_0(A)$.

(ii) $a\mathcal{R}^{BC} = \overline{\mathrm{span}} \bigcup_{\lambda > \omega} \mathrm{rg}(B_\lambda)$ for every $\omega > \omega_0(A)$.

(iii) $a\mathcal{R}^{BC} \subseteq \overline{\mathrm{span}} \bigcup_{\lambda > \omega_0(A)} \ker(\lambda - A_m)$.

[2]By $\mathrm{rg}(T)$ we denote the range $TX \subseteq Y$ of an operator $T : X \to Y$.

Part (iii) shows that there is an upper bound for the reachability space depending on the eigenvectors of A_m only, independent of the control operator B. This justifies the following notion.

Definition 2.10. The *maximal reachability space* of $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ is defined by

$$\mathcal{R}^{\mathrm{BC}}_{\max} := \overline{\operatorname{span}} \bigcup_{\lambda > \omega_0(A)} \ker(\lambda - A_m).$$

The system $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ is called *maximally controllable* if $e\mathcal{R}^{\mathrm{BC}} = \mathcal{R}^{\mathrm{BC}}_{\max}$.

We stress again that $\mathcal{R}^{\mathrm{BC}}_{\max} \neq X$ may happen (cf. [16, Sec. 5] for an example), hence the relevant question about exact or approximate controllability for boundary systems is indeed to compare $e\mathcal{R}^{\mathrm{BC}}$ or $a\mathcal{R}^{\mathrm{BC}}$ to the space $\mathcal{R}^{\mathrm{BC}}_{\max}$ and not to the whole space X, as it is usually done in the classical situation in systems theory (see [10]).

After this short summary on boundary control systems $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ taken mainly from [17] in the context of approximate controllability, we now turn our attention to the case of *exact* controllability.

3. Exact controllability.

We start this section by giving two characterizations of p-boundary admissibility for a control operator B which frequently also simplifies the explicit computation of the associated controllability map $\mathcal{B}^{\mathrm{BC}}_t$. Here for $\lambda \in \mathbb{C}$ we introduce the function $\varepsilon_\lambda : \mathbb{R} \to \mathbb{C}$ by $\varepsilon_\lambda(s) := e^{\lambda s}$. Moreover, for $f \in \mathrm{L}^p[0,t]$ and $u \in U$ we define

$$f \otimes u \in \mathrm{L}^p\big([0,t], U\big) \quad \text{by} \quad (f \otimes u)(s) := f(s) \cdot u.$$

Finally, we denote by $\mathbb{1}_{[\alpha,\beta]}$ the characteristic function of the interval $[\alpha, \beta] \subset [0,t]$.

Proposition 3.1. *For a control operator $B \in \mathcal{L}(U, \partial X)$ the following are equivalent.*

(a) B is p-boundary admissible.

(b) There exist $\lambda \in \rho(A)$, $t > 0$ and $M \in \mathcal{L}\big(\mathrm{L}^p\big([0,t], U\big), X\big)$ such that for all $0 \le \alpha \le \beta \le t$ and $v \in U$

$$\big(e^{\lambda\beta}\, T(t-\beta) - e^{\lambda\alpha}\, T(t-\alpha)\big) B_\lambda v = M(\varepsilon_\lambda \cdot \mathbb{1}_{[\alpha,\beta]} \otimes v). \tag{6}$$

(c) There exist $t > 0$, $\lambda_0 > \omega_0(A)$ and $M \in \mathcal{L}\big(\mathrm{L}^p\big([0,t], U\big), X\big)$ such that for all $\lambda \ge \lambda_0$ and $v \in U$

$$\big(e^{\lambda t} - T(t)\big) B_\lambda v = M(\varepsilon_\lambda \otimes v). \tag{7}$$

Moreover, in this case the controllability map is given by $\mathcal{B}^{BC}_t = M$.

Proof. Let $u = \varepsilon_\lambda \cdot \mathbb{1}_{[\alpha,\beta]} \otimes v$ for some $\lambda \in \rho(A)$, $0 \le \alpha \le \beta \le t$ and $v \in U$. Then

$$\begin{aligned} \int_0^t T(t-s) B_\lambda u(s)\, ds &= e^{\lambda t} \int_\alpha^\beta e^{-\lambda(t-s)}\, T(t-s) B_\lambda v\, ds \\ &= e^{\lambda t} \int_{t-\beta}^{t-\alpha} e^{-\lambda s}\, T(s) B_\lambda v\, ds \\ &= R(\lambda, A) \cdot \big(e^{\lambda\beta}\, T(t-\beta) - e^{\lambda\alpha}\, T(t-\alpha)\big) B_\lambda v. \end{aligned} \tag{8}$$

(a)$\Rightarrow$(b). Since by assumption B is p-boundary admissible we have $\mathcal{B}^{\mathrm{BC}}_t \in \mathcal{L}(\mathrm{L}^p\big([0,t], U\big), X)$. Hence, (5) and (8) imply (6) for $M = \mathcal{B}^{\mathrm{BC}}_t$.

(b)⇒(a). We start by proving (4). The idea is to show this first for functions of the type $u = \varepsilon_\lambda \cdot \mathbb{1}_{[\alpha,\beta]} \otimes v$. Then by linearity it also holds for linear combinations of such functions and a density argument implies (4) for arbitrary $u \in \mathrm{L}^p\big([0,t],U\big)$. To this end let $u = \varepsilon_\lambda \cdot \mathbb{1}_{[\alpha,\beta]} \otimes v$ for $[\alpha,\beta] \subset [0,t]$ and $v \in U$. Then (6) and (8) imply

$$\begin{aligned}\int_0^t T(t-s)B_\lambda u(s)\,ds &= R(\lambda,A)\cdot M(\varepsilon_\lambda \cdot \mathbb{1}_{[\alpha,\beta]} \otimes v)\\ &= R(\lambda,A)\cdot Mu. \end{aligned} \tag{9}$$

Note that the multiplication operator $\mathcal{M}_\lambda \in \mathcal{L}\big(\mathrm{L}^p\big([0,t],U\big)\big)$ defined by $\mathcal{M}_\lambda u := \varepsilon_\lambda \cdot u$ is an isomorphism (with bounded inverse $\mathcal{M}_{-\lambda}$). Hence, it maps dense sets of $\mathrm{L}^p\big([0,t],U\big)$ into dense sets. Since the step functions are dense in $\mathrm{L}^p\big([0,t],U\big)$ (see [3, p.14]), the linear combinations of functions of the type $\varepsilon_\lambda \cdot \mathbb{1}_{[\alpha,\beta]} \otimes v$ for $[\alpha,\beta] \subset [0,t]$ and $v \in U$ form a dense subspace of $\mathrm{L}^p\big([0,t],U\big)$. Thus, we conclude that (9) holds for all $u \in \mathrm{L}^p\big([0,t],U\big)$. Clearly this implies that B is p-boundary admissible and $\mathcal{B}_t^{\mathrm{BC}} = M$.

Recall that $B_A = (\lambda - A_{-1})B_\lambda$ is independent of $\lambda \in \rho(A)$. Hence, (a) $\iff$ (c) follows as before by replacing the total set $\{\varepsilon_\lambda \cdot \mathbb{1}_{[\alpha,\beta]} \otimes v : 0 \le \alpha < \beta \le t, v \in U\}$ by the set $\{\varepsilon_\lambda \otimes v : \lambda \ge \lambda_0, v \in U\}$ which by the Stone–Weierstraß theorem is total as well in $\mathrm{L}^p\big([0,t],U\big)$ for all $\lambda_0 > \omega_0(A)$. □

We note that by linearity it would suffice that Part (b) of Proposition 3.1 is satisfied for $\alpha = 0$ and all $0 \le \beta \le t$ (or for all $0 \le \alpha \le t$ and $\beta = t$).

