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AN EPIDEMIC MODEL WITH NONLOCAL DIFFUSION ON NETWORKS

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Abstract. We consider a SIS system with nonlocal diffusion which is the continuous version of a discrete model for the propagation of epidemics on a metapopulation network. Under the assumption of limited transmission, we prove the global existence of a unique solution for any diffusion coefficients. We investigate the existence of an endemic equilibrium and prove its linear stability, which corresponds to the loss of stability of the disease-free equilibrium. In the case of equal diffusion coefficients, we reduce the system to a Fisher-type equation with nonlocal diffusion, which allows us to study the large time behaviour of the solutions. We show large time convergence to either the disease-free or the endemic equilibrium.

1. Introduction.

In this paper, we propose to consider a new model for the spread of epidemics on heterogeneous networks, which we formally derive from a discrete model proposed in [17, 18]. Each node of the network corresponds to a patch of the metapopulation and is characterised by its degree $x > 0$. In contrast with [17, 18], the degree is viewed here as a continuous variable and the structure of the network is encapsulated in its degree distribution. We assume namely that the density of this distribution is given by a smooth function $p : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies

$$\forall x \geq 0, \quad p(x) \geq 0, \; p(0) = 0 \text{ and } \int_0^\infty p(x)\, dx = 1. \tag{1.1}$$

The condition $p(0) = 0$ corresponds to a connected network, with no isolated patch. Moreover, we restrict to the case of uncorrelated network, as will be explained below. This continuous approximation has already been used in the context of complex networks (see [5], [6]) but to our knowledge it has not appeared when coupled with epidemic process. For any nonnegative measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, we denote by $\langle \phi \rangle \in [0, \infty]$ its mean value on the network defined by

$$\langle \phi \rangle = \int_0^\infty \phi(x)\, p(x) dx. \tag{1.2}$$

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We assume that the mean value of the degree is finite, so that

$$\text{there exists } m \in (0,\infty) \text{ such that } m = \langle x \rangle = \int_0^\infty x\, p(x)dx. \tag{1.3}$$

Let us denote by $S(x,t)$ (resp. $I(x,t)$) the density of susceptible (resp. infected) individuals of degree $x \geq 0$ at time $t \geq 0$. In our model, their evolution in time is given by the solution of the following nonlinear and nonlocal system

$$(C)\begin{cases} \dfrac{\partial S}{\partial t} &= I\,(\mu - \beta S) - D_S\left(S - \dfrac{x}{m}s(t)\right) \quad x>0, t>0 \\ \dfrac{\partial I}{\partial t} &= I\,(-\mu + \beta S) - D_I\left(I - \dfrac{x}{m}i(t)\right) \quad x>0, t>0 \\ S(x,0) &= S_0(x), \quad x \geq 0 \\ I(x,0) &= I_0(x), \quad x \geq 0, \end{cases}$$

with $s(t)$ and $i(t)$ given by

$$\forall t > 0, \quad s(t) = \langle S(.,t)\rangle, \quad i(t) = \langle I(.,t)\rangle. \tag{1.4}$$

The constant $\mu > 0$ is the recovery rate of the epidemics. The function $\beta = \beta(x,t)$ is the transmission rate which according to standard epidemiological modelling is of one of the following forms, either

(H_1) limited transmission

$$\beta(x,t) = \frac{\beta_0}{N(x,t)}, \tag{1.5}$$

where $N(x,t)$ is the total density of individuals of degree x at time t defined by

$$N(x,t) = S(x,t) + I(x,t) \tag{1.6}$$

or

(H_2) nonlimited transmission

$$\beta(x,t) = \beta_0,$$

where $\beta_0 > 0$ is a given constant. In this paper, we deal with the case of limited transmission and assume in the sequel that β is given by (1.5). The companion paper [13] is devoted to the analysis of the model under the assumption of nonlimited transmission (H_2).

This system is obtained as a continous version of the discrete SIS metapopulation model proposed in [17].

The classical compartmental models of epidemiology, among which the most basic are the SIS and SIR models (see [9], [4]) usually assume homogeneous, well-mixed population. More recently, there is a vast literature devoted to epidemic models in the case where the population is displaying heterogeneous properties, regarding either its geographical distribution and density or its epidemic parameters, such as the transmission coefficient (see [11]). Simultaneously the recent years have witnessed an impressive body of works investigating complex networks using the tools and methods of statistical physics (see [14] for a survey). The main interest lies in understanding dynamical processes taking place on those networks and the impact of the topology of the network on their qualitative behaviour.

One of the main topic of interest concerns epidemic networks that have attracted a lot of attention (see [11], [16], and [15] for a recent review).

Our work originated from a series of papers by R.Pastor-Satorras, A.Vespignani, V. Collizza which focuses on the propagation of epidemics in “metapopulation”.

The concept of metapopulation has been introduced in ecology by R. Levins in [12] and metapopulation dynamics has been studied subsequently ([7], [8]). It aims at describing a population in which individuals are spatially distributed in their habitat, thus forming subpopulations. Here the metapopulation is mapped onto a network, where nodes correspond to patches of subpopulations that migrate along the edges. The epidemic transmission takes place in each patch.

Following ([1], [2], [3]), J. Saldana proposed in [17] and [18] a discrete model for the propagation of epidemics in a heterogeneous network, with the additional feature that reaction and diffusion processes take place simultaneously. This assumption allowed him to derive a time-continuous model.

In this paper, we propose to consider a continuous version of this model, where both time and degree take values in $\mathbb{R}_+$. The resulting model is a system of differential equations with nonlocal diffusion. We show that it retains the main features of the discrete ones, while providing rigorous and more precise results.

Let us namely describe the results contained in this paper. In section 2, we present the discrete model and recall the results obtained in [17, 18, 19]. In section 3, we show that the system (C) admits a unique, global in time solution for any diffusion coefficients $D_S, D_I > 0$ and for any smooth, nonnegative initial data (S_0, I_0) with finite mean values. The result is obtained using fixed point methods and a priori estimates on the solutions. In section 4, we prove the existence of an endemic equilibrium (EE) if and only if $\beta_0 > \mu$. We then prove that the disease-free equilibrium (DFE) is linearly stable as long as the (EE) does not exist and that it loses its stability as soon as the (EE) does exist, in which case the latter is stable. Next in section 5, we study large-time asymptotics of the solutions to (C). In the case where (DFE) is linearly stable, we prove that it is also the limit of any solution of system (C) for large time. Next in the case of equal diffusion coefficients, we reduce the system to a Fisher-type equation with nonlocal diffusion. By using a comparison principle satisfied by this equation, we obtain the large-time behavior of the solutions to System (C) and prove that (EE) is globally asymptotically stable whenever it exists.

2. The discrete model.

Our model is based on a model for the spread of infectious diseases in heterogeneous metapopulations proposed in [17] and [18] that we briefly recall here.

The author considers a biological population living in separated patches that are connected by pathways (roads, transportation lines, etc...). This is mapped onto a network, whose nodes are the patches and whose edges are the links connecting them. Such a network is called a metapopulation - a “population of populations” - network.

His model is based on previous descriptions of complex network using methods of statistical physics ([14], [16]), more precisely on a model proposed in ([1], [2], [3]). Applying a formalism from statistical mechanics to complex networks, the novelty of these models is to focus on the degree, as the main variable. Let us recall that the degree of a node is defined in graph theory as an integer which is the number of its nearest neighbors measured by the number of edges attached to it. The main assumption is that, after taking statistical averages on configurations, the density of individuals of a certain type, whether susceptibles or infected, in a node of degree k only depends on k.

Precisely, the complex network is characterized by the degree distribution of its nodes. The topology of the network is encoded in the probability distribution of degrees $(p(k))$, $k \in \mathbb{N}$ that satisfies

$$\forall k \geq 0, \;\; p(k) \geq 0 \text{ and } \sum_{0}^{\infty} p(k) = 1,$$

with a finite mean value $m > 0$ defined by

$$m = \sum_{0}^{\infty} k\, p(k) < \infty.$$

Diffusion is modeled as a migration process along the edges, which tends to balance the outflow from the node of degree k and the inflow from the neighboring nodes of degree j. Hence it depends on the degree correlation function in the network, $(p(j|k))_{j\geq 0}$ for each $k \geq 0$, defined as the conditional probability distribution of a neighboring node of a node of degree k. Note that $p(j|k)kp(k)$ is the probability that an edge departing from a node of degree k is connected to a node of degree j. Therefore this quantity is a symmetric function of (j,k), which imposes the consistency condition

$$\forall (j,k) \in \mathbb{N}^2, \quad kp(j|k)p(k) = jp(k|j)p(j). \tag{2.1}$$

In order to consider the epidemic process taking place on the network, it is assumed that each node contains two types of individuals: susceptibles S and infected I. The epidemic transmission occurs in the nodes. In the case of the classical SIS model, it corresponds to the bosonic reactions scheme

$$S + I \xrightarrow{\beta} 2I, \quad I \xrightarrow{\mu} S,$$

where $\mu > 0$ is the recovery rate and $\beta > 0$ is the transmission rate.

In the papers ([17]), [19]), the additional assumption is that reaction and diffusion occur simultaneously, which allows the author to derive the following system. Let $S_k(t)$ (resp. $I_k(t)$) denote the (statistical) density of susceptibles (resp. infected) in a node of degree k. Combining reaction in the nodes and diffusion along the edges, Saldana's model proposes the following time-evolution model for (S_k, I_k), for any fixed $k \in \mathbb{N}$,

$$\left\{ \begin{array}{rcl} \dfrac{\partial S_k}{\partial t} & = & I_k\,(\mu - \beta_k S_k) - D_S \left(S_k - k \displaystyle\sum_j p(j|k) \frac{1}{j} S_j \right) \\ \dfrac{\partial I_k}{\partial t} & = & I_k\,(-\mu + \beta_k S_k) - D_I \left(I_k - k \displaystyle\sum_j p(j|k) \frac{1}{j} I_j \right), \end{array} \right.$$

where $D_S > 0$ (resp. $D_I > 0$) is the diffusion coefficient of susceptible (resp. infected) individuals.

J. Saldana then focuses on the particular case of "uncorrelated networks", where $p(j|k)$ is independent of k. In view of the consistency condition (2.1), it is therefore given by $p(j|k) = \frac{jp(j)}{m}$. In this case, the above model simplifies into

$$\left\{ \begin{array}{rcl} \dfrac{\partial S_k}{\partial t} & = & I_k\,(\mu - \beta_k S_k) - D_S \left(S_k - \dfrac{k}{m} \langle S \rangle \right) \\ \dfrac{\partial I_k}{\partial t} & = & I_k\,(-\mu + \beta_k S_k) - D_I \left(I_k - \dfrac{k}{m} \langle I \rangle \right), \end{array} \right.$$

where

$$\langle S\rangle = \sum_{k\geq 0} p(k)S_k, \quad \langle I\rangle = \sum_{k\geq 0} p(k)I_k$$

are the mean values of S_k and I_k on the network. The results established in ([17], [18]) concern the existence of equilibria and their linear stability, which is investigated using matrix analysis. The authors also carry out Monte-Carlo simulations in [19] in order to test the effects of different probability distributions on the existence and stability of the equilibria.

