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THE EXPONENTIAL DECAY RATE OF GENERIC TREE OF 1-D WAVE EQUATIONS WITH BOUNDARY FEEDBACK CONTROLS

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Abstract. In this paper, we study the exponential decay rate of generic tree of 1-d wave equations with boundary feedback controls. For the networks, there are some results on the exponential stability, but no result on estimate of the decay rate. The present work mainly estimates the decay rate for these systems, including signal wave equation, serially connected wave equations, and generic tree of 1-d wave equations. By defining the weighted energy functional of the system, and choosing suitable weighted functions, we obtain the estimation value of decay rate of the systems.

1. **Introduction.**

In this paper, our aim is to estimate decay rate of some concrete 1-d wave network systems. Before going on, we introduce some notation. Let $\mathbb{X}$ be a Banach space and $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a closed and densely defined linear operator. Let us consider the abstract differential equation in $\mathbb{X}$:

$$\begin{cases} \dot{x}(t) = Ax(t),\ t > 0, \\ x(0) = x_0. \end{cases} \tag{1}$$

Suppose that A generates a C_0 semigroup $T(t)$, then the solution to (1) is given by $x(t) = T(t)x_0$, and there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$||x(t)|| \le Me^{\omega t}||x_0||,$$

where $\omega \ge \omega_0(A)$ and the scalar defined by

$$\omega_0(A) = \lim_{t\to\infty} \frac{\ln ||T(t)||}{t}$$

is called the growth order of the semigroup $T(t)$.

If $\omega_0(A) = -\beta_0 < 0$ with $\beta_0 > 0$, then the system (1) is said to be the exponentially stable, and β_0 is called the exponential decay rate of the system (1).

It is well known that for given A, to determine $\omega_0(A)$ has been a difficult topic in mathematical system theory. Since A is known, we can calculate the spectrum of A, and determine the scalar

$$s(A) = \sup\{\Re\lambda, \lambda \in \sigma(A)\}.$$

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In general, it holds that $s(A) \leq \omega_0(A)$. If it holds equality, i.e., $s(A) = \omega_0(A)$, then the system (1) is said to satisfy the spectrum determined growth assumption. When the system (1) satisfies the spectrum determined growth assumption, we can obtain $\omega_0(A)$ by $s(A)$. For example, if $T(t)$ is eventually norm continuous semigroup, differentiable semigroup or analytic semigroup, it holds that $s(A) = \omega_0(A)$(see, [37]). In particular, if A is resolvent compact, and its eigenvectors system forms a uncondition basis for $\mathbb{X}$, then system (1) satisfies the spectrum determined growth assumption (see,[11]).

However, in practice, even if $s(A) = \omega_0(A)$, we cannot calculate the exact value of $\omega_0(A)$, this is because we cannot calculate the exact values of $\sigma(A)$. In most case, we only obtain the asymptotical values of $\sigma(A)$. So, to obtain the approximation of $\omega_0(A)$ has been an interesting topic in practice. In this paper we concern with the estimate problem of $\omega_0(A)$ for the 1-d wave networks with boundary controls.

Let us recall briefly research development of 1-d wave networks. The study of control problem on single 1-d wave equation or string started early in 1970's [39]. Rusell [38], Loins [30] obtained the stabilization result for it. In [41], Shubov studied the spectral property of the damping string system. Cox and Zuazua [12] gave the decay rate of the energy functional of a damped string. Xu et al. in [45] studied the stability of a string with feedback time delay, and in [46] they studied the general linear feedback on the boundary and calculated all spectrum of the closed loop system. Krstic et al. in [27] studied the output feedback of an unstable wave equation, and obtained the exponential stability. Ammari in [5] studied the large time behavior of the solutions and optimal location of a homogenous string equation.

Although single string is a simple system, under the boundary velocity feedback, the closed loop system is a more complex system. We can prove the system is exponentially stable, only a few systems can be determined the explicit decay rate.

