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HOMOGENIZATION OF NONLINEAR DISSIPATIVE HYPERBOLIC PROBLEMS EXHIBITING ARBITRARILY MANY SPATIAL AND TEMPORAL SCALES

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Abstract. This paper concerns the homogenization of nonlinear dissipative hyperbolic problems

$$\partial_{tt}u^{\varepsilon}(x,t)-\nabla\cdot\left(a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}(x,t)\right)$$
$$+g\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}},u^{\varepsilon}(x,t),\nabla u^{\varepsilon}(x,t)\right)=f(x,t)$$

where both the elliptic coefficient a and the dissipative term g are periodic in the $n+m$ first arguments where n and m may attain any non-negative integer value. The homogenization procedure is performed within the framework of evolution multiscale convergence which is a generalization of two-scale convergence to include several spatial and temporal scales. In order to derive the local problems, one for each spatial scale, the crucial concept of very weak evolution multiscale convergence is utilized since it allows less benign sequences to attain a limit. It turns out that the local problems do not involve the dissipative term g even though the homogenized problem does and, due to the nonlinearity property, an important part of the work is to determine the effective dissipative term. A brief illustration of how to use the main homogenization result is provided by applying it to an example problem exhibiting six spatial and eight temporal scales in such a way that a and g have disparate oscillation patterns.

1. **Introduction.** The framework of hyperbolic equations has applications in a wide range of areas such as, e.g., electromagnetics, acoustics, hyperbolic heat conduction and dynamic elasticity. Such physical phenomena may exhibit a multiscale behavior with respect to both space and time, for example in electromagnetism by a vibrating heterogeneous medium. Furthermore, if a physical system suffers from energy losses these may be accounted for by a so-called dissipative term. In this contribution we homogenize problems showing these three physical notions as we study the homogenization of hyperbolic problems having a nonlinear dissipative term and where there is an arbitrary number of both spatial and temporal scales involved.

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1.1. **An outline of the problem and the homogenization results.** In this paper we investigate the hyperbolic problem

$$\begin{aligned}\partial_{tt}u^{\varepsilon}(x,t)-\nabla\cdot\left(a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}(x,t)\right)&\\ +g\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{q_1}},\ldots,\frac{t}{\varepsilon^{r_m}},u^{\varepsilon}(x,t),\nabla u^{\varepsilon}(x,t)\right)&=f(x,t)\text{ in }\Omega_T,\\ u^{\varepsilon}(x,t)&=0\text{ on }\partial\Omega\times(0,T),\\ u^{\varepsilon}(x,0)&=0\text{ in }\Omega,\\ \partial_t u^{\varepsilon}(x,0)&=0\text{ in }\Omega,\end{aligned}\tag{1}$$

where $\varepsilon>0$ tends to zero, $0<q_1<\cdots<q_n$ and $0<r_1<\cdots<r_m$. Here $\Omega_T=\Omega\times(0,T)$, where Ω is an open bounded subset of $\mathbb{R}^N$ with C^2 boundary, and a and g are periodic with respect to the unit cube $Y=(0,1)^N$ in $\mathbb{R}^N$ in the n first variables and with respect to the interval $S=(0,1)$ in the following m variables. Observe that it is not necessarily the case that all the scales give rise to actual oscillations, i.e., a and g may have disparate oscillation patterns.

In order to homogenize (1) we study the asymptotic behavior of the corresponding sequence of weak solutions $\{u^{\varepsilon}\}$ and derive the homogenized problem uniquely solved by the limit of this sequence. Along with the homogenized problem we obtain for each spatial scale present in a an associated local problem solved by so-called correctors. These generative local problems are either hyperbolic or elliptic and in the former case we say that we have resonance. It turns out that resonance occurs when the spatial oscillation mode in question coincides with some temporal oscillation mode, i.e., when the i-th spatial scale satisfies $\varepsilon^{q_i}=\varepsilon^{r_j}$ for some j. Concerning the dissipative term g we find that it does not appear in the local problems and hence spatial scales only present in g merely give rise to non-generative local problems, i.e., problems with vanishing solutions.

1.2. **Background and theoretical pivotals.** For the homogenization of (1) we employ in a general sense the concept of multiscale convergence, a technique introduced in [18] for two spatial scales, formalized in [1] and generalized to an arbitrary number of spatial scales in [2]. In [14] a compactness result akin to multiscale convergence was proven in order to overcome the impediments appearing when there are rapid oscillations in time present as well, see also [24] for a related result. In [8] the notion of very weak multiscale convergence was introduced making it possible to handle fast temporal oscillations for any number of spatial scales. A linear parabolic problem with an arbitrary number of scales in both space and time was studied in [10] employing results, originally presented in [28], generalizing very weak multiscale convergence to the evolution setting. In the present paper we use a version of the compactness result concerning very weak evolution multiscale convergence found in [10] which together with a corresponding evolution multiscale compactness result for gradients are the key results in the homogenization of (1).

A few papers concerning homogenization problems exhibiting both a rapid spatial and a rapid temporal scale are, e.g., [13], [17], [23], [27] and [31] treating various parabolic problems, and [21] where a linear hyperbolic problem is studied.

Homogenization of certain nonlinear hyperbolic problems has been performed, e.g., in [4], [26] and [32] having one rapid spatial scale, in [30] with two rapid spatial scales, in [25] having one rapid scale in both space and time, in [22] where two rapid spatial and one rapid temporal scale appear, and in [33] which involves one rapid spatial and two rapid temporal scales. The key homogenization tool in [33], making rapid temporal scales possible to treat, is a compactness result of very weak evolution multiscale convergence type. Since in the present paper we provide a general very weak evolution multiscale compactness result it is possible to formulate a homogenization result for nonlinear hyperbolic problems having an arbitrary number of spatial and temporal scales.

1.3. **Novelties.** The present paper deals with the homogenization of a hyperbolic problem that exhibits an arbitrary number of scales in both space and time. This particular combination has, up to the authors' knowledge, never been studied in detail before even though the parabolic setting has been investigated in [10]. Moreover, the evolution equation in (1) carries a somewhat tricky nonlinear dissipative term equipped with the full set of spatial and temporal scales. One of the main challenges in the homogenization procedure is the process of passing to the limit for the dissipative term since it requires nontrivial techniques, developed in this contribution, to handle.

1.4. **Organization of the paper.** This paper is organized in the following way. In Section 2 we introduce evolution multiscale convergence and its related very weak version and give some results which prove useful in the procedure of homogenizing (1). Section 3 begins by establishing existence, uniqueness and an a priori estimate for sequences of solutions to (1). The section proceeds by unraveling the relevant convergence properties of the dissipative term and is concluded by formulating and proving the main result of the paper, i.e., deriving the homogenized problem and the local problems for (1). In the final part, Section 4, we provide an illustrative example of the use of the general homogenization result given in this paper.

Notation. Denote $Y_k = Y$ for $k = 1,\dots,n$, $Y^n = Y_1 \times \cdots \times Y_n$, $y^n = (y_1,\dots,y_n)$, $dy^n = dy_1 \cdots dy_n$; $S_j = S$ for $j = 1,\dots,m$, $S^m = S_1 \times \cdots \times S_m$, $s^m = (s_1,\dots,s_m)$, $ds^m = ds_1 \cdots ds_m$; and $\mathcal{Y}_{n,m} = Y^n \times S^m$ supplemented by $\mathcal{Y}_{0,m} = S^m$. We let $\varepsilon'_k(\varepsilon)$, $k = 1,\dots,n$, and $\varepsilon''_j(\varepsilon)$, $j = 1,\dots,m$, be strictly positive functions such that $\varepsilon'_k(\varepsilon)$ and $\varepsilon''_j(\varepsilon)$ go to zero when ε does. We say that $\{\varepsilon'_1,\dots,\varepsilon'_n\}$ and $\{\varepsilon''_1,\dots,\varepsilon''_m\}$ are lists of spatial and temporal scales, respectively. Furthermore, we introduce $\frac{x}{\varepsilon'^n} = (\frac{x}{\varepsilon'_1},\dots,\frac{x}{\varepsilon'_n})$ and $\frac{t}{\varepsilon''^m} = (\frac{t}{\varepsilon''_1},\dots,\frac{t}{\varepsilon''_m})$ and, in a similar manner, given the exponents $0 < q_1 < \cdots < q_n$ and $0 < r_1 < \cdots < r_m$ we have $\frac{x}{\varepsilon^{q^n}} = (\frac{x}{\varepsilon^{q_1}},\dots,\frac{x}{\varepsilon^{q_n}})$ and $\frac{t}{\varepsilon^{r^m}} = (\frac{t}{\varepsilon^{r_1}},\dots,\frac{t}{\varepsilon^{r_m}})$.

We let $F_\sharp(Y)$ be the space of all functions in $F_{\text{loc}}(\mathbb{R}^N)$ that are Y-periodic repetitions of some function in $F(Y)$. Moreover, for any function $\phi : \Omega_T \times \mathbb{R}^{nN+m} \to \mathbb{R}^K$, where $K = 1, N$ or $N \times N$, such that $\phi(x,t,\cdot)$ is $\mathcal{Y}_{n,m}$-periodic we introduce $[\phi]^\varepsilon(x,t) = \phi(x,t,\frac{x}{\varepsilon'^n},\frac{t}{\varepsilon''^m})$, a notation that will be used whenever convenient.

Finally, suppose that $F_1(U_1),\dots,F_\mu(U_\mu)$ are function spaces over $U_i \subset \mathbb{R}^{M_i}$ for $i = 1,\dots,\mu$, then the so-called tensor product space $F_1(U_1) \otimes \cdots \otimes F_\mu(U_\mu)$ over $U_1 \times \cdots \times U_\mu$ is the space of all finite linear combinations of functions on the form $\phi_1 \cdots \phi_\mu$ where $\phi_i \in F_i(U_i)$ for $i = 1,\dots,\mu$.

2. **Preliminaries.** This section deals with evolution multiscale convergence and the related concept of very weak evolution multiscale convergence. We give some results essential for the homogenization of the hyperbolic problem (1) which is performed in Section 3.

2.1. **Evolution multiscale convergence.** Since we study the problem (1) exhibiting an arbitrary number of spatial and temporal scales we invoke the concept of evolution multiscale convergence, a generalization of Nguetseng's classical two-scale convergence [18]. We give the following definition, also exploited in, e.g., [9] and [10].

Definition 2.1. Let $\{\varepsilon'_1, \ldots, \varepsilon'_n\}$ and $\{\varepsilon''_1, \ldots, \varepsilon''_m\}$ be given lists of spatial and temporal scales, respectively. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1, m+1)$-scale converge to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} u^\varepsilon(x,t) v\left(x, t, \frac{x}{\varepsilon'^n}, \frac{t}{\varepsilon''^m}\right) dxdt$$
$$= \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x,t,y^n,s^m)\, v(x,t,y^n,s^m)\, dy^n ds^m dxdt$$

for any $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. This convergence is denoted

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m).$$

In order to give the relevant compactness results we need to introduce a certain separatedness property imposed on the spatial and temporal scales involved.