The system (C) considered here is the formal continuous limit of the above system. Namely, we replace $k \in \mathbb{N}$ by $x \in \mathbb{R}_+$, and assume that the network's degree distribution is a density p satisfying (1.1). Denoting by $S(x,t)$ (resp. $I(x,t)$) the corresponding number of susceptibles (resp. infected) individuals at time $t \geq 0$ and defining the mean values by (1.2) yields the evolution system (C).

3. **Existence and uniqueness of a solution to system (C).** In the sequel, we consider a couple of initial data (S_0, I_0) satisfying the following assumptions.

Assumption 1. *The initial data $S_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and $I_0 : \mathbb{R}_+ \to \mathbb{R}_+$ are smooth, nonnegative functions on $\mathbb{R}_+$ satisfying $S_0(0) = 0$ and $I_0(0) = 0$.*

Assumption 2. *Let $s_0 = \langle S_0\rangle$ and $i_0 = \langle I_0\rangle$ be the mean values of S_0 and I_0 respectively. Assume that $0 < s_0, i_0 < \infty$.*

Under these assumptions, we show that system (C) admits a unique, global in time solution and establish the following theorem

Theorem 3.1. *There exists a unique solution (S, I) to System (C) on $\mathbb{R}_+ \times \mathbb{R}_+$.*

Proof. We denote by $n(t)$ the mean value of the functions $N(.,t)$ at time $t \geq 0$ so that in view of (1.6) and (1.4)

$$n(t) = \langle N(.,t)\rangle = \int_0^\infty N(x,t)\, p(x)dx = s(t) + i(t) \tag{3.1}$$

and by $n_0 = s_0 + i_0 > 0$ the initial total mean population. □

3.1. **A priori estimates.** We first establish a priori estimates on the solutions of (C).

Proposition 1. *Assume that (S_0, I_0) satisfy assumptions 1 and 2 above. Let (S, I) be a smooth solution to system (C) on $\mathbb{R}_+ \times \mathbb{R}_+$. The following properties are satisfied.*

(i) The function n is constant in time: $\forall t > 0, n(t) = n_0$
(ii) $\forall t > 0$, $\forall x > 0$, $I(x,t) > 0$
(iii) $\forall t > 0$, $\forall x > 0$, $S(x,t) > 0$
(iv) $\forall t > 0$, $S(0,t) = I(0,t) = 0$

Remark 1. For definiteness it is assumed that $\beta(x,t)S(x,t) = 0$ in system (C) whenever $N(x,t) = 0$. However it follows from Proposition 1 that $N(x,t) > 0$ for all $t > 0$ and $x > 0$.

Proof. To establish (i), note that in view of the definitions of $s(t)$ and $i(t)$ in (1.4) and of m in (1.3), we have that for all $t > 0$,

$$\langle S(.,t) - \frac{x}{m}s(t)\rangle = \langle I(.,t) - \frac{x}{m}i(t)\rangle = 0.$$

Adding up the two equations in system (C) shows that $N = S + I$ satisfies for all $x > 0$,

$$\forall t > 0, \ \ \frac{\partial N}{\partial t}(x,t) = -D_S\left(S(x,t) - \frac{x}{m}s(t)\right) - D_I\left(I(x,t) - \frac{x}{m}i(t)\right)$$

so that since the function N is smooth,

$$n'(t) = \frac{d}{dt}\left(\int_0^\infty N(x,t)\, p(x)\, dx\right) = \langle \frac{\partial N(.,t)}{\partial t}\rangle = 0$$

for all $t > 0$. Hence the function n is constant in time,

$$\forall t \geq 0, \ \ n(t) = n(0) = s_0 + i_0 = n_0.$$

To establish (ii), note that the function I satisfies for all $x > 0$

$$\forall t > 0, \ \ \frac{\partial I}{\partial t} = I\,(-\mu + \beta S - D_I) + D_I \frac{x}{m} i(t). \tag{3.2}$$

Let us first show that

$$\forall t \geq 0, \ \ \ i(t) > 0 \tag{3.3}$$

Let us denote $A = \{t \geq 0 /\ \forall s \in [0,t],\ i(s) > 0\}$. Since $i_0 > 0$, by continuity there exists $\varepsilon_0 > 0$ such that $[0, \varepsilon_0] \subset A$.

Let us define $T^* = \sup(A)$ so that $T^* \geq \varepsilon_0 > 0$. If we assume by contradiction that A is bounded by above, then $T^* < \infty$, $A = [0, T^*)$ and $i(T^*) = 0$.

We define the functions f and F by

$$\forall x > 0, \ \ \forall t > 0, \ \ f(x,t) = -(\mu + D_I) + \beta(x,t)S(x,t) \text{ and } F(x,t) = \int_0^t f(x,s)ds$$

so that in view of (3.2), we have that

$$\frac{\partial\left(e^{-F} I\right)}{\partial t} = e^{-F}\left(\frac{\partial I}{\partial t} - fI\right) = e^{-F} D_I \frac{x}{m} i(t).$$

Hence we have that

$$\frac{\partial\left(e^{-F} I\right)}{\partial t}(x,t) > 0 \text{ for all } t \in (0, T*) \text{ and } x > 0.$$

Since $I(x,0) \geq 0$, we deduce that for all $x > 0$ and $0 < t < t' < T^*$,

$$0 \leq I(x,0) < e^{-F(x,t)} I(x,t) < e^{-F(x,t')} I(x,t') \tag{3.4}$$

so that $I(x,t) > 0$ for $t \in (0, T^*)$ and $x > 0$. Letting $t' \to T^*$, this implies that $I(x, T^*) > 0$ for all $x > 0$, which contradicts $i(T^*) = 0$. Hence $T^* = \infty$ which proves (3.3). We then argue as above and deduce (ii) from (3.4).

To establish (iii), note that the function S satisfies for all $x > 0$

$$\forall t > 0, \ \ \frac{\partial S}{\partial t} = S\,(-\beta I - D_S) + \mu I + D_S \frac{x}{m} s(t). \tag{3.5}$$

Since by (ii), we have that $I \geq 0$ on $\mathbb{R}_+ \times \mathbb{R}_+$, we follow the same method as in the proof of (ii) to show that $s(t) > 0$ for all $t \geq 0$ and to conclude that $S(x,t) > 0$ for all $t > 0$ and $x > 0$.

To establish (iv), note that $\alpha(t) = I(0,t)$ satifies (3.2) with $x = 0$ which is the linear equation $\alpha' = f(0,t)\alpha$. Since $\alpha(0) = 0$ by Assumption 1, it follows that $\alpha(t) = 0$ for all $t \geq 0$. Next using (3.5) with $x = 0$, we prove similarly that $S(0,t) = 0$ for all $t \geq 0$. □

3.2. **Fixed point procedure.** Let (s, i) be a given couple of continuous nonnegative functions defined on $\mathbb{R}_+$ with $(s(0), i(0)) = (s_0, i_0)$. To this couple (s, i), we associate the following evolution system

$$\left(\widehat{C}\right) \begin{cases} \dfrac{\partial \widehat{S}}{\partial t} = \widehat{I}\left(\mu - \widehat{\beta}(x,t)\widehat{S}\right) - D_S\left(\widehat{S} - \dfrac{x}{m}s(t)\right) & x > 0, t > 0 \\ \dfrac{\partial \widehat{I}}{\partial t} = \widehat{I}\left(-\mu + \widehat{\beta}(x,t)\widehat{S}\right) - D_I\left(\widehat{I} - \dfrac{x}{m}i(t)\right) & x > 0, t > 0 \\ \widehat{S}(x,0) = S_0(x), & x \geq 0 \\ \widehat{I}(x,0) = I_0(x), & x \geq 0 \end{cases}$$

and denote by $\left(\widehat{S}, \widehat{I}\right)$ its solution. We prove below that it is defined on $\mathbb{R}_+ \times \mathbb{R}_+$. We also define

$$\forall x > 0, \;\; \forall t > 0, \;\; \widehat{N}(x,t) = \widehat{S}(x,t) + \widehat{I}(x,t)$$

and, in view of (1.5), we assume here that $\widehat{\beta}(x,t) = \frac{\beta_0}{\widehat{N}(x,t)}$ (with the convention that $\widehat{\beta}(x,t)\widehat{S}(x,t) = 0$ whenever $\widehat{N}(x,t) = 0$). Next we define the functions $\left(\widehat{s}, \widehat{i}\right)$ by

$$\forall t > 0, \;\; \widehat{s}(t) = \langle \widehat{S}(.,t) \rangle, \;\; \widehat{i}(t) = \langle \widehat{I}(.,t) \rangle \tag{3.6}$$

and $\widehat{n} = \widehat{s} + \widehat{i}$.

Note that $\left(\widehat{S}, \widehat{I}\right)$ is a solution of (C) if and only if $\widehat{s} = s$ and $\widehat{i} = i$ so that the existence of a solution to (C) relies on the existence of a fixed point to the operator $(s, i) \to (\widehat{s}, \widehat{i})$.

3.2.1. *A priori estimates on* $\left(\widehat{S}, \widehat{I}\right)$. The well-posedness of system $\left(\widehat{C}\right)$ follows from classical results on system of odes. To establish global existence of a solution $\left(\widehat{S}, \widehat{I}\right)$ on $\mathbb{R}_+$, we first prove the following a priori estimates of the solutions and obtain upper bounds in the next lemma. Define

$$d = \min(D_I, D_S), \quad D = \max(D_I, D_S) \tag{3.7}$$

Lemma 3.2. *Let $T > 0$ be given and consider 2 smooth nonnegative functions (s, i) defined on $\mathbb{R}_+$ such that $(s(0), i(0)) = (s_0, i_0)$. Let $(\widehat{S}, \widehat{I})$ be a smooth solution to system $(\widehat{C})$ on $\mathbb{R}_+ \times [0, T]$. Then the following properties are satisfied.*

(i) $\forall t \in [0, T]$, $\forall x > 0$, $\widehat{I}(x,t) \geq 0$
(ii) $\forall t \in [0, T]$, $\forall x > 0$, $\widehat{S}(x,t) \geq 0$
(iii) $\forall t \in [0, T]$, $\widehat{I}(0,t) = \widehat{S}(0,t) = 0$
(iv) $\forall t \in [0, T]$, $\forall x \in \mathbb{R}_+$,

$$0 \leq \widehat{N}(x,t) \leq N_+(x,t) = N(x,0)e^{-dt} + \frac{D}{d}\frac{x}{m}(1 - e^{-dt}) \max_{s \in [0,T]} n(s) \tag{3.8}$$

Proof. To establish (i), note that $\widehat{I}$ satifies for all $x > 0$

$$\forall t \in [0, T], \;\; \frac{\partial \widehat{I}}{\partial t} = \widehat{I}\left(-\mu - D_I + \widehat{\beta}\widehat{S}\right) + D_I \frac{x}{m} i(t)$$

with $i(t) \geq 0$ so that

$$\forall t \in [0, T], \;\; \frac{\partial \widehat{I}}{\partial t} \geq \widehat{I}\left(-\mu - D_I + \widehat{\beta}\widehat{S}\right) \text{ and } \widehat{I}(.,0) = I_0 \geq 0.$$

It is standard to deduce that $\widehat{I}(.,t) \geq 0$ for all $t \in [0,T]$.