For multi-link system, Liu et al. in [31] studied the stabilization problem of a serially connected strings and proved that the closed loop system is exponentially stable. Since then, the modelling and control for the wave equations, for instance, see [42, 32, 33, 29, 28], became gradually a hot topic in the world. Dager and Zuazua [15, 14, 13] studied the controllability of star-shaped and tree-shaped networks of strings; Ammari and Jellouli [1, 2] studied the stabilization of star-shaped tree and generic tree of strings. Recently Ammari in [3] studied a chain of serially connected strings using a frequency domain method and a special analysis for the resolvent. In [4] the authors analysed the spectrum of the dissipative Schrodinger operator on binary tree-shaped networks, and proved the Riesz basis property of the system. Hence the system satisfies the spectrum determined growth assumption.

Jellouli [26] analyzed the spectrum of a degenerate tree and by the spectral decomposition. He proved the best decay rate identifying with the spectral abscissa of the system. However, they do not give the decay rate of the systems due to difficulty of spectrum exact calculation.

To study the decay rate, Xu, Guo et al in [20, 47] studied the Riesz basis property of the closed loop system. Under certain conditions, they proved that the closed loop systems have Riesz basis property, and hence the systems satisfy the spectrum determine growth assumption, i.e., $s(A) = \omega_0(A)$. Since then, there are many papers studying the Riesz basis property of the closed loop system for different 1-d wave networks, for example, see, [34, 50, 48, 17, 25, 24, 23]. These results show that the closed loop systems satisfy $s(A) = \omega_0(A)$. However, due to the difficulty of spectrum calculation of the wave networks, they do not give the estimate of the decay rate of the systems.

Moreover, Nicaise and Valein in [36] studied the stabilization of the 1-d wave networks with a delay in the feedbacks. Under certain conditions, they proved that the networks system is exponentially stable, but have no estimate of the decay rate. More recent results on stabilization and supper-stability of the 1-d wave networks, we refer to [53, 54, 52],[19, 18]. About research development for the general 1-d wave network, we refer to literatures [51] and [49].

Stabilization is one of the most important problems in the research of 1-d wave networks. Under suitable feedback control laws, we can use the different approach, such as multiplier method [51], spectral analysis method [46], as well as resolvent estimate method [22], to prove the exponential stability of the closed loop systems. But there is no result on the estimate of the decay rate.

The estimate problem of the decay rate of the system appears not only in the 1-d wave networks, but also in the other networks, for instance, the first hyperbolic systems, [16, 10, 9, 8] for 1-d linear hyperbolic systems, [43] for thermo-elastic networks, [21] for gas networks and others [6, 35]. Based on the reasons above, in this paper, we concentrate our attention on the estimate problem of the 1-d wave networks. Our approach is inspired by the works [16, 10]. The most important thing is that we find out the conditions which make the inequalities hold and hence get the decay rate estimate.

The rest of this paper is organized as follows: In section 2, we discuss the decay rate of a serially connected 1-d wave equations. At first we discuss a single 1-d wave equation, and from it we will obtain some information about the decay rate of the weighted energy functional and the spectrum of the system. Using this information we can assert the decay rate of the serially connected wave system. In section 3, we discuss the decay rate of the generic tree of 1-d wave networks. Herein we will extend the approach used in section 2 to the generic tree of 1-d wave networks. At first, we discuss the simple tree of 1-d wave network. From this simple model, we will find out some rule of the parameter choices. After then we use this rule to get the estimate of the decay rate of the generic tree of 1-d wave networks. Finally, in section 4, we conclude this paper.

2. The decay rate of serially connected wave equations.

In this section we estimate the decay rate of serially connected 1-d wave equations. At first, we study a control problem of 1-d wave equation. By defining a weighted energy functional of the system, we get a feedback control law. Furthermore, we study the decay rate of the closed loop system. For a single wave equation, Loins [30] proved the exponential stability and Xu [46] proved the system has the Riesz basis by the spectral analysis. Next, we study the serially connected strings, and determine its decay rate. This model was studied early in 1989 by Liu et al in [31], they proved the exponential stability, but they had not given the decay rate of the system.