Corollary 3.2. *Let*[3] $n \in \mathbb{N}_1$ *and assume that B is p-boundary admissible. Then for all* $u \in \mathrm{L}^p\big([0,nt],U\big)$

$$\mathcal{B}_{nt}^{BC}u = \sum_{k=0}^{n-1} T(t)^k M u_k \tag{10}$$

where $u_k \in \mathrm{L}^p\big([0,t],U\big)$ *is defined by*

$$u_k(s) = u\big((n-k-1)t + s\big) \tag{11}$$

and $M \in \mathcal{L}\big(\mathrm{L}^p\big([0,t],U\big),X\big)$ *is the operator introduced in Proposition 3.1.*

Proof. Let $u \in \mathrm{L}^p\big([0,nt],U\big)$. Then by (5)

$$\begin{aligned}\mathcal{B}_{nt}^{\mathrm{BC}}u &= (\lambda - A)\int_0^{nt} T(nt-s)B_\lambda\, u(s)\,ds\\ &= (\lambda - A)\sum_{k=1}^{n} T\big((n-k)t\big)\int_{(k-1)t}^{kt} T(kt-s)B_\lambda\, u(s)\,ds\\ &= \sum_{k=1}^{n} T\big((n-k)t\big)\cdot(\lambda - A)\int_0^t T(t-s)B_\lambda\, u_{n-k}(s)\,ds\\ &= \sum_{k=0}^{n-1} T(t)^k\, \mathcal{B}_t^{\mathrm{BC}} u_k. \end{aligned}$$

□

[3] We use the notation $\mathbb{N}_l := \{l, l+1, l+2, \ldots\}$ for the set of natural numbers starting at $l \in \mathbb{N}$.

In Section 5 we will see that (6), (7), and (10) allow us to easily compute the controllability map in the situations studied in [16, Sect. 4] and [17, Sect. 3] dealing with the control of flows in networks.

Corollary 3.3. *If B is p-boundary admissible, then the exact reachability space in time nt for $n \in \mathbb{N}_1$ is given by*

$$e\mathcal{R}_{nt}^{BC} = \left\{ \sum_{k=0}^{n-1} T(t)^k M u_k : u_k \in \mathrm{L}^p\big([0,t],U\big),\ 1 \le k \le n-1 \right\},$$

where $M \in \mathcal{L}\big(\mathrm{L}^p\big([0,t],U\big),X\big)$ is the operator from Proposition 3.1.

4. **Positive controllability.** In this section we are interested in positive control functions yielding positive states. To this end we will make the following

Additional Assumption 4.1. *The spaces X and U are Banach lattices.*

Moreover, by $Y^+ := \{y \in Y : Y \ge 0\}$ we denote the positive cone in a Banach lattice Y. For a detailed account of the theory of semigroups of positive linear operators we refer to [2, 6].

Note that in the sequel we do *not* make any positivity assumptions on $(T(t))_{t\ge 0}$, B or Q_λ if not stated otherwise.

Definition 4.2. (a) The *exact positive reachability space in time $t \ge 0$* of system $\Sigma_{\mathrm{BC}}(A_m,B,Q)$ is defined by

$$e^+\mathcal{R}_t^{\mathrm{BC}} := \Big\{\mathcal{B}_t^{\mathrm{BC}} u : u \in \mathrm{L}^p\big([0,t],U^+\big)\Big\}.$$

Moreover, we define the *exact positive reachability space* (in arbitrary time) by

$$e^+\mathcal{R}^{\mathrm{BC}} := \bigcup_{t\ge 0} e^+\mathcal{R}_t^{\mathrm{BC}}$$

and call $\Sigma_{\mathrm{BC}}(A_m,B,Q)$ *exactly positive controllable* (in arbitrary time) if $e^+\mathcal{R}^{\mathrm{BC}} = X^+$.

(b) The *approximate positive reachability space in time $t \ge 0$* of $\Sigma_{\mathrm{BC}}(A_m,B,Q)$ is defined by

$$a^+\mathcal{R}_t^{\mathrm{BC}} := \overline{e^+\mathcal{R}_t^{\mathrm{BC}}}.$$

Moreover, we define the *approximate positive reachability space* (in arbitrary time) by

$$a^+\mathcal{R}^{\mathrm{BC}} := \overline{\bigcup_{t\ge 0} a^+\mathcal{R}_t^{\mathrm{BC}}}$$

and call $\Sigma_{\mathrm{BC}}(A_m,B,Q)$ *approximately positive controllable* if $a^+\mathcal{R}^{\mathrm{BC}} = X^+$.

First we give necessary and sufficient conditions implying that starting from the initial state $x_0 = 0$ positive controls result in positive states.

Proposition 4.3. *Assume that $B \in \mathcal{L}(U,\partial X)$ is p-boundary admissible. Then*

$$e^+\mathcal{R}_t^{BC} \subset X^+ \tag{12}$$

if and only if

$$a^+\mathcal{R}_t^{BC} \subset X^+ \tag{13}$$

if and only if there exists $\lambda \in \mathbb{R} \cap \rho(A)$ such that

$$\big(e^{\lambda\beta}\,T(t-\beta) - e^{\lambda\alpha}\,T(t-\alpha)\big)B_\lambda \ge 0 \qquad \textit{for all } 0 \le \alpha \le \beta \le t. \tag{14}$$

Moreover, if $(T(t))_{t\ge 0}$ is positive, then the above assertions are satisfied if and only if

$$e^{+}\mathcal{R}^{BC} \subset X^{+} \tag{15}$$

if and only if

$$a^{+}\mathcal{R}^{BC} \subset X^{+} \tag{16}$$

if and only if there exists $\lambda > \omega_0(A)$ and $t > 0$ such that

$$\big(e^{\lambda s} - T(s)\big)B_\lambda \ge 0 \qquad \text{for all } 0 \le s \le t \tag{17}$$

if and only if there exists $\lambda_0 > \omega_0(A)$ such that

$$B_\lambda \ge 0 \qquad \text{for all } \lambda \ge \lambda_0. \tag{18}$$

Proof. The equivalence of (12) and (13) follows from the closedness of X^+. To show the equivalence of (12) and (14) recall that by [3, p.14] the step functions are dense in $\mathrm{L}^p\big([0,t],U\big)$. Since the map $u \mapsto u^+$ on $\mathrm{L}^p\big([0,t],U\big)$ is continuous, we conclude that the positive step functions are dense in $\mathrm{L}^p\big([0,t],U^+\big)$. The claim then follows from (the proof of) Proposition 3.1 using the boundedness of the controllability map $\mathcal{B}_t^{\mathrm{BC}}$.

Now assume that $(T(t))_{t\ge 0}$ is positive. Then the equivalences of (12), (13) with (15), (16) follow from Corollary 3.3 using the fact that the reachability spaces are growing in time. In particular, this implies that if (14) holds for some $t > 0$ it holds for arbitrary $t > 0$ and choosing $\beta = t$ and $\alpha = 0$ we obtain (17) for arbitrary $t > 0$.