To establish (ii), note that the function $\widehat{S}$ satisfies for all $x > 0$

$$\forall t \in [0,T], \;\; \frac{\partial \widehat{S}}{\partial t} = \widehat{S}\left(-\widehat{\beta}\widehat{I} - D_S\right) + \mu \widehat{I} + D_S \frac{x}{m} s(t). \tag{3.9}$$

with $s(t) \geq 0$ and $\widehat{I} \geq 0$ on $\mathbb{R}_+ \times [0,T]$ by (i). Thus

$$\forall t \in [0,T], \;\; \frac{\partial \widehat{S}}{\partial t} \geq \widehat{S}\left(-\widehat{\beta}\widehat{I} - D_S\right) \text{ with } \widehat{S}(.,0) = S_0 \geq 0.$$

As for (i), it is then standard to deduce that $\widehat{S}(.,t) \geq 0$ for all $t \in [0,T]$.

The proof of (iii) is the same as the one of Proposition 1 (iv).

Finally, note that $\widehat{N}$ satisfies for all $x > 0$

$$\forall t \in [0,T], \;\; \frac{\partial \widehat{N}}{\partial t} = -D_S\left(\widehat{S} - \frac{x}{m} s(t)\right) - D_I\left(\widehat{I} - \frac{x}{m} i(t)\right). \tag{3.10}$$

so that

$$\forall t \in [0,T], \;\; \frac{\partial \widehat{N}}{\partial t} \leq -d\widehat{N} + D\frac{x}{m} n(t) \tag{3.11}$$

for all $x > 0$. A straightforward integration yields that for all $x > 0$

$$\forall t \in [0,T], \;\; 0 \leq \widehat{N}(x,t)e^{dt} \leq N(x,0) + D\frac{x}{m}\int_0^t e^{ds} n(s) ds$$

which yields (iv) and completes the proof of Lemma 3.2. □

Finally note that it follows from the estimates below that for all $x > 0$, $\widehat{I}(x,.)$ and $\widehat{S}(x,.)$ are bounded on $[0,T]$ for any fixed $T > 0$. This implies the existence of a unique global solution $\left(\widehat{S},\widehat{I}\right)$ to system $\left(\widehat{C}\right)$ on $\mathbb{R}_+$.

3.2.2. *Proof of the contraction.* In this section, we consider any fixed $T > 0$ and define the set

$$E = \{(i,n) \in C^0([0,T],\mathbb{R}^2), \; (i(0),n(0)) = (i_0,n_0) \text{ and } \forall t \in [0,T], \, 0 \leq i(t) \leq n(t)\},$$

where $C^0([0,T],\mathbb{R}^2)$ is equipped with the norm $\| (f,g) \| = \sup\limits_{t\in[0,T]} e^{-\lambda t} \left(|f(t)| + |g(t)|\right)$ for a suitable $\lambda > 0$. This set is a closed, convex subset of $C^0([0,T],\mathbb{R}^2)$. The corresponding function $s : [0,T] \to \mathbb{R}_+$ is then defined by $s + i = n$ on $[0,T]$.

Proposition 2. *We define*

$$\phi : \;\; E \longrightarrow C^0([0,T],\mathbb{R}^2)$$
$$(i,n) \longmapsto \left(\widehat{i},\widehat{n}\right),$$

Then $\phi(E) \subset E$ and ϕ is a contraction in E.

Proof. We first show that $\phi(E) \subset E$. Let us consider $(i,n) \in E$. Since $s = n - i$, we have that $s, i \geq 0$ on $[0,T]$. Thus by Lemma 3.2, it follows that $\widehat{i}, \widehat{s} \geq 0$ on $[0,T]$. Therefore $\widehat{n} = \widehat{i} + \widehat{s} \geq \widehat{i} \geq 0$ on $[0,T]$ so that $\left(\widehat{i},\widehat{n}\right) \in E$.

Next, let us consider $(i_1,n_1) \in E$ (resp. $(i_2,n_2) \in E$) and denote by $\left(\widehat{S}_1,\widehat{I}_1\right)$ (resp. $\left(\widehat{S}_2,\widehat{I}_2\right)$) the corresponding solutions of system $\left(\widehat{C}\right)$ and define accordingly for $j = 1, 2$

$$\forall x > 0,\ \ \forall t \in [0, T],\ \ \widehat{N}_j(x,t) = \widehat{S}_j(x,t) + \widehat{I}_j(x,t) \text{ and } \widehat{\beta}_j(x,t) = \frac{\beta_0}{\widehat{N}_j(x,t)}.$$

Define next the functions y and z for all $(x,t) \in \mathbb{R}_+ \times [0,T]$ by

$$y(x,t) = \left(\widehat{I}_2 - \widehat{I}_1\right)(x,t),\ \ z(x,t) = \left(\widehat{N}_2 - \widehat{N}_1\right)(x,t).$$

We establish the following estimates on y and z.

Lemma 3.3. *There exists $C > 0$ such that for all $(x,t) \in \mathbb{R}_+ \times [0,T]$,*

$$\left|\frac{\partial y}{\partial t}\right| + \left|\frac{\partial z}{\partial t}\right| \le C(|y| + |z|) + C\frac{x}{m}(|i_2 - i_1|(t) + |n_2 - n_1|(t)) \tag{3.12}$$

Proof. In a first step, we prove that for all $(x,t) \in \mathbb{R}_+ \times [0,T]$,

$$\left|\frac{\partial y}{\partial t}\right| \le C_1(|y| + |z|) + D_I\frac{x}{m}|i_2 - i_1|(t) \tag{3.13}$$

with $C_1 = 2\beta_0 + \mu + D_I$.

Note that $\widehat{I}_j$ $(j = 1, 2)$ satisfies

$$\frac{\partial \widehat{I}_j}{\partial t} = \widehat{I}_j\left(-\mu + \widehat{\beta}_j\widehat{S}_j\right) - D_I\left(\widehat{I}_j - \frac{x}{m}i_j(t)\right).$$

Thus, if we define Δ by

$$\forall (x,t) \in \mathbb{R}_+ \times [0,T],\ \ \Delta(x,t) = \left(\widehat{\beta}_2\widehat{S}_2\widehat{I}_2 - \widehat{\beta}_1\widehat{S}_1\widehat{I}_1\right)(x,t)$$

then y satisfies on $\mathbb{R}_+ \times [0,T]$

$$\frac{\partial y}{\partial t} = \Delta - (\mu + D_I)y + D_I\frac{x}{m}\left(i_2(t) - i_1(t)\right), \tag{3.14}$$

We omit (x,t) and rewrite

$$\begin{aligned} \Delta &= \frac{\beta_0}{\widehat{N}_1\widehat{N}_2}\left(\widehat{N}_1\widehat{S}_2\widehat{I}_2 - \widehat{N}_2\widehat{S}_1\widehat{I}_1\right) \\ &= \frac{\beta_0}{\widehat{N}_1\widehat{N}_2}\left(\widehat{I}_1\widehat{I}_2(\widehat{S}_2 - \widehat{S}_1) + \widehat{S}_1\widehat{S}_2(\widehat{I}_2 - \widehat{I}_1)\right) \end{aligned}$$

so that, since $0 \le \widehat{I}_j, \widehat{S}_j \le \widehat{N}_j$ for $j = 1, 2$,

$$|\Delta| \le \beta_0\left(|\widehat{S}_2 - \widehat{S}_1| + |\widehat{I}_2 - \widehat{I}_1|\right).$$

Thus, using that $\left|\widehat{S}_2 - \widehat{S}_1\right| \le |y| + |z|$, we deduce from (3.14) that

$$\left|\frac{\partial y}{\partial t}\right| \le (2\beta_0 + \mu + D_I)|y| + \beta_0|z| + D_I\frac{x}{m}|i_2 - i_1|(t)$$

which yields (3.13).

In a second step, we prove a similar differential inequality on $z = \widehat{N}_2 - \widehat{N}_1$ and show that for all $(x,t) \in \mathbb{R}_+ \times [0,T]$,

$$\left|\frac{\partial z}{\partial t}\right| \le D(|y| + |z|) + D\frac{x}{m}(|i_2 - i_1|(t) + |n_2 - n_1|(t)) \tag{3.15}$$

with $D = \max(D_I, D_S)$.

Note that adding up the two equations in system $(\widehat{C})$ shows that $\widehat{N}_j$ $(j = 1, 2)$ satisfies

$$\begin{aligned}\frac{\partial \widehat{N}_j}{\partial t} &= - \quad D_S(\widehat{S}_j - \frac{x}{m}s_j(t)) - D_I(\widehat{I}_j(x,t) - \frac{x}{m}i_j(t)) \\ &= - \quad D_S(\widehat{N}_j - \frac{x}{m}n_j(t)) - (D_I - D_S)(\widehat{I}_j(x,t) - \frac{x}{m}i_j(t)).\end{aligned}$$

By substraction, it follows that

$$\frac{\partial z}{\partial t} = -D_S z - (D_I - D_S)y + D_S\frac{x}{m}(n_2 - n_1)(t) + (D_I - D_S)\frac{x}{m}(i_2 - i_1)(t)$$

which yields (3.15) using that $0 \le D_S, |D_I - D_S| \le D$.

Adding up (3.13) and (3.15), we obtain (3.12) for a suitable $C > 0$, which completes the proof of Lemma 3.3. □

Let us define the functions A and F for $(x,t) \in \mathbb{R}_+ \times [0,T]$ by

$$A(x,t) = C\frac{x}{m}\left[|i_2 - i_1|(t) + |n_2 - n_1|(t)\right]$$

and

$$F(x,t) = \int_0^t \left|\frac{\partial y}{\partial t}(x,\tau)\right| + \left|\frac{\partial z}{\partial t}(x,\tau)\right| \, d\tau$$

so that $A, F \ge 0$ and $F(x,0) = 0$. Since $y(.,0) = z(.,0) = 0$, it follows that

$$\forall (x,t) \in \mathbb{R}_+ \times [0,T], \;\; |y(x,t)| + |z(x,t)| \le F(x,t) \tag{3.16}$$

so that the inequality (3.12) implies that

$$\forall (x,t) \in \mathbb{R}_+ \times [0,T], \;\; \frac{\partial F}{\partial t}(x,t) \le CF(x,t) + A(x,t). \tag{3.17}$$

By Gronwall's lemma, this implies that

$$\forall (x,t) \in \mathbb{R}_+ \times [0,T], \;\; F(x,t) \le \int_0^t e^{C(t-s)}A(x,s)ds.$$

Since for all $s \in \times[0,T]$,

$$|i_2(s) - i_1(s)| + |n_2(s) - n_1(s)| \le e^{\lambda s}\|(i_2 - i_1, n_2 - n_1)\|,$$

it follows that if we choose $\lambda > C$,

$$\forall (x,t) \in \mathbb{R}_+ \times [0,T], \;\; F(x,t) \le C\frac{x}{m}\|(i_2 - i_1, n_2 - n_1)\|\frac{e^{\lambda t} - e^{Ct}}{\lambda - C}$$

which in view of (3.16) implies that for all $(x,t) \in \mathbb{R}_+ \times [0,T]$,

$$|y(x,t)| + |z(x,t)| \le C\frac{x}{m}\|(i_2 - i_1, n_2 - n_1)\|\frac{e^{\lambda t} - e^{Ct}}{\lambda - C} \tag{3.18}$$