Although some results of this section are known, we hope to find a general approach which can be apply to more complex system.

2.1. The decay rate of single wave equation.

In this subsection, we study the decay rate of signal 1-d wave equation. Although it is a simple model and has been studied in [46] by the spectral analysis method, we hope one can find a general approach from it.

We begin with recalling a control problem of 1-d wave equation:

$$\begin{cases} w_{tt}(x,t) = c^2 w_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ w(0,t) = 0, \quad cw_x(1,t) = -\alpha u(t), \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x). \end{cases} \tag{2}$$

where $x \in (0,1)$ is the space variable, and $t > 0$ is the time variable. c is a positive real number, which presents the wave-speed. The function $u(t)$ is exterior force (control).

In the sequel, we always use the abbreviations w_t, w_{tt}, w_x and w_{xx} to represent $\frac{\partial w}{\partial t}$, $\frac{\partial^2 w}{\partial t^2}$, $\frac{\partial w}{\partial x}$ and $\frac{\partial^2 w}{\partial x^2}$, respectively.

For (2), we introduce new functions

$$\begin{cases} \xi(x,t) = w_t(x,t) + cw_x(x,t), \\ \eta(x,t) = w_t(x,t) - cw_x(x,t). \end{cases} \tag{3}$$

Using (2), we can find out

$$\begin{cases} \xi_t(x,t) = c\xi_x(x,t), \\ \eta_t(x,t) = -c\eta_x(x,t). \end{cases}$$

Now we define a weighted energy functional of (2) by

$$V(t) = \int_0^1 [p(x)\xi^2(x,t) + q(x)\eta^2(x,t)]dx, \tag{4}$$

where the weighting functions $p(x)$ and $q(x)$ is defined as follows:

$$p(x) = p_1 e^{\gamma x}, \quad q(x) = q_1 e^{-\gamma x}, \tag{5}$$

where γ, p_1 and q_1 are positive constants, they are determined later. Obviously,

$$p'(x) = \gamma p(x), \quad q'(x) = -\gamma q(x).$$

Differentiating $V(t)$ leads to

$$\begin{aligned} \dot{V}(t) &= 2\int_0^1 [p(x)\xi(x,t)\xi_t(x,t) + q(x)\eta(x,t)\eta_t(x,t)]dx \\ &= 2c\int_0^1 [p(x)\xi(x,t)\xi_x(x,t) - q(x)\eta(x,t)\eta_x(x,t)]dx \\ &= c\int_0^1 [p(x)\partial_x\xi^2(x,t) - q(x)\partial_x\eta^2(x,t)]dx. \end{aligned}$$

Integration by parts, we obtain

$$\begin{aligned} \dot{V}(t) &= c[p(1)\xi^2(1,t) - q(1)\eta^2(1,t)] - c[p(0)\xi^2(0,t) - q(0)\eta^2(0,t)] \\ &\quad -c\gamma\int_0^1 (p(x)\xi^2(x,t) + q(x)\eta^2(x,t))dx. \end{aligned}$$

Using (3) (4) and (5) we get

$$\begin{aligned} \dot{V}(t) &= cp_1 e^{\gamma}(w_t + cw_x)^2(1,t) - cq_1 e^{-\gamma}(w_t - cw_x)^2(1,t) \\ &\quad -cp_1(w_t + cw_x)^2(0,t) + cq_1(w_t - cw_x)^2(0,t) - \gamma cV(t) \\ &= cp_1 e^{\gamma}(w_t(1,t) - \alpha u(t))^2 - cq_1 e^{-\gamma}(w_t(1,t) + \alpha u(t))^2 \\ &\quad +c^3(q_1 - p_1)w_x^2(0,t) - \beta V(t) \\ &= -\beta V(t) + B(t), \end{aligned}$$

where $\beta = \gamma c$.

