NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **11**, Number **4**, December **2016**
doi:10.3934/nhm.2016009
pp. **545–562**

HOMOGENIZATION OF A THERMAL PROBLEM WITH FLUX JUMP

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(Communicated by Dag Lukkassen)

Abstract. The goal of this paper is to analyze, through homogenization techniques, the effective thermal transfer in a periodic composite material formed by two constituents, separated by an imperfect interface where both the temperature and the flux exhibit jumps. Following the hypotheses on the flux jump, two different homogenized problems are obtained. These problems capture in various ways the influence of the jumps: in the homogenized coefficients, in the right-hand side of the homogenized problem, and in the correctors.

1. **Introduction.** In the last two decades, the study of the macroscopic properties of heterogeneous composite materials with imperfect contact between their constituents has been a subject of major interest for engineers, mathematicians, physicists. In particular, the problem of thermal transfer in such heterogeneous media has attracted the attention of a broad category of researchers, due to the fact that the macroscopic properties of a composite can be strongly influenced by the imperfect bonding between its components. This imperfect contact can be generated by various causes: the presence of impurities at the boundaries, the presence of a thin interphase, the interface damage, chemical processes.

The homogenization of a thermal problem in a two-component composite with interfacial barrier, with jump of the temperature and continuity of the flux, was studied for the first time in the pioneering work [4], where the asymptotic expansion method was used. Many mathematical studies were performed since then, in order to rigorously justify the convergence results. Various mathematical methods were used: the energy method in [17] and [31], the two-scale convergence method in [19], and more recently the unfolding method for periodic homogenization in [16] and [35], to cite just a few of them. The main common point of all these studies is the fact that at the interface between the two components the flux of the temperature is continuous, the temperature field has a jump and the flux is proportional to this jump. Several cases are studied, following the order of magnitude with respect to the small parameter ε of the resistance generated by the imperfect contact between

2010 *Mathematics Subject Classification.* Primary: 35B27, 80M35; Secondary: 80M40.
Key words and phrases. Homogenization, imperfect interface, the periodic unfolding method.

the constituents, leading to completely different macroscopic problems. Here, ε is a small real parameter related to the characteristic size of the two constituents. In some cases, an effect of the imperfect conditions is observed in the coefficients of the homogenized matrix, via the local problems; in other cases, there is no effect at all in the homogenized problem.

For similar homogenization problems of parabolic or hyperbolic type, we refer the reader to [14] and [28]. For the case when both components of the composite material are connected, we refer to [30], [32], [33], [34], [35] and [36]. Also, for problems involving jumps in the solution in other contexts, such as heat transfer in polycrystals with interfacial resistance, linear elasticity problems or problems modeling the electrical conduction in biological tissues, see [1], [2], [18], [20], [21], [22], [26] and [37].

Our goal in this paper is to analyze the effective thermal transfer in a periodic composite material formed by two constituents, one connected and the other one disconnected, separated by an imperfect interface where both the temperature and the flux exhibit jumps. This mathematical model is not restricted to the thermal transfer, but can be used in other contexts, too. Transmission problems involving jumps in the solutions or in the fluxes are encountered in various domains, such as linear elasticity, theory of semiconductors, the study of photovoltaic systems or problems in media with cracks (see, for instance, [3], [6], [7], [25] and [29]). Formal methods of averaging were widely used in the literature to deal with such imperfect transmission problems. Still, obtaining rigorous results based on the homogenization theory is a difficult task in many cases. Some results were nevertheless obtained for problems with flux jump, by using homogenization techniques. We mention here the results obtained in [24] for reaction-diffusion problems in porous media, in [27] for problems arising in the combustion theory and in [23] for a problem corresponding to the Gouy-Chapman-Stern model for an electric double layer.

Here, we consider a composite material occupying an open bounded set Ω in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz-continuous boundary $\partial\Omega$. We assume that Ω is formed by two parts denoted Ω_1^ε and Ω_2^ε, occupied by two materials with different thermal characteristics, separated by an imperfect interface Γ^ε. We also assume that the phase Ω_1^ε is connected and reaches the external fixed boundary $\partial\Omega$ and that Ω_2^ε is disconnected: it is the union of domains of size ε, periodically distributed in Ω with periodicity ε. In such a domain, we study the asymptotic behavior, as ε tends to zero, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\text{div } (A^\varepsilon \nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\text{div } (A^\varepsilon \nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \dfrac{h^\varepsilon}{\varepsilon} (u_1^\varepsilon - u_2^\varepsilon) - G^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \dfrac{h^\varepsilon}{\varepsilon} (u_1^\varepsilon - u_2^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The main novelty brought by our paper consists in allowing the presence, apart from the discontinuity in the temperature field, of a jump in the thermal flux across the imperfect interface Γ^ε, given by the function G^ε. Two different representative cases are studied here, following the conditions imposed on G^ε (stated explicitly in Section 2, Case 1 and Case 2). Let us mention that such functions were already encountered in a different context, more precisely in [9] and [13] for the case of the perforated domains with non homogeneous Neumann boundary conditions on the perforations. For a similar problem of homogenization in a perforated domain with Fourier boundary conditions we refer the reader to [5]. After passage to the limit with the unfolding method, we obtain here two different unfolded problems (stated in Theorem 4.1 and Theorem 4.7), corresponding to the above mentioned cases for the flux jump function G^ε. In Corollary 4.2 and Theorem 4.3, we then give the corresponding homogenized problems. In both situations, the homogenized matrix A^{hom} is constant and it depends on the function describing the jump of the solution. This phenomenon was already observed in some cases without flux jump. Moreover, for the first case studied here, we notice in the right-hand side the presence of a new source term distributed all over the domain Ω and depending on the flux jump function. For the second case, we notice that the influence of the jump in the flux is captured by the correctors only and so this jump plays no role in the homogenized problem; nevertheless, in Remark 8 we mention a case when the homogenized problem depends on this jump, too. This type of result is to be compared with the Neumann problem in perforated domains (see [9], [13]), where similar phenomena occur.

Various other scalings of the temperature and flux jumps, leading to different macroscopic problems, will be studied in a forthcoming paper. Moreover, for the study of the homogenization of a two permeability problem with flux jump, we refer to [8].

The paper is organized as follows: in Section 2, we introduce the microscopic problem and we fix the notation. In Section 3, we review the definition and the basic properties of the unfolding operators we shall use in our proofs. We state and prove the main homogenization results of this paper in Section 4. Corrector results are given in Section 5. We end our paper by a few concluding remarks and some references.

2. **Setting of the problem.**

Let Ω be a bounded open set in $\mathbb{R}^N$ ($N \geq 2$), with a Lipschitz continuous boundary $\partial\Omega$ and $Y = (0,1)^N$ the reference cell in $\mathbb{R}^N$. We assume that Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2 \subset Y$ and $Y = Y_1 \cup \overline{Y}_2$. We also suppose that $\Gamma = \partial Y_2$ is Lipschitz continuous and that Y_2 is connected. In fact, our results can be extended to the case in which the set Y_2 has a finite number of connected components, as in [16].

Throughout the paper, the small parameter ε will take its values in a positive real sequence tending to zero and C will be a positive constant independent of ε, whose value can change from line to line.