To show the remaining assertions we fix some $\lambda > \omega_0(A)$ and define on $\mathcal{X} := X \times X$ the operator matrix

$$\mathcal{A} := \begin{pmatrix} A-\lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad D(\mathcal{A}) := \Big\{ \tbinom{x}{y} \in D(A_m) \times \partial X : Qx = By \Big\}.$$

Then by [14, Cor. 3.4] the matrix $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\ge 0}$ given by

$$\mathcal{T}(s) = \begin{pmatrix} e^{-\lambda s}T(s) & \big(I - e^{-\lambda s}T(s)\big)B_\lambda \\ 0 & I \end{pmatrix}, \quad s \ge 0.$$

Moreover, by [14, Lem. 3.1] we have $(0,+\infty) \subset \rho(\mathcal{A})$ and

$$R(\mu,\mathcal{A}) = \begin{pmatrix} R(\mu+\lambda, A) & \frac{1}{\mu}B_{\mu+\lambda} \\ 0 & \frac{1}{\mu} \end{pmatrix} \quad \text{for } \mu > 0. \tag{19}$$

Now, if (17) holds then $\mathcal{T}(s) \ge 0$ for all $0 \le s \le t$ which implies that $(\mathcal{T}(t))_{t\ge 0}$ is positive which is equivalent to the fact that $\mathcal{A}$ is resolvent positive. However, by (19) the latter is the case if and only if (18) is satisfied which shows the equivalence of (17) and (18). Finally, if (17) holds, then

$$\begin{aligned}\big(e^{\lambda\beta}\,T(t-\beta) - e^{\lambda\alpha}\,T(t-\alpha)\big)B_\lambda &= e^{\lambda\beta}\,T(t-\beta)\cdot\big(I - e^{-\lambda(\beta-\alpha)}\,T(\beta-\alpha)\big)B_\lambda \\ &\ge 0\end{aligned}$$

for all $0 \le \alpha \le \beta \le t$. This proves (14) and completes the proof. □

In the sequel we use the notation $\operatorname{co} M$ and $\overline{\operatorname{co}}\, M$ to indicate the convex hull and the closed convex hull of a set $M \subset X$, respectively.

Proposition 4.4. *Assume that $B \in \mathcal{L}(U,\partial X)$ is p-boundary admissible and that $e^{+}\mathcal{R}_t^{BC} \subset X^{+}$. Then the following holds.*

(i) $a^{+}\mathcal{R}^{BC}$ is a closed convex cone, invariant under $(T(t))_{t\ge 0}$ and $R(\lambda, A)$ for $\lambda > \omega_0(A)$.

(ii) $a^+\mathcal{R}^{BC} = \overline{\text{co}}\,\big\{\big(e^{\lambda\beta}T(t-\beta) - e^{\lambda\alpha}T(t-\alpha)\big)B_\lambda v : 0 \le \alpha \le \beta \le t,\, v \in U^+\big\}$ *for all* $\lambda > \omega_0(A)$.

(iii) $a^+\mathcal{R}^{BC} = \overline{\text{co}}\,\{T(t)B_\lambda v : t \ge 0,\, \lambda > w,\, v \in U^+\}$ *for all* $w > \omega_0(A)$.

(iv) $a^+\mathcal{R}^{BC} = \overline{\text{co}}\,\{R(\lambda, A)^n B_\lambda v : n \in \mathbb{N}_0,\, \lambda > w,\, v \in U^+\}$ *for some (and hence for all)* $w > \omega_0(A)$.

Proof. (i). Clearly, $a^+\mathcal{R}^{\text{BC}}$ is a closed convex cone. Its invariance under $(T(t))_{t\ge0}$ and $R(\lambda, A)$ for $\lambda > \omega_0(A)$ follows from the representations in (iii) and (iv).

To show (ii) we note that by (5) and (8) the inclusion "$\supseteq$" holds. Now recall that the positive step functions are dense in $\mathrm{L}^p\big([0,t], U^+\big)$ and invariant under positive convex combinations. Hence, the boundedness of the controllability maps implies equality of the spaces in (ii).

To obtain (iii) we note that by (5) and (8) we have

$$\big(e^{\lambda\beta}\,T(t-\beta) - e^{\lambda\alpha}\,T(t-\alpha)\big)B_\lambda v \in e^+\mathcal{R}^{\text{BC}}$$

for all $0 \le \alpha \le \beta \le t$ and $v \in U^+$. Multiplying this inclusion by $e^{-\lambda\beta} > 0$ and putting $s := t-\beta$ and $r := t-\alpha$ implies

$$\big(T(s) - e^{\lambda(s-r)}\,T(r)\big)B_\lambda v \in e^+\mathcal{R}^{\text{BC}}$$

for all $0 \le s \le r$ and $v \in U^+$. Since $\lambda > \omega_0(A)$ we obtain

$$\lim_{r\to+\infty} e^{\lambda(s-r)}\,\|T(r)\| = 0$$

and hence

$$T(s)B_\lambda v \in a^+\mathcal{R}^{\text{BC}}$$

for all $s \ge 0$ and $v \in U^+$. This shows the inclusion "$\supseteq$" in (iii). For the converse inclusion in (iii) it suffices to prove that

$$e^+\mathcal{R}_t^{\text{BC}} \subset \overline{\text{co}}\,\big\{T(s)B_\mu y : s \ge 0,\, \mu > w,\, y \in U^+\big\} \tag{20}$$

for all $t > 0$ and $w > \omega_0(A)$. Since $B \in \mathcal{L}(U, \partial X)$ is p-boundary admissible the controllability map $\mathcal{B}_t^{\text{BC}}$ is continuous. Moreover, the positive step functions are dense in $\mathrm{L}^p\big([0,t], U^+\big)$ and $\overline{\text{co}}\,\{T(s)B_\mu y : s \ge 0,\, \mu > w,\, y \in U^+\}$ is a convex cone. Combining these facts and (8) it follows that (20) holds if

$$\big(e^{\lambda\beta}T(t-\beta) - e^{\lambda\alpha}T(t-\alpha)\big)B_\lambda v \in \overline{\text{co}}\,\big\{T(s)B_\mu y : s \ge 0,\, \mu > w,\, y \in U^+\big\} \tag{21}$$

for all $0 \le \alpha \le \beta \le t$, $k \in \mathbb{N}_0$ and $v \in X^+$. Since $(T(t))_{t\ge0}$ is strongly continuous the following integral is the limit of Riemann sums, hence for $\nu > \max\{0, w\}$ we obtain using Lemma 2.3.(iii)

$$\begin{aligned}
\overline{\text{co}}\,\big\{T(s)B_\mu y : s \ge 0,\, \mu > w,\, y \in U^+\big\} \ni \nu \int_{t-\beta}^{t-\alpha} e^{\lambda(t-r)}T(r)B_\nu v\,dr \\
= \big(e^{\lambda\beta}\,T(t-\beta) - e^{\lambda\alpha}\,T(t-\alpha)\big)\nu R(\lambda, A)B_\nu v \\
= \big(e^{\lambda\beta}\,T(t-\beta) - e^{\lambda\alpha}\,T(t-\alpha)\big)\nu R(\nu, A)B_\lambda v \\
\to \big(e^{\lambda\beta}\,T(t-\beta) - e^{\lambda\alpha}\,T(t-\alpha)\big)B_\lambda v,
\end{aligned}$$

as $\nu \to +\infty$. This proves (21) and completes the proof of (iii).