Multiply (3.18) by $p(x)$ and integrate on $[0,\infty)$ to obtain that

$$\int_0^\infty |y(x,t)|\, p(x)dx + \int_0^\infty |z(x,t)|\, p(x)dx \le C\|(i_2 - i_1, n_2 - n_1)\|\frac{e^{\lambda t} - e^{Ct}}{\lambda - C}$$

Note that by definition of $\widehat{i}$ and $\widehat{n}$, we have that for all $t \in [0,T]$,

$$\left|\widehat{i}_2(t) - \widehat{i}_1(t)\right| = \left|\int_0^\infty y(x,t)p(x)dx\right| \le \int_0^\infty |y(x,t)|\, p(x)dx$$

and similarly

$$|\widehat{n}_2(t) - \widehat{n}_1(t)| = |\int_0^\infty z(x,t)p(x)dx| \leq \int_0^\infty |z(x,t)|p(x)dx$$

so that for all $t \in [0,T]$,

$$e^{-\lambda t}\left[|\widehat{i}_2(t) - \widehat{i}_1(t)| + |\widehat{n}_2(t) - \widehat{n}_1(t)|\right] \leq C[\| (i_2 - i_1, n_2 - n_1) \| \frac{1 - e^{(C-\lambda)t}}{\lambda - C}.$$

Hence

$$\| \left(\widehat{i}_2 - \widehat{i}_1, \widehat{n}_2 - \widehat{n}_1\right) \| \leq \frac{C}{\lambda - C} \| (i_2 - i_1, n_2 - n_1) \|.$$

We finally choose $\lambda > 2C$ and define $k = \frac{C}{\lambda - C}$ so that $0 < k < 1$ and

$$\| \left(\widehat{i}_2 - \widehat{i}_1, \widehat{n}_2 - \widehat{n}_1\right) \| \leq k \| (i_2 - i_1, n_2 - n_1) \|,$$

which proves that ϕ is a contraction in E. □

In conclusion, we established that ϕ admits a unique fixed point, i.e. a couple of functions $i : [0,T] \to \mathbb{R}_+$ and $n : [0,T] \to \mathbb{R}_+$ with $0 \leq i \leq n$ on $[0,T]$ such that $\widehat{i} = i$ and $\widehat{n} = n$ on $[0,T]$. It follows that $\widehat{s} = \widehat{n} - \widehat{i} = s$ on $[0,T]$. Hence the corresponding function pair $(\widehat{S}, \widehat{I}) = (S, I)$ satisfies system (C) on $[0,T]$. This holds for an arbitrary $T > 0$, thus proving the existence of a unique, global solution to system (C) on $\mathbb{R}_+$, which completes the proof of Theorem 3.1.

4. **Equilibria: Existence and linear stability.** In this section, we are concerned with equilibrium solutions to system (C). Such a solution is given by a couple of nonnegative, C^1 functions

$$(S^*, I^*) : \mathbb{R}_+ \to \mathbb{R}_+^2 \quad \text{such that for all } x > 0,\ \ N^*(x) = S^*(x) + I^*(x) > 0$$

which satisfies system (C^*) on $\mathbb{R}_+^*$

$$(C^*) \begin{cases} I^*(x)\,(\mu - \beta^*(x)S^*(x)) &= D_S\left(S^*(x) - \frac{x}{m}s^*\right) \\ I^*(x)\,(-\mu + \beta^*(x)S^*(x)) &= D_I\left(I^*(x) - \frac{x}{m}i^*\right). \end{cases}$$

with

$$s^* = \int_0^\infty S^*(x)\,p(x)\,dx \text{ and } i^* = \int_0^\infty I^*(x)\,p(x)\,dx, \tag{4.1}$$

where $\beta^*(x) = \dfrac{\beta_0}{N^*(x)}$ for all $x > 0$.

4.1. **DFE and EE.** The disease-free and endemic equilibria are defined as follows.

Definition 4.1. For a given $n_0 > 0$, any equilibrium solution such that $\langle N^*(x)\rangle = n_0$ is of one of the two following types,

- The disease-free equilibrium (DFE), given by

$$\forall x \geq 0,\ \ I^*(x) = 0,\ \ S^*(x) = n_0\frac{x}{m},$$

- An endemic equilibrium (EE), which is a nonnegative solution $(S^*(x), I^*(x))$ of system (C^*) on $\mathbb{R}_+^*$ such that $i^* > 0$, $s^* \geq 0$ and $s^* + i^* = n_0$.

We prove the following result.

Theorem 4.2. *There exists an endemic equilibrium if and only if* $\beta_0 > \mu$.

In this case, for any $n_0 > 0$*, there exists a unique endemic equilibrium such that* $\langle N^*(x)\rangle = n_0$ *given by*

$$I^*(x) = i^* \frac{x}{m},\ \ S^*(x) = s^* \frac{x}{m} \tag{4.2}$$

with

$$i^* = n_0(1 - \frac{\mu}{\beta_0}),\ \ s^* = n_0 \frac{\mu}{\beta_0}. \tag{4.3}$$

Proof. Adding the two lines of system (C^*) shows that any solution satisfies

$$\forall x \geq 0,\ \ D_S S^*(x) + D_I I^*(x) = a\frac{x}{m} \tag{4.4}$$

with $a = D_S s^* + D_I i^*$. Taking the limit at $x = 0$ implies that $S^*(0) = I^*(0) = 0$. Therefore we define the C^0 functions $\sigma : \mathbb{R}_+^* \to \mathbb{R}_+$ and $\tau : \mathbb{R}_+^* \to \mathbb{R}_+$ by

$$\forall x > 0,\ \ S^*(x) = \frac{x}{m}\sigma(x),\ \ \ I^*(x) = \frac{x}{m}\tau(x) \tag{4.5}$$

Substituting (S^*, I^*) in (4.4) and in the second equation of system (C^*) yields the following system (c^*) on $\mathbb{R}_+^*$,

$$(c^*) \left\{ \begin{array}{lcl} D_S\sigma(x) + D_I\tau(x) & = & a \\ \tau(x)(\mu + D_I - \beta_0 \frac{\sigma(x)}{\sigma(x)+\tau(x)}) & = & D_I i^*, \end{array} \right.$$

with

$$s^* = \langle \frac{x}{m}\sigma(x)\rangle,\ \ i^* = \langle \frac{x}{m}\tau(x)\rangle \text{ and } a = D_S s^* + D_I i^*. \tag{4.6}$$

From the first equation in system (c^*), we deduce that

$$\forall x > 0,\ \ \sigma(x) = \frac{a}{D_S} - \frac{D_I}{D_S}\tau(x)$$

with the second equation in system (c^*) reading

$$\tau(x)[(\mu + D_I)\tau(x) + (\mu + D_I - \beta_0)\sigma(x)] = D_I i^*[\sigma(x) + \tau(x)]. \tag{4.7}$$

Thus substituting the above expression of σ reduces equation (4.7) to a quadratic equation with constant coefficients for $\tau(x)$. Therefore the only possible solutions are constant so that there exist $(\sigma, \tau) \in \mathbb{R}_+^2$ such that

$$\forall x > 0,\ \ \sigma(x) = \sigma \text{ and } \tau(x) = \tau.$$

Note that in view of (4.6) and the definition of $m = \langle x\rangle$, it follows that

$$s^* = \langle \frac{x}{m}\sigma(x)\rangle = \sigma\langle \frac{x}{m}\rangle = \sigma \text{ and } i^* = \langle \frac{x}{m}\tau(x)\rangle = \tau.$$

Coming back to equation (4.7), this yields

$$i^*[(\mu + D_I)i^* + (\mu + D_I - \beta_0)s^*] = D_I i^*[s^* + i^*]$$

which, using that $i^* > 0$ and $s^* + i^* = n_0$, simplifies as

$$(\mu + D_I)n_0 - \beta_0 s^* = D_I n_0$$

so that s^* and consequently i^* are given by (4.3). The corresponding (EE) is given by (4.2) under the necessary and sufficient condition $\beta_0 > \mu$. □

Remark 2. Note that in the case of nonlimited transmission, the necessary and sufficient condition of existence of an (EE) reads $n_0 \geq \underline{N}$, where $\underline{N}$ also depends on the structure of the network, through the probability distribution p (see [13]), and not only on the epidemic parameters β_0 and μ as in the case considered here.

4.2. **Linear stability.** We address here the issue of linear stability of the equilibria defined above. The results stated here on the continuous model coincide with the ones obtained by Saldana on the discrete model in [17] and [18].

Let $(S^*(x), I^*(x))$ be an equilibrium solution to system (C), with $s^* = \langle S^* \rangle$, $i^* = \langle I^* \rangle$ and $n_0 = s^* + i^*$. The linearized system around $(S^*(x), I^*(x))$ is given for all $x > 0$ by

$$(L)\left\{\begin{array}{rcl} \dfrac{\partial f}{\partial t} & = & -\beta_0 \left(I^*/N^*\right)^2 f + \Big(\mu - \beta_0 \left(S^*/N^*\right)^2\Big)\, g - D_S\Big(f - \dfrac{x}{m}F(t)\Big) \\ \dfrac{\partial g}{\partial t} & = & \beta_0 \left(I^*/N^*\right)^2 f + \Big(-\mu + \beta_0 \left(S^*/N^*\right)^2\Big)\, g - D_I\Big(g - \dfrac{x}{m}G(t)\Big), \end{array}\right.$$

with initial data $(f(x,0), g(x,0)) = (f_0(x), g_0(x))$ and where we used the notation

$$\forall t \geq 0, \;\; F(t) = \langle f(.,t) \rangle, \;\; G(t) = \langle g(.,t) \rangle. \tag{4.8}$$

We also define the functions $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $H : \mathbb{R}_+ \to \mathbb{R}$ by

$$\forall t \geq 0, \;\; h(.,t) = f(.,t) + g(.,t) \text{ and } H(t) = \langle h(.,t) \rangle. \tag{4.9}$$

We make the assumption that $f_0, g_0 : \mathbb{R}_+ \to \mathbb{R}$ are C^1 with

$$\int_0^\infty \left(|f_0(x)| + |g_0(x)|\right) p(x) dx < \infty \text{ and } \langle f_0 \rangle + \langle g_0 \rangle = 0. \tag{4.10}$$

Note that this last assumption is imposed by the conservation law for the total mean density $n(t) = s(t) + i(t)$ in system (C) and the corresponding following property for system (L).