For each $k \in \mathbb{Z}^N$, we denote $Y^k = k + Y$ and $Y^k_\alpha = k + Y_\alpha$, for $\alpha = 1, 2$. For each ε, we define, $\mathbb{Z}_\varepsilon = \left\{k \in \mathbb{Z}^N : \ \varepsilon\overline{Y}^k_2 \subset \Omega\right\}$ and we set $\Omega^\varepsilon_2 = \bigcup_{k\in\mathbb{Z}_\varepsilon} \left(\varepsilon Y^k_2\right)$ and $\Omega^\varepsilon_1 = \Omega \setminus \overline{\Omega}^\varepsilon_2$. The boundary of Ω^ε_2 is denoted by Γ^ε and n^ε is the unit outward normal to Ω^ε_2.

Our goal is to describe the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$ of the following problem:

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u^\varepsilon_1) = f \quad \text{in } \Omega^\varepsilon_1, \\ -\text{div}\,(A^\varepsilon \nabla u^\varepsilon_2) = f \quad \text{in } \Omega^\varepsilon_2, \\ A^\varepsilon \nabla u^\varepsilon_1 \cdot n^\varepsilon = \dfrac{h^\varepsilon}{\varepsilon}\,(u^\varepsilon_1 - u^\varepsilon_2) - G^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u^\varepsilon_2 \cdot n^\varepsilon = \dfrac{h^\varepsilon}{\varepsilon}\,(u^\varepsilon_1 - u^\varepsilon_2) \quad \text{on } \Gamma^\varepsilon, \\ u^\varepsilon_1 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Remark 1. We notice that

$$A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon - A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = -G^\varepsilon,$$

which clearly shows that the flux of the solution exhibits a jump across Γ^ε.

The function $f \in L^2(\Omega)$ is given. Let h be a Y–periodic function in $L^\infty(\Gamma)$ such that there exists $h_0 \in \mathbb{R}$ with $0 < h_0 < h(y)$ a.e. on Γ. We set

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \Gamma^\varepsilon.$$

For $\alpha, \beta \in \mathbb{R}$, with $0 < \alpha \leq \beta$, let $\mathcal{M}(\alpha, \beta, Y)$ be the set of all the matrices $A \in (L^\infty(Y))^{N\times N}$ with the property that, for any $\xi \in \mathbb{R}^N$, $\alpha|\xi|^2 \leq (A(y)\xi, \xi) \leq \beta|\xi|^2$, almost everywhere in Y. For a Y-periodic symmetric matrix $A \in \mathcal{M}(\alpha, \beta, Y)$, we set

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega.$$

Let g be a Y-periodic function that belongs to $L^2(\Gamma)$. We define

$$g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \Gamma^\varepsilon.$$

For the given function G^ε in (1), we consider the following two relevant forms (see [13]):

Case 1. $G^\varepsilon = \varepsilon g\Big(\dfrac{x}{\varepsilon}\Big)$, if $\mathcal{M}_\Gamma(g) \neq 0$.

Case 2. $G^\varepsilon = g\Big(\dfrac{x}{\varepsilon}\Big)$, if $\mathcal{M}_\Gamma(g) = 0$.

Here, $\mathcal{M}_\Gamma(g) = \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma g(y)\,\mathrm{d}y$ denotes the mean value of the function g on Γ.

In order to write the variational formulation of problem (1), we introduce, for every positive $\varepsilon < 1$, the Hilbert space

$$H^\varepsilon = V^\varepsilon \times H^1(\Omega_2^\varepsilon).$$

The space $V^\varepsilon = \big\{v \in H^1(\Omega_1^\varepsilon),\ v = 0 \text{ on } \partial\Omega\big\}$ is endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}$, for any $v \in V^\varepsilon$, and the space $H^1(\Omega_2^\varepsilon)$ is equipped with the usual norm. On the space H^ε, we consider the scalar product

$$(u, v)_{H^\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_1 \nabla v_1\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \nabla u_2 \nabla v_2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} (u_1 - u_2)(v_1 - v_2)\,\mathrm{d}\sigma_x \tag{2}$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ belong to the space H^ε. The norm generated by the scalar product (2) is given by

$$\|v\|_{H^\varepsilon}^2 = \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \frac{1}{\varepsilon}\|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

The variational formulation of problem (1) is the following one: find $u^\varepsilon \in H^\varepsilon$ such that

$$a(u^\varepsilon, v) = l(v), \quad \forall v \in H^\varepsilon, \tag{3}$$

where the bilinear form $a : H^\varepsilon \times H^\varepsilon \to \mathbb{R}$ and the linear form $l : H^\varepsilon \to \mathbb{R}$ are given by

$$a(u, v) = \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1 \nabla v_1\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2 \nabla v_2\,\mathrm{d}x + \int_{\Gamma^\varepsilon} \frac{h^\varepsilon}{\varepsilon}(u_1 - u_2)(v_1 - v_2)\,\mathrm{d}\sigma_x$$

and

$$l(v) = \int_{\Omega_1^\varepsilon} f v_1\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f v_2\,\mathrm{d}x + \int_{\Gamma^\varepsilon} G^\varepsilon v_1\,\mathrm{d}\sigma_x,$$

respectively.

We state now an existence and uniqueness result and some necessary a priori estimates for the solution of the variational problem (3).

Theorem 2.1. *For any $\varepsilon \in (0,1)$, the variational problem (3) has a unique solution $u^\varepsilon \in H^\varepsilon$. Moreover, there exists a constant $C > 0$, independent of ε, such that*

$$\|\nabla u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leq C, \quad \|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \leq C$$

and

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C\varepsilon^{1/2}.$$

Proof. The proof of this theorem follows exactly the same steps as in [16] and [28]. □

3. **The periodic unfolding method for a two-component domain.** In this section, we shall briefly recall the definitions and the main properties of the unfolding operators $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$, introduced, for a two-component domain, by P. Donato et al. in [16] (see, also, [9], [10], [11] and [15]) and of the boundary unfolding operator $\mathcal{T}_b^\varepsilon$, introduced in [11] and [12]. The main feature of these operators is that they map functions defined on the oscillating domains Ω_1^ε, Ω_2^ε and, respectively, Γ^ε, into functions defined on the fixed domains $\Omega \times Y_1$, $\Omega \times Y_2$ and $\Omega \times \Gamma$, respectively.

For $x \in \mathbb{R}^N$, we denote by $[x]_Y$ its integer part $k \in \mathbb{Z}^N$, such that $x - [x]_Y \in Y$ and we set $\{x\}_Y = x - [x]_Y$ for $x \in \mathbb{R}^N$. So, for $x \in \mathbb{R}^N$, we have $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right] + \left\{\frac{x}{\varepsilon}\right\}\right)$. For defining the above mentioned periodic unfolding operators, we consider the following sets (see [16]):

$$\widehat{\mathbb{Z}}_\varepsilon = \left\{k \in \mathbb{Z}^N \mid \varepsilon Y^k \subset \Omega\right\}, \quad \widehat{\Omega}^\varepsilon = \operatorname{int} \bigcup_{k \in \widehat{\mathbb{Z}}_\varepsilon} \left(\varepsilon \overline{Y}^k\right), \quad \Lambda^\varepsilon = \Omega \setminus \widehat{\Omega}^\varepsilon,$$

$$\widehat{\Omega}_\alpha^\varepsilon = \bigcup_{k \in \widehat{\mathbb{Z}}_\varepsilon} \left(\varepsilon Y_\alpha^k\right), \quad \Lambda_\alpha^\varepsilon = \Omega_\alpha^\varepsilon \setminus \widehat{\Omega}_\alpha^\varepsilon, \quad \widehat{\Gamma}^\varepsilon = \partial \widehat{\Omega}_2^\varepsilon.$$

Definition 3.1. For any Lebesgue measurable function φ on $\Omega_\alpha^\varepsilon$, $\alpha \in \{1,2\}$, we define the periodic unfolding operators by the formula

$$\mathcal{T}_\alpha^\varepsilon(\varphi)(x,y) = \begin{cases} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}^\varepsilon \times Y_\alpha, \\ 0 & \text{for a.e. } (x,y) \in \Lambda^\varepsilon \times Y_\alpha. \end{cases}$$

If φ is a function defined in Ω, for simplicity, we write $\mathcal{T}_\alpha^\varepsilon(\varphi)$ instead of $\mathcal{T}_\alpha^\varepsilon(\varphi|_{\Omega_\alpha^\varepsilon})$.