That the right-hand-sides of the equalities in (iii) and (iv) coincide follows from the integral representation of the resolvent (see [18, Cor. II.1.11]) and the Post–Widder inversion formula (see [18, Cor. III.5.5]). For the details we refer to the proof of [7, Prop. 3.3]. □

Corollary 4.5. *Assume that $B \in \mathcal{L}(U, \partial X)$ is p-boundary admissible and that $a^+\mathcal{R}^{BC} \subset X^+$. Then the following are equivalent.*

(a) The system $\Sigma_{BC}(A_m, B, Q)$ is approximately positive controllable.
(b) There exists $w > \omega_0(A)$ such that the following implication holds for all $\varphi \in X'$

$$\big\langle T(s)B_\lambda v, \varphi\big\rangle \geq 0 \text{ for all } v \in U^+,\ s \geq 0 \text{ and } \lambda > w \Rightarrow \varphi \geq 0.$$

(c) There exists $w > \omega_0(A)$ such that the following implication holds for all $\varphi \in X'$

$$\big\langle R(\lambda, A)^n B_\lambda v, \varphi\big\rangle \geq 0 \text{ for all } v \in U^+,\ n \in \mathbb{N} \text{ and } \lambda > w \Rightarrow \varphi \geq 0.$$

Proof. This follows from the proof of [7, Thm. 3.4] by replacing [7, Prop. 3.3] with our Proposition 4.4. □

Remark 4.6. The previous two results generalize [7, Prop. 3.3 and Thm. 3.4], respectively, where it is assumed that $(T(t))_{t\geq 0}$, B and Q_λ for all $\lambda > \lambda_0$ are all positive and, in particular, where the additional hypothesis

(H) *There exists $\gamma > 0$ and $\lambda_0 \in \mathbb{R}$ such that $\|Qx\| \geq \gamma\lambda\|x\|$ for all $\lambda > \lambda_0$ and $x \in \ker(\lambda - A_m)$*

is made. We note that Hypothesis (H) is quite strong, e.g., in reflexive state spaces X it implies that $A = A_m$, cf. [1, Lem. A.1]. Hence, the results of [7] are not applicable to state spaces like $X = \mathrm{L}^p([a, b], Y)$ for $p \in (1, +\infty)$ and reflexive Y.

Combining Corollary 3.2 and Proposition 4.3 we finally obtain the following characterization of an exact positive reachability space.

Corollary 4.7. *Assume that B is p-boundary admissible, $t > 0$ and $n \in \mathbb{N}_1$. Then the exact positive reachability space in time nt is given by*

$$e^+\mathcal{R}^{BC}_{nt} = \left\{ \sum_{k=0}^{n-1} T(t)^k M u_k : u_k \in \mathrm{L}^p\big([0, t], U^+\big),\ 1 \leq k \leq n - 1 \right\},$$

where $M \in \mathcal{L}\big(\mathrm{L}^p\big([0, t], U^+\big), X\big)$ is the operator from Proposition 3.1. Moreover, the operator M is positive if and only if $a^+\mathcal{R}^{BC}_t \subset X^+$.

5. **Examples.**

In this section we will show how our abstract results can be applied to a linear transport equation with boundary control and to vertex control of linear flows in networks subject to static and dynamic boundary conditions.

5.1. **Exact & positive boundary controllability of a transport equation.**

In this subsection we study the controlled transport equation in $\mathbb{C}^m$ given by[4]

$$\begin{cases} \dot{x}(t, s) = x'(t, s), & s \in [0, 1],\ t \geq 0, \\ x(t, 1) = \mathbb{B}x(t, 0) + u(t) \cdot b, & t \geq 0, \\ x(0, s) = 0, & s \in [0, 1]. \end{cases} \tag{22}$$

Here $x : \mathbb{R}_+ \times [0, 1] \to \mathbb{C}^m$ (i.e., $x(t) = (x_j(t, s))_{j=1}^m$), $\mathbb{B} \in \mathrm{M}_m(\mathbb{C})$ implements the boundary conditions for the functions $x_j(t, s)$, $u : \mathbb{R}_+ \to \mathbb{C}$ is a control function, and $b \in \mathbb{C}^m$ is the vector that assigns the control. Roughly spoken, $\mathbb{B}$ determines how the material (flowing on the s-interval $[0, 1]$ from right to left) leaving the system at the left end point $s = 0$ is again fed into the system at the right end point $s = 1$.

[4] We denote by $x'(t, s)$ the derivative of $x(t, s)$ with respect to the "space" variable s.

We note that by our approach one can also deal with other boundary conditions and controls. However, the above choice is useful for studying the network example in Section 5.2.

In order to fit the system (22) in our general framework we choose

- the state space $X := \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)$ for some $1 \le p < +\infty$,
- the boundary space $\partial X := \mathbb{C}^m$,
- the control space $U := \mathbb{C}$,
- the control operator $B := b \in \mathbb{C}^m \simeq \mathcal{L}(U,\partial X) = \mathcal{L}(\mathbb{C},\mathbb{C}^m)$,
- the system operator

$$A_m := \operatorname{diag}\big(\tfrac{d}{ds}\big)_{m\times m} \quad \text{with domain} \quad D(A_m) := \mathrm{W}^{1,p}\big([0,1],\mathbb{C}^m\big),$$

- the boundary operator $Q : \mathrm{W}^{1,p}\big([0,1],\mathbb{C}^m\big) \to \mathbb{C}^m$, $Qf := f(1) - \mathbb{B}f(0)$,
- the operator $A \subset A_m$ with domain $D(A) = \ker Q$,
- the state trajectory $x : \mathbb{R}_+ \to \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)$, $x(t) := x(t,\cdot)$.

For these choices the controlled transport equation (22) can be reformulated as an abstract Cauchy problem with boundary control of the form (1). Clearly, the above boundary operator Q is surjective.

Observe that the operator A is a difference operator as considered in [12, 25, 15, 6]. By [6, Cor. 18.4] we know that for $\lambda \in \mathbb{C}$ and $A = A_m|_{\ker(Q)}$ as above we have

$$\lambda \in \rho(A) \iff e^{\lambda} \in \rho(\mathbb{B}).$$

Moreover, by [6, Prop. 18.7] the operator A generates a strongly continuous semigroup given by

$$\big(T(t)f\big)(s) = \mathbb{B}^k f(t+s-k) \quad \text{if} \ \ t+s \in [k,k+1) \text{ for } k \in \mathbb{N}_0, \tag{23}$$

where $\mathbb{B}^0 := Id$. This shows that the Assumptions 2.2 are satisfied. To proceed we have to compute the associated Dirichlet operator.

Lemma 5.1. *For $\lambda \in \rho(A)$ the Dirichlet operator $Q_\lambda \in \mathcal{L}\big(\mathbb{C}^m, \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)\big)$ is given by*

$$Q_\lambda = \varepsilon_\lambda \otimes R(e^{\lambda},\mathbb{B}). \tag{24}$$

Proof. By Lemma 2.3.(ii) we know that $Q : \ker(\lambda - A_m) \to \partial X$ is invertible. Moreover, for $d \in \mathbb{C}^m = \partial X$ we have

$$Q\big(\varepsilon_\lambda \otimes R(e^{\lambda},\mathbb{B})\, d\big) = e^{\lambda} \cdot R(e^{\lambda},\mathbb{B})\, d - \mathbb{B} \cdot R(e^{\lambda},\mathbb{B})\, d = d$$

which proves (24). □

In order to apply Proposition 3.1 to the present situation we need the following.