Lemma 4.3. *Let (f, g) be a smooth solution to the linearized system (L) on $\mathbb{R}_+ \times \mathbb{R}_+$. Then*

$$\forall t > 0, \;\; H(t) = H(0).$$

Proof. Using that for all $t > 0$

$$\langle f(.,t) - \frac{x}{m}F(t) \rangle = \langle g(.,t) - \frac{x}{m}G(t) \rangle = 0$$

and that h satisfies

$$\forall t > 0, \;\; \frac{\partial h}{\partial t} = -D_S\left(f - \frac{x}{m}F(t)\right) - D_I\left(g - \frac{x}{m}G(t)\right)$$

it follows that

$$H'(t) = \frac{d}{dt}\left(\int_0^\infty h(x,t)\, p(x)\, dx\right) = \langle \frac{\partial h(.,t)}{\partial t} \rangle = 0$$

for all $t > 0$ which proves that the function H is constant in time. □

This lemma shows that orbital stability imposes $\lim_{t\to\infty} H(t) = H(0) = 0$ which is condition (4.10). In this case, any solution (f, g) to (L) with (f_0, g_0) satisfying (4.10) has the property that

$$\forall t \geq 0, \;\; H(t) = F(t) + G(t) = 0. \tag{4.11}$$

We'll make use of the auxiliary following result.

Lemma 4.4. *For any $\alpha \in \mathbb{R}$, the solution of the linear nonlocal equation*

$$\frac{\partial \phi}{\partial t} = \alpha\phi - D(\phi - \frac{x}{m}\langle \phi(.,t) \rangle), \;\; x > 0, \;\; t > 0 \tag{4.12}$$

is given for all $x > 0$ and $t > 0$ by

$$\phi(x,t) = (\phi(x,0) - \frac{x}{m}\langle\phi(.,0)\rangle)e^{(\alpha-D)t} + \frac{x}{m}\langle\phi(.,0)\rangle e^{\alpha t}. \tag{4.13}$$

Moreover,

$$\forall t > 0,\ \ \langle\phi(.,t)\rangle = \langle\phi(.,0)\rangle e^{\alpha t}. \tag{4.14}$$

Proof. Let us define for all $x > 0$ and $t > 0$

$$\Phi(t) = \langle\phi(.,t)\rangle \text{ and } \tilde{\phi}(x,t) = \phi(x,t) - \frac{x}{m}\Phi(t).$$

Multiplying (4.12) by $p(x)$ and integrating on $\mathbb{R}_+$ shows that $\Phi'(t) = \alpha\Phi(t)$ for all $t > 0$ which yields (4.14). Next we deduce from equation (4.12) that $\frac{\partial\tilde{\phi}}{\partial t} = (\alpha - D)\tilde{\phi}$ so that

$$\forall x > 0, \forall t > 0,\ \ \tilde{\phi}(x,t) = (\phi(x,0) - \frac{x}{m}\langle\phi(.,0)\rangle)e^{(\alpha-D)t}$$

which implies that $\phi(x,t) = \tilde{\phi}(x,t) + \frac{x}{m}\Phi(t)$ is given by (4.13) in view of (4.14). □

We investigate below the linear stability of the (DFE) and of the (EE). To this end, let us define for all $x > 0$ and $t > 0$

$$\tilde{f}(x,t) = f(x,t) - \frac{x}{m}F(t) \text{ and } \tilde{g}(x,t) = g(x,t) - \frac{x}{m}G(t). \tag{4.15}$$

4.2.1. *Disease free equilibrium.* We consider the above linearized system (L^0) in the case that $I^*(x) = 0$ and $S^*(x) = N^*(x) = n_0\frac{x}{m}$, namely

$$(L^0)\begin{cases} \dfrac{\partial f}{\partial t} &= (\mu - \beta_0)g - D_S\Big(f - \dfrac{x}{m}\langle f(.,t)\rangle\Big) \\ \dfrac{\partial g}{\partial t} &= (-\mu + \beta_0)g - D_I\Big(g - \dfrac{x}{m}\langle g(.,t)\rangle\Big) \end{cases}$$

and establish the following result.

Proposition 3. *Let (f_0, g_0) satisfy (4.10) with $\langle f_0\rangle \neq 0$. Then*

$$\lim_{t\to\infty} f(x,t) = \lim_{t\to\infty} g(x,t) = 0 \text{ if and only if } \beta_0 < \mu.$$

Thus the disease free equilibrium is linearly stable if and only if there is no endemic equilibrium.

Proof. The function g satisfies

$$\frac{\partial g}{\partial t} = (-\mu + \beta_0)g - D_I(g - \frac{x}{m}\langle g(.,t)\rangle). \tag{4.16}$$

Using Lemma 4.4 with $\alpha = \beta_0 - \mu$ and the expression (4.13), we have that for all $x > 0$ and $t > 0$,

$$g(x,t) = (g_0(x) - \frac{x}{m}\langle g_0\rangle)e^{(\beta_0-\mu-D_I)t} + \frac{x}{m}\langle g_0\rangle e^{(\beta_0-\mu)t}.$$

Consequently, we have

$$[\forall x \geq 0,\ \ \lim_{t\to\infty} g(x,t) = 0] \text{ if and only if } \beta_0 - \mu < 0.$$

Since $F(t) + G(t) = 0$ by (4.11), it follows in view of (4.10) and (4.14) that

$$\forall t > 0,\ \ F(t) = \langle f_0\rangle e^{(\beta_0-\mu)t}.$$

The first equation of (L^0) yields

$$\begin{aligned}\frac{\partial \tilde f}{\partial t} &= -D_S\tilde f + (\mu-\beta_0)\tilde g\\ &= -D_S\tilde f + (\mu-\beta_0)(g_0(x) - \frac{x}{m}\langle g_0\rangle)e^{(\beta_0-\mu-D_I)t}\end{aligned}$$

If $\beta_0 - \mu < 0$, a straightforward computation shows therefore that $\lim_{t\to\infty} \tilde f(x,t) = 0$ which in view of the above expression of $F(t)$ shows that

$$\forall x \geq 0, \quad \lim_{t\to\infty} f(x,t) = 0.$$

□

4.2.2. *Endemic equilibrium.* We consider here the linearized system (L^*) around the endemic equilibrium $(S^*(x), I^*(x))$ given by (4.2), which reads as follows.

$$(L^*)\left\{\begin{aligned}\frac{\partial f}{\partial t} &= -\beta_0(1-\tfrac{\mu}{\beta_0})^2 f + \mu(1-\tfrac{\mu}{\beta_0})g - D_S(f - \frac{x}{m}F(t))\\ \frac{\partial g}{\partial t} &= \beta_0(1-\tfrac{\mu}{\beta_0})^2 f - \mu(1-\tfrac{\mu}{\beta_0})g - D_I(g - \frac{x}{m}G(t)).\end{aligned}\right.$$

We establish the following result.

Proposition 4. *Let (f_0, g_0) satisfy (4.10) with $\langle f_0\rangle \neq 0$. Then*

$$\lim_{t\to\infty} f(x,t) = \lim_{t\to\infty} g(x,t) = 0 \textit{ if and only if } \beta_0 > \mu.$$

Thus as soon as there exists an endemic equilibrium, it is linearly stable

Proof. Replacing $f = h - g$ and using that $F(t) + G(t) = 0$ for all $t > 0$, we obtain the equivalent linear system (E^*) for the functions (g, h),

$$(E^*)\left\{\begin{aligned}\frac{\partial g}{\partial t} &= (\mu-\beta_0)g + \frac{(\beta_0-\mu)^2}{\beta_0}h - D_I(g - \frac{x}{m}G(t))\\ \frac{\partial h}{\partial t} &= -D_S h - (D_I - D_S)(g - \frac{x}{m}G(t)).\end{aligned}\right.$$

Let us multiply the first equation of the system (E^*) by $p(x)$ and integrate on R_+. Since $H(t) = \langle h(.,t)\rangle = 0$, we obtain that $G'(t) = (-\beta_0+\mu)G(t)$ so that $G(t) = \langle g_0\rangle e^{(\mu-\beta_0)t}$. Thus $G \to 0$ when $t\to\infty$ if and only if $-\beta_0+\mu < 0$. Assuming that this necessary condition of stability is satisfied, let us consider the equivalent system $(\tilde E)$ for the functions $(\tilde g, h)$,

$$(\tilde E)\left\{\begin{aligned}\frac{\partial \tilde g}{\partial t} &= (\mu-\beta_0-D_I)\tilde g + \tfrac{(\beta_0-\mu)^2}{\beta_0}h,\\ \frac{\partial h}{\partial t} &= -D_S h - (D_I - D_S)\tilde g.\end{aligned}\right.$$

Hence $(\tilde E)$ is an homogeneous linear system of the form $Y' = AY$, with $Y = (\tilde g, h)$ and the constant matrix A given by

$$A = \begin{pmatrix} \mu-\beta_0-D_I & \dfrac{(\beta_0-\mu)^2}{\beta_0}\\ D_S - D_I & -D_S\end{pmatrix}$$

Note that since $\mu - \beta_0 < 0$, $\mathrm{Tr}(A) = \mu - \beta_0 - D_I - D_S < 0$ and

$$\det(A) = D_S\mu\frac{(\beta_0-\mu)}{\beta_0} + D_I D_S + D_I\frac{(\beta_0-\mu)^2}{\beta_0} > 0,$$

which implies that both eigenvalues of A are strictly negative. Thus for all $x \geq 0$, $h(x,t), \tilde{g}(x,t) \to 0$ when $t \to \infty$. This implies that $g(x,t) = \tilde{g}(x,t) + \frac{x}{m}G(t) \to 0$ and $f(x,t) = h(x,t) - g(x,t) \to 0$ when $t \to \infty$ at an exponential rate for x bounded in $\mathbb{R}_+$. □

5. **Large time asymptotic behaviour.** Confirming the linear stability results proved in section 4, we show below that the (DFE) is globally asymptotically stable as long as the (EE) does not exist and that if the (EE) does exist, it is globally asymptotically stable in the case $D_S = D_I$.

Precisely, we consider any initial data (S_0, I_0) satisfying the assumptions 1 and 2 and denote by $n_0 = \langle S_0 \rangle + \langle I_0 \rangle$ the total mean density. We make the additional assumption that $\frac{N(x,0)}{x}$ is bounded so that there exists $k \geq 1$ such that

$$\forall x \geq 0, \;\; 0 \leq N(x,0) \leq k n_0 \frac{x}{m}. \tag{5.1}$$

Let (S, I) be the corresponding solution to system (C) on $\mathbb{R}_+ \times \mathbb{R}_+$. We first show in the next Lemma that it is sufficient to obtain the limiting behavior of $I(x,t)$ as $t \to \infty$.

Lemma 5.1. *Assume that the initial data (S_0, I_0) satisfy assumptions 1 and 2 and (5.1). Then the following properties hold.*

1. *There exists $C \geq 1$ such that for all $x \geq 0$ and for all $t \geq 0$,*

$$0 \leq N(x,t) \leq C n_0 \frac{x}{m}. \tag{5.2}$$

2. *For any fixed $x \geq 0$, we have that*

$$\lim_{t\to\infty} (I(x,t) - \frac{x}{m} i(t)) = 0 \quad \Rightarrow \lim_{t\to\infty} (N(x,t) - n_0 \frac{x}{m}) = 0$$
$$\Rightarrow \lim_{t\to\infty} (S(x,t) - \frac{x}{m} s(t)) = 0$$

Proof. 1. Note that it follows from (3.8) applied to $N = \widehat{N}$ that for all $x, t > 0$,

$$0 \leq N(x,t) \leq N(x,0)e^{-dt} + n_0 \frac{D}{d} \frac{x}{m} (1 - e^{-dt}), \tag{5.3}$$

with d and D defined in (3.7). Thus if $C = \max(k, \frac{D}{d})$, using (5.1), we have that

$$0 \leq N(x,t) \leq \max(N(x,0), n_0 \frac{D}{d} \frac{x}{m}) \leq C n_0 \frac{x}{m},$$

which proves (5.2).