For any function ϕ which is Lebesgue-measurable on Γ^ε, the periodic boundary unfolding operator $\mathcal{T}_b^\varepsilon$ is defined by

$$\mathcal{T}_b^\varepsilon(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}^\varepsilon \times \Gamma, \\ 0 & \text{for a.e. } (x,y) \in \Lambda^\varepsilon \times \Gamma. \end{cases}$$

Remark 2. We notice that if $\varphi \in H^1(\Omega_\alpha^\varepsilon)$, then $\mathcal{T}_b^\varepsilon(\varphi) = \mathcal{T}_\alpha^\varepsilon(\varphi)|_{\widehat{\Omega}^\varepsilon \times \Gamma}$.

We recall here some useful properties of these operators (see, for instance, [9], [15] and [16]).

Proposition 1. *For $p \in [1,\infty)$ and $\alpha = 1,2$, the operators $\mathcal{T}^\varepsilon_\alpha$ are linear and continuous from $L^p(\Omega^\varepsilon_\alpha)$ to $L^p(\Omega \times Y_\alpha)$ and*

(i) if φ and ψ are two Lebesgue measurable functions on $\Omega^\varepsilon_\alpha$, one has

$$\mathcal{T}^\varepsilon_\alpha(\varphi\psi) = \mathcal{T}^\varepsilon_\alpha(\varphi)\mathcal{T}^\varepsilon_\alpha(\psi);$$

(ii) for every $\varphi \in L^1(\Omega^\varepsilon_\alpha)$, one has

$$\frac{1}{|Y|}\int_{\Omega\times Y_\alpha} \mathcal{T}^\varepsilon_\alpha(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\widehat{\Omega}^\varepsilon_\alpha} \varphi(x)\,\mathrm{d}x = \int_{\Omega^\varepsilon_\alpha} \varphi(x)\,\mathrm{d}x - \int_{\Lambda_\varepsilon} \varphi(x)\,\mathrm{d}x;$$

(iii) if $\{\varphi^\varepsilon\}_\varepsilon \subset L^p(\Omega)$ is a sequence such that $\varphi^\varepsilon \longrightarrow \varphi$ strongly in $L^p(\Omega)$, then

$$\mathcal{T}^\varepsilon_\alpha(\varphi^\varepsilon) \longrightarrow \varphi \quad \textit{strongly in } L^p(\Omega \times Y_\alpha);$$

(iv) if $\varphi \in L^p(Y_\alpha)$ is Y-periodic and $\varphi^\varepsilon(x) = \varphi(x/\varepsilon)$, then

$$\mathcal{T}^\varepsilon_\alpha(\varphi^\varepsilon) \longrightarrow \varphi \quad \textit{strongly in } L^p(\Omega \times Y_\alpha);$$

(v) if $\varphi \in W^{1,p}(\Omega^\varepsilon_\alpha)$, then $\nabla_y\left(\mathcal{T}^\varepsilon_\alpha(\varphi)\right) = \varepsilon\mathcal{T}^\varepsilon_\alpha(\nabla\varphi)$ and $\mathcal{T}^\varepsilon_\alpha(\varphi)$ belongs to $L^2\left(\Omega; W^{1,p}(Y_\alpha)\right)$. Moreover, for every $\varphi \in L^1(\Gamma^\varepsilon)$, one has

$$\int_{\widehat{\Gamma}^\varepsilon} \varphi(x)\,\mathrm{d}\sigma_x = \frac{1}{\varepsilon|Y|}\int_{\Omega\times\Gamma} \mathcal{T}^\varepsilon_b(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y.$$

The following result was proven, for our geometry, in [16].

Lemma 3.2. *If $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$ is a sequence in H^ε, then*

$$\frac{1}{\varepsilon|Y|}\int_{\Omega\times\Gamma} |\mathcal{T}^\varepsilon_1(u^\varepsilon_1) - \mathcal{T}^\varepsilon_2(u^\varepsilon_2)|^2\,\mathrm{d}x\,\mathrm{d}\sigma_y \le \int_{\Gamma^\varepsilon} |u^\varepsilon_1 - u^\varepsilon_2|^2\,\mathrm{d}\sigma_x.$$

Moreover, if $\varphi \in \mathcal{D}(\Omega)$, then, for ε small enough, we have

$$\varepsilon\int_{\Gamma^\varepsilon} h^\varepsilon(u^\varepsilon_1 - u^\varepsilon_2)\varphi\,\mathrm{d}\sigma_x = \int_{\Omega\times\Gamma} h(y)\left(\mathcal{T}^\varepsilon_1(u^\varepsilon_1) - \mathcal{T}^\varepsilon_2(u^\varepsilon_2)\right)\mathcal{T}^\varepsilon_\alpha(\varphi)\,\mathrm{d}x\,\mathrm{d}\sigma_y,$$

with $\alpha = 1$ or $\alpha = 2$.

We also recall here some general compactness results obtained in [16] for bounded sequences in H^ε.

Lemma 3.3. *Let $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$ be a bounded sequence in H^ε. Then, there exists a constant $C > 0$, independent of ε, such that*

$$\begin{aligned}
\|\mathcal{T}^\varepsilon_1(\nabla u^\varepsilon_1)\|_{L^2(\Omega\times Y_1)} &\le C,\\
\|\mathcal{T}^\varepsilon_2(\nabla u^\varepsilon_2)\|_{L^2(\Omega\times Y_2)} &\le C,\\
\|\mathcal{T}^\varepsilon_2(u^\varepsilon_1) - \mathcal{T}^\varepsilon_1(u^\varepsilon_2)\|_{L^2(\Omega\times\Gamma)} &\le C\varepsilon.
\end{aligned}$$

Theorem 3.4. *Let $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$ be a bounded sequence in H^ε. Then, up to a subsequence, still denoted by ε, there exist $u_1 \in H^1_0(\Omega)$, $u_2 \in L^2(\Omega)$, $\widehat{u}_1 \in$*

$L^2\left(\Omega, H^1_{per}(Y_1)\right)$ *and* $\widehat{u}_2 \in L^2\left(\Omega, H^1(Y_2)\right)$ *such that*

$$\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \longrightarrow u_1 \text{ strongly in } L^2\left(\Omega, H^1(Y_1)\right),$$

$$\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \text{ weakly in } L^2(\Omega \times Y_1),$$

$$\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_2 \text{ weakly in } L^2(\Omega, H^1(Y_2)),$$

$$\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \text{ weakly in } L^2\left(\Omega \times Y_2\right),$$

$$\widetilde{u}_\alpha^\varepsilon \rightharpoonup \frac{|Y_\alpha|}{|Y|} u_\alpha \quad \text{weakly in } L^2(\Omega), \quad \alpha = 1, 2,$$

where $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ *for almost every* $x \in \Omega$ *and* $\tilde{\ }$ *denotes the extension by zero of a function to the whole of the domain* Ω*. Moreover, we have* $u_1 = u_2$ *and*

$$\frac{1}{\varepsilon}\left[\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{M}_\Gamma(\mathcal{T}_1^\varepsilon(u_1^\varepsilon))\right] \rightharpoonup y_\Gamma \nabla u_1 + \widehat{u}_1 \quad \text{weakly in } L^2\left(\Omega, H^1(Y_1)\right),$$

with $y_\Gamma = y - \mathcal{M}_\Gamma(y)$ *and*

$$\frac{1}{\varepsilon}\left[\mathcal{T}_2^\varepsilon(u_2^\varepsilon) - \mathcal{M}_\Gamma(\mathcal{T}_2^\varepsilon(u_2^\varepsilon))\right] \rightharpoonup \widehat{u}_2 \quad \text{weakly in } L^2\left(\Omega, H^1(Y_2)\right).$$

4. **Homogenization results.** In this section, we pass to the limit, with $\varepsilon \to 0$, in the variational formulation (3) of the problem (1). To this end, we use the periodic unfolding method and the general compactness results given in Section 3.