Lemma 5.2. *Let $\lambda \in \rho(A)$. Then for all $0 \le \alpha \le 1$*

$$\big(e^{\lambda\alpha} \cdot T(1-\alpha)B_\lambda\big)(s) = \begin{cases} \varepsilon_\lambda(1+s) \cdot R(e^{\lambda},\mathbb{B})\, b & \text{if } 0 \le s < \alpha, \\ \varepsilon_\lambda(1+s) \cdot R(e^{\lambda},\mathbb{B})\, b - \varepsilon_\lambda(s) \cdot b & \text{if } \alpha \le s \le 1. \end{cases}$$

Hence, (6) *is satisfied for*

$$M = b \in \mathcal{L}\big(\mathrm{L}^p[0,1], \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)\big),\ (Mu)(s) = u(s) \cdot b.$$

Proof. The claim follows from (24) and (23) by the following simple computation.

$$\begin{aligned}\big(e^{\lambda\alpha}\cdot T(1-\alpha)B_\lambda\big)(s) &= e^{\lambda\alpha}\cdot\Big(T(1-\alpha)\big(\varepsilon_\lambda\otimes R(e^\lambda,\mathbb{B})\,b\big)\Big)(s)\\ &= e^{\lambda\alpha}\cdot\begin{cases}\varepsilon_\lambda(1-\alpha+s)\cdot R(e^\lambda,\mathbb{B})\,b & \text{if } 0\le s<\alpha,\\ \varepsilon_\lambda(s-\alpha)\cdot\mathbb{B}R(e^\lambda,\mathbb{B})\,b & \text{if } \alpha\le s\le 1,\end{cases}\\ &= \begin{cases}\varepsilon_\lambda(1+s)\cdot R(e^\lambda,\mathbb{B})\,b & \text{if } 0\le s<\alpha,\\ \varepsilon_\lambda(1+s)\cdot R(e^\lambda,\mathbb{B})\,b-\varepsilon_\lambda(s)\cdot b & \text{if } \alpha\le s\le 1.\end{cases}\end{aligned}$$ □

Thus, by Proposition 3.1 the operator B is p-boundary admissible. Next we compute the appropriate reachability space.

Corollary 5.3. *If $t\ge m$ then the exact reachability space of the controlled transport equation* (22) *is given by*

$$e\mathcal{R}_t^{BC}=e\mathcal{R}^{BC}=\mathrm{L}^p[0,1]\otimes\operatorname{span}\left\{b,\mathbb{B}b,\dots,\mathbb{B}^{m-1}b\right\}.$$

Proof. Note that by (23) we have $T(1)f=\mathbb{B}f$. Hence, for $t=m$ the assertion follows immediately from Corollary 3.3 and Lemma 5.2. Clearly, $e\mathcal{R}_t^{\mathrm{BC}}$ increases in time $t\ge 0$. However, by the Cayley–Hamilton theorem $\operatorname{span}\{b,\mathbb{B}b,\dots,\mathbb{B}^l b\}=\operatorname{span}\{b,\mathbb{B}b,\dots,\mathbb{B}^{m-1}b\}$ for all $l\ge m-1$ and the claim follows. □

Remark 5.4. Let $l\le m$ be the degree of the minimal polynomial of $\mathbb{B}$. Then the previous proof shows that for all $t\ge l$ we even have

$$e\mathcal{R}_t^{\mathrm{BC}}=e\mathcal{R}^{\mathrm{BC}}=\mathrm{L}^p[0,1]\otimes\operatorname{span}\left\{b,\mathbb{B}b,\dots,\mathbb{B}^{l-1}b\right\}.$$

Corollary 5.5. *The following assertions are equivalent.*

(a) Equation (22) *is exactly boundary controllable in time $t\ge m$, i.e., $e\mathcal{R}_t^{BC}=X$.*
(b) Equation (22) *is maximally controllable in time $t\ge m$, i.e., $e\mathcal{R}_t^{BC}=\mathcal{R}_{max}^{BC}$.*
(c) $\operatorname{span}\left\{b,\mathbb{B}b,\dots,\mathbb{B}^{m-1}b\right\}=\mathbb{C}^m$.

Proof. Note that $\ker(\lambda-A_m)=\varepsilon_\lambda\otimes\mathbb{C}^m$. Since by the Stone–Weierstraß theorem we have

$$\overline{\operatorname{span}}\bigcup_{\lambda>\omega_0(A)}\{\varepsilon_\lambda\}=\mathrm{L}^p[0,1],$$

the maximal reachability space equals

$$\mathcal{R}_{\max}^{\mathrm{BC}}=\mathrm{L}^p[0,1]\otimes\mathbb{C}^m=X$$

and the assertions follow immediately from Corollary 5.3. □

Remark 5.6. The previous result characterizes the *exact* maximal boundary controllability by a one-dimensional control in terms of a Kalman-type condition which is well-known in control theory.

Combining Remark 5.4 and Corollary 5.5 we furthermore obtain the following

Corollary 5.7. *Let $l\in\mathbb{N}$ be the degree of the minimal polynomial of $\mathbb{B}$. If $l<m$, the transport equation* (22) *is not maximally controllable, i.e., $e\mathcal{R}^{BC}\subsetneq\mathcal{R}_{max}^{BC}$.*

Finally, we investigate positive controllability and consider

- the positive cone $X^+:=\mathrm{L}^p\big([0,1],\mathbb{R}_+^m\big)$ in the state space X,
- the positive cone $U^+:=\mathbb{R}_+$ in the control space U,
- a positive matrix $\mathbb{B}\in\mathrm{M}_m(\mathbb{R}_+)$,
- a positive control operator $B:=b\in\mathbb{R}_+^m$.

Then by (23)–(24) the operators $T(t) \in \mathcal{L}(X)$ for $t \geq 0$ and $B_\lambda \in \mathcal{L}(U, X)$ for $\lambda > \omega_0(A)$ are positive. Thus arguing as above using Proposition 4.3 and Corollary 4.7 we obtain the following.

Corollary 5.8. *The exact positive reachability space of the controlled transport equation* (22) *is given by*

$$e^{+}\mathcal{R}^{BC} = \mathrm{L}^p\big([0,1], \mathbb{R}_+\big) \otimes \mathrm{co}\big\{\mathbb{B}^k b \;:\; k \in \mathbb{N}_0\big\}.$$

Hence, the problem is exactly positive controllable if and only if

$$\mathrm{co}\big\{\mathbb{B}^k b \;:\; k \in \mathbb{N}_0\big\} = \mathbb{R}_+^m.$$

5.2. **Vertex control of flows in networks.** The previous example can be easily adapted to cover a transport problem on a network controlled in a single vertex. More precisely, consider a network consisting of n vertices $\{v_1, \ldots, v_n\}$ and m edges $\{e_1, \ldots, e_m\}$. As shown in [6, Sec. 18.1], its structure can be described by either the transposed weighted adjacency matrix $\mathbb{A} \in \mathrm{M}_n(\mathbb{C})$ given by

$$\mathbb{A}_{ij} := \begin{cases} w_{jk} & \text{if } v_j \xrightarrow{e_k} v_i, \\ 0 & \text{otherwise,} \end{cases}$$

or by the transposed weighted adjacency matrix of the line graph $\mathbb{B} \in \mathrm{M}_m(\mathbb{C})$ where

$$\mathbb{B}_{ij} := \begin{cases} w_{ki} & \text{if } \xrightarrow{e_j} v_k \xrightarrow{e_i}\ , \\ 0 & \text{otherwise.} \end{cases}$$

We also need the transposed weighted outgoing incidence matrix $(\Phi_w^-)^\top =: \Psi \in \mathrm{M}_{m\times n}(\mathbb{C})$ defined by

$$\Psi_{ij} := \begin{cases} w_{ij} & \text{if } v_j \xrightarrow{e_i}\ , \\ 0 & \text{otherwise} \end{cases}$$

and the corresponding unweighted outgoing incidence matrix denoted by $\Phi^- \in \mathrm{M}_{n\times m}(\mathbb{C})$. For the weights we assume $0 \leq w_{ij} \leq 1$, thus all these matrices are positive. Moreover, we assume that Ψ is column stochastic, i.e., the weights on all outgoing edges from a given vertex sum up to 1. This implies that $\mathbb{B}$ is column stochastic as well. For a detailed account of the various graph matrices we refer to [6, Sec. 18.1]. Here we only mention the following relations

$$\Psi\mathbb{A} = \mathbb{B}\Psi, \quad \Psi R(\lambda, \mathbb{A}) = R(\lambda, \mathbb{B})\Psi, \quad \text{and} \quad \Phi^-\Psi = Id_{\mathbb{C}^n} \tag{25}$$

which we will need in the sequel.