2. Assume that for some fixed $x \geq 0$, $\lim_{t\to\infty} (I(x,t) - \frac{x}{m} i(t)) = 0$ so that for all $\epsilon > 0$, there exists $A > 0$ such that

$$\forall t \geq A, \;\; |I(x,t) - \frac{x}{m} i(t)| \leq \epsilon. \tag{5.4}$$

Note that N satisfies

$$\frac{\partial N}{\partial t} = -D_S (S - \frac{x}{m} s(t)) - D_I (I - \frac{x}{m} i(t)).$$

We substitute $S(x,t) = (N - I)(x,t)$ and $s(t) = n_0 - i(t)$ and rewrite this equation as

$$\frac{\partial N}{\partial t} = -D_S (N - n_0 \frac{x}{m}) - (D_I - D_S)(I - \frac{x}{m} i(t)). \tag{5.5}$$

By integration on $[t, 2t]$ after multiplication by $e^{D_S t}$, we obtain that

$$e^{2D_S t}(N(x,2t) - n_0 \tfrac{x}{m}) = e^{D_S t}(N(x,t) - n_0 \tfrac{x}{m}) \\ +(D_S - D_I)\int_t^{2t} e^{D_S \tau}(I(x,\tau) - \tfrac{x}{m} i(\tau))d\tau.$$

Using (5.2), there exists $\tilde{C} > 0$ such that

$$\forall x \geq 0, \;\; |N(x,t) - n_0 \frac{x}{m}| \leq \tilde{C} n_0 \frac{x}{m}$$

and using (5.4), we have that for all $t \geq A$,

$$\left| \int_t^{2t} e^{D_S \tau}(I(x,\tau) - \frac{x}{m} i(\tau))d\tau \right| \leq \frac{\epsilon}{D_S} e^{2D_S t}$$

Hence

$$|N(x,2t) - n_0 \frac{x}{m}| \leq \tilde{C} e^{-D_S t} n_0 \frac{x}{m} + \epsilon \left| 1 - \frac{D_I}{D_S} \right|$$

which proves that

$$\lim_{t\to\infty} \left(N(x,t) - n_0 \frac{x}{m} \right) = 0$$

and consequently

$$\lim_{t\to\infty} \left(S(x,t) - s(t) \frac{x}{m} \right) = 0.$$

□

Based on this lemma, in the sequel we only consider the asymptotic behavior of $I(x,t)$ at $t \to \infty$. The equation for I reads

$$\frac{\partial I}{\partial t} = I(-\mu + \beta S) - D_I(I - \frac{x}{m} i(t))$$

with $i(t) = \langle I(.,t) \rangle$. Since $\beta = \dfrac{\beta_0}{N(x,t)}$ and $S(x,t) = N(x,t) - I(x,t)$, it is equivalent to

$$\frac{\partial I}{\partial t} = I(-\mu + \beta_0 - \frac{\beta_0}{N(x,t)} I) - D_I(I - \frac{x}{m} i(t)). \tag{5.6}$$

Integrating this equation on $\mathbb{R}_+$ after multiplication by $p(x)$ shows that

$$\forall t > 0, \;\; i'(t) = (-\mu + \beta_0) i(t) - \int_0^\infty \frac{\beta_0}{N(x,t)} I^2(x,t) p(x) dx. \tag{5.7}$$

In the sequel, we distinguish the two cases $0 < \beta_0 \leq \mu$ and $\beta_0 > \mu$.

5.1. **Convergence to the (DFE).** We assume here that $0 < \beta_0 \leq \mu$ and establish the following result.

Proposition 5. *For all* $x \geq 0$,

$$\lim_{t\to\infty} I(x,t) = 0, \;\; \lim_{t\to\infty} S(x,t) = n_0 \frac{x}{m}. \tag{5.8}$$

Proof. Since $I(x,t), N(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, it follows from (5.7) that

$$\forall t \geq 0, \;\; 0 \leq i(t) \leq i_+(t),$$

where $i_+(t) = i(0)e^{(\beta_0 - \mu)t}$ is the solution of $i'_+(t) = (-\mu + \beta_0) i_+(t)$ with $i_+(0) = i(0)$. Thus in view of (5.6),

$$\forall t \geq 0, \;\; \forall x \geq 0, \;\; 0 \leq I(x,t) \leq I_+(x,t),$$

where $I_+(x,t)$ is the solution of

$$\frac{\partial I_+}{\partial t} = I_+(\beta_0 - \mu - D_I) + D_I \frac{x}{m} i(0) e^{(\beta_0-\mu)t}. \tag{5.9}$$

with $I_+(x,0) = I_0(x)$. A straightforward computation yields

$$\forall (x,t) > 0,\ I_+(x,t) = (I_0(x) - i(0)\frac{x}{m})e^{(\beta_0-\mu-D_I)t} + i(0)\frac{x}{m}e^{(\beta_0-\mu)t}.$$

Consequently, if $\beta_0 - \mu < 0$, it follows that for all $x \geq 0$

$$\lim_{t\to\infty} I(x,t) = \lim_{t\to\infty} I_+(x,t) = 0$$

at an exponential convergence rate $\geq \beta_0 - \mu$.

In the limit case $\beta_0 = \mu$, the proof is modified as follows. First note that (5.6) reads in this case

$$\frac{\partial I}{\partial t} = -\frac{\beta_0}{N(x,t)} I^2 - D_I[I - \frac{x}{m} i(t)] \tag{5.10}$$

and in view of (5.7), the function i satisfies

$$\forall t > 0,\ i'(t) = -\int_0^\infty \frac{\beta_0}{N(x,t)} I^2(x,t) p(x) dx.$$

By Schwarz's inequality,

$$\int_0^\infty I(x,t)p(x)dx \leq \left(\int_0^\infty \frac{I^2(x,t)}{N(x,t)} p(x)dx\right)^{1/2} \left(\int_0^\infty N(x,t)p(x)dx\right)^{1/2}$$

or equivalently, since $i(t) \geq 0$ and $n(t) = \langle N(.,t)\rangle = n_0$,

$$\forall t > 0,\ i(t)^2 \leq n_0 \left(\int_0^\infty \frac{I^2(x,t)}{N(x,t)} p(x)dx\right). \tag{5.11}$$

it follows that

$$\forall t > 0,\ i'(t) \leq -\frac{\beta_0}{n_0} i(t)^2 < 0 \tag{5.12}$$

which proves that

$$\forall t > 0,\ 0 \leq i(t) \leq \frac{n_0}{\beta_0 t + \frac{n_0}{i_0}} \text{ and } \lim_{t\to\infty} \searrow i(t) = 0. \tag{5.13}$$

Next it follows from (5.10) that

$$\forall t \geq 0,\quad \forall x \geq 0,\ 0 \leq I(x,t) \leq I_+(x,t),$$

where $I_+(x,t)$ is the solution of

$$\frac{\partial I_+}{\partial t} = -D_I I_+ + D_I \frac{x}{m} i(t) \tag{5.14}$$

with $I_+(x,0) = I_0(x)$. Thus

$$(e^{D_I t} I_+(x,t))_t = D_I e^{D_I t} \frac{x}{m} i(t).$$

After integration on $[t, 2t]$ with $t > 0$ for any fixed $x \geq 0$, we use (5.13) and obtain that

$$e^{2D_I t} I_+(x,2t) = e^{D_I t} I_+(x,t) + \frac{x}{m}\int_t^{2t} D_I e^{D_I s} i(s) ds$$

$$\leq e^{D_I t} I_+(x,t) + \frac{x}{m} i(t) \int_t^{2t} D_I e^{D_I s} ds \leq e^{D_I t} I_+(x,t) + \frac{x}{m} i(t) e^{2D_I t}$$

which proves that for all $x \geq 0$

$$0 \leq I_+(x, 2t) \leq e^{-D_I t} I_+(x,t) + \frac{x}{m} i(t).$$

Since $I_+(x,t)$ is bounded by $I_0(x) + D_I \frac{x}{m} i_0 t$, this implies that

$$\lim_{t \to \infty} I(x,t) = 0 \tag{5.15}$$

for all $x \geq 0$, with a convergence rate smaller than $\frac{1}{t}$ by (5.13).

From property 2 in Lemma 5.1, we then obtain the limit of $S(.,t)$ as $t \to \infty$ stated in (5.8) which completes the proof of Proposition 5. □

5.2. **Convergence to the (EE).** We assume here that $\beta_0 > \mu$ and establish the following result.

Proposition 6. *Assume that $D_I = D_S$ and suppose that (5.1) holds. Then for all $x \geq 0$,*

$$\lim_{t \to \infty} I(x,t) = I^*(x) = i^* \frac{x}{m} \tag{5.16}$$

$$\lim_{t \to \infty} S(x,t) = S^*(x) = s^* \frac{x}{m}, \tag{5.17}$$

with (i^*, s^*) given in (4.3).

Proof. Since $I(0,t) \equiv 0$ for all $t \geq 0$, we make a change of functions and define the C^1 function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\forall x \in \mathbb{R}_+, \ \ \forall t \geq 0, \ \ I(x,t) = i^* \frac{x}{m} u(x,t). \tag{5.18}$$

Similarly since $N(0,t) \equiv 0$ for all $t \geq 0$, we define a C^1 function $w : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\forall x \in \mathbb{R}_+, \ \ \forall t \geq 0, \ \ N(x,t) = n_0 \frac{x}{m} w(x,t). \tag{5.19}$$

Let us also consider the distribution probability q defined on $\mathbb{R}_+$ by

$$\forall x \geq 0, \ \ q(x) = \frac{x p(x)}{m}$$

and associate to any function $v : \mathbb{R}_+ \to \mathbb{R}$ with $v \in L^1(\mathbb{R}_+, q(x)dx)$ its mean value with respect to q denoted by

$$\langle v \rangle_q = \int_0^\infty v(x) q(x) dx. \tag{5.20}$$

In view of (5.6) and of (4.3), the function u satisfies

$$\frac{\partial u}{\partial t} = (\beta_0 - \mu) u \left(1 - \frac{u}{w}\right) - D_I (u - \langle u(.,t) \rangle_q), \ \ t > 0, \ \ x \geq 0. \tag{5.21}$$

We now restrict to the case $D_I = D_S$. In this case, equation (5.5) reduces to

$$\forall x > 0, \ \ \forall t > 0, \ \ \frac{\partial N}{\partial t} = -D_I (N - \frac{x}{m} \langle N(.,t) \rangle) = -D_I (N - n_0 \frac{x}{m}).$$