We start by emphasizing again that by applying the general results stated in Theorem 3.4 to the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (3), which is bounded in H^ε, we obtain, at the macroscale, $u_1 = u_2$. In what follows, we shall denote their common value by u. We notice that u belongs to $H^1_0(\Omega)$.

Moreover, using the priori estimates from Theorem 2.1 and the general compactness results from Theorem 3.4, we know that there exist $u \in H^1_0(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y_1))$, $\widehat{u}_2 \in L^2(\Omega, H^1(Y_2))$ such that $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ and up to a subsequence, for $\varepsilon \to 0$, we have:

$$\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \to u \quad \text{strongly in } L^2(\Omega, H^1(Y_1)), \tag{4}$$

$$\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1), \tag{5}$$

$$\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u \quad \text{weakly in } L^2(\Omega, H^1(Y_2)), \tag{6}$$

$$\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega \times Y_2), \tag{7}$$

$$\widetilde{u}_\alpha^\varepsilon \rightharpoonup \frac{|Y_\alpha|}{|Y|} u \quad \text{weakly in } L^2(\Omega), \quad \alpha = 1, 2. \tag{8}$$

Moreover, one has

$$\frac{\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)}{\varepsilon} \rightharpoonup \widehat{u}_1 - \overline{u}_2 \quad \text{weakly in } L^2(\Omega \times \Gamma), \tag{9}$$

where $\overline{u}_2 \in L^2(\Omega, H^1(Y_2))$ is defined by

$$\overline{u}_2 = \widehat{u}_2 - y_\Gamma \nabla u - \xi_\Gamma,$$

for some $\xi_\Gamma \in L^2(\Omega)$.

Let $W_{\mathrm{per}}(Y_1) = \{v \in H^1_{\mathrm{per}}(Y_1) \,|\, \mathcal{M}_\Gamma(v) = 0\}$. We consider the space

$$\mathcal{V} = H^1_0(\Omega) \times L^2\left(\Omega; W_{\mathrm{per}}(Y_1)\right) \times L^2\left(\Omega, H^1(Y_2)\right),$$

endowed with the norm

$$\|V\|^2_{\mathcal{V}} = \|\nabla v + \nabla_y \widehat{v}_1\|^2_{L^2(\Omega\times Y_1)} + \|\nabla v + \nabla_y \overline{v}_2\|^2_{L^2(\Omega\times Y_2)} + \|\widehat{v}_1 - \overline{v}_2\|^2_{L^2(\Omega\times\Gamma)},$$

for all $V = (v, \widehat{v}_1, \overline{v}_2) \in \mathcal{V}$.

For the passage to the limit, we have to distinguish between two cases, following the form of the function G^ε.

Case 1. $G^\varepsilon = \varepsilon g\left(\dfrac{x}{\varepsilon}\right)$, if $\mathcal{M}_\Gamma(g) \neq 0$.

Theorem 4.1. *The unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (3) converges, in the sense of Theorem 3.4, to the unique solution $(u, \widehat{u}_1, \overline{u}_2) \in \mathcal{V}$ of the following unfolded limit problem:*

$$\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y_1} A(y)(\nabla u + \nabla_y \widehat{u}_1)(\nabla\varphi + \nabla_y \Phi_1)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \frac{1}{|Y|}\int_{\Omega\times Y_2} A(y)(\nabla u + \nabla_y \overline{u}_2)(\nabla\varphi + \nabla_y \Phi_2)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \frac{1}{|Y|}\int_{\Omega\times\Gamma} h(y)(\widehat{u}_1 - \overline{u}_2)(\Phi_1 - \Phi_2)\,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \\
&+ \frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(g)\int_\Omega \varphi(x)\,\mathrm{d}x,
\end{aligned} \tag{10}$$

for all $\varphi \in H_0^1(\Omega)$, $\Phi_1 \in L^2(\Omega, H^1_{per}(Y_1))$ and $\Phi_2 \in L^2(\Omega, H^1(Y_2))$.

Proof. In order to obtain the limit problem (10), we first unfold the variational formulation (3) and by using Lemma 3.2 we get

$$\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y_1} \mathcal{T}_1^\varepsilon(A^\varepsilon)\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon)\mathcal{T}_1^\varepsilon(\nabla v_1)\,\mathrm{d}x \\
&+ \frac{1}{|Y|}\int_{\Omega\times Y_2} \mathcal{T}_2^\varepsilon(A^\varepsilon)\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\mathcal{T}_2^\varepsilon(\nabla v_2)\,\mathrm{d}x \\
&+ \frac{1}{|Y|}\int_{\Omega\times\Gamma} h(y)\frac{\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)}{\varepsilon}\,\frac{\mathcal{T}_1^\varepsilon(v_1) - \mathcal{T}_2^\varepsilon(v_2)}{\varepsilon}\,\mathrm{d}\sigma_x \\
=&\frac{1}{|Y|}\int_{\Omega\times Y_1} \mathcal{T}_1^\varepsilon(f)\mathcal{T}_1^\varepsilon(v_1)\,\mathrm{d}x \\
&+ \frac{1}{|Y|}\int_{\Omega\times Y_2} \mathcal{T}_2^\varepsilon(f)\mathcal{T}_2^\varepsilon(v_2)\,\mathrm{d}x + \frac{1}{\varepsilon}\frac{1}{|Y|}\int_{\Omega\times\Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_b^\varepsilon(v_1)\,\mathrm{d}\sigma_x.
\end{aligned}$$

For $\alpha = 1, 2$, we choose in this unfolded problem the admissible test functions

$$v_\alpha = \varphi(x) + \varepsilon\omega_\alpha(x)\psi_\alpha\left(\frac{x}{\varepsilon}\right), \tag{11}$$

with $\varphi, \omega_\alpha \in \mathcal{D}(\Omega)$, $\psi_1 \in H^1_{\mathrm{per}}(Y_1)$, $\psi_2 \in H^1(Y_2)$ and for which we obviously have

$$\mathcal{T}_\alpha^\varepsilon(v_\alpha) \to \varphi(x) \quad \text{strongly in } L^2(\Omega\times Y_\alpha) \tag{12}$$

and

$$\mathcal{T}_\alpha^\varepsilon(\nabla v_\alpha) \to \nabla\varphi(x) + \nabla_y\Phi_\alpha \quad \text{strongly in } L^2(\Omega\times Y_\alpha), \tag{13}$$

where $\Phi_\alpha(x, y) = \omega_\alpha(x)\psi_\alpha(y)$.