We then consider a transport equation on the m edges, which are all parametrized on the interval $[0,1]$, imposing n boundary conditions in the vertices, controlled in a single vertex v_i, i.e.,

$$\begin{cases} \dot{x}(t,s) = x'(t,s), & s \in [0,1],\ t \geq 0, \\ x(t,1) = \mathbb{B}x(t,0) + u(t)\cdot \Psi v, & t \geq 0, \\ x(0,s) = 0, & s \in [0,1]. \end{cases} \tag{26}$$

Here $x : \mathbb{R}_+ \times [0,1] \to \mathbb{C}^m$, that is, $x(t) = (x_j(t,\cdot))_{j=1}^m$ consists of functions on the parametrized edges, and $u : \mathbb{R}_+ \to \mathbb{C}$ is a control function acting on the vertex $v = v_i$, which is represented by the i-th canonical basis vector in $\mathbb{C}^n$. The matrix Ψ applied to v then takes the control with the appropriate weights to the outgoing edges. Moreover, the boundary conditions are encoded into the matrix $\mathbb{B}$. Note that by applying this adjacency matrix we "glue" together the relevant values of the functions $x_j(t)$ at the endpoints $s = 0$ and $s = 1$ of the edges that share a common vertex. Since $\mathbb{B}$ is column stochastic, this also implements conservation of mass in every vertex (the so-called Kirchhoff's condition).

To rewrite equation (26) in our abstract form we take as in Section 5.1 the state space $X := \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)$, the control space $U := \mathbb{C}$ and the boundary space $\partial X := \mathbb{C}^m$. Adapting the domain of A_m as

$$D(A_m) := \big\{f \in \mathrm{W}^{1,p}\big([0,1],\mathbb{C}^m\big) : f(1) \in \operatorname{rg}\Psi\big\}$$

and choosing the control operator $B = b := \Psi v \in \mathbb{C}^m$ we are in the situation considered in [16] and [7], see also [6, Sec. 18.4].

Then the *approximate* controllability space for the network flow problem computed in [16, Cor. 4.3] by Corollary 5.3 above indeed coincides with the *exact* controllability space.

Corollary 5.9. *If $t \geq \min\{m,n\} =: l$ then the exact reachability space of the controlled transport in network problem* (26) *equals*

$$\begin{aligned} e\mathcal{R}_t^{BC} = e\mathcal{R}^{BC} &= \mathrm{L}^p[0,1] \otimes \operatorname{span}\big\{\Psi v, \mathbb{B}\Psi v, \dots, \mathbb{B}^{l-1}\Psi v\big\} \\ &= \mathrm{L}^p[0,1] \otimes \Psi \operatorname{span}\big\{v, \mathbb{A}v, \dots, \mathbb{A}^{l-1}v\big\}. \end{aligned}$$

Note that in big connected networks one usually has $n \leq m$, hence the latter space is more relevant for applications.

Positive control for this problem was already studied in [7] and the approximate positive reachability space was computed. However, our approach even yields in this context the *exact* reachability space.

Corollary 5.10. *The exact positive reachability space of the controlled transport in network problem* (26) *is given by*

$$\begin{aligned} e^{+}\mathcal{R}^{BC} &= \mathrm{L}^p\big([0,1],\mathbb{R}_+\big) \otimes \operatorname{co}\big\{\mathbb{B}^k\Psi v : k \in \mathbb{N}_0\big\} \\ &= \mathrm{L}^p\big([0,1],\mathbb{R}_+\big) \otimes \Psi \operatorname{co}\big\{\mathbb{A}^k v : k \in \mathbb{N}_0\big\}. \end{aligned}$$

Remark 5.11. Let us revisit the motivating questions from the introduction. We have answered the first and the third one by giving explicit descriptions of the appropriate reachability spaces. From these descriptions one can also see that the choice of the vertex v where the control acts is important. This answers also the second question. For concrete examples we refer to [16, Sec. 5]. A characterization of a vertex yielding maximal control remains open.

Finally, we note that by adding more vertices where the control takes place the reachability space increases since the appropriate span or convex hull in Corollaries 5.9 and 5.10, respectively, obtain additional terms in the added vertex. Hence, in this way it is easier to achieve maximal controllability.

5.3. **Exact & positive boundary controllability of flows in networks with dynamical boundary conditions.** In this subsection we investigate exact and positive controllability in the situation of [17, Sect. 3]. Without going much into details we only introduce the necessary facts to state the problem and to compute the corresponding reachability spaces.

We start from the transport problem on the network introduced in the previous example, but now change the transmission process in the vertices allowing for dynamical boundary conditions. To encode the structure of the underlying network and the imposed boundary conditions we use the incidence matrices introduced above as well as the weighted incoming incidence matrix Φ_w^+ given by

$$\left(\Phi_w^+\right)_{ij} := \begin{cases} w_{ij}^+ & \text{if } \xrightarrow{e_j} v_i \ , \\ 0 & \text{otherwise,} \end{cases}$$

for some $0 \le w_{ij}^+ \le 1$. Defining

$$\mathbb{A} := \Phi_w^+ \Psi \quad \text{and} \quad \mathbb{B} := \Psi \Phi_w^+ \tag{27}$$

we obtain the adjacency matrices as above (with different nonzero weights). We mention that the relations (25) remain valid also in this case.

We are then interested in the network transport problem with dynamical boundary conditions in $s = 1$ considered already in [30] and [17, Sect. 3], i.e.,

$$\begin{cases} \dot{x}(t,s) = x'(t,s), & s \in [0,1],\ t \ge 0, \\ \dot{x}(t,1) = \mathbb{B}x(t,0) + u(t) \cdot \Psi v, & t \ge 0, \\ x(0,s) = 0, & s \in [0,1], \\ \Phi^- x(1,0) = 0. \end{cases} \tag{28}$$

To embed this example in our setting we introduce

- the state space $X := \mathrm{L}^p\big([0,1], \mathbb{C}^m\big) \times \mathbb{C}^n$ where $1 \le p < +\infty$,
- the boundary space $\partial X := \mathbb{C}^m$,
- the control space $U := \mathbb{C}$,
- the control operator $B := \Psi v \in \mathbb{C}^m \simeq \mathcal{L}(U, \partial X) = \mathcal{L}(\mathbb{C}, \mathbb{C}^m)$ where $v = v_i$ denotes the i-th canonical basis vector of $\mathbb{C}^n$ meaning that the control acts in the i-th vertex of the network,
- the system operator[5]

$$A_m := \begin{pmatrix} \operatorname{diag}\left(\frac{d}{ds}\right)_{m\times m} & 0 \\ \Phi_w^+ \delta_0 & 0 \end{pmatrix} \quad \text{with domain}$$

$$D(A_m) := \left\{ \tbinom{f}{d} \in \mathrm{W}^{1,p}\big([0,1], \mathbb{C}^m\big) \times \mathbb{C}^n : f(1) \in \operatorname{rg}\Psi \right\},$$

- the boundary operator $Q : D(A_m) \times \mathbb{C}^n \to \mathbb{C}^m$, $Q\tbinom{f}{d} := \Phi^- f(1) - d$,
- the operator $A \subset A_m$ with domain $D(A) = \ker Q$.