Hence we obtain that for all $x > 0$ and $t > 0$,

$$N(x,t) = (N(x,0) - n_0 \frac{x}{m}) e^{-D_I t} + n_0 \frac{x}{m}. \tag{5.22}$$

Consequently by definition of w in (5.19), we have that

$$\forall x > 0,\ \ \forall t \geq 0,\ \ \ w(x,t) = \left(\frac{N(x,0)}{n_0 \frac{x}{m}} - 1 \right) e^{-D_I t} + 1,$$

so that in view of (5.1),

$$\forall x \in \mathbb{R}_+,\ \ |w(x,t) - 1| \leq (k+1) e^{-D_I t}. \tag{5.23}$$

Thus for any fixed $\epsilon \in (0,1)$, there exists $T^\varepsilon > 0$ such that

$$\forall t \geq T^\varepsilon,\ \ \forall x \in \mathbb{R}_+,\ \ \frac{1}{1+\varepsilon}\ \leq\ w(x,t)\ \leq \frac{1}{1-\varepsilon}. \tag{5.24}$$

In the sequel, we prove that

$$\forall x \in \mathbb{R}_+,\ \ \lim_{t\to\infty} u(x,t) = 1 \tag{5.25}$$

which is equivalent to (5.16). To establish this result, we study below the properties of (5.21). This leads us to consider the following nonlocal initial-value problem

$$(NL) \begin{cases} \dfrac{\partial f}{\partial t} & = & -D(f - \langle f \rangle_q) + \phi(f), & x \geq 0, t \in (0,T] \\ f(x,0) & = & f_0(x), & x \geq 0, \end{cases}$$

where $f_0 \in L^1(\mathbb{R}_+, q(x)dx)$. Here we assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $\phi(0) = 0$. □

5.2.1. *Comparison principle.* For any $T > 0$, let us define

$$X_T = \{v \in C^1(\mathbb{R}_+ \times [0,T], \mathbb{R}),\ \ \ v(.,t) \in L^1(\mathbb{R}_+, q(x)dx) \text{ for all } t \in [0,T]\}$$

By definition, a solution f to (NL) on $\mathbb{R}_+ \times [0,T]$ belongs to X_T. We establish a comparison principle for Problem (NL).

Theorem 5.2. *For any $v \in X_T$, define the function $\mathcal{N}[v] : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ by*

$$\mathcal{N}[v] := \frac{\partial v}{\partial t} + D(v - \langle v(.,t) \rangle_q) - \phi(v),$$

(i) Assume that $f \in X_T$ and satisfies

$$\forall x \geq 0,\ \ \forall t \in (0,T],\ \ \mathcal{N}[f](x,t) \geq 0 \tag{5.26}$$

$$\forall x \geq 0,\ \ \ f(x,0) \geq 0 \text{ with } \langle f(.,0) \rangle_q > 0. \tag{5.27}$$

Then

$$\forall x \geq 0,\ \ \forall t \in (0,T],\ \ f(x,t) > 0.$$

(ii) Consider two functions $(f^-, f^+) \in X_T^2$ such that for all $x \geq 0$,

$$\forall t \in [0,T],\ \ \mathcal{N}[f^-](x,t) \leq \mathcal{N}[f^+](x,t) \tag{5.28}$$

$$f^-(x,0) \leq f^+(x,0) \text{ with } \langle f^-(.,0) \rangle_q < \langle f^+(.,0) \rangle_q \tag{5.29}$$

Then

$$\forall x \geq 0,\ \ \forall t \in (0,T],\ \ \ f^-(x,t) < f^+(x,t).$$

Proof. It relies on the following strong positivity principle in the linear case.

Lemma 5.3. *Let $c : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ be a continous function, with $T > 0$. For any $v \in X_T$, define the function $\mathcal{L}[v] : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ by*

$$\mathcal{L}[v] = \frac{\partial v}{\partial t} + D(v - \langle v(.,t)\rangle_q) - c(x,t)v,$$

for all $x \geq 0$ and $t \in [0,T]$. Assume that $f \in X_T$ satisfies (5.27) and that

$$\forall x \geq 0, \;\; \forall t \in (0,T], \;\; \mathcal{L}[f](x,t) \geq 0. \tag{5.30}$$

Then for all $x \geq 0$ and $t \in (0,T]$, $f(x,t) > 0$.

Proof. Let us define $K(x,t) = \int_0^t (D - c(x,s))ds$ for all $x \geq 0$ and for all $t \in [0,T]$. It follows from (5.30) that for all $x \geq 0$ and for all $t \in (0,T]$,

$$\frac{\partial}{\partial t}[e^{K(x,t)} f(x,t)] \geq De^{K(x,t)} \langle f(.,t)\rangle_q. \tag{5.31}$$

We consider the set

$$E = \{t \in [0,T], \;\forall s \in [0,t], \;\; \langle f(.,s)\rangle_q > 0\}.$$

Since $\langle f(.,0)\rangle_q > 0$ by (5.27), there exists $\varepsilon_0 \in (0,T)$ such that $[0,\varepsilon_0] \subset E$. Let us define $T^* = \sup(E)$ so that $T^* \geq \varepsilon_0 > 0$. In view of (5.31), we have that

$$\forall t \in (0,T^*), \;\; \frac{\partial}{\partial t}[e^{K(x,t)} f(x,t)] > 0,$$

so that the function $t \to e^{K(x,t)} f(x,t)$ is strictly increasing on $[0,T^*)$. Thus for all $(x,t) \in \mathbb{R}_+ \times (0,T^*)$, $f(x,t) > 0$. Moreover,

$$\forall (x,t) \in \mathbb{R}_+ \times [\varepsilon_0,T^*), \;\; e^{K(x,t)} f(x,t) \geq e^{K(x,\varepsilon_0)} f(x,\varepsilon_0) > 0$$

Letting $t \to T^*$, this implies that $f(x,T^*) > 0$ for all $x \in \mathbb{R}_+$. Hence $\langle f(.,T^*)\rangle_q > 0$ which proves that $T^* = T$. Thus $f(x,t) > 0$ for all $t > 0$ and $x \in \mathbb{R}_+$. □

We now complete the proof of Theorem 5.2.

(i) Since $\phi(0) = 0$, we define a continuous function $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$\forall v \in \mathbb{R}_+, \;\; \phi(v) = vg(v).$$

Thus in view of (5.26), f satisfies (5.30) with $c(x,t) = g(f(x,t))$. Using (5.27), we conclude from Lemma 5.3 that $\forall x \geq 0$ and $\forall t > 0$, $f(x,t) > 0$.

(ii) Note that it follows from (5.28) that the function $f = f^+ - f^-$ satisfies

$$\forall x \geq 0, \;\; \forall t \in (0,T], \;\; \frac{\partial f}{\partial t} + D(f - \langle f\rangle_q) - c(x,t)f \geq 0$$

where

$$c(x,t) = \begin{cases} \dfrac{\phi(f^+(x,t)) - \phi(f^-(x,t))}{f(x,t)} & \text{if } f(x,t) \neq 0 \\ \phi'(f^-(x,t)) & \text{if } f(x,t) = 0. \end{cases}$$

In view of (5.29), the conclusion follows again from Lemma 5.3.

□

5.2.2. *Construction of sub-supersolutions to the nonlocal Fisher equation.* In the sequel, we consider the particular case of a Fisher-type, monostable nonlinearity. Namely, let $\phi : [0,1] \to \mathbb{R}$ be a C^1 function such that

$$\phi(0) = \phi(1) = 0 \text{ with } \phi'(0) > 0, \;\; \phi'(1) < 0, \tag{5.32}$$

a typical example of which given by

$$\forall u \in [0,1], \;\; \phi(u) = \lambda u(1-u) \tag{5.33}$$

with $\lambda > 0$. Note that equation (5.21) is of this type if w is replaced by 1. We first extend ϕ to a C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\phi < 0 \text{ on } (-\infty, 0) \cup (1, \infty), \;\; \phi > 0 \text{ on } (0,1) \tag{5.34}$$

$$\exists C_0 > 0 \text{ such that for all } f \in \mathbb{R}, \;\; \phi(f) \le C_0(1-f) \tag{5.35}$$

Next we establish the following result which lies at the core of the proof.

Proposition 7. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^1 function satisfying (5.32)-(5.34)-(5.35). Let $f_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, nonnegative function with $\langle f_0 \rangle_q > 0$.*

Let f be the solution of (NL) on $\mathbb{R}_+ \times \mathbb{R}_+$. Then

$$\forall x \ge 0, \;\; \lim_{t\to\infty} f(x,t) = 1$$

Proof. In view of (5.32), we have the following properties.

$$\exists C_1 > 0 \text{ such that for all } f \in [0,1/2], \;\; \phi(f) \ge C_1 f, \tag{5.36}$$

$$\exists C_2 > 0 \text{ such that for all } f \in [0,1], \;\; 0 \le \phi(f) \le C_2 f, \tag{5.37}$$

It follows from Theorem 5.2 (i) that

$$\forall x \ge 0, \;\; \forall t > 0, \;\; f(x,t) > 0. \tag{5.38}$$

Let us denote by $i(t) = \langle f(.,t) \rangle_q$ the average value of f so that $i(t) > 0$ for all $t > 0$. We establish the proof in two steps.

Step A. We prove that

$$\forall x \ge 0, \;\; \liminf_{t\to\infty} f(x,t) \ge 1. \tag{5.39}$$

The proof relies on the construction of 2 successive subsolutions to (NL). The first subsolution guarantees that $f(x,t)$ is bounded from below by a strictly positive constant uniformly on $\mathbb{R}_+$ at some time $T > 0$. After time T, the solution of the ODE with a strictly positive initial data provides a second subsolution that converges to 1 as $t \to \infty$.

Remark 3. Note that since $0 \le u \le w$, it follows from (5.23) that the function u is bounded. In this case, a strictly positive lower bound for u can be obtained using the standard comparison principle for ODE. In contrast, the proof below does not require the function $f(x,t)$ to be bounded in x.

We first choose an integer $n \in \mathbb{N}$ large enough so that

$$n > D \text{ and } n > 2 + 2\frac{D}{C_1} \tag{5.40}$$

and choose $A_n > 0$ large enough so that

$$\int_0^{A_n/2} q(x)dx \ge 1 - \frac{1}{n}. \tag{5.41}$$

In view of (5.38), $f(x,1) > 0$ for all $x \geq 0$ so that we can define $m_n > 0$ by

$$m_n = \min_{x \in [0,A_n]} f(x,1) > 0$$

and choose $\delta_n > 0$ small enough so that

$$0 < \delta_n < \min(m_n, \frac{1}{8}). \tag{5.42}$$

Next we consider a smooth function $\psi_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\psi_n \in C^\infty(\mathbb{R}_+), \quad \psi_n \geq 0 \text{ and } \psi_n' \leq 0 \text{ on } \mathbb{R}_+ \tag{5.43}$$

$$\forall x \in [0, A_n/2], \quad \psi_n(x) = \delta_n \text{ and } \forall x \geq A_n, \quad \psi_n(x) = 0 \tag{5.44}$$

so that by construction using (5.41) we have that

$$\delta_n(1 - \frac{1}{n}) \leq \delta_n \int_0^{A_n/2} q(x)dx \leq \langle \psi_n \rangle_q \leq \delta_n. \tag{5.45}$$

Moreover in view of the definition of δ_n in (5.42), we have that

$$\forall x \geq 0, \quad f(x,1) > \psi_n(x). \tag{5.46}$$

Next we prove the following lemma.