Now, the passage to the limit with $\varepsilon \to 0$ is classical, by using convergences (4)-(9), (12)-(13) and the ideas in [15]. The only term which requires more attention is in the right-hand side, the integral term involving the function G^ε. For this term, we have:

$$
\begin{aligned}
&\frac{1}{\varepsilon}\frac{1}{|Y|}\int_{\Omega\times\Gamma}\mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_b^\varepsilon(v_1)\,\mathrm{d}\sigma_x\\
=&\frac{1}{|Y|}\int_{\Omega\times\Gamma}\mathcal{T}_b^\varepsilon\left(g\left(\frac{x}{\varepsilon}\right)\right)\mathcal{T}_b^\varepsilon\left(\varphi(x)+\varepsilon\omega_1(x)\psi_1\left(\frac{x}{\varepsilon}\right)\right)\mathrm{d}\sigma_x\\
=&\frac{1}{|Y|}\int_{\Omega\times\Gamma}g(y)\mathcal{T}_b^\varepsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\
+&\varepsilon\frac{1}{|Y|}\int_{\Omega\times\Gamma}g(y)\mathcal{T}_b^\varepsilon(\omega_1)(x,y)\mathcal{T}_b^\varepsilon(\psi_1)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y\to\frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(g)\int_\Omega\varphi(x)\,\mathrm{d}x.
\end{aligned}
\tag{14}
$$

By the density of $\mathcal{D}(\Omega)\otimes H^1_{\mathrm{per}}(Y_1)$ in $L^2(\Omega,H^1_{\mathrm{per}}(Y_1))$ and of $\mathcal{D}(\Omega)\otimes H^1(Y_2)$ in $L^2(\Omega,H^1(Y_2))$, we get (10).

We notice that our limit problem (10) is similar with the one obtained in [16] (see relation (3.43)), the only difference being the right-hand side, in which an extra term involving the function g arises. More precisely, our right-hand side writes

$$\int_\Omega F(x)\varphi(x)\,\mathrm{d}x,$$

with

$$F(x)=f(x)+\frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(g).$$

The extra term is, in fact, just a real constant and this allows us to prove the uniqueness of the solution of problem (10) exactly as in [16], since the presence of this constant term does not change the linearity nor the continuity of its right-hand side. Thus, due to the uniqueness of $(u,\widehat{u}_1,\overline{u}_2)\in\mathcal{V}$, all the above convergences hold true for the whole sequence, which ends the proof of the theorem. □

Corollary 1. *The function $u\in H^1_0(\Omega)$ defined by (4) is the unique solution of the following homogenized equation:*

$$-\,div(A^{hom}\nabla u)=f+\frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(g)\quad in\ \Omega,\tag{15}$$

where A^{hom} is the homogenized matrix whose entries are given, for $i,j=1,\dots,N$, by

$$A^{hom}_{ij}=\frac{1}{|Y|}\int_{Y_1}\left(a_{ij}-\sum_{k=1}^N a_{ik}\frac{\partial\chi_1^j}{\partial y_k}\right)\mathrm{d}y+\frac{1}{|Y|}\int_{Y_2}\left(a_{ij}-\sum_{k=1}^N a_{ik}\frac{\partial\chi_2^j}{\partial y_k}\right)\mathrm{d}y,\tag{16}$$

in terms of $\chi_1^j\in H^1_{per}(Y_1)$ and $\chi_2^j\in H^1(Y_2)$, $j=1,\dots,N$, the weak solutions of the following cell problems:

$$
\begin{cases}
-div_y(A(y)(\nabla_y\chi_1^j-e_j))=0 & in\ Y_1,\\
-div_y(A(y)(\nabla_y\chi_2^j-e_j))=0 & in\ Y_2,\\
(A(y)\nabla_y\chi_1^j)\cdot n=(A(y)\nabla_y\chi_2^j)\cdot n & on\ \Gamma,\\
(A(y)(\nabla_y\chi_1^j-e_j))\cdot n=h(y)(\chi_1^j-\chi_2^j) & on\ \Gamma,\\
\mathcal{M}_\Gamma(\chi_1^j)=0,
\end{cases}
\tag{17}
$$

where n denotes the unit outward normal to Y_2.

Proof. The proof of this result is classical. Indeed, by choosing successively $\varphi = 0$ and $\Phi_1 = \Phi_2 = 0$ in (10), we obtain:

$$\begin{aligned} &\frac{1}{|Y|}\int_{\Omega\times Y_1} A(y)(\nabla u + \nabla_y \widehat{u}_1)\nabla_y \Phi_1 \,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{|Y|}\int_{\Omega\times Y_2} A(y)(\nabla u + \nabla_y \overline{u}_2)\nabla_y \Phi_2 \,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{|Y|}\int_{\Omega\times \Gamma} h(y)(\widehat{u}_1 - \overline{u}_2)(\Phi_1 - \Phi_2)\,\mathrm{d}x\,\mathrm{d}\sigma_y = 0 \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\frac{1}{|Y|}\int_{\Omega\times Y_1} A(y)(\nabla u + \nabla_y \widehat{u}_1)\nabla \varphi \,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{|Y|}\int_{\Omega\times Y_2} A(y)(\nabla u + \nabla_y \overline{u}_2)\nabla \varphi \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega f(x)\varphi(x)\,\mathrm{d}x + \frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(g)\int_\Omega \varphi(x)\,\mathrm{d}x. \end{aligned} \tag{19}$$

We search now $\widehat{u}_1$ and $\overline{u}_2$ in the usual form

$$\widehat{u}_1(x,y) = -\sum_{j=1}^N \frac{\partial u}{\partial x_j}(x)\chi_1^j(y), \tag{20}$$

$$\overline{u}_2(x,y) = -\sum_{j=1}^N \frac{\partial u}{\partial x_j}(x)\chi_2^j(y). \tag{21}$$

Standard computations lead to the homogenized limit problem, in which the term containing g gives a contribution only to the right-hand side, without affecting the cell problems and the homogenized matrix. □

Remark 3. The right scaling ε in front of the function g^ε prescribed at the interface Γ^ε leads in the limit to the presence of a new source term distributed all over the domain Ω.

Remark 4. It is possible to study our initial problem (1) also for a nonzero function g with mean-value $\mathcal{M}_\Gamma(g)$ equal to zero. But, in this situation, there is no contribution of g in the right-hand side of the homogenized equation and, thus, the limit problem is the same as in the case with no g at all in the microscopic problem.

Remark 5. We remark that the homogenized matrix A^{hom} depends on the function h. So, the effect of the two jumps involved in our microscopic problem is recovered in the homogenized problem, in the right-hand side and also in the left-hand side (through the homogenized coefficients).

Remark 6. All the previous results still hold true for the case in which the set Y_2 is not connected, but consists on a finite number of connected components.

Case 2. $G^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right)$, if $\mathcal{M}_\Gamma(g) = 0$.

Theorem 4.2. *The unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (3) converges, in the sense of Theorem 3.4, to the unique solution $(u, \widehat{u}_1, \overline{u}_2) \in \mathcal{V}$ of the* *following unfolded limit problem:*

$$\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y_1} A(y)(\nabla u+\nabla_y \widehat{u}_1)(\nabla\varphi+\nabla_y\Phi_1)\,\mathrm{d}x\,\mathrm{d}y\\
&+\frac{1}{|Y|}\int_{\Omega\times Y_2} A(y)(\nabla u+\nabla_y \overline{u}_2)(\nabla\varphi+\nabla_y\Phi_2)\,\mathrm{d}x\,\mathrm{d}y\\
&+\frac{1}{|Y|}\int_{\Omega\times \Gamma} h(y)(\widehat{u}_1-\overline{u}_2)(\Phi_1-\Phi_2)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\
=&\int_{\Omega} f(x)\varphi(x)\,\mathrm{d}x+\frac{1}{|Y|}\int_{\Omega\times\Gamma} g(y)\Phi_1(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y,
\end{aligned}\tag{22}$$

for all $\varphi\in H^1_0(\Omega)$, $\Phi_1\in L^2(\Omega,H^1_{per}(Y_1))$, $\Phi_2\in L^2(\Omega,H^1(Y_2))$.