Definition 2.2. Let $\{\varepsilon'_1, \ldots, \varepsilon'_n\}$ and $\{\varepsilon''_1, \ldots, \varepsilon''_m\}$ be lists of well-separated scales, i.e., there exist positive integers k and l such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon'_i}\left(\frac{\varepsilon'_{i+1}}{\varepsilon'_i}\right)^k = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon''_j}\left(\frac{\varepsilon''_{j+1}}{\varepsilon''_j}\right)^l = 0$$

for $i = 1, \ldots, n-1$ and $j = 1, \ldots, m-1$. Consider all elements from both lists. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each pair is removed and the list in order of magnitude of all the remaining elements is well-separated, then the lists $\{\varepsilon'_1, \ldots, \varepsilon'_n\}$ and $\{\varepsilon''_1, \ldots, \varepsilon''_m\}$ are said to be jointly well-separated.

See Section 2.4 in [28] for a more technical definition and some illustrative examples regarding the notion of joint well-separatedness.

Remark 1. Clearly, the lists $\{\varepsilon^{q_1}, \ldots, \varepsilon^{q_n}\}$ and $\{\varepsilon^{r_1}, \ldots, \varepsilon^{r_m}\}$ in (1) are jointly well-separated since $0 < q_1 < \cdots < q_n$ and $0 < r_1 < \cdots < r_m$.

We have the following fundamental results.

Proposition 1. *Suppose that the lists $\{\varepsilon'_1, \ldots, \varepsilon'_n\}$ and $\{\varepsilon''_1, \ldots, \varepsilon''_m\}$ of scales are jointly well-separated and that $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. Then*

$$\|[v]^\varepsilon\|_{L^2(\Omega_T)} \to \|v\|_{L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))}$$

as ε goes to zero.

Proof. This follows readily from the corresponding result concerning an arbitrarily number of spatial scales as given in, e.g., Section 2 in [2]. □

Proposition 2. *Suppose that the lists $\{\varepsilon'_1, \ldots, \varepsilon'_n\}$ and $\{\varepsilon''_1, \ldots, \varepsilon''_m\}$ of scales are jointly well-separated and that $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. Then*

$$v\left(x, t, \frac{x}{\varepsilon'^n}, \frac{t}{\varepsilon''^m}\right) \overset{n+1,m+1}{\rightharpoonup} v(x,t,y^n,s^m)$$

as ε goes to zero.

Proof. This is a straightforward generalization of a special case of a classical result found in, e.g., Theorem 2 in [16]. □

The following compactness result is a direct consequence of Theorem 2.9 in [29]. For a detailed proof see, e.g., the proof of Theorem A.1 in [10].

Theorem 2.3. *Let* $\{u^\varepsilon\}$ *be a bounded sequence in* $L^2(\Omega_T)$ *and suppose that the lists* $\{\varepsilon_1', \ldots, \varepsilon_n'\}$ *and* $\{\varepsilon_1'', \ldots, \varepsilon_m''\}$ *are jointly well-separated. Then, up to a subsequence,*

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m)$$

for some $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$.

Remark 2. The Propositions 1 and 2 and Theorem 2.3 actually hold also for mere joint separatedness whose definition amounts to an obvious modification of joint well-separatedness.

In order to characterize the evolution multiscale limit of bounded sequences of gradients in the context of hyperbolic problems we introduce a convenient function space. First fix $\gamma > 0$ such that $\gamma < 2/(N-2)$ if $N \geq 3$. Let $V(0,T;H_0^1(\Omega),L^2(\Omega))$ be the space of all functions in $L^{2(\gamma+1)}(0,T;H_0^1(\Omega))$ such that the first and second temporal derivatives belong to $L^2(\Omega_T)$ and $L^2(0,T;H^{-1}(\Omega))$, respectively. The norm of this Banach space is given by

$$\|v\|_{V(0,T;H_0^1(\Omega),L^2(\Omega))} = \|v\|_{L^{2(\gamma+1)}(0,T;H_0^1(\Omega))} + \|\partial_t v\|_{L^2(\Omega_T)} + \|\partial_{tt} v\|_{L^2(0,T;H^{-1}(\Omega))}.$$

Observe that throughout the present paper we assume that γ is given as above. We will later see how γ is connected to the hyperbolic problem studied in the present paper.

We are now prepared to formulate and prove the theorem below concerning in particular the evolution multiscale convergence of sequences of gradients where the so-called correctors emerge in the limit.

Theorem 2.4. *Let* $\{u^\varepsilon\}$ *be a bounded sequence in* $V(0,T;H_0^1(\Omega),L^2(\Omega))$ *and suppose that the lists* $\{\varepsilon_1', \ldots, \varepsilon_n'\}$ *and* $\{\varepsilon_1'', \ldots, \varepsilon_m''\}$ *are jointly well-separated. Then, up to a subsequence,*

$$\begin{aligned} u^\varepsilon(x,t) &\to u(x,t) \text{ in } L^{2(\gamma+1)}(\Omega_T), \\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \text{ in } L^{2(\gamma+1)}\left(0,T;H_0^1(\Omega)\right), \\ \partial_t u^\varepsilon(x,t) &\rightharpoonup \partial_t u(x,t) \text{ in } L^2(\Omega_T), \\ \partial_{tt} u^\varepsilon(x,t) &\rightharpoonup \partial_{tt} u(x,t) \text{ in } L^2(0,T;H^{-1}(\Omega)) \end{aligned}$$

and

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j\left(x,t,y^j,s^m\right) \tag{2}$$

where $u \in V(0,T;H_0^1(\Omega),L^2(\Omega))$ *and* $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1,m};H_\sharp^1(Y_j)/\mathbb{R})$, $j = 1, \ldots, n$.

Proof. Since $\{u^\varepsilon\}$ is bounded in $V(0,T;H_0^1(\Omega),L^2(\Omega))$ it clearly holds, up to a subsequence, that

$$\begin{aligned} u^\varepsilon(x,t) &\rightharpoonup u(x,t) \text{ in } L^{2(\gamma+1)}\left(0,T;H_0^1(\Omega)\right), \\ \partial_t u^\varepsilon(x,t) &\rightharpoonup \partial_t u(x,t) \text{ in } L^2(\Omega_T) \end{aligned}$$

and

$$\partial_{tt}u^{\varepsilon}(x,t) \rightharpoonup \partial_{tt}u(x,t) \text{ in } L^2(0,T;H^{-1}(\Omega))$$

where $u \in V(0,T;H_0^1(\Omega),L^2(\Omega))$. Since $H_0^1(\Omega)$ is continuously embedded in $L^{2(\gamma+1)}(\Omega)$ we have by, e.g., Theorem 5.1 in [15] that $V(0,T;H_0^1(\Omega),L^2(\Omega))$ is compactly embedded in $L^{2(\gamma+1)}(\Omega_T)$. Thus, since

$$u^{\varepsilon}(x,t) \rightharpoonup u(x,t) \text{ in } V(0,T;H_0^1(\Omega),L^2(\Omega))$$

we get that

$$u^{\varepsilon}(x,t) \to u(x,t) \text{ in } L^{2(\gamma+1)}(\Omega_T).$$

Following along the lines of the proof of Theorem 4 in [10], also found as Theorem 2.10 in [29], we have (2). □

2.2. **Very weak evolution multiscale convergence.** In [14] the homogenization of a linear parabolic problem with oscillations in one spatial and one temporal scale was studied. In that context a compactness result was presented which can be seen, to the best of the authors' knowledge, as the first result of very weak convergence type. This primordial version of very weak convergence was employed also in [11] and [12] where a spatial and a temporal scale, respectively, was added to the scales in [14]. In [8] very weak multiscale convergence was introduced where an arbitrary number of spatial scales is allowed. This elaborates an idea in [24] where a modified version of the result in [14] was presented.

In the process of homogenizing (1) we will need a convenient generalization, called very weak evolution multiscale convergence, of the space-exclusive concept in [8] which also takes rapid temporal oscillations into consideration in an explicit manner.

Definition 2.5. Let $\{\varepsilon'_1,\ldots,\varepsilon'_n\}$ and $\{\varepsilon''_1,\ldots,\varepsilon''_m\}$ be given lists of spatial and temporal scales, respectively. A sequence $\{u^{\varepsilon}\}$ in $L^1(\Omega_T)$ is said to $(n+1,m+1)$-scale converge very weakly to $u_0 \in L^1(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} u^{\varepsilon}(x,t)v\left(x,\frac{x}{\varepsilon'^{n-1}}\right)c\left(t,\frac{t}{\varepsilon''^{m}}\right)\varphi\left(\frac{x}{\varepsilon'_n}\right)dxdt \\ = \int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} u_0(x,t,y^n,s^m)v(x,y^{n-1})c(t,s^m)\varphi(y_n)\,dy^n ds^m dxdt\end{aligned}$$

for any $v \in D(\Omega;C_\sharp^\infty(Y^{n-1})), \varphi \in C_\sharp^\infty(Y_n)/\mathbb{R}$ and $c \in D(0,T;C_\sharp^\infty(S^m))$. This convergence is denoted

$$u^{\varepsilon}(x,t) \overset{n+1,m+1}{\underset{\text{vw}}{\rightharpoonup}} u_0(x,t,y^n,s^m).$$

Remark 3. A unique very weak evolution multiscale limit is provided by requiring that

$$\int_{Y_n} u_0(x,t,y^n,s^m)\,dy_n = 0,$$

i.e. $u_0 \in L^1(\Omega_T \times \mathcal{Y}_{n-1,m};L^1(Y_n)/\mathbb{R})$, which is explained in detail in Proposition 2.26 in [28] for the case of very weak two-scale convergence. The generalization to very weak evolution multiscale convergence is straightforward.

The following compactness result concerning very weak evolution multiscale convergence will be a key result in the homogenization of (1). The result is a special case of Theorem 7 in [10] and Theorem 2.78 in [28].

Theorem 2.6. *Let $\{u^\varepsilon\}$ be a bounded sequence in $V(0,T;H_0^1(\Omega),L^2(\Omega))$ and assume that the lists $\{\varepsilon_1',\dots,\varepsilon_n'\}$ and $\{\varepsilon_1'',\dots,\varepsilon_m''\}$ are jointly well-separated. Then, up to a subsequence,*

$$\frac{u^\varepsilon(x,t)}{\varepsilon_n'} \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} u_n(x,t,y^n,s^m),$$

where $u_n \in L^2(\Omega_T \times \mathcal{Y}_{j-1,m};H_\sharp^1(Y_j)/\mathbb{R})$ is the same as in Theorem 2.4.

Apart from being crucial in the homogenization procedure the theorem above illustrates the fundamental merit of the very weak evolution multiscale convergence in that it permits certain $L^2(\Omega_T)$-unbounded sequences to attain a limit, possibly allowing new paths to non-trivial limits, something that ordinary evolution multiscale convergence does not.

3. **Homogenization of the hyperbolic problem.** We are now prepared to homogenize the nonlinear dissipative hyperbolic problem (1). The present section commences by a more precise formulation of the homogenization problem giving structure conditions on the coefficient of the elliptic term and on the nonlinear dissipative term. In the first Subsection we prove a crucial convergence result for sequences of dissipative terms and in the second Subsection we formulate and prove the main homogenization result.