Obviously, if we can choose $u(t)$ and p_1, q_1 and γ such that the boundary parts $B(t)$ satisfy

$$\begin{aligned} B(t) &= cp_1e^{\gamma}(w_t(1,t)-\alpha u(t))^2 - cq_1e^{-\gamma}(w_t(1,t)+\alpha u(t))^2 \\ &\quad +c^3(q_1-p_1)w_x^2(0,t) \\ &\leq 0, \end{aligned}$$

then the system (2) can be exponentially stable.

For example, we take $u(t) = w_t(1,t)$, $p_1 = q_1$, then the boundary parts have the form

$$\begin{aligned} B(t) &= cp_1w_t^2(1,t)[e^{\gamma}(1-\alpha)^2 - e^{-\gamma}(1+\alpha)^2] \\ &= cp_1e^{-\gamma}(1-\alpha)^2w_t^2(1,t)\left[e^{2\gamma} - \frac{(1+\alpha)^2}{(1-\alpha)^2}\right]. \end{aligned}$$

Therefore, the γ satisfying $0 < \gamma \leq \ln\left|\frac{1+\alpha}{1-\alpha}\right|$ is desired. In this case, we have $\dot{V}(t) \leq -\beta V(t)$ and hence

$$V(t) \leq V(0)e^{-\beta t}.$$

The decay rate of the weighted functional of the system (2) is at least $\beta = \gamma c$.

Note that according to above choice of $u(t)$, the closed loop system corresponding to (2) is

$$\begin{cases} w_{tt}(x,t) = c^2 w_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ w_x(0,t) = 0, \\ cw_x(1,t) = -\alpha w_t(1,t), \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x). \end{cases} \tag{6}$$

Note that the weighted energy functional satisfies the following inequality

$$\begin{aligned} &2e^{-\gamma}p_1\int_0^1 [|cw_x(x,t)|^2 + |w_t(x,t)|^2]dx \\ &\leq V(t) \leq 2e^{\gamma}p_1\int_0^1 [|cw_x(x,t)|^2 + |w_t(x,t)|^2]dx. \end{aligned}$$

So the decay rate of the system (6) is $\frac{\beta}{2}$.

Summarizing above discussion, we have proved the following result.

Theorem 2.1. *The closed loop system (6) is exponentially stable, and the decay rate is at least* $\frac{\beta}{2} = \frac{\gamma c}{2}$ *where* $0 < \gamma \leq \ln\left|\frac{1+\alpha}{1-\alpha}\right|$.

Remark 1. By the spectral analysis, we know that all eigenvalues of the system (6) are located in the line $\Re\lambda = -\frac{c}{2}\ln\left|\frac{1+\alpha}{1-\alpha}\right|$, obviously,

$$\frac{c}{2}\ln\left|\frac{1+\alpha}{1-\alpha}\right| = \frac{c}{2}\max\left\{\gamma; 0 < \gamma \leq \ln\left|\frac{1+\alpha}{1-\alpha}\right|\right\}.$$

This means that $\gamma = \ln\left|\frac{1+\alpha}{1-\alpha}\right|$ is the optimal value, and the maximal decay rate of (6) is $\frac{\beta}{2} = \frac{c}{2}\ln\left|\frac{1+\alpha}{1-\alpha}\right|$.

Note that the result of Theorem 2.1 is obtained under the condition $\alpha \neq 1$. If $\alpha = 1$, we have a better result.

Theorem 2.2. *If* $\alpha = 1$, *the closed loop system (6) is super-stable. That is , there exist a positive constant* τ, *when* $t \geq \tau$, $(w(x,t), w_t(x,t)) \equiv (0,0)$.

Proof. If $\alpha = 1$, we can take $p_1 > q_1$. In this case, the boundary parts always satisfy

$$-4cq_1 w_t^2(1,t) - c^3(p_1 - q_1)w_x^2(0,t) \le 0.$$

Thus

$$\dot{V}(t) + \beta V(t) = -4cq_1 w_t^2(1,t) - c^3(p_1 - q_1)w_x^2(0,t).$$

From above we get

$$e^{\beta t}V(t) \quad = \quad V(0) - 4cq_1 \int_0^t e^{\beta s} w_s^2(1,s)ds + c^3(p_1 - q_1)\int_0^t e^{\beta s} w_x^2(0,s)]ds,$$

that holds for all $\beta = \gamma c$. Therefore, it must exist a τ such that $V(t) = 0$, $\forall t \ge \tau$. Hence when $t \ge \tau$, we have $(w(x,t), w_t(x,t)) \equiv (0,0)$. □

Remark 2. We can prove that $\tau = \frac{2}{c}$.

Remark 3. The super-stability problem of a system was studied by [7]. The similar questions for the 1-d wave network were studied in [46, 40, 54].

2.2. **The decay rate of serially connected 1-d wave equations.** Let us recall the model studied in [31]. Under suitable change, the model can be written as follows:

$$\begin{cases} w_{i,tt}(x,t) = c_i^2 w_{i,xx}(x,t), x \in (0,1), t > 0, i = 1,2,\cdots,n, \\ w_1(0,t) = 0, \\ w_i(1,t) = w_{i+1}(0,t), \ i = 1,2,\cdots,n-1, \\ c_i w_{i,x}(1,t) = c_{i+1} w_{i+1,x}(0,t), i = 1,\cdots,n-1, \\ c_n w_{n,x}(1,t) = -\alpha w_{n,t}(1,t), \\ w_i(x,0) = w_{i,0}(x), \\ w_{i,t}(x,0) = w_{i,1}(x), x \in (0,1), \ i = 1,2,\cdots,n. \end{cases} \tag{7}$$

Similarly, we take transform of the variable:

$$\xi_i(x,t) = w_{i,t}(x,t) + c_i w_{i,x}(x,t),$$

$$\eta_i(x,t) = w_{i,t}(x,t) - c_i w_{i,x}(x,t).$$

Under the transform the equations become

$$\begin{cases} \xi_{i,t}(x,t) = c_i \xi_{i,x}(x,t), \\ \eta_{i,t}(x,t) = -c_i \eta_{i,x}(x,t), i = 1,2,\cdots,n, \\ \xi_1(0,t) = -\eta_1(0,t), \\ \xi_i(1,t) = \xi_{i+1}(0,t), \\ \eta_i(1,t) = \eta_{i+1}(0,t), i = 1,2,\cdots,n-1, \\ (1+\alpha)\xi_n(1,t) = (1-\alpha)\eta_n(1,t), \\ \xi(x,0) = \xi_0(x), \quad \eta(x,0) = \eta_0(x). \end{cases} \tag{8}$$

We define the weighted energy functional of the system (7) as

$$V(t) = \sum_{i=1}^n V_i(t)$$

where

$$V_i(t) = \int_0^1 [p_i(x)\xi_i^2(x,t) + q_i(x)\eta_i^2(x,t)]dx, \tag{9}$$

and

$$p_i(x) = p_i e^{\gamma_i x}, \quad q_i(x) = q_i e^{-\gamma_i x},$$

with $p_i, q_i, \gamma_i > 0$.

Since

$$\dot{V}_i(t) = c_i[p_i(x)\xi_i^2(x,t) - q_i(x)\eta_i^2(x,t)]|_0^1 - \gamma_i c_i \int_0^1 [p_i(x)\xi_i^2(x,t) + q_i(x)\eta_i^2(x,t)]dx$$
$$= -\beta_i V_i(t) + c_i[p_i(x)\xi_i^2(x,t) - q_i(x)\eta_i^2(x,t)]|_0^1,$$

where $\beta_i = \gamma_i c_i$, we have

$$\dot{V}(t) = \sum_{i=1}^{n} \dot{V}_i(t) = -\sum_{i=1}^{n} \beta_i V_i(t) + B(t),$$

where

$$B(t) = c_n[p_n(1)\xi_n^2(1,t) - q_n(1)\eta_n^2(1,t)] + \sum_{i=1}^{n-1} c_i[p_i(1)\xi_i^2(1,t) - q_i(1)\eta_i^2(1,t)]$$
$$- \sum_{i=2}^{n} c_i[p_i(0)\xi_i^2(0,t) - q_i(0)\eta_i^2(0,t)] - c_1[p_1(0)\xi_1^2(0,t) - q_1(0)\eta_1^2(0,t)].$$

Using the boundary and connection conditions we get

$$B(t) = c_n[p_n e^{\gamma_n} - q_n e^{-\gamma_n}(\frac{1+\alpha}{1-\alpha})^2]\xi_n^2(1,t) + \sum_{i=1}^{n-1} c_i[p_i e^{\gamma_i}\xi_{i+1}^2(0,t) - q_i e^{-\gamma_i}\eta_{i+1}^2(0,t)]$$
$$- \sum_{i=1}^{n-1} c_{i+1}[p_{i+1}\xi_{i+1}^2(0,t) - q_{i+1}\eta_{i+1}^2(0,t)] - c_1(p_1 - q_1)\eta_1^2(0,t)$$
$$= c_n[p_n e^{\gamma_n} - q_n e^{-\gamma_n}(\frac{1+\alpha}{1-\alpha})^2]\xi_n^2(1,t) + \sum_{i=1}^{n-1}(c_i p_i e^{\gamma_i} - c_{i+1}p_{i+1})\xi_{i+1}^2(0,t)$$
$$+ \sum_{i=1}^{n-1}(c_{i+1}q_{i+1} - c_i q_i e^{-\gamma_i})\eta_{i+1}^2(0,t) - c_1(p_1 - q_1)\eta_1^2(0,t).$$

The following theorem gives the conditions for p_i, q_i and γ_i, $i = 1, 2, \cdots, n$.

Theorem 2.3. *Suppose that the parameters p_i, q_i and γ_i, $i = 1, 2, \cdots, n$, satisfy the following conditions:*

$$\begin{cases} p_1 = q_1 = p, \\ c_{i+1}q_{i+1} = c_i q_i e^{-\gamma_i}, \\ c_{i+1}p_{i+1} = c_i p_i e^{\gamma_i}, i = 1, 2, \cdots, n-1, \\ \gamma_i = \frac{\frac{1}{c_i}}{\sum_{i=1}^{n} \frac{1}{c_i}} \ln\left|\frac{1+\alpha}{1-\alpha}\right|, i = 1, 2, \cdots, n. \end{cases} \tag{10}$$

then the decay rate of the system (7) is at least $\frac{\beta}{2}$, where

$$\beta = \min_{i=1,2,\cdots,n}\{\beta_i = \gamma_i c_i\} = \frac{1}{\sum_{i=1}^{n} \frac{1}{c_i}} \ln\left|\frac{1+\alpha}{1-\alpha}\right|.$$

Proof. If $p_1 = q_1 = p$, and $c_{i+1}q_{i+1} = c_i q_i e^{-\gamma_i}$, $c_{i+1}p_{i+1} = c_i p_i e^{\gamma_i}$ for $i = 1, 2, \cdots, n-1$, then we have recursion relationship

$$p_{i+1} = \frac{c_i}{c_{i+1}} e^{\gamma_i} p_i, \quad q_{i+1} = \frac{c_i}{c_{i+1}} e^{-\gamma_i} q_i.$$

Therefore,

$$p_n=\frac{c_1}{c_n}e^{\sum\limits_{i=1}^{n-1}\gamma_i}p,\quad q_n=\frac{c_1}{c_n}e^{-\sum\limits_{i=1}^{n-1}\gamma_i}p.$$

In this case, we have

$$\begin{aligned}B(t)&=&c_n[p_ne^{\gamma_n}-q_ne^{-\gamma_n}(\frac{1+\alpha}{1-\alpha})^2]\xi_n^2(1,t)\\&&=c_1p\left[e^{\sum\limits_{i=1}^{n}\gamma_i}-e^{-\sum\limits_{i=1}^{n}\gamma_i}(\frac{1+\alpha}{1-\alpha})^2\right]\xi_n^