Proof. In order to get the problem (22), we pass to the limit in the unfolded form of the variational formulation (3) with the same test functions (11) as in Theorem 4.1, which verify (12) and (13). The only difference is that now the limit of the term involving the function G^ε is different. More precisely, we have:

$$\begin{aligned}
&\frac{1}{\varepsilon}\frac{1}{|Y|}\int_{\Omega\times\Gamma}\mathcal{T}^\varepsilon_b(G^\varepsilon)\mathcal{T}^\varepsilon_b(v_1)\,\mathrm{d}\sigma_x\\
=&\frac{1}{\varepsilon}\frac{1}{|Y|}\int_{\Omega\times\Gamma}\mathcal{T}^\varepsilon_b\left(g\left(\frac{x}{\varepsilon}\right)\right)\mathcal{T}^\varepsilon_b\left(\varphi(x)+\varepsilon\omega_1(x)\psi_1\left(\frac{x}{\varepsilon}\right)\right)\,\mathrm{d}\sigma_x\\
=&\frac{1}{\varepsilon}\frac{1}{|Y|}\int_{\Omega\times\Gamma} g(y)\mathcal{T}^\varepsilon_b(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\
&+\frac{1}{|Y|}\int_{\Omega\times\Gamma} g(y)\mathcal{T}^\varepsilon_b(\omega_1)(x,y)\mathcal{T}^\varepsilon_b(\psi_1)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\
=&\frac{1}{\varepsilon}\frac{|\Gamma|}{|Y|}\mathcal{M}_\Gamma(g)\int_\Omega\varphi(x)\,\mathrm{d}x+\frac{1}{|Y|}\int_{\Omega\times\Gamma} g(y)\omega_1(x)\psi_1(y)\,\mathrm{d}x\,\mathrm{d}\sigma_y.
\end{aligned}$$

Then, since $\mathcal{M}_\Gamma(g)=0$, by using the density of $\mathcal{D}(\Omega)\otimes H^1_{\mathrm{per}}(Y_1)$ in the space $L^2(\Omega,H^1_{\mathrm{per}}(Y_1))$ and of $\mathcal{D}(\Omega)\otimes H^1(Y_2)$ in the space $L^2(\Omega,H^1(Y_2))$, we get the unfolded limit problem (22).

Due to the uniqueness of $(u,\widehat{u}_1,\overline{u}_2)\in\mathcal{V}$, which is proven by the Lax-Milgram theorem, all the above convergences hold true for the whole sequence, which ends the proof of the theorem. □

Remark 7. Let us point out that the term $\frac{1}{|Y|}\int_{\Omega\times\Gamma} g(y)\Phi_1(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y$ in (22) represents the main difference with respect to the unfolded equation (10), where the term involving g is a nonzero constant, recovered explicitly in the right-hand side of the homogenized equation (15). This cannot be the case here, since this term involves now explicitly both variables x and y. We have to understand the contribution in the homogenized problem of this nonstandard term generated by the discontinuity of the flux in the initial problem. Actually, it will be seen in the next theorem that, apart from the classical solutions χ^j_1 and χ^j_2 of the cell problems (17), we are led to introduce in (30)-(31) two additional scalar terms η_1 and η_2, verifying a new imperfect transmission cell problem (see (32)).

Theorem 4.3. *The solution* $(u, \widehat{u}_1, \overline{u}_2) \in \mathcal{V}$ *of (22) is such that:*

$$\widehat{u}_1(x,y) = -\sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\chi_1^j(y) + \eta_1(y),$$

$$\overline{u}_2(x,y) = -\sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\chi_2^j(y) + \eta_2(y),$$

where χ_1^j *and* χ_2^j *are defined by (17) and the function* (η_1, η_2) *is the unique solution of the cell problem*

$$\begin{cases} -div_y(A(y)\nabla\eta_1) = 0 \quad \text{in } Y_1, \\ -div_y(A(y)\nabla\eta_2) = 0 \quad \text{in } Y_2, \\ A(y)\nabla\eta_1 \cdot n = h(y)(\eta_1 - \eta_2) - g(y) \quad \text{on } \Gamma, \\ A(y)\nabla\eta_2 \cdot n = h(y)(\eta_1 - \eta_2) \quad \text{on } \Gamma, \\ \mathcal{M}_\Gamma(\eta_1) = 0. \end{cases}$$

The function $u \in H_0^1(\Omega)$ *is the unique solution of the following homogenized equation*

$$-\, div(A^{hom}\nabla u) = f \quad \text{in } \Omega, \tag{23}$$

where A^{hom} *is the homogenized matrix whose entries are given in (16).*

Proof. By choosing $\varphi = 0$ in (22), we obtain:

$$\begin{aligned} &\frac{1}{|Y|}\int_{\Omega\times Y_1} A(y)(\nabla u + \nabla_y \widehat{u}_1)\nabla_y \Phi_1 \,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{|Y|}\int_{\Omega\times Y_2} A(y)(\nabla u + \nabla_y \overline{u}_2)\nabla_y \Phi_2 \,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{|Y|}\int_{\Omega\times\Gamma} h(y)(\widehat{u}_1 - \overline{u}_2)(\Phi_1 - \Phi_2)\,\mathrm{d}x\,\mathrm{d}\sigma_y = \frac{1}{|Y|}\int_{\Omega\times\Gamma} g(y)\Phi_1(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y. \end{aligned} \tag{24}$$

We point out that the presence of the term $\frac{1}{|Y|}\int_{\Omega\times\Gamma} g(y)\Phi_1(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y$ in this equation represents the main difference with respect to the previous case.

By choosing now suitable test functions Φ_1 and Φ_2 in (24), we obtain

$$-\operatorname{div}_y(A(y)\nabla_y \widehat{u}_1) = \operatorname{div}_y(A(y)\nabla u) \quad \text{in } \Omega \times Y_1, \tag{25}$$

$$-\operatorname{div}_y(A(y)\nabla_y \overline{u}_2) = \operatorname{div}_y(A(y)\nabla u) \quad \text{in } \Omega \times Y_2, \tag{26}$$

$$A(y)(\nabla u + \nabla_y \overline{u}_2)\cdot n = h(y)(\widehat{u}_1 - \overline{u}_2) \quad \text{on } \Omega\times\Gamma, \tag{27}$$

$$A(y)(\nabla u + \nabla_y \widehat{u}_1)\cdot n = h(y)(\widehat{u}_1 - \overline{u}_2) - g(y) \quad \text{on } \Omega\times\Gamma. \tag{28}$$

We point out here that we also have a discontinuity type condition:

$$A(y)(\nabla u + \nabla_y \widehat{u}_1)\cdot n = A(y)(\nabla u + \nabla_y \overline{u}_2)\cdot n - g(y) \quad \text{on } \Omega\times\Gamma. \tag{29}$$