As is shown in [17, Prop. 3.4] these spaces and operators satisfy all assumptions of Section 2. To proceed we first need to compute the associated Dirichlet operator Q_λ and an explicit representation of the semigroup operators $T(t)$ for $t \in [0,1]$.

Lemma 5.12. *(i) For each $0 \ne \lambda \in \rho(A)$, the Dirichlet operator $Q_\lambda \in \mathcal{L}(\mathbb{C}^n, X)$ is given by*

$$Q_\lambda = \begin{pmatrix} \lambda\varepsilon_\lambda \otimes \Psi R(\lambda e^\lambda, \mathbb{A}) \\ \mathbb{A} R(\lambda e^\lambda, \mathbb{A}) \end{pmatrix}.$$

(ii) The semigroup $(T(t))_{t\ge 0}$ generated by A is given by[6]

$$\left[T(t)\tbinom{f}{d}\right]_1 (s) = \begin{cases} f(t+s) & \text{if} \quad 0 \le t < 1-s, \\ \mathbb{B}\, V_{t+s-1} f + \Psi d & \text{if } 1-s \le t \le 1, \end{cases} \tag{29}$$

$$\left[T(t)\tbinom{f}{d}\right]_2 = \Phi_w^+ V_t f + d \qquad \text{for } 0 \le t \le 1, \tag{30}$$

[5] By δ_s we denote the point evaluation in $s \in [0,1]$, i.e., $\delta_s(f) = f(s)$.

[6] We use the notations $\left[\tbinom{f}{d}\right]_1 := f$ and $\left[\tbinom{f}{d}\right]_2 := d$ for the canonical projections of $\tbinom{f}{d} \in X$.

where

$$V_s f := \int_0^s f(r)\,dr \quad \text{for } f \in \mathrm{L}^p\big([0,1],\mathbb{C}^m\big). \tag{31}$$

Proof. Assertion (i) is proved in [17, Prop. 3.8]. Equation (30) is shown in the proof of [17, Prop. 3.4.(iii)]. The statement (29) for the first coordinate then follows from [30, Lem. 6.1]. □

Next we apply Proposition 3.1 to the present situation.

Lemma 5.13. *Let $\lambda \in \rho(A)$. Then for all $0 \le \alpha \le 1$*

$$\big[e^{\lambda\alpha}\cdot T(1-\alpha)B_\lambda\big]_1(s) = \begin{cases} \lambda\varepsilon_\lambda(1+s)\cdot \Psi R(\lambda e^\lambda,\mathbb{A})\,v & \text{if } 0 \le s < \alpha,\\ \lambda\varepsilon_\lambda(1+s)\cdot \Psi R(\lambda e^\lambda,\mathbb{A})\,v - \varepsilon_\lambda(s)\cdot\Psi v & \text{if } \alpha \le s \le 1.\end{cases}$$

$$\big[e^{\lambda\alpha}\cdot T(1-\alpha)B_\lambda\big]_2 = e^\lambda \mathbb{A}R(\lambda e^\lambda,\mathbb{A})\,v$$

Hence the equality in (6) is satisfied for

$$M = \begin{pmatrix}\Psi v\\ 0\end{pmatrix} \in \mathcal{L}\Big(\mathrm{L}^p[0,1], \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)\times\mathbb{C}^n\Big),\ (Mu)(s) = \begin{pmatrix}u(s)\cdot\Psi v\\ 0\end{pmatrix}.$$

Proof. Using the explicit representations of Q_λ and $T(t)$ given in Lemma 5.12 and the relations (25) we obtain

$$\begin{aligned}
&\big[e^{\lambda\alpha}\cdot T(1-\alpha)B_\lambda\big]_1(s) =\\
&\quad= e^{\lambda\alpha}\cdot\begin{cases}\lambda\varepsilon_\lambda(1-\alpha+s)\cdot\Psi R(\lambda e^\lambda,\mathbb{A})\,v & \text{if } 0 \le s < \alpha,\\ \lambda\mathbb{B}\,V_{s-\alpha}\,\varepsilon_\lambda\cdot\Psi R(\lambda e^\lambda,\mathbb{A})\,v + \Psi\mathbb{A}R(\lambda e^\lambda,\mathbb{A})v & \text{if } \alpha \le s \le 1,\end{cases}\\
&\quad= e^{\lambda\alpha}\cdot\begin{cases}\lambda\varepsilon_\lambda(1-\alpha+s)\cdot\Psi R(\lambda e^\lambda,\mathbb{A})\,v & \text{if } 0 \le s < \alpha,\\ \big(\varepsilon_\lambda(s-\alpha)-1\big)\cdot\Psi\,\mathbb{A}R(\lambda e^\lambda,\mathbb{A})\,v + \Psi\mathbb{A}R(\lambda e^\lambda,\mathbb{A})v & \text{if } \alpha \le s \le 1,\end{cases}\\
&\quad= \begin{cases}\lambda\varepsilon_\lambda(1+s)\cdot\Psi R(\lambda e^\lambda,\mathbb{A})\,v & \text{if } 0 \le s < \alpha,\\ \varepsilon_\lambda(s)\cdot\Psi\big(\lambda e^\lambda R(\lambda e^\lambda,\mathbb{A}) - Id\big)\,v & \text{if } \alpha \le s \le 1.\end{cases}\\
&\quad= \begin{cases}\lambda\varepsilon_\lambda(1+s)\cdot\Psi R(\lambda e^\lambda,\mathbb{A})\,v & \text{if } 0 \le s < \alpha,\\ \lambda\varepsilon_\lambda(1+s)\cdot\Psi R(\lambda e^\lambda,\mathbb{A})\,v - \varepsilon_\lambda(s)\cdot\Psi v & \text{if } \alpha \le s \le 1.\end{cases}
\end{aligned}$$

Similarly, for the second coordinate we have

$$\begin{aligned}
\big[e^{\lambda\alpha}\cdot T(1-\alpha)B_\lambda\big]_2 &= e^{\lambda\alpha}\Big(\lambda\Phi_w^+\,V_{1-\alpha}\,\varepsilon_\lambda\cdot\Psi R(\lambda e^\lambda,\mathbb{A})\,v + \mathbb{A}R(\lambda e^\lambda,\mathbb{A})\,v\Big)\\
&= e^{\lambda\alpha}\Big(\big(\varepsilon_\lambda(1-\alpha)-1\big)\cdot\mathbb{A}R(\lambda e^\lambda,\mathbb{A})\,v + \mathbb{A}R(\lambda e^\lambda,\mathbb{A})\,v\Big)\\
&= e^\lambda\mathbb{A}R(\lambda e^\lambda,\mathbb{A})\,v,
\end{aligned}$$

where we used (27). □

We note that by [17, Prop. 3.5] the states of the controlled flow at time $t \ge 0$ are given by the first coordinate of the states in our "extended" state space $X = \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)\times\mathbb{C}^n$. For this reason we also need to compute the first coordinate of $T(1)^k\binom{\Psi g}{0}$.