Consider (1), i.e., the sequence of nonlinear dissipative hyperbolic problems

$$\begin{aligned}
\partial_{tt}u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}}\right)\nabla u^\varepsilon(x,t)\right)&\\
+g\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}},u^\varepsilon(x,t),\nabla u^\varepsilon(x,t)\right) &= f(x,t) \text{ in } \Omega_T,\\
u^\varepsilon(x,t) &= 0 \text{ on } \partial\Omega\times(0,T),\\
u^\varepsilon(x,0) &= 0 \text{ in } \Omega,\\
\partial_t u^\varepsilon(x,0) &= 0 \text{ in } \Omega,
\end{aligned} \tag{3}$$

where $0<q_1<\dots<q_n$, $0<r_1<\dots<r_m$ and $f\in L^2(\Omega_T)$. For the elliptic coefficient a we assume that there exists $\alpha>0$ such that the conditions

- **(A1):** $a\in C_\sharp^1(\mathcal{Y}_{n,m})^{N\times N}$
- **(A2):** $a_{ij}=a_{ji}$ for $i,j=1,\dots,N$, i.e., a is symmetric
- **(A3):** $a(y^n,s^m)\,\xi\cdot\xi\ge\alpha|\xi|^2$

are satisfied for all $(y^n,s^m)\in\mathbb{R}^{nN}\times\mathbb{R}^m$ and all $\xi\in\mathbb{R}^N$. Moreover, suppose that for the dissipative term g there are positive constants $C_0,\dots,C_5$ for which

- **(B1):** $g\in C^1(\mathbb{R}^{nN+m}\times\mathbb{R}\times\mathbb{R}^N)$ and $g(\cdot,k,\xi)$ is $\mathcal{Y}_{n,m}$-periodic
- **(B2):** $|g(y^n,s^m,k,\xi)|\le C_0(1+|k|^{\gamma+1}+|\xi|)$
- **(B3):** $|\partial_k g(y^n,s^m,k,\xi)|\le C_0(1+|k|^\gamma)$
- **(B4):** $g(y^n,s^m,k,\xi)\,\eta\ge C_1|k|^\gamma k\eta - C_2(1+|\eta||\xi|)$
- **(B5):** $|\nabla_\xi g(y^n,s^m,k,\xi)|\le C_3$
- **(B6):** $(g(y^n,s^m,k,\xi)-g(y^n,s^m,k',\xi'))\,\eta$ $\ge -\left(C_4\left(|k|^\gamma+|k'|^\gamma\right)|k-k'|+C_5|\xi-\xi'|\right)|\eta|$

hold for all $(y^n,s^m)\in\mathbb{R}^{nN}\times\mathbb{R}^m$, $\eta\in\mathbb{R}$ and $(k,\xi),(k',\xi')\in\mathbb{R}\times\mathbb{R}^N$. Recall that $\gamma>0$ and, if $N\ge3$, $\gamma<2/(N-2)$.

Remark 4. As an example of a non-trivial dissipative term satisfying (B1)–(B6) we have

$$g\left(y^n, s^m, k, \xi\right) = |k|^{\gamma} k + \psi\left(y^n, s^m\right) \sum_{l=1}^{N} \sin \xi_l$$

where ψ is non-negative and belongs to $C^1_\sharp\left(\mathcal{Y}_{n,m}\right)$. See Section 1 in [33] and Section 2 in [3] for similar examples, though in different settings, of dissipative terms.

Introduce

$$V_0(0,T; H^1_0\left(\Omega\right), L^2\left(\Omega\right)) \\ = \left\{v \in V(0,T; H^1_0\left(\Omega\right), L^2\left(\Omega\right)) : v(x,0) = \partial_t v(x,0) = 0 \text{ a.e. in } \Omega\right\}$$

which is a real reflexive Banach space with the same norm as $V(0,T; H^1_0\left(\Omega\right), L^2\left(\Omega\right))$, see, e.g., p. 104 in [20]. The weak formulation of (3) is thus that we search for solutions $u^\varepsilon \in V_0(0,T; H^1_0\left(\Omega\right), L^2\left(\Omega\right))$ such that

$$\int_{\Omega_T} u^\varepsilon\left(x,t\right) v\left(x\right) \partial_{tt} c\left(t\right) dxdt + \int_{\Omega_T} a\left(\frac{x}{\varepsilon^{q^n}}, \frac{t}{\varepsilon^{r^m}}\right) \nabla u^\varepsilon\left(x,t\right) \cdot \nabla v\left(x\right) c\left(t\right) dxdt \\ + \int_{\Omega_T} g\left(\frac{x}{\varepsilon^{q^n}}, \frac{t}{\varepsilon^{r^m}}, u^\varepsilon(x,t), \nabla u^\varepsilon(x,t)\right) v(x)\, c(t)\, dxdt = \int_{\Omega_T} f\left(x,t\right) v(x)\, c(t)\, dxdt \tag{4}$$

for all $v \in H^1_0(\Omega)$ and $c \in C(0,T)$. We give the following existence and uniqueness theorem.

Theorem 3.1. *Suppose that the structure conditions (A1)–(A3) and (B1)–(B6) are satisfied. Then there exists a unique weak solution* $u^\varepsilon \in V_0(0,T; H^1_0\left(\Omega\right), L^2\left(\Omega\right))$ *to (3) for every fixed* $\varepsilon > 0$.

Proof. The claim follows readily from Theorem 2.1 in [3]. □

Next, we give some crucial a priori estimates in the proposition below.

Proposition 3. *Suppose that the structure conditions (A1)–(A3) and (B1)–(B6) are satisfied. Then the sequence* $\{u^\varepsilon\}$ *of unique weak solutions to (3) and its spatio-temporally differentiated sequence* $\{\nabla \partial_t u^\varepsilon\}$ *are bounded in* $V_0(0,T; H^1_0\left(\Omega\right), L^2\left(\Omega\right))$ *and* $L^2\left(\Omega_T\right)^N$*, respectively.*

Proof. Observe that Theorem 3.1 guarantees the existence of unique weak solutions in $V_0(0,T; H^1_0\left(\Omega\right), L^2\left(\Omega\right))$ to (3). The a priori estimates follow readily from the results in the paragraph *A priori estimates* in Section 3 in [3]. □

Finally, we provide a result concerning strong convergence of sequences of temporally differentiated weak solutions to the hyperbolic problem (3). This improves a corresponding weak convergence result found in Theorem 2.4.

Proposition 4. *Suppose that the structure conditions (A1)–(A3) and (B1)–(B6) are satisfied. Then for the sequence* $\{u^\varepsilon\}$ *of unique weak solutions to (3) we have that*

$$\partial_t u^\varepsilon\left(x,t\right) \to \partial_t u\left(x,t\right) \text{ in } L^2\left(\Omega_T\right) \tag{5}$$

holds up to a subsequence.

Proof. Following along the same lines as in the proof of Theorem 4.1 in [30], the Aubin–Lions Lemma together with the a priori estimates of Proposition 3 imply that (5) holds up to a subsequence. □

3.1. **A convergence result for the dissipative term.** In the proof of the main result, i.e. Theorem 3.2 in Subsection 3.2, we need to establish a limit of evolution multiscale type for sequences of dissipative terms in order to obtain the homogenized problem. This work is carried out in the present preparative Subsection for the purpose of lightening the presentation of the proof of the main homogenization result. Moreover, the obtained convergence results in the present section may have some independent value within the theory of asymptotic analysis of nonlinear evolution multiscale functions.

To begin with we formulate the following preliminary proposition.

Proposition 5. *Suppose that $g : \mathbb{R}^{nN+m} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies (B1), (B2) and (B6). Let $\phi, \phi_j \in D(\Omega_T)$ and $\psi_j \in C_\sharp^\infty(\mathcal{Y}_{j,m})$ for $j = 1, \ldots, n$ and let $\{u^\varepsilon\}$ be a sequence in $L^{2(\gamma+1)}(\Omega_T)$ such that $u^\varepsilon \to u$. Then, up to a subsequence,*

$$g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla\left(\phi(x,t) + \sum_{j=1}^{n} \varepsilon^{q_j}\phi_j(x,t)\psi_j\left(\frac{x}{\varepsilon^{q_j}}, \frac{t}{\varepsilon^{r_m}}\right)\right)\right) \tag{6}$$

$$\overset{n+1,m+1}{\rightharpoonup} g\left(y^n, s^m, u(x,t), \nabla\phi(x,t) + \sum_{j=1}^{n} \phi_j(x,t)\nabla_{y_j}\psi_j\left(y^j, s^m\right)\right).$$

Proof. We first prove the auxiliary result

$$g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla\phi(x,t) + \sum_{j=1}^{n} \phi_j(x,t)\nabla_{y_j}\psi_j\left(\frac{x}{\varepsilon^{q_j}}, \frac{t}{\varepsilon^{r_m}}\right)\right) \tag{7}$$

$$\overset{n+1,m+1}{\rightharpoonup} g\left(y^n, s^m, u(x,t), \nabla\phi(x,t) + \sum_{j=1}^{n} \phi_j(x,t)\nabla_{y_j}\psi_j\left(y^j, s^m\right)\right).$$

By (B1), (B2) and the properties of u, ϕ, ϕ_j and ψ_j for $j = 1, \ldots, n$ it follows that $g(y^n, s^m, u(x,t), \nabla\phi(x,t) + \sum_j \phi_j(x,t)\nabla_{y_j}\psi_j(y^j, s^m))$ belongs to $L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$ which implies, by Proposition 2, that

$$g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u(x,t), \nabla\phi(x,t) + \sum_{j=1}^{n} \phi_j(x,t)\nabla_{y_j}\psi_j\left(\frac{x}{\varepsilon^{q_j}}, \frac{t}{\varepsilon^{r_m}}\right)\right)$$

$$\overset{n+1,m+1}{\rightharpoonup} g\left(y^n, s^m, u(x,t), \nabla\phi(x,t) + \sum_{j=1}^{n} \phi_j(x,t)\nabla_{y_j}\psi_j\left(y^j, s^m\right)\right).$$

Introduce the sequence $\{\delta^\varepsilon\}$ defined according to

$$\delta^\varepsilon(x,t) = g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla\phi(x,t) + \sum_{j=1}^{n} \phi_j(x,t)\nabla_{y_j}\psi_j\left(\frac{x}{\varepsilon^{q_j}}, \frac{t}{\varepsilon^{r_m}}\right)\right)$$

$$-g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u(x,t), \nabla\phi(x,t) + \sum_{j=1}^{n} \phi_j(x,t)\nabla_{y_j}\psi_j\left(\frac{x}{\varepsilon^{q_j}}, \frac{t}{\varepsilon^{r_m}}\right)\right).$$

In order to prove the auxiliary result (7) we must thus show that

$$\delta^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} 0$$

when $\varepsilon \to 0$. Since (B2) is satisfied and due to the fact that $u^\varepsilon \to u$ in $L^{2(\gamma+1)}(\Omega_T)$ we have that δ^ε is bounded in $L^2(\Omega_T)$. Using Theorem 2.3 there exists a limit $\delta_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ such that, up to a subsequence,

$$\delta^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \delta_0(x,t,y^n,s^m) \tag{8}$$

as ε tends to zero. We have, for any $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$,

$$\begin{aligned} &-\int_{\Omega_T} \delta^\varepsilon(x,t)\, v\left(x,t,\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}}\right) dxdt \qquad (9)\\ &\le C_4\left(\|u^\varepsilon\|^\gamma_{L^{2(\gamma+1)}(\Omega_T)} + \|u\|^\gamma_{L^{2(\gamma+1)}(\Omega_T)}\right)\|u^\varepsilon - u\|_{L^{2(\gamma+1)}(\Omega_T)}\,\|[v]^\varepsilon\|_{L^2(\Omega_T)} \end{aligned}$$

by (B6) and the Generalized Hölder Inequality which is applicable since it holds that $\frac{1}{2(\gamma+1)/\gamma} + \frac{1}{2(\gamma+1)} + \frac{1}{2} = 1$. Letting $\varepsilon \to 0$ and using (8) in (9) we obtain

$$-\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} \delta_0(x,t,y^n,s^m)\, v(x,t,y^n,s^m)\, dy^n ds^m dxdt \le 0 \tag{10}$$

and due to the fact that this holds for any $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$ we have equality in (10), i.e., $\delta_0 = 0$. The auxiliary result (7) is proven.

We are prepared to prove (6). Observe first that the final argument in the left-hand side of (6) is $\nabla\phi(x,t) + \sum_{j=1}^n \phi_j(x,t)\nabla_{y_j}\psi_j(\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}) + \kappa^\varepsilon(x,t)$ where the remainder κ^ε is given according to

$$\begin{aligned} \kappa^\varepsilon(x,t) &= \sum_{j=1}^n \varepsilon^{q_j}\nabla\phi_j(x,t)\,\psi_j\left(\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)\\ &+\sum_{j=2}^n\sum_{i=1}^{j-1}\varepsilon^{q_j-q_i}\phi_j(x,t)\,\nabla_{y_i}\psi_j\left(\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right). \end{aligned}$$

Taking the auxiliary result (7) into consideration what remains to show is that

$$\eta^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} 0$$

where

$$\begin{aligned} \eta^\varepsilon(x,t) =& g\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}},u^\varepsilon(x,t),\nabla\phi(x,t)+\sum_{j=1}^n\phi_j(x,t)\nabla_{y_j}\psi_j\left(\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)+\kappa^\varepsilon(x,t)\right)\\ &-g\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}},u^\varepsilon(x,t),\nabla\phi(x,t)+\sum_{j=1}^n\phi_j(x,t)\nabla_{y_j}\psi_j\left(\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)\right). \end{aligned}$$

Proceeding as in the verification of the auxiliary result we deduce that there exists a limit $\eta_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ such that, up to a subsequence,

$$\eta^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \eta_0(x,t,y^n,s^m)$$

as $\varepsilon \to 0$. We get, for every $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$,

$$
\begin{aligned}
&-\int_{\Omega_T} \eta^\varepsilon(x,t)\, v\left(x,t,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_m}}\right) dxdt \\
&\leq C_5 \left(\sum_{j=1}^{n} \varepsilon^{q_j} \left\|\nabla\phi_j [\psi_j]^\varepsilon\right\|_{L^2(\Omega_T)^N} \left\|[v]^\varepsilon\right\|_{L^2(\Omega_T)} \right. \\
&\left. + \sum_{j=2}^{n}\sum_{i=1}^{j-1} \varepsilon^{q_j - q_i} \left\|\phi_j [\nabla_{y_i}\psi_j]^\varepsilon\right\|_{L^2(\Omega_T)^N} \left\|[v]^\varepsilon\right\|_{L^2(\Omega_T)} \right)
\end{aligned}
$$

where we have used (B6) and the Hölder Inequality. Observe that the sequences $\{\nabla\phi_j[\psi_j]^\varepsilon\}$, $\{\phi_j[\nabla_{y_i}\psi_j]^\varepsilon\}$ and $\{[v]^\varepsilon\}$ are all L^2-bounded. This together with the fact that ε^{q_j} and $\varepsilon^{q_j-q_i}$ appearing in the sums tend to zero for all i and j means that we arrive at

$$
\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} \eta_0(x,t,y^n,s^m)\, v(x,t,y^n,s^m)\, dy^n ds^m dxdt = 0
$$

as ε goes to zero, and hence $\eta_0 = 0$. The proof is complete. □

Remark 5. It is not difficult to verify that Proposition 5 actually holds also for $\phi_j\psi_j$ replaced by any finite linear combination of such products.

We are now prepared to state the main result of this Subsection.

Proposition 6. *Suppose that $g : \mathbb{R}^{nN+m} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies (B1), (B2) and (B6). Let $\{u^\varepsilon\}$ be a bounded sequence in $V(0,T;H_0^1(\Omega), L^2(\Omega))$ such that*

$$
\lim_{\varepsilon\to 0} \left\|\nabla u^\varepsilon\right\|_{L^2(\Omega_T)^N} = \left\| \nabla u + \sum_{j=1}^{n} \nabla_{y_j} u_j \right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N} \tag{11}
$$

where u and u_j, $j = 1, \ldots, n$, are given according to Theorem 2.4. Then, up to a subsequence,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T} g\left(\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla u^\varepsilon(x,t)\right) w(x,t)\, dxdt \\
&= \int_{\Omega_T}\left(\int_{\mathcal{Y}_{n,m}} g\left(y^n, s^m, u(x,t), \nabla u(x,t) + \sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right) dy^n ds^m\right) \\
&\quad \times w(x,t)\, dxdt
\end{aligned} \tag{12}
$$

for any $w \in D(\Omega_T)$.

Remark 6. In practice, the assumption (11) amounts to excluding the possibility that

$$
\lim_{\varepsilon\to 0} \left\|\nabla u^\varepsilon\right\|_{L^2(\Omega_T)^N} > \left\| \nabla u + \sum_{j=1}^{n} \nabla_{y_j} u_j \right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}
$$

since for any sequence $\{v^\varepsilon\}$ in $L^2(\Omega_T)$ two-scale converging to $v_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ it holds that

$$
\lim_{\varepsilon\to 0} \left\|v^\varepsilon\right\|_{L^2(\Omega_T)} \geq \left\|v_0\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})}. \tag{13}
$$

The inequality (13) is a straightforward generalization of a result found in, e.g., Theorem 0.2 in [1].

Proof of Proposition 6. The sequence $\{g(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla u^\varepsilon(x,t))\}$ is clearly bounded in $L^2(\Omega_T)$ and hence, by Theorem 2.3,

$$g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla u^\varepsilon(x,t)\right) \overset{n+1,m+1}{\rightharpoonup} g_0(x,t,y^n,s^m) \tag{14}$$

up to a subsequence for some $g_0 \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. We need to identify the limit g_0 in terms of g as given in (12). By (B6) and the Hölder Inequality we get

$$\begin{aligned}\int_{\Omega_T} &\left(g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla u^\varepsilon(x,t)\right)\right.\\ &\left.- g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla\left(\phi(x,t) + \sum_{j=1}^n \varepsilon^{q_j}\psi_j\left(x,t,\frac{x}{\varepsilon^{q_j}}, \frac{t}{\varepsilon^{r_m}}\right)\right)\right)\right) \\ &\times\left(u^\varepsilon(x,t) - \left(\phi(x,t) + \sum_{j=1}^n \varepsilon^{q_j}\psi_j\left(x,t,\frac{x}{\varepsilon^{q_j}}, \frac{t}{\varepsilon^{r_m}}\right)\right)\right) dxdt \\ &+ C_5\left\|u^\varepsilon - \left(\phi + \sum_{j=1}^n \varepsilon^{q_j}[\psi_j]^\varepsilon\right)\right\|_{L^2(\Omega_T)} \left\|\nabla u^\varepsilon - \nabla\left(\phi + \sum_{j=1}^n \varepsilon^{q_j}[\psi_j]^\varepsilon\right)\right\|_{L^2(\Omega_T)^N} \geq 0\end{aligned} \tag{15}$$

for any $\phi \in D(\Omega_T)$ and, for each $j = 1, \ldots, n$, any finite linear combination ψ_j of products on the form $\phi_j v_{1,j} \cdots v_{j,j} c_{1,j} \cdots c_{m,j}$ where $\phi_j \in D(\Omega_T)$, $v_{i,j} \in C_\sharp^\infty(Y_i)$ for $i = 1, \ldots, j-1$, $v_{j,j} \in C_\sharp^\infty(Y_j)/\mathbb{R}$ and $c_{k,j} \in C_\sharp^\infty(S_k)$ for $k = 1, \ldots, m$.

In order to proceed we employ Evans's perturbed test function method, see [6],[7]. Introduce the arbitrary but fixed parameter $\theta > 0$ and the sequences $\{\phi^\mu\}$ and $\{\psi_j^\mu\}$ of such test functions satisfying

$$\phi^\mu \to u - \theta w \text{ in } L^2(0,T; H_0^1(\Omega))$$

and

$$\psi_j^\mu \to u_j - \theta w_j \text{ in } L^2(\Omega_T \times \mathcal{Y}_{j-1,m}; H_\sharp^1(Y_j)/\mathbb{R})$$

and a.e. in Ω_T and $\Omega_T \times \mathcal{Y}_{j,m}$, respectively, as $\mu \to \infty$ where w and w_j belong to the same function spaces as ϕ and ψ_j, respectively, for $j = 1, \ldots, n$. Using the introduced sequences we investigate the asymptotic behavior when $\varepsilon \to 0$ of the two norms in the last term of the left-hand side of (15). By the reverse and ordinary triangle inequalities,

$$\begin{aligned}&\left| \left\|u^\varepsilon - \left(\phi^\mu + \sum_{j=1}^n \varepsilon^{q_j}\left[\psi_j^\mu\right]^\varepsilon\right)\right\|_{L^2(\Omega_T)} - \|u - \phi^\mu\|_{L^2(\Omega_T)}\right| \\ &\leq \|u^\varepsilon - u\|_{L^2(\Omega_T)} + \sum_{j=1}^n \varepsilon^{q_j}\left\|\left[\psi_j^\mu\right]^\varepsilon\right\|_{L^2(\Omega_T)},\end{aligned}$$

and since $u^\varepsilon \to u$ in $L^{2(\gamma+1)}(\Omega_T)$ for some subsequence, see Theorem 2.4, it holds that $u^\varepsilon \to u$ in $L^2(\Omega_T)$ implying

$$\left\|u^\varepsilon - \left(\phi^\mu + \sum_{j=1}^n \varepsilon^{q_j}\left[\psi_j^\mu\right]^\varepsilon\right)\right\|_{L^2(\Omega_T)} \to \|u - \phi^\mu\|_{L^2(\Omega_T)} \tag{16}$$

as ε tends to zero and we have taken care of the first norm. Expanding the square of the second norm and using (11), Theorem 2.4 and Proposition 1 we deduce that, up to a sub sequence,

$$\begin{aligned}&\left\|\nabla u^{\varepsilon}-\nabla\left(\phi^{\mu}+\sum_{j=1}^{n}\varepsilon^{q_j}\left[\psi_j^{\mu}\right]^{\varepsilon}\right)\right\|_{L^2(\Omega_T)^N} \qquad (17)\\ \to\quad &\left\|\left(\nabla u+\sum_{j=1}^{n}\nabla_{y_j}u_j\right)-\left(\nabla\phi^{\mu}+\sum_{j=1}^{n}\nabla_{y_j}\psi_j^{\mu}\right)\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\end{aligned}$$

when $\varepsilon\to 0$.

Passing to the limit by letting $\varepsilon\to 0$ in (15) using (16), (17), (14) and Proposition 5, taking Remark 5 into consideration, we get

$$\begin{aligned}&\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\Bigg(g_0\left(x,t,y^n,s^m\right)\\ &-g\left(y^n,s^m,u\left(x,t\right),\nabla\phi^{\mu}\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}\psi_j^{\mu}\left(x,t,y^j,s^m\right)\right)\Bigg) \qquad (18)\\ &\times\left(u\left(x,t\right)-\phi^{\mu}\left(x,t\right)\right)dy^n ds^m dxdt\\ &+C_5\left\|u-\phi^{\mu}\right\|_{L^2(\Omega_T)}\left\|\left(\nabla u+\sum_{j=1}^{n}\nabla_{y_j}u_j\right)-\left(\nabla\phi^{\mu}+\sum_{j=1}^{n}\nabla_{y_j}\psi_j^{\mu}\right)\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\geq 0\end{aligned}$$

up to a subsequence. Observe that by (B2) we have the majorization

$$\begin{aligned}&\left|g\left(y^n,s^m,u\left(x,t\right),\nabla\phi^{\mu}\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}\psi_j^{\mu}\left(x,t,y^j,s^m\right)\right)\left(u\left(x,t\right)-\phi^{\mu}\left(x,t\right)\right)\right|\\ &\leq C_0\left(1+\left|u\left(x,t\right)\right|^{\gamma+1}+\left|\nabla\phi^{\mu}\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}\psi_j^{\mu}\left(x,t,y^j,s^m\right)\right|\right)\\ &\times\left|\left(u\left(x,t\right)-\phi^{\mu}\left(x,t\right)\right)\right|,\end{aligned}$$

and using Lebesgue's Generalized Majorized Convergence Theorem we get

$$\begin{aligned}&\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}g\left(y^n,s^m,u\left(x,t\right),\nabla\phi^{\mu}+\sum_{j=1}^{n}\nabla_{y_j}\psi_j^{\mu}\right)\\ &\times\left(u\left(x,t\right)-\phi^{\mu}\left(x,t\right)\right)dy^n ds^m dxdt\\ &\to\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}g\left(y^n,s^m,u\left(x,t\right),\nabla\left(u-\theta w\right)+\sum_{j=1}^{n}\nabla_{y_j}\left(u_j-\theta w_j\right)\right)\\ &\times\theta w\left(x,t\right)dy^n ds^m dxdt\end{aligned}$$

as $\mu \to \infty$. Hence, (18) becomes

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\Bigg(g_0\left(x,t,y^n,s^m\right)$$

$$-g\Bigg(y^n,s^m,u(x,t),\nabla(u(x,t)-\theta w(x,t))+\sum_{j=1}^{n}\nabla_{y_j}\big(u_j(x,t,y^j,s^m)-\theta w_j(x,t,y^j,s^m)\big)\Bigg)\Bigg)$$

$$\times\theta w\left(x,t\right)dy^n ds^m dxdt+C_5\left\|\theta w\right\|_{L^2(\Omega_T)}\left\|\theta\nabla w+\theta\sum_{j=1}^{n}\nabla_{y_j}w_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\geq 0,$$

and dividing by θ followed by letting θ tend to zero we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\left(g_0\left(x,t,y^n,s^m\right)-g\left(y^n,s^m,u\left(x,t\right),\nabla u\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right)\right)$$
$$\times w\left(x,t\right)dy^n ds^m dxdt\geq 0$$

where we have used Lebesgue's Generalized Majorized Convergence Theorem in a similar manner as in the limit process with respect to μ above. Since $w \in D(\Omega_T)$ is arbitrary the inequality is in fact an equality and we have

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}g_0\left(x,t,y^n,s^m\right)w\left(x,t\right)dy^n ds^m dxdt$$

$$=\int_{\Omega_T}\left(\int_{\mathcal{Y}_{n,m}}g\left(y^n,s^m,u\left(x,t\right),\nabla u\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right)dy^n ds^m\right)$$
$$\times w\left(x,t\right)dxdt$$

for any $w \in D(\Omega_T)$. In particular, the evolution multiscale convergence (14) holds with respect to test functions in the subspace $D(\Omega_T)$ of $L^2\left(\Omega_T; C_\sharp\left(\mathcal{Y}_{n,m}\right)\right)$ and thus the proof is complete. □

3.2. **The main homogenization result.** Before stating the main homogenization result we introduce the characteristic numbers d_i and ρ_i for $i = 1,\ldots,n$ defined with respect to the scale exponents $0 < q_1 < \cdots < q_n$ and $0 < r_1 < \cdots < r_m$ appearing in the problem (3) studied:

- If $q_i < r_1$, then $d_i = m$; if $r_j \leq q_i < r_{j+1}$ for some $j = 1,\ldots,m-1$, then $d_i = m - j$; and if $q_i \geq r_m$, then $d_i = 0$.
- If $q_i = r_j$ for some $j = 1,\ldots,m$ we let $\rho_i = 1$, otherwise $\rho_i = 0$.

Remark 7. The number ρ_i shows whether there is resonance ($\rho_i = 1$) or not ($\rho_i = 0$) with respect to the i-th spatial scale. The meaning of d_i is simply that its value is the number of temporal scales "faster" than the spatial scale in question.

We are now prepared to formulate the main result of the paper.

Theorem 3.2. *Let* $\{u^\varepsilon\}$ *be the sequence of unique weak solutions in* $V_0(0,T;H_0^1(\Omega), L^2(\Omega))$ *to* (3). *Then it holds that*

$$\begin{aligned} u^\varepsilon(x,t) &\to u(x,t) \text{ in } L^{2(\gamma+1)}(\Omega_T), \\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \text{ in } L^{2(\gamma+1)}(0,T;H_0^1(\Omega)), \\ \partial_t u^\varepsilon(x,t) &\to \partial_t u(x,t) \text{ in } L^2(\Omega_T), \\ \partial_{tt} u^\varepsilon(x,t) &\rightharpoonup \partial_{tt} u(x,t) \text{ in } L^2(0,T;H^{-1}(\Omega)) \end{aligned}$$

and

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^{m-d_j}\right),$$

where $u \in V_0(0,T;H_0^1(\Omega), L^2(\Omega))$ *is the unique weak solution to*

$$\begin{aligned} \partial_{tt} u(x,t) - \nabla\cdot(b(x,t)\nabla u(x,t)) & \\ +h(x,t,u(x,t),\nabla u(x,t)) &= f(x,t) \text{ in } \Omega_T, \\ u(x,t) &= 0 \text{ on } \partial\Omega\times(0,T), \\ u(x,0) &= 0 \text{ in } \Omega, \\ \partial_t u(x,0) &= 0 \text{ in } \Omega, \end{aligned} \tag{19}$$

with

$$b(x,t)\nabla u(x,t) = \int_{\mathcal{Y}_{n,m}} a(y^n,s^m)\left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^{m-d_j}\right)\right) dy^n ds^m \tag{20}$$

and

$$h(x,t,u(x,t),\nabla u(x,t)) = \int_{\mathcal{Y}_{n,m}} g\left(y^n,s^m,u(x,t),\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^{m-d_j}\right)\right) dy^n ds^m. \tag{21}$$

Here $u_j \in L^2(\Omega_T\times\mathcal{Y}_{j-1,m-d_j}; H_\sharp^1(Y_j)/\mathbb{R})$, $j=1,\dots,n$, *are the unique weak solutions to the system of local problems*

$$\rho_i \partial_{s_{m-d_i}s_{m-d_i}} u_i\left(x,t,y^i,s^{m-d_i}\right) - \nabla_{y_i}\cdot\int_{S_{m-d_i+1}}\cdots\int_{S_m}\int_{Y_{i+1}}\cdots\int_{Y_n} a(y^n,s^m) \times\left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^{m-d_j}\right)\right) dy_n\cdots dy_{i+1} ds_m\cdots ds_{m-d_i+1} = 0, \tag{22}$$

for $i=1,\dots,n$.

Remark 8. When $d_i = 0$ in (22) the interpretation is that there is no local temporal integration involved and that there is no established independence of any local temporal variable. Analogously, there is no local spatial integration if $i=n$.

Remark 9. The local problems do not involve the dissipative term g. This implies that the correctors are independent of local spatial variables not present in a, i.e., those exclusive to g. In particular this means that the correctors associated with such local spatial variables vanish due to the mean value zero property. Hence, only local problems corresponding to local spatial scales present in a are generative, i.e., generates actual contribution to the system of local problems.

For solutions to (3) it is possible to verify assumption (11) of Proposition 6, utilized in the proof of Theorem 3.2, according to the lemma below.

Lemma 3.3. *Let $\{u^\varepsilon\}$ be the sequence of unique weak solutions in $V_0(0,T;H_0^1(\Omega), L^2(\Omega))$ to (3). Then, up to a subsequence,*

$$\lim_{\varepsilon\to 0}\|\nabla u^\varepsilon\|_{L^2(\Omega_T)^N} = \left\|\nabla u + \sum_{j=1}^{n}\nabla_{y_j}u_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N} \tag{23}$$

where u and u_j, $j=1,\dots,n$, are given according to Theorem 2.4.

Proof. To begin with, by Theorem 3.1 there in fact exists a unique weak solution $u^\varepsilon \in V_0(0,T;H_0^1(\Omega), L^2(\Omega))$ to (3). Introduce $\omega \in D(\Omega_T)$ and, for each index $j=1,\dots,n$, let σ_j be a finite linear combination of products on the form $\omega_j v_{1,j}\cdots v_{j,j}c_{1,j}\cdots c_{m,j}$ where $\omega_j \in D(\Omega_T)$, $v_{i,j}\in C_\sharp^\infty(Y_i)$ for $i=1,\dots,j-1$, $v_{j,j}\in C_\sharp^\infty(Y_j)/\mathbb{R}$ and $c_{k,j}\in C_\sharp^\infty(S_k)$ for $k=1,\dots,m$. We then have that

$$\begin{aligned}
&\left|\|\nabla u^\varepsilon\|_{L^2(\Omega_T)^N} - \left\|\nabla u + \sum_{j=1}^{n}\nabla_{y_j}u_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\right| \\
&\leq \left|\|\nabla u^\varepsilon\|_{L^2(\Omega_T)^N} - \left\|\nabla\left(\omega + \sum_{j=1}^{n}\varepsilon^{q_j}[\sigma_j]^\varepsilon\right)\right\|_{L^2(\Omega_T)^N}\right| \\
&+ \left|\left\|\nabla\left(\omega + \sum_{j=1}^{n}\varepsilon^{q_j}[\sigma_j]^\varepsilon\right)\right\|_{L^2(\Omega_T)^N} - \left\|\nabla\omega + \sum_{j=1}^{n}\nabla_{y_j}\sigma_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\right| \\
&+ \left|\left\|\nabla\omega + \sum_{j=1}^{n}\nabla_{y_j}\sigma_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N} - \left\|\nabla u + \sum_{j=1}^{n}\nabla_{y_j}u_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\right|.
\end{aligned} \tag{24}$$

We investigate the right-hand side with respect to its limit as $\varepsilon\to 0$ term by term and then add up the result.

The first term. For the first term we have

$$\begin{aligned}
&\left|\|\nabla u^\varepsilon\|_{L^2(\Omega_T)^N} - \left\|\nabla\left(\omega + \sum_{j=1}^{n}\varepsilon^{q_j}[\sigma_j]^\varepsilon\right)\right\|_{L^2(\Omega_T)^N}\right| \\
&\leq \left\|\nabla u^\varepsilon - \nabla\left(\omega + \sum_{j=1}^{n}\varepsilon^{q_j}[\sigma_j]^\varepsilon\right)\right\|_{L^2(\Omega_T)^N}
\end{aligned} \tag{25}$$

by the reverse triangle inequality. By condition (A3) and the weak formulation (4) we have

$$\left\|\nabla u^\varepsilon - \nabla\left(\omega + \sum_{j=1}^{n}\varepsilon^{q_j}[\sigma_j]^\varepsilon\right)\right\|^2_{L^2(\Omega_T)^N}$$

$$\leq \frac{1}{\alpha}\int_{\Omega_T} a\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}}\right)\left(\nabla u^\varepsilon(x,t)-\nabla\left(\omega(x,t)+\sum_{j=1}^n \varepsilon^{q_j}\sigma_j\left(x,t,\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)\right)\right)$$

$$\cdot\left(\nabla u^\varepsilon(x,t)-\nabla\left(\omega(x,t)+\sum_{j=1}^n \varepsilon^{q_j}\sigma_j\left(x,t,\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)\right)\right)dxdt$$

$$= \frac{1}{\alpha}\left(\int_{\Omega_T} f(x,t)\,u^\varepsilon(x,t)\,dxdt - \int_0^T \langle \partial_{tt}u^\varepsilon(t), u^\varepsilon(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt \right. \tag{26}$$

$$- \int_{\Omega_T} g\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}},u^\varepsilon(x,t),\nabla u^\varepsilon(x,t)\right)u^\varepsilon(x,t)\,dxdt$$

$$+\int_{\Omega_T} a\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}}\right)\nabla\left(\omega(x,t)+\sum_{j=1}^n \varepsilon^{q_j}\sigma_j\left(x,t,\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)\right)$$

$$\cdot\nabla\left(\omega(x,t)+\sum_{j=1}^n \varepsilon^{q_j}\sigma_j\left(x,t,\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)\right)dxdt$$

$$\left. -2\int_{\Omega_T} a\left(\frac{x}{\varepsilon^{q^n}},\frac{t}{\varepsilon^{r^m}}\right)\nabla u^\varepsilon(x,t)\cdot\nabla\left(\omega(x,t)+\sum_{j=1}^n \varepsilon^{q_j}\sigma_j\left(x,t,\frac{x}{\varepsilon^{q^j}},\frac{t}{\varepsilon^{r^m}}\right)\right)dxdt\right).$$

In order to proceed with the inequality we must verify that

$$\lim_{\varepsilon\to 0}\int_0^T \langle \partial_{tt}u^\varepsilon(t), u^\varepsilon(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt = \int_0^T \langle \partial_{tt}u(t), u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt. \tag{27}$$

We first observe that

$$\int_0^T \langle \partial_{tt}u^\varepsilon(t), u^\varepsilon(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt = \langle \partial_t u^\varepsilon(T), u^\varepsilon(T)\rangle_{L^2(\Omega),L^2(\Omega)} - \|\partial_t u^\varepsilon\|^2_{L^2(\Omega_T)},$$

see e.g. Section 24.3 in [34]. Let us introduce the space

$$\Lambda(0,T,\Omega) = \left\{v \in C([0,T];H_0^1(\Omega)) : \partial_t v \in C([0,T];L^2(\Omega))\right\}$$

equipped with the norm

$$\|v\|_{\Lambda(0,T,\Omega)} = \sup_{0\leq t\leq T}\|v(t)\|_{H_0^1(\Omega)} + \sup_{0\leq t\leq T}\|\partial_t v(t)\|_{L^2(\Omega)}.$$

We have that $\{u^\varepsilon\}$ is bounded in $V_0(0,T;H_0^1(\Omega),L^2(\Omega))$ due to Proposition 3 and, as noted in e.g. Subsection 2.2 in [22], that $V_0(0,T;H_0^1(\Omega),L^2(\Omega))$ is continuously embedded in $\Lambda(0,T,\Omega)$ which implies that $\{u^\varepsilon\}$ is bounded in $\Lambda(0,T,\Omega)$. Hence, $\{\partial_t u^\varepsilon(T)\}$ and $\{u^\varepsilon(T)\}$ are bounded in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Thus

$$\partial_t u^\varepsilon(T) \rightharpoonup \partial_t u(T) \text{ in } L^2(\Omega)$$

and, since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$,

$$u^\varepsilon(T) \to u(T) \text{ in } L^2(\Omega);$$

both convergences are up to subsequences. According to the 'weak-strong' convergence property, see e.g. Proposition 1.19 in [5], we have

$$\lim_{\varepsilon\to 0}\langle \partial_t u^\varepsilon(T), u^\varepsilon(T)\rangle_{L^2(\Omega),L^2(\Omega)} = \langle \partial_t u(T), u(T)\rangle_{L^2(\Omega),L^2(\Omega)}. \tag{28}$$

Moreover, using Proposition 4,

$$\lim_{\varepsilon\to 0}\|\partial_t u^\varepsilon\|_{L^2(\Omega_T)}=\|\partial_t u\|_{L^2(\Omega_T)} \tag{29}$$

holds up to a subsequence. Combining (28) and (29) we get (27).

Letting ε tend to zero and employing (27) we get for the right-hand side of (26) that its limit is equal to

$$\begin{aligned}
&\frac{1}{\alpha}\Bigg(\int_{\Omega_T} f(x,t)\,u(x,t)\,dxdt-\int_0^T\langle\partial_{tt}u(t),u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt\\
&\quad-\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} g_0(x,t,y^n,s^m)\,u(x,t)\,dy^nds^mdxdt\\
&\quad+\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} a(y^n,s^m)\nabla\left(\omega(x,t)+\sum_{j=1}^n\sigma_j\left(x,t,y^j,s^m\right)\right)\\
&\quad\cdot\nabla\left(\omega(x,t)+\sum_{j=1}^n\sigma_j\left(x,t,y^j,s^m\right)\right)dy^nds^mdxdt\\
&\quad-2\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} a(y^n,s^m)\left(\nabla u(x,t)+\sum_{j=1}^n\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right)\\
&\quad\cdot\nabla\left(\omega(x,t)+\sum_{j=1}^n\sigma_j\left(x,t,y^j,s^m\right)\right)dy^nds^mdxdt\Bigg)\\
&=\frac{1}{\alpha}\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} a(y^n,s^m)\left(\left(\nabla u(x,t)+\sum_{j=1}^n\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right)\right.\\
&\left.-\nabla\left(\omega(x,t)+\sum_{j=1}^n\sigma_j\left(x,t,y^j,s^m\right)\right)\right)\cdot\left(\left(\nabla u(x,t)+\sum_{j=1}^n\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right)\right.\\
&\left.-\nabla\left(\omega(x,t)+\sum_{j=1}^n\sigma_j\left(x,t,y^j,s^m\right)\right)\right)dy^nds^mdxdt.
\end{aligned}$$

Fix some $\delta>0$. Then, for some $\varepsilon_1>0$, it holds that

$$\begin{aligned}
&\left\|\nabla u^\varepsilon-\nabla\left(\omega+\sum_{j=1}^n\varepsilon^{q_j}\left[\sigma_j\right]^\varepsilon\right)\right\|^2_{L^2(\Omega_T)^N}\\
&\le\frac{1}{\alpha}\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} a(y^n,s^m)\left(\left(\nabla u(x,t)+\sum_{j=1}^n\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right)\right.\\
&\left.-\nabla\left(\omega(x,t)+\sum_{j=1}^n\sigma_j\left(x,t,y^j,s^m\right)\right)\right)\cdot\left(\left(\nabla u(x,t)+\sum_{j=1}^n\nabla_{y_j}u_j\left(x,t,y^j,s^m\right)\right)\right.
\end{aligned}$$

$$-\nabla\left(\omega\left(x,t\right)+\sum_{j=1}^{n}\sigma_{j}\left(x,t,y^{j},s^{m}\right)\right)\Bigg)\,dy^{n}ds^{m}dxdt+\frac{3\delta^{2}}{16}$$

for all $0<\varepsilon\leq\varepsilon_1$ (with respect to the extracted subsequence). Due to density we may choose ω and σ_j, $j=1,\dots,n$, such that

$$\begin{aligned}&\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}a\left(y^{n},s^{m}\right)\left(\left(\nabla u\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^{j},s^{m}\right)\right)\right.\\&-\left.\left(\nabla\omega\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}\sigma_j\left(x,t,y^{j},s^{m}\right)\right)\right)\cdot\left(\left(\nabla u\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^{j},s^{m}\right)\right)\right.\\&-\left.\left(\nabla\omega\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}\sigma_j\left(x,t,y^{j},s^{m}\right)\right)\right)dy^{n}ds^{m}dxdt\leq\frac{\alpha\delta^{2}}{16}.\end{aligned}\tag{30}$$

Hence, by (25) and (30) we have

$$\left|\left\|\nabla u^{\varepsilon}\right\|_{L^2(\Omega_T)^N}-\left\|\nabla\left(\omega+\sum_{j=1}^{n}\varepsilon^{q_j}\left[\sigma_j\right]^{\varepsilon}\right)\right\|_{L^2(\Omega_T)^N}\right|\leq\frac{\delta}{2}\tag{31}$$

for every $0<\varepsilon\leq\varepsilon_1$.

The second term. For the second term on the right-hand side of (24) we have that there exists an $\varepsilon_2>0$ such that

$$\left|\left\|\nabla\left(\omega+\sum_{j=1}^{n}\varepsilon^{q_j}\left[\sigma_j\right]^{\varepsilon}\right)\right\|_{L^2(\Omega_T)^N}-\left\|\nabla\omega+\sum_{j=1}^{n}\nabla_{y_j}\sigma_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\right|\leq\frac{\delta}{4}\tag{32}$$

for any $0<\varepsilon\leq\varepsilon_2$ by the fundamental convergence result of Proposition 1.

The third term. For the third term we get

$$\begin{aligned}&\left|\left\|\nabla\omega+\sum_{j=1}^{n}\nabla_{y_j}\sigma_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}-\left\|\nabla u+\sum_{j=1}^{n}\nabla_{y_j}u_j\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})^N}\right|^{2}\\&\leq\frac{1}{\alpha}\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}a\left(y^{n},s^{m}\right)\left(\left(\nabla u\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^{j},s^{m}\right)\right)\right.\\&-\left.\left(\nabla\omega\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}\sigma_j\left(x,t,y^{j},s^{m}\right)\right)\right)\cdot\left(\left(\nabla u\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^{j},s^{m}\right)\right)\right.\\&-\left.\left(\nabla\omega\left(x,t\right)+\sum_{j=1}^{n}\nabla_{y_j}\sigma_j\left(x,t,y^{j},s^{m}\right)\right)\right)dy^{n}ds^{m}dxdt\leq\frac{\delta^{2}}{16}\end{aligned}\tag{33}$$

by the reverse triangle inequality, (A3) and (30).

Adding up the terms. Utilizing (31), (32) and (33) in (24) we obtain

$$\left| \|\nabla u^{\varepsilon}\|_{L^2(\Omega_T)^N} - \left\| \nabla u + \sum_{j=1}^{n} \nabla_{y_j} u_j \right\|_{L^2(\Omega_T \times \mathcal{Y}_{n,m})^N} \right| \leq \delta \tag{34}$$

for every $0 < \varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$. This means that the left-hand side of (34) tends to zero as ε does. Thus, we have verified (23) and the proof is complete. □

Remark 10. The density property referred to in the proof is that $D(\Omega_T)$ and $D(\Omega_T) \otimes C_{\sharp}^{\infty}(Y_1) \otimes \cdots \otimes C_{\sharp}^{\infty}(Y_{j-1}) \otimes C_{\sharp}^{\infty}(Y_j)/\mathbb{R} \otimes C_{\sharp}^{\infty}(S_1) \otimes \cdots \otimes C_{\sharp}^{\infty}(S_m)$ are dense in $V_0(0,T; H_0^1(\Omega), L^2(\Omega))$ and $L^2(\Omega_T \times \mathcal{Y}_{j-1,m}; H_{\sharp}^1(Y_j)/\mathbb{R})$, respectively, for $j = 1, \ldots, n$.

We also need the following lemma, henceforth referred to as the Periodic Generalized Variational Lemma.

Lemma 3.4. *Let G be a non-empty open set in $\mathbb{R}$ and assume that $w \in L_{\text{loc}}^1(G)$ and suppose that*

$$\int_G w(s) \frac{d^{\mu}}{ds^{\mu}} \phi(s)\, ds = 0$$

for all $\phi \in C_{\sharp}^{\infty}(S)$. Then, for some constant $C \in \mathbb{R}$, $w(s) = C$ for a.e. $s \in G$ where $C = 0$ if $\mu = 0$.

Proof. Suppose $\mu = 0$. The statement in this case follows by the Generalized Variational Lemma, see e.g. Proposition 18.36 in [34], since the set of S-periodic repetitions of functions in $D(S)$ is a mere subset of $C_{\sharp}^{\infty}(S)$.

Suppose $\mu > 0$. Since the space of all ϕ spans $C_{\sharp}^{\infty}(S)$ we have that the corresponding derivatives $\frac{d}{ds}\phi$ will span $C_{\sharp}^{\infty}(S)/\mathbb{R}$. Hence, by induction, we have that the set of all $\frac{d^{\mu}}{ds^{\mu}}\phi$ spans $C_{\sharp}^{\infty}(S)/\mathbb{R}$ for any $\mu > 0$. Then, the claim follows by Corollary 18.37 in [34] where $C_{\sharp}^{\infty}(S)/\mathbb{R}$ is used instead of $D(S)/\mathbb{R}$ as the space of test functions similar to the argument in the case $\mu = 0$. □

We are now prepared to carry out the proof of the main result.

Proof of Theorem 3.2. To begin with, as already noted in the proof of Lemma 3.3, there exists a bounded sequence $\{u^{\varepsilon}\}$ in $V_0(0,T; H_0^1(\Omega), L^2(\Omega))$ of unique weak solutions. Hence, by Theorem 2.4 it holds up to a subsequence that

$$\begin{aligned}
u^{\varepsilon}(x,t) &\to u(x,t) \text{ in } L^{2(\gamma+1)}(\Omega_T), \\
u^{\varepsilon}(x,t) &\rightharpoonup u(x,t) \text{ in } L^{2(\gamma+1)}(0,T; H_0^1(\Omega)), \\
\partial_t u^{\varepsilon}(x,t) &\rightharpoonup \partial_t u(x,t) \text{ in } L^2(\Omega_T), \\
\partial_{tt} u^{\varepsilon}(x,t) &\rightharpoonup \partial_{tt} u(x,t) \text{ in } L^2(0,T; H^{-1}(\Omega))
\end{aligned}$$

and

$$\nabla u^{\varepsilon}(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j\left(x,t,y^j,s^m\right)$$

where $u \in V_0(0,T; H_0^1(\Omega), L^2(\Omega))$ and $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1,m}; H_{\sharp}^1(Y_j)/\mathbb{R})$ for $j = 1, \ldots, n$. Moreover, Proposition 4 guarantees that the convergence of $\{\partial_t u^{\varepsilon}\}$ in $L^2(\Omega_T)$ is in fact strong.

Let us consider the weak formulation (4) of (3), i.e.,

$$\int_{\Omega_T} u^{\varepsilon}(x,t)\, v(x)\, \partial_{tt} c(t)\, dxdt + \int_{\Omega_T} a\left(\frac{x}{\varepsilon^{q^n}}, \frac{t}{\varepsilon^{r^m}}\right) \nabla u^{\varepsilon}(x,t)\, \nabla v(x)\, c(t)\, dxdt \quad (35)$$
$$+\int_{\Omega_T} g\left(\frac{x}{\varepsilon^{q^n}}, \frac{t}{\varepsilon^{r^m}}, u^{\varepsilon}(x,t), \nabla u^{\varepsilon}(x,t)\right) v(x)\, c(t)\, dxdt = \int_{\Omega_T} f(x,t)\, v(x)\, c(t)\, dxdt$$

where $v \in H_0^1(\Omega)$ and $c \in D(0,T)$.

First we derive the homogenized problem (19). Passing to the limit in (35) using Theorem 2.4 and Proposition 6, which is applicable due to Lemma 3.3, we obtain

$$\int_{\Omega_T} u(x,t)\, v(x)\, \partial_{tt} c(t)$$
$$+\left(\int_{\mathcal{Y}_{n,m}} a(y^n, s^m)\left(\nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j\left(x,t,y^j,s^m\right)\right) \nabla v(x)\, c(t)\right.$$
$$\left.+g\left(y^n, s^m, u(x,t), \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j\left(x,t,y^j,s^m\right)\right) dy^n ds^m\right) v(x)\, c(t)\, dxdt$$
$$= \int_{\Omega_T} f(x,t)\, v(x)\, c(t)\, dxdt$$

which is the weak form of (19).

Our next aim is to extract the local problem (22) for each $i = 1, \dots, n$ and the associated independencies with respect to the local variables. In order to do so fix $i = 1, \dots, n$ and introduce the test functions

$$v(x) = \varepsilon^p v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right),\ p > 0$$

and

$$c(t) = c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right),\ \lambda = 1, \dots, m$$

with $v_1 \in D(\Omega)$, $v_j \in C_\sharp^\infty(Y_{j-1})$ for $j = 2, \dots, i$, $v_{i+1} \in C_\sharp^\infty(Y_i)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_l \in C_\sharp^\infty(S_{l-1})$ for $l = 2, \dots, \lambda+1$. We choose p and λ later. With these test functions used in (35) we obtain

$$\int_{\Omega_T} u^{\varepsilon}(x,t)\, \varepsilon^p v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times\left(\partial_{tt} c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\right.$$
$$+2\sum_{l=2}^{\lambda+1} \varepsilon^{-r_{l-1}} \partial_t c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots \partial_{s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)$$
$$+\sum_{l=2}^{\lambda+1} \varepsilon^{-2r_{l-1}} c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots \partial_{s_{l-1}s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)$$
$$+2\sum_{l=3}^{\lambda+1}\sum_{\mu=2}^{l-1} \varepsilon^{-(r_{\mu-1}+r_{l-1})} c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots \partial_{s_{\mu-1}} c_\mu\left(\frac{t}{\varepsilon^{r_{\mu-1}}}\right)$$

$$\cdots \partial_{s_{l-1}} c_l \left(\frac{t}{\varepsilon^{r_{l-1}}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\Bigg)$$
$$+a\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}\right) \nabla u^\varepsilon(x,t) \cdot \Bigg(\varepsilon^p \nabla v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$+\sum_{j=2}^{i+1} \varepsilon^{p-q_{j-1}} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots \nabla_{y_{j-1}} v_j\left(\frac{x}{\varepsilon^{q_{j-1}}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\Bigg)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)$$
$$+g\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}, u^\varepsilon(x,t), \nabla u^\varepsilon(x,t)\right) \varepsilon^p v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dxdt$$
$$= \int_{\Omega_T} f(x,t)\, \varepsilon^p v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dxdt$$

and passing to the limit using Theorem 2.4 we immediately arrive, up to a subsequence, at

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \varepsilon^{-q_i} u^\varepsilon(x,t)\, v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times \Bigg(2\sum_{l=2}^{\lambda+1} \varepsilon^{p+q_i-r_{l-1}} \partial_t c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots \partial_{s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)$$
$$+\sum_{l=2}^{\lambda+1} \varepsilon^{p+q_i-2r_{l-1}} c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots \partial_{s_{l-1}s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)$$
$$+2\sum_{l=3}^{\lambda+1}\sum_{\mu=2}^{l-1} \varepsilon^{p+q_i-(r_{\mu-1}+r_{l-1})} c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots \partial_{s_{\mu-1}} c_\mu\left(\frac{t}{\varepsilon^{r_{\mu-1}}}\right)$$
$$\cdots \partial_{s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\Bigg) + a\left(\frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_m}}\right) \nabla u^\varepsilon(x,t)$$
$$\cdot \sum_{j=2}^{i+1} \varepsilon^{p-q_{j-1}} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots \nabla_{y_{j-1}} v_j\left(\frac{x}{\varepsilon^{q_{j-1}}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dxdt = 0$$

where we have extracted a factor ε^{-q_i} in the part involving the time derivatives. Assume that $p + q_i - 2r_\lambda \geq 0$ and $p - q_i \geq 0$. Employing Theorems 2.6 and 2.4 we get, up to a subsequence,

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \varepsilon^{-q_i} u^\varepsilon(x,t)\, \varepsilon^{p+q_i-2r_\lambda} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_\lambda\left(\frac{t}{\varepsilon^{r_{\lambda-1}}}\right) \partial_{s_\lambda s_\lambda} c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) \tag{36}$$

$$+a\left(\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}\left(x,t\right)\cdot\varepsilon^{p-q_i}v_1(x)v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i}v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t)c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)dxdt=0$$

which is our starting point for the remainder of the proof.

Let us first derive the independencies of the local temporal variables. Suppose $d_i>0$. In what follows let λ start at the value m and run down to $m-d_i+1$ step by step. For each fixed λ choose $p=2r_\lambda-q_i$ giving $p+q_i-2r_\lambda=0$. Moreover, $p-q_i=2\left(r_\lambda-q_i\right)>0$ since $\lambda>m-d_i$. Hence, by (36) we get, up to a subsequence,

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\varepsilon^{-q_i}u^{\varepsilon}(x,t)\varepsilon^{0}v_1(x)v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t)c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_\lambda\left(\frac{t}{\varepsilon^{r_{\lambda-1}}}\right)\partial_{s_\lambda s_\lambda}c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)$$
$$+a\left(\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}\left(x,t\right)\varepsilon^{2(r_\lambda-q_i)}v_1(x)v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i}v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t)c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)dxdt=0,$$

utilizing Theorems 2.6 and 2.4 yields

$$\int_{\Omega_T}\int_{\mathcal{Y}_{i,\lambda}}u_i(x,t,y^i,s^\lambda)v_1(x)v_2\left(y_1\right)\cdots v_{i+1}\left(y_i\right)$$
$$\times c_1(t)c_2\left(s_1\right)\cdots c_\lambda\left(s_{\lambda-1}\right)\partial_{s_\lambda s_\lambda}c_{\lambda+1}\left(s_\lambda\right)dy^i ds^\lambda dxdt=0,$$

and using the Periodic Generalized Variational Lemma, i.e. Lemma 3.4, we have left

$$\int_{S_\lambda}u_i(x,t,y^i,s^\lambda)\partial_{s_\lambda s_\lambda}c_{\lambda+1}\left(s_\lambda\right)ds_\lambda=0,$$

a.e. in $\Omega_T\times\mathcal{Y}_{i,\lambda-1}$ and for all $c_{\lambda+1}\in C_\sharp^\infty\left(S_\lambda\right)$. Hence, by the same lemma, this means that u_i is independent of s_λ. Thus, we conclude that u_i does not depend on any of the local temporal variables $s_{m-d_i+1},\ldots,s_m$. Consequently, we have $u_i\in L^2(\Omega_T\times\mathcal{Y}_{i-1,m-d_i};H_\sharp^1(Y_i)/\mathbb{R})$ for $i=1,\ldots,n$.

The next step is to determine the local problems, one for each spatial scale. In order to derive the i-th local problem we choose $p=q_i$ and $\lambda=m-d_i$ where $d_i\geq 0$. We have $p+q_i-2r_\lambda=2\left(q_i-r_{m-d_i}\right)\geq 0$, with equality for $\rho_i=1$ (i.e., resonance), and $p-q_i=0$. This means that (36) is valid also for these choices of p and λ. We have, up to a subsequence,

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\varepsilon^{-q_i}u^{\varepsilon}(x,t)\varepsilon^{2(q_i-r_{m-d_i})}v_1(x)v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t)c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{m-d_i}\left(\frac{t}{\varepsilon^{r_{m-d_i-1}}}\right)\partial_{s_{m-d_i}s_{m-d_i}}c_{m-d_i+1}\left(\frac{t}{\varepsilon^{r_{m-d_i}}}\right)$$
$$+a\left(\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}\left(x,t\right)\cdot\varepsilon^{0}v_1(x)v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i}v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t)c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{m-d_i+1}\left(\frac{t}{\varepsilon^{r_{m-d_i}}}\right)dxdt=0,$$

using Theorems 2.6 and 2.4 and observing that $\varepsilon^{2(q_i - r_{m-d_i})} \to \rho_i$ we get

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} & \rho_i u_i(x,t,y^i,s^{m-d_i}) v_1(x) v_2(y_1)\cdots v_{i+1}(y_i) \\ & \times c_1(t)c_2(s_1)\cdots c_{m-d_i}(s_{m-d_i-1})\,\partial_{s_{m-d_i}s_{m-d_i}} c_{m-d_i+1}(s_{m-d_i}) \\ & + a(y^n,s^m)\left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^{m-d_i}\right)\right) \\ & \cdot v_1(x)v_2(y_1)\cdots v_i(y_{i-1})\nabla_{y_i} v_{i+1}(y_i) \\ & \times c_1(t)c_2(s_1)\cdots c_{m-d_i+1}(s_{m-d_i})\,dy^n ds^m dx dt = 0,\end{aligned}$$

and by the Periodic Generalized Variational Lemma we finally we arrive at

$$\begin{aligned}\int_{S_{m-d_i}}\cdots\int_{S_m}\int_{Y_i}\cdots\int_{Y_n} & \rho_i u_i(x,t,y^i,s^{m-d_i}) v_{i+1}(y_i)\,\partial_{(s_{m-d_i})(s_{m-d_i})} c_{m-d_i+1}(s_{m-d_i}) \\ & + a(y^n,s^m)\left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^{m-d_i}\right)\right) \\ & \cdot\nabla_{y_i} v_{i+1}(y_i)\,c_{m-d_i+1}(s_{m-d_i})\,dy_n\cdots dy_i ds_m\cdots ds_{m-d_i} = 0\end{aligned}$$

a.e. in $\Omega_T \times \mathcal{Y}_{i-1,m-d_i-1}$ and for all $v_{i+1} \in C_\sharp^\infty(Y_i)/\mathbb{R}$ and $c_{m-d_i+1} \in C_\sharp^\infty(S_{m-d_i})$. By density this holds for all $v_{i+1} \in H_\sharp^1(Y_i)/\mathbb{R}$ which means that we have obtained the weak form of the local problem. □

4. **An illustration of the use of the main homogenization result.** To instantiate the usefulness of Theorem 3.2 we look at an illustrative example problem exhibiting six spatial and eight temporal scales, i.e., $n = 5$ and $m = 7$. We study (3) with the evolution equation given by

$$\begin{aligned}\partial_{tt}u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{x}{\varepsilon^4},\frac{t}{\varepsilon},\frac{t}{\varepsilon^4},\frac{t}{\varepsilon^{\frac{9}{2}}}\right)\nabla u^\varepsilon(x,t)\right) & \qquad (37) \\ +g\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^3},\frac{x}{\varepsilon^{\frac{15}{2}}},\frac{t}{\varepsilon^{\frac{3}{2}}},\frac{t}{\varepsilon^{\frac{10}{3}}},\frac{t}{\varepsilon^6},\frac{t}{\varepsilon^7},u^\varepsilon(x,t),\nabla u^\varepsilon(x,t)\right) & = f(x,t) \text{ in } \Omega_T\end{aligned}$$

where $f \in L^2(\Omega_T)$ and the conditions (A1)–(A3) and (B1)–(B6) are assumed to be fulfilled for $\tilde{a}(y^5,s^7) = a(y^2,y_4,s_1,s_4,s_5)$ and $\tilde{g}(y^5,s^7) = g(y_1,y_3,y_5,s_2,s_3,s_6,s_7)$. In terms of $\tilde{a}$ and $\tilde{g}$ the equation (37) may be written as

$$\begin{aligned}\partial_{tt}u^\varepsilon(x,t) - \nabla\cdot\left(\tilde{a}\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{x}{\varepsilon^3},\frac{x}{\varepsilon^4},\frac{x}{\varepsilon^{\frac{15}{2}}},\frac{t}{\varepsilon},\frac{t}{\varepsilon^{\frac{3}{2}}},\frac{t}{\varepsilon^{\frac{10}{3}}},\frac{t}{\varepsilon^4},\frac{t}{\varepsilon^{\frac{9}{2}}},\frac{t}{\varepsilon^6},\frac{t}{\varepsilon^7}\right)\nabla u^\varepsilon(x,t)\right) & \\ +\tilde{g}\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{x}{\varepsilon^3},\frac{x}{\varepsilon^4},\frac{x}{\varepsilon^{\frac{15}{2}}},\frac{t}{\varepsilon},\frac{t}{\varepsilon^{\frac{3}{2}}},\frac{t}{\varepsilon^{\frac{10}{3}}},\frac{t}{\varepsilon^4},\frac{t}{\varepsilon^{\frac{9}{2}}},\frac{t}{\varepsilon^6},\frac{t}{\varepsilon^7},u^\varepsilon(x,t),\nabla u^\varepsilon(x,t)\right) & \qquad (38) \\ = f(x,t) \text{ in } \Omega_T. &\end{aligned}$$

Clearly, (38) is on a form for which Theorem 3.2 is directly applicable.

By Theorem 3.2 we get that $u \in V_0(0,T;H_0^1(\Omega),L^2(\Omega))$ solves the homogenized problem (19) with the effective coefficient b and dissipative term h given according to (20) and (21), respectively, for $n = 5$ and $m = 7$. Moreover, the correctors $u_i \in L^2(\Omega_T \times \mathcal{Y}_{3,7-d_i}; H_\sharp^1(Y_4)/\mathbb{R})$, $i = 1,\ldots,5$, are given by five local problems determined by the characteristic numbers d_i and ρ_i. Note that it is only local problems arising from spatial scales present in a that are generative, see Remark 9.

For $i = 1$ the number of temporal scales faster than the first, i.e. slowest, spatial scale ε is 6, i.e. $d_1 = 6$. The spatial scale in question coincides with a temporal scale which means that $\rho_1 = 1$. Furthermore, this spatial scale is present in a, i.e., the local problem is generative. In the same way d_i, ρ_i and the generative property for each $i = 2, \ldots, 5$ are derived and we summarize the result in Table 1.

i	d_i	ρ_i	Generative
1	6	1	Yes
2	5	0	Yes
3	5	0	No
4	3	1	Yes
5	0	0	No

Table 1

We now have the information required in order to extract all local problems. Since the procedure is straightforward we only carry out the details for one of them. If we, e.g., consider $i = 4$ then a glance in Table 1 gives that the relevant characteristic numbers are $d_4 = 3$ and $\rho_4 = 1$ and that the local problem is generative. Hence, $u_4 \in L^2(\Omega_T \times \mathcal{Y}_{3,4}; H^1_\sharp(Y_4)/\mathbb{R})$ and (22) becomes

$$\partial_{s_4 s_4} u_4\left(x,t,y^4,s^4\right) - \nabla_{y_4} \cdot \int_{S_5}\int_{S_6}\int_{S_7}\int_{Y_5} \tilde{a}\left(y^5, s^7\right) \tag{39}$$

$$\times\left(\nabla u\left(x,t\right) + \sum_{j=1}^{5} \nabla_{y_j} u_j\left(x,t,y^j,s^{7-d_j}\right)\right) dy_5 ds_7 ds_6 ds_5 = 0.$$

With the definition of $\tilde{a}$ in terms of a and using Table 1 again, equation (39) yields

$$\partial_{s_4 s_4} u_4\left(x,t,y^2,y_4,s_1,s_4\right) - \nabla_{y_4} \cdot \left(\int_{S_5} a\left(y^2,y_4,s_1,s_4,s_5\right) ds_5\right.$$

$$\times\Big(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right)$$

$$\left.\left. + \nabla_{y_4} u_4\left(x,t,y^2,y_4,s_1,\right)\right)\right) = 0$$

where we have used the fact that a only depends on y^2, y_4, s_1, s_4 and s_5. Clearly, this means that the correctors only depend on these local variables and hence we have that $u_4 \in L^2(\Omega_T \times Y^2 \times S_1 \times S_4; H^1_\sharp(Y_4)/\mathbb{R})$. The two remaining generative local problems can be determined in a similar manner.

What the illustrative example of this section demonstrates, apart from showing how to proceed in a special case, is that in spite of the fact that it may seem like that the oscillations of a and g match in the general homogenization result, it is possible to treat also problems with disparate oscillation patterns. In practice this is achieved by numbing out excessive oscillation modes.

Remark 11. An alternative approach in the study of problems of the type studied in the present paper involving several structure functions would have been to let the functions have formally different sets of local spatial and temporal scales but allowing any number of them to coincide. The reason we have chosen the method used here, i.e. to consider structure functions a and g with formally identical oscillation modes, is because it is more convenient to implement in practise.

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