In the classical case with jump in the solution and with continuity of the flux, the use of the standard correctors χ_1^j and χ_2^j defined in (17) is enough in order to express the functions $\widehat{u}_1$ and $\overline{u}_2$ in terms of the function ∇u. The presence of the function g in relations (28) and (29) suggests us to search $\widehat{u}_1$ and $\overline{u}_2$ in the following nonstandard form:

$$\widehat{u}_1(x,y) = -\sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\chi_1^j(y) + \eta_1(y), \tag{30}$$

$$\overline{u}_2(x,y) = -\sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\chi_2^j(y) + \eta_2(y), \tag{31}$$

where χ_1^j and χ_2^j are defined by (17) and the functions η_1, η_2 have to be found. To this end, we introduce (30) and (31) in (25)-(29) and we obtain:

$$\begin{cases} -\mathrm{div}_y(A(y)\nabla\eta_1) = 0 \quad \text{in } Y_1, \\ -\mathrm{div}_y(A(y)\nabla\eta_2) = 0 \quad \text{in } Y_2, \\ A(y)\nabla\eta_1 \cdot n = h(y)(\eta_1 - \eta_2) - g(y) \quad \text{on } \Gamma, \\ A(y)\nabla\eta_2 \cdot n = h(y)(\eta_1 - \eta_2) \quad \text{on } \Gamma, \\ \mathcal{M}_\Gamma(\eta_1) = 0. \end{cases} \tag{32}$$

We obviously have

$$A(y)\nabla\eta_1 \cdot n - A(y)\nabla\eta_2 \cdot n = -g(y) \tag{33}$$

and then we notice that the new local problem (32) is an imperfect transmission problem, involving both the discontinuities in the solution and in the flux, given in terms of h and g, respectively.

By the Lax-Milgram theorem, the problem (32) has a unique solution in the space

$$H = W_{\mathrm{per}}(Y_1) \times H^1(Y_2),$$

endowed with the scalar product

$$(\eta,\zeta)_H = (\nabla\eta_1, \nabla\zeta_1)_{L^2(Y_1)} + (\nabla\eta_2, \nabla\zeta_2)_{L^2(Y_2)} + (\eta_1 - \eta_2, \zeta_1 - \zeta_2)_{L^2(\Gamma)}.$$

By choosing now $\Phi_1 = \Phi_2 = 0$ in (22), we get:

$$\begin{aligned} &\frac{1}{|Y|}\int_{\Omega\times Y_1} A(y)(\nabla u + \nabla_y \widehat{u}_1)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{|Y|}\int_{\Omega\times Y_2} A(y)(\nabla u + \nabla_y \overline{u}_2)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}y \\ = &\int_\Omega f(x)\varphi(x)\,\mathrm{d}x. \end{aligned} \tag{34}$$

Integrating by parts with respect to x, we obtain, for $x \in \Omega$,

$$-\mathrm{div}_x\left(\frac{1}{|Y|}\int_{Y_1} A(y)(\nabla u + \nabla_y\widehat{u}_1)\,\mathrm{d}y + \frac{1}{|Y|}\int_{Y_2} A(y)(\nabla u + \nabla_y\overline{u}_2)\,\mathrm{d}y\right) = f(x).$$

By using here the particular form (30) and (31) of the functions $\widehat{u}_1$ and $\overline{u}_2$ and the definition (16) of the matrix A^{hom}, we get:

$$\begin{aligned} &-\mathrm{div}_x\,(A^{\mathrm{hom}}\nabla u) \\ =&f + \mathrm{div}_x\left(\frac{1}{|Y|}\int_{Y_1} A(y)\nabla\eta_1(y)dy + \frac{1}{|Y|}\int_{Y_2} A(y)\nabla\eta_2(y)dy\right) \quad \text{in } \Omega, \end{aligned} \tag{35}$$

which leads immediately to the homogenized problem (23). We notice that this problem does not involve the function g, because the second term of the right-hand side in (35) actually vanishes. □

Remark 8. All the above results can be extended to the case in which A^ε is a sequence of matrices in $\mathcal{M}(\alpha,\beta,\Omega)$ such that

$$\mathcal{T}_\alpha^\varepsilon(A^\varepsilon) \to A \text{ strongly in } L^1(\Omega \times Y), \tag{36}$$

for some matrix $A = A(x,y)$ in $\mathcal{M}(\alpha,\beta,\Omega\times Y)$. The heterogeneity of the medium modeled by such a matrix induces different effects in our limit problems (10) and

(22) respectively. In both cases, since the correctors χ_α^j depend also on x, the new homogenized matrix $A_x^{\hom}$ is no longer constant, but it depends on x. A more interesting effect arises in the second case. As we have seen in Theorem 4.3, if the matrix A depends only on the variable y, the functions η_α are independent of x and there is no contribution of the term containing g in the decoupled form of the limit problem. So, the limit equation is the same as that corresponding to the case with no jump on the flux in the microscopic problem. Now, the dependence of A on x prevents this phenomenon to occur, and, hence, the function g brings an explicit contribution in the homogenized problem, which becomes

$$
\begin{aligned}
&-\operatorname{div}_x\left(A_x^{\hom}\nabla u\right) = f\\
&+\operatorname{div}_x\left(\frac{1}{|Y|}\int_{Y_1} A(x,y)\nabla\eta_1(x,y)dy + \frac{1}{|Y|}\int_{Y_2} A(x,y)\nabla\eta_2(x,y)dy\right) \quad \text{in } \Omega.
\end{aligned}
$$

A similar effect was observed in the homogenization of the Neumann problem in perforated domains (see [9]).

5. **Corrector results.** Our goal in this section is to state corrector results for the problem (1). To this end, let us start by recalling the definition of the adjoints of the unfolding operators $\mathcal{T}_\alpha^\varepsilon$, for $\alpha = 1, 2$ (see [9] and [16]).

Definition 5.1. For $p \in [1, \infty)$, the averaging operators $\mathcal{U}_\alpha^\varepsilon : L^p(\Omega\times Y_\alpha) \to L^p(\Omega_\alpha^\varepsilon)$, with $\alpha = 1, 2$, are defined by

$$
\mathcal{U}_\alpha^\varepsilon(\phi)(x) = \begin{cases} \dfrac{1}{|Y|}\displaystyle\int_Y \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) \mathrm{d}z & \text{for a.e. } x \in \widehat{\Omega}_\alpha^\varepsilon,\\ 0 & \text{for a.e. } x \in \Lambda_\alpha^\varepsilon. \end{cases}
$$

It is not difficult to see that these averaging operators are almost left-inverses of the corresponding unfolding operators $\mathcal{T}_\alpha^\varepsilon$, i.e., for any $\varphi \in L^p(\Omega_\alpha^\varepsilon)$, one has

$$
\mathcal{U}_\alpha^\varepsilon(\mathcal{T}_\alpha^\varepsilon(\varphi))(x) = \begin{cases} \varphi(x) & \text{for a.e. } x \in \widehat{\Omega}_\alpha^\varepsilon,\\ 0 & \text{for a.e. } x \in \Lambda_\alpha^\varepsilon. \end{cases}
$$

We recall now some useful properties of the averaging operators $\mathcal{U}_\alpha^\varepsilon$ (see [9] and [16]).

Proposition 2. *For $p \in [1, \infty)$ and $\alpha = 1, 2$, the operators $\mathcal{U}_\alpha^\varepsilon$ are linear and continuous from $L^p(\Omega \times Y_\alpha)$ to $L^p(\Omega_\alpha^\varepsilon)$ and*

(i) $\|\mathcal{U}_\alpha^\varepsilon(\phi) - \phi\|_{L^p(\Omega_\alpha^\varepsilon)} \to 0$ for every $\phi \in L^p(\Omega)$;

(ii) if $\varphi_\varepsilon \in L^p(\Omega_\alpha^\varepsilon)$, then the following statements are equivalent:

- *$\mathcal{T}_\alpha^\varepsilon(\varphi_\varepsilon) \to \widehat{\varphi}$ strongly in $L^p(\Omega \times Y_\alpha)$ and $\displaystyle\int_{\Lambda_\alpha^\varepsilon} |\varphi_\varepsilon|^p \,\mathrm{d}x \to 0$;*
- *$\|\varphi_\varepsilon - \mathcal{U}_\alpha^\varepsilon(\widehat{\varphi})\|_{L^p(\Omega_\alpha^\varepsilon)} \to 0$.*

We are now in the position to state the convergence of the energy and corrector results for the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of problem (1). We shall consider separately the above mentioned representative cases corresponding to the hypotheses imposed on the jump function G^ε.