Lemma 5.14. *We have*

$$\Big(T(1)\tbinom{f}{d}\Big)(s) = \begin{pmatrix}\Psi\,\Phi_w^+\,V_s & \Psi\\ \Phi_w^+\,V_1 & Id\end{pmatrix}\begin{pmatrix}f\\ d\end{pmatrix} = \begin{pmatrix}\mathbb{B}\,V_s f + \Psi d\\ \Phi_w^+\,V_1 f + d\end{pmatrix},$$

where the operator $V_s \in \mathcal{L}\big(\mathrm{L}^p\big([0,1],\mathbb{C}^m\big),\mathrm{W}^{1,p}\big([0,1],\mathbb{C}^m\big)\big)$ *is defined in* (31). *Moreover, for* $k \in \mathbb{N}_1$ *we have*

$$\Big[T(1)^k \tbinom{\Psi g}{0}\Big]_1(s) = \Psi(\mathbb{A}V_s + \delta_1)^{k-1}\mathbb{A}\,V_s g = (\mathbb{B}V_s + \delta_1)^{k-1}\mathbb{B}\Psi\,V_s g. \tag{32}$$

Proof. The formula for $T(1)$ follows immediately from Lemma 5.12.(ii). Since $\Psi\mathbb{A} = \mathbb{B}\Psi$, it suffices to show the second equality in (32). Obviously this equation holds for $k = 1$. To verify it for $k > 1$ we note that by (25) the matrix Ψ is left invertible with left inverse Φ^-. Hence, we obtain

$$\Big[T(1)\tbinom{f}{d}\Big]_2 = \Phi^-\delta_1\Big[T(1)\tbinom{f}{d}\Big]_1.$$

If $\binom{f}{d} \in \operatorname{rg} T(1)$ we can write $f = \Psi h$ and the previous equation implies

$$\Big[T(1)\tbinom{f}{d}\Big]_1(s) = \Big[T(1)\tbinom{\Psi h}{\delta_1 h}\Big]_1(s) = \mathbb{B}V_s\Psi h + \Psi\delta_1 h = (\mathbb{B}V_s + \delta_1)f.$$

Now assume that (32) holds for some $k \ge 1$. Then for $\binom{f}{d} = T(1)^k\binom{\Psi g}{0} \in \operatorname{rg} T(1)$ we conclude

$$\begin{aligned}\Big[T(1)^{k+1}\tbinom{\Psi g}{0}\Big]_1(s) &= \Big[T(1)\cdot T(1)^k\tbinom{\Psi g}{0}\Big](s)\\ &= (\mathbb{B}V_s + \delta_1)\cdot(\mathbb{B}V_s + \delta_1)^{k-1}\,\mathbb{B}\Psi\,V_s g\\ &= (\mathbb{B}V_s + \delta_1)^k\,\mathbb{B}\Psi\,V_s g.\end{aligned}$$ □

The previous two lemmas together with Corollary 3.2 imply the following result.

Corollary 5.15. *For* $l \in \mathbb{N}_2$ *and* $u \in \mathrm{L}^p[0,l]$ *we have*

$$\begin{aligned}\big[\mathcal{B}_l^{BC}u\big]_1(s) &= \Psi\Bigg(u_0\otimes v + \sum_{k=1}^{l-1}\big(\mathbb{A}V_s + \delta_1\big)^{k-1}V_s(u_k\otimes\mathbb{A}v)\Bigg)\\ &= u_0\otimes\Psi v + \sum_{k=1}^{l-1}\big(\mathbb{B}V_s + \delta_1\big)^{k-1}V_s(u_k\otimes\mathbb{B}\Psi\,v)\end{aligned} \tag{33}$$

where $u_k \in \mathrm{L}^p[0,1]$ *is defined as in* (11).

Using this explicit representation of the controllability map we now compute the exact reachability space for the control problem given in (28).

Corollary 5.16. *If* $t \ge \min\{m,n\} =: l$ *then the exact reachability space of the controlled flow with dynamic boundary conditions* (28) *is given by*[7]

$$\begin{aligned}\big[e\mathcal{R}_t^{BC}\big]_1 &\subseteq \left\{\Psi\sum_{k=0}^{l}\big(u_k\otimes\mathbb{A}^k\,v\big) : u_k\in\mathrm{W}^{k,p}[0,1] \text{ for } 0\le k\le l\right\}\\ &= \left\{\sum_{k=0}^{l}\big(u_k\otimes\mathbb{B}^k\Psi\,v\big) : u_k\in\mathrm{W}^{k,p}[0,1] \text{ for } 0\le k\le l\right\}.\end{aligned}$$

Proof. The equality of the two sets on the right-hand-side follows immediately from (25). To show the inclusion in the second set we combine Corollaries 3.3 and 5.15. First observe, that for the operators $\mathbb{B}$, V_s, and δ_1 we have

$$\mathbb{B}V_s f = V_s\mathbb{B}f, \quad \mathbb{B}\delta_1 f = \delta_1\mathbb{B}f, \quad \delta_1 V_s f = V_1 f$$

[7]Here we define $\mathrm{W}^{0,p}[0,1] := \mathrm{L}^p[0,1]$.

for every $f \in \mathrm{L}^p\big([0,1],\mathbb{C}^m\big)$ while

$$\delta_1^k f = \delta_1 f = f(1) \quad \text{for } k \geq 1.$$

So, when expanding $(\mathbb{B}V_s + \delta_1)^{k-1}V_s$ we can rearrange the terms to obtain expressions of the form

$$\alpha_i \mathbb{B}^i V_{s_1} \cdots V_{s_{i+1}}, \quad 0 \leq i \leq k-1,$$

where α_i are scalar coefficients and $s_j \in \{s,1\}$, $1 \leq j \leq i+1$. Next, for arbitrary $u \in \mathrm{L}^p[0,1]$ and $0 \leq k \leq l$ we have

$$V_{s_1} \cdots V_{s_k} u \in \mathrm{W}^{k,p}[0,1], \quad s_j \in \{s,1\}, 1 \leq j \leq k.$$

Combining these facts we obtain the desired result by considering (33) for all $u \in \mathrm{L}^p[0,l]$. □

By the previous Corollary we immediately obtain the following result which improves [17, Thm. 3.10] and shows that $\big[a\mathcal{R}_t^{\mathrm{BC}}\big]_1$ is constant for $t \geq \min\{m,n\} =: l$.

Corollary 5.17. *If $t \geq \min\{m,n\} =: l$ then the approximate controllability space of the controlled flow with dynamic boundary conditions* (28) *is given by*

$$\begin{aligned}\big[a\mathcal{R}_t^{BC}\big]_1 &= \mathrm{L}^p[0,1] \otimes \operatorname{span}\big\{\Psi v, \mathbb{B}\,\Psi v, \ldots, \mathbb{B}^{l-1}\Psi v\big\} \\ &= \mathrm{L}^p[0,1] \otimes \Psi \operatorname{span}\big\{v, \mathbb{A}v, \ldots, \mathbb{A}^{l-1}v\big\}.\end{aligned}$$

In the same manner as before we also obtain the following result on positive controllability.

Corollary 5.18. *The approximate positive controllability space of the controlled flow with dynamic boundary conditions* (28) *is given by*

$$\begin{aligned}\big[a^+\mathcal{R}^{BC}\big]_1 &= \mathrm{L}^p[0,1] \otimes \overline{\mathrm{co}}\,\big\{\mathbb{B}^k\Psi v : k \in \mathbb{N}_0\big\} \\ &= \mathrm{L}^p[0,1] \otimes \Psi\,\overline{\mathrm{co}}\,\big\{\mathbb{A}^k v : k \in \mathbb{N}_0\big\}.\end{aligned}$$

Conclusion. Using a new characterization of admissible boundary control operators (see Proposition 3.1) we are able to describe explicitly the *exact* reachability space of the abstract boundary control system $\Sigma_{\mathrm{BC}}(A_m, B, Q)$, cf. (1). Moreover, this approach allows also to determine the *positive* reachability space obtained considering only positive control functions. Our results generalize and improve the ones obtained in the former works [7, 16, 17] where only approximate controllability or positive controllability under quite restrictive assumptions are studied.

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