Lemma 5.4. *Define $T_n = \frac{n}{D}\frac{1}{4\delta_n}$, then $T_n > 2$ and*

$$\forall x \geq 0, \forall t \in [1, T_n], \quad f(x,t) \geq \psi_n(x) + D\frac{\delta_n}{n}(t-1). \tag{5.47}$$

Proof. Note that $T_n > 2\frac{n}{D} > 2$ by (5.40)-(5.42) and let us define the function $f_- : \mathbb{R}_+ \times [1, T_n] \to \mathbb{R}_+$ by

$$\forall t \in [1, T_n], \quad \forall x \geq 0, \quad f_-(x,t) = \psi_n(x) + D\frac{\delta_n}{n}(t-1).$$

We prove below that

$$\forall x \geq 0, \forall t \in [1, T_n], \quad \mathcal{N}[f_-](x,t) \leq 0. \tag{5.48}$$

In view of the definition of ψ_n and of (5.42), we have that

$$\forall x \geq 0, \quad 0 \leq \psi_n(x) \leq \delta_n < 1/8, \tag{5.49}$$

so that for all $t \in [1, T_n]$ and for all $x \geq 0$,

$$0 \leq f_-(x,t) \leq \delta_n + D\frac{\delta_n}{n}T_n \leq \delta_n + \frac{1}{4} \leq \frac{3}{8} < \frac{1}{2} \tag{5.50}$$

which shows that

$$\forall t \in [1, T_n], \quad \forall x \geq 0, \quad \phi(f_-(x,t)) \geq 0. \tag{5.51}$$

We have that for all $(x,t) \in \mathbb{R}_+ \times [1, T_n]$,

$$\mathcal{N}[f_-](x,t) = D\frac{\delta_n}{n} + D(\psi_n(x) - \langle \psi_n \rangle_q) - \phi(f_-(x,t)).$$

We distinguish two cases depending on the values taken by $\psi_n(x)$.

1st case $0 \leq \psi_n(x) \leq \delta_n(1 - \frac{2}{n})$. Using (5.45) and (5.51), we have that

$$\begin{aligned} \mathcal{N}[f_-](x,t) &\leq D\frac{\delta_n}{n} + D\psi_n(x) - D\langle \psi_n \rangle_q - \phi(f_-(x,t)) \\ &\leq D\frac{\delta_n}{n} + D\delta_n(1 - \frac{2}{n}) - D\delta_n(1 - \frac{1}{n}) = 0. \end{aligned}$$

2nd case $\delta_n(1-\frac{2}{n}) \le \psi_n(x) \le \delta_n$. Using (5.36) and (5.50), we have that

$$\forall x \ge 0, \forall t \in [1, T_n], \;\; \phi(f_-(x,t)) \ge C_1 f_-(x,t) \ge C_1 \psi_n(x)$$

so that in view of (5.45)

$$\begin{aligned}
\mathcal{N}[f_-](x,t) &\le D\frac{\delta_n}{n} + D\psi_n(x) - D\langle \psi_n \rangle_q - C_1\psi_n(x) \\
&\le D\frac{\delta_n}{n} + D\delta_n - D\delta_n(1-\frac{1}{n}) - C_1\psi_n(x) \\
&\le 2D\frac{\delta_n}{n} - C_1\delta_n(1-\frac{2}{n}) \\
&\le \frac{\delta_n}{n}(2D - nC_1 + 2C_1) < 0
\end{aligned}$$

by choice of n in (5.40).

Thus we have established inequality (5.48). Using (5.46), we conclude from the comparison principle stated in Theorem 5.2 that

$$\forall t \in [1, T_n], \;\; \forall x \ge 0, \;\; f(x,t) \ge f_-(x,t)$$

which is inequality (5.47) and completes the proof of Lemma 5.4. □

Since $\psi_n \ge 0$ on $\mathbb{R}_+$, it follows that

$$\forall x \ge 0, \;\; f(x,T_n) \ge f_-(x,T_n) \ge D\frac{\delta_n}{n}(T_n - 1) = \frac{1}{4} - D\frac{\delta_n}{n} > \frac{1}{8}. \tag{5.52}$$

Let us now consider the solution of the ODE

$$\begin{cases} \dfrac{dy}{dt} = \phi(y(t)), \quad t > T_n \\ y(T_n) = \frac{1}{8}, \end{cases}$$

so that it follows from the comparison principle for (NL) that

$$\forall t \in [T_n, \infty), \;\; \forall x \ge 0, \;\; f(x,t) \ge y(t). \tag{5.53}$$

Since $\lim_{t\to\infty} y(t) = 1$, passing to the limit implies (5.39) and concludes step A.

Step B. Next we prove that

$$\forall x \ge 0, \;\; \limsup_{t\to\infty} f(x,t) \le 1 \tag{5.54}$$

It relies on the construction of a supersolution to (NL).

Note that using property (5.35), we have that

$$\forall t \ge 0, \;\; \forall x \ge 0, \;\; f(x,t) \le f_+(x,t), \tag{5.55}$$

where f_+ is the solution of

$$(NL_+)\begin{cases} \dfrac{\partial f_+}{\partial t} &= -D(f_+ - \langle f_+ \rangle_q) + C_0(1-f_+), \quad x \ge 0, t > 0 \\ f_+(x,0) &= f_0(x), \quad x \ge 0. \end{cases}$$

We compute f_+ explicitely. Note that $g = f_+ - 1$ satisfies

$$\forall t > 0, \;\; \forall x > 0, \;\; \frac{\partial g}{\partial t} = -D(g - \langle g(.,t) \rangle_q) - C_0 g$$

Thus the function $\phi = \frac{x}{m} g$ satisfies equation (4.12) with $\alpha = -C_0$ where the mean value is taken with respect to the probability distribution p defined by $q(x) = \frac{x}{m} p(x)$

for $x \geq 0$, since $\langle \phi(.,t)\rangle_p = \langle g(.,t)\rangle_q$. It follows from (4.13) that for all $x > 0$ and $t > 0$,

$$\phi(x,t) = (\phi(x,0) - \frac{x}{m}\langle \phi(.,0)\rangle_p)e^{-(C_0+D)t} + \frac{x}{m}\langle \phi(.,0)\rangle_p e^{-C_0 t}$$

so that

$$f_+(x,t) = (f_0(x) - \langle f_0\rangle_q)e^{-(C_0+D)t} + (\langle f_0\rangle_q - 1)e^{-C_0 t} + 1.$$

Hence

$$\forall x \geq 0, \ \lim_{t\to\infty} f_+(x,t) = 1$$

which implies (5.54) in view or (5.55). □

5.2.3. *Conclusion.* To complete the proof of (5.25), note that it follows from equations (5.21) and (5.24) that

$$\forall x \geq 0, \ \forall t \geq T^{\varepsilon}, \ \frac{\partial u}{\partial t} \leq -D(u - \langle u(.,t)\rangle_q) + (\beta_0 - \mu)u(1 - (1-\varepsilon)u)$$

and that

$$\forall x \geq 0, \ \forall t \geq T^{\varepsilon}, \ \frac{\partial u}{\partial t} \geq -D(u - \langle u(.,t)\rangle_q) + (\beta_0 - \mu)u(1 - (1+\varepsilon)u).$$

Thus we apply Theorem 5.2 (ii) to conclude that

$$\forall x \geq 0, \ \forall t \geq T^{\varepsilon}, \ u_-(x,t) \leq u(x,t) \leq u_+(x,t), \tag{5.56}$$

where u_- is the solution of

$$\begin{cases} \dfrac{\partial u_-}{\partial t} = -D(u_- - \langle u_-(.,t)\rangle_q) + (\beta_0 - \mu)u_-(1 - (1+\varepsilon)u_-) & x \geq 0, t \geq T^{\varepsilon} \\ u_-(x,T^{\varepsilon}) = u(x,T^{\varepsilon}), & x \geq 0 \end{cases}$$

and u_+ is the solution of

$$\begin{cases} \dfrac{\partial u_+}{\partial t} = -D(u_+ - \langle u_+(.,t)\rangle_q) + (\beta_0 - \mu)u_+(1 - (1-\varepsilon)u_+) & x \geq 0, t \geq T^{\varepsilon} \\ u_+(x,T^{\varepsilon}) = u(x,T^{\varepsilon}), & x \geq 0. \end{cases}$$

Note that $v_+ = (1-\varepsilon)u_+$ satisfies

$$\begin{cases} \dfrac{\partial v_+}{\partial t} = -D(v_+ - \langle v_+(.,t)\rangle_q) + \phi(v_+) & x \geq 0, t \geq T^{\varepsilon} \\ v_+(x,T^{\varepsilon}) = (1-\varepsilon)u(x,T^{\varepsilon}), & x \geq 0, \end{cases}$$

with $\phi(v) = (\beta_0 - \mu)v(1-v)$. We perform a time-translation and rewrite

$$\forall t \geq 0, \forall x \geq 0, \ v_+(x, t + T^{\varepsilon}) = f_+(x,t),$$

where f_+ is a solution of system (NL) on $\mathbb{R}_+ \times \mathbb{R}_+$ with initial data

$$f_+(x,0) = (1-\varepsilon)u(x,T^{\varepsilon}).$$

Thus applying the result of Proposition 7 to f_+ yields that

$$\forall x \geq 0, \ \lim_{t\to\infty} f_+(x,t) = 1.$$

Since in view of (5.56), we have that

$$\forall x \geq 0, \ \forall t \geq T^{\varepsilon}, \ u(x,t) \leq \frac{v_+(x,t)}{1-\varepsilon} = \frac{f_+(x,t-T^{\varepsilon})}{1-\varepsilon},$$

it follows that

$$\forall \varepsilon \in (0,1), \ \limsup_{t\to\infty} u(x,t) \ \leq \ \frac{1}{1-\varepsilon}.$$

A similar argument about u_- yields finally

$$\forall \varepsilon \in (0,1), \frac{1}{1+\varepsilon} \leq \liminf_{t\to\infty} u(x,t) \leq \limsup_{t\to\infty} u(x,t) \leq \frac{1}{1-\varepsilon}$$

which proves (5.25).

Remark 4. Note that equation (5.21) as well as more generally the nonlocal diffusion equation

$$\frac{\partial u}{\partial t} = -D[u - \langle u(.,t)\rangle_q] + \phi(x,u), \ \ t > 0, \ \ x \geq 0 \tag{5.57}$$

can be interpreted as the gradient flow for the nonlocal energy functional

$$E[u] = \int_0^\infty [\frac{D}{2}(u - \langle u\rangle_q)^2 + \Phi(x,u)]\, q(x)dx,$$

where the potential Φ is defined by $\frac{\partial \Phi}{\partial u}(x,u) = -\phi(x,u)$. Namely, if u the solution of (5.57) on $\mathbb{R}_+ \times [0,T]$, then

$$\forall t \in [0,T], \ \ \frac{d}{dt}E[u(.,t)] = -\int_0^\infty u_t(x,t)^2 q(x)dx \leq 0.$$

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