Case 1. In this case, since the term containing the flux jump gives a contribution only to the right-hand side of the limit equation, without affecting the solutions of the cell problems and the homogenized matrix, we are led to standard corrector results, whose proof follows exactly the same steps as in [16], Theorem 4.4.

Theorem 5.2. *Under the assumptions of Theorem 4.1, if $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the unique solution of problem (1), then*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \,\mathrm{d}x\right)\\
=&\frac{1}{|Y|}\int_{\Omega\times Y_1} A(y)(\nabla u + \nabla_y \widehat{u}_1)(\nabla u + \nabla_y \widehat{u}_1)\,\mathrm{d}x\,\mathrm{d}y \qquad (37)\\
&+\frac{1}{|Y|}\int_{\Omega\times Y_2} A(y)(\nabla u + \nabla_y \overline{u}_2)(\nabla u + \nabla_y \overline{u}_2)\,\mathrm{d}x\,\mathrm{d}y \le \int_\Omega A^{hom}\nabla u \nabla u \,\mathrm{d}x,
\end{aligned}$$

$$\lim_{\varepsilon\to 0}\left(\int_{\Lambda_1^\varepsilon} |\nabla u_1^\varepsilon|^2 \,\mathrm{d}x + \int_{\Lambda_2^\varepsilon} |\nabla u_2^\varepsilon|^2 \,\mathrm{d}x\right) = 0, \qquad (38)$$

$$\mathcal{T}_1^\varepsilon(\nabla\ u_1^\varepsilon) \to \nabla\, u + \nabla_y \widehat{u}_1 \quad \textit{strongly in } L^2(\Omega\times Y_1) \qquad (39)$$

and

$$\mathcal{T}_2^\varepsilon(\nabla\ u_2^\varepsilon) \to \nabla_y \widehat{u}_2 \quad \textit{strongly in } L^2(\Omega\times Y_2). \qquad (40)$$

Moreover, the following corrector result holds true:

$$\left\|\nabla u_1^\varepsilon - \nabla u + \sum_{j=1}^N \mathcal{U}_1^\varepsilon\left(\frac{\partial u}{\partial x_j}\right)\mathcal{U}_1^\varepsilon\left(\nabla_y \chi_1^j\right)\right\|_{L^2(\Omega_1^\varepsilon)} \longrightarrow 0 \qquad (41)$$

and

$$\left\|\nabla u_2^\varepsilon - \nabla u + \sum_{j=1}^N \mathcal{U}_2^\varepsilon\left(\frac{\partial u}{\partial x_j}\right)\mathcal{U}_2^\varepsilon\left(\nabla_y \chi_2^j\right)\right\|_{L^2(\Omega_2^\varepsilon)} \longrightarrow 0. \qquad (42)$$

Case 2. In this case, the result is similar to the one stated in Theorem 5.2, but now the functions η_α, solution of the local problem (32), appear in the correctors of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of problem (1), too.

Theorem 5.3. *Under the assumptions of Theorem 4.3, if $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the unique solution of problem (1), then (37)-(40) hold true. Moreover, we have the following corrector result:*

$$\left\|\nabla u_1^\varepsilon - \nabla u + \sum_{j=1}^N \mathcal{U}_1^\varepsilon\left(\frac{\partial u}{\partial x_j}\right)\mathcal{U}_1^\varepsilon\left(\nabla_y \chi_1^j\right) - \mathcal{U}_1^\varepsilon\left(\nabla_y \eta_1\right)\right\|_{L^2(\Omega_1^\varepsilon)} \longrightarrow 0 \qquad (43)$$

and

$$\left\|\nabla u_2^\varepsilon - \nabla u + \sum_{j=1}^N \mathcal{U}_2^\varepsilon\left(\frac{\partial u}{\partial x_j}\right)\mathcal{U}_2^\varepsilon\left(\nabla_y \chi_2^j\right) - \mathcal{U}_2^\varepsilon\left(\nabla_y \eta_2\right)\right\|_{L^2(\Omega_2^\varepsilon)} \longrightarrow 0. \qquad (44)$$

Proof. The proof of relations (37)-(40) follows the classical steps. The main difference with respect to the previous case is that, in order to get relation (37), we need the following convergence result:

$$\frac{1}{\varepsilon}\frac{1}{|Y|}\int_{\Omega\times\Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_b^\varepsilon(u_1^\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma_y \to \frac{1}{|Y|}\int\limits_{\Omega\times\Gamma} g(y)\widehat{u}_1(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y.$$

This result is a direct consequence of Proposition 3.5 in [9], the fact that $\mathcal{M}_\Gamma(g)$ is zero and that the function u belongs to the space $H_0^1(\Omega)$.

We now notice that

$$\begin{aligned}
&||\nabla u_1^\varepsilon - \nabla u - \mathcal{U}_1^\varepsilon(\nabla_y \widehat{u}_1)||_{L^2(\Omega_1^\varepsilon)} \\
&\le ||\nabla u_1^\varepsilon - \mathcal{U}_1^\varepsilon(\nabla u) - \mathcal{U}_1^\varepsilon(\nabla_y \widehat{u}_1)||_{L^2(\Omega_1^\varepsilon)} + ||\mathcal{U}_1^\varepsilon(\nabla u) - \nabla u||_{L^2(\Omega_1^\varepsilon)}
\end{aligned}$$

and

$$\begin{aligned}
||\nabla u_2^\varepsilon - \nabla u - \mathcal{U}_2^\varepsilon(\nabla_y \overline{u}_2)||_{L^2(\Omega_2^\varepsilon)} &= ||\nabla u_2^\varepsilon - \nabla u - \mathcal{U}_2^\varepsilon(\nabla_y \widehat{u}_2 - \nabla u)||_{L^2(\Omega_2^\varepsilon)} \\
&\le ||\nabla u_2^\varepsilon - \mathcal{U}_2^\varepsilon(\nabla_y \widehat{u}_2)||_{L^2(\Omega_1^\varepsilon)} + ||\mathcal{U}_2^\varepsilon(\nabla u) - \nabla u||_{L^2(\Omega_2^\varepsilon)}.
\end{aligned}$$

Then, we derive (43) and (44) by using convergences (39), (40), Proposition 2 and relations (30), (31). □

Remark 9. Let us notice that similar corrector results can be stated in the case in which the matrix A depends both on x and y, as in Remark 8.

6. **Conclusions.** Via the periodic unfolding method, the effective thermal transfer in a periodic composite material formed by two constituents, separated by an imperfect interface where both the temperature and the flux exhibit jumps, was analyzed. Depending on the hypotheses imposed on the jump of the flux, two different homogenized problems were obtained. The influence of the jumps of the flux and of the temperature field is captured at the limit in various ways: in the homogenized coefficients, in the right-hand side of the homogenized problem, and in the correctors.

Acknowledgments. Both authors acknowledge the support from the Laboratoire Éuropéen Associé CNRS Franco-Roumain "Math-Mode".

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Received for publication December 2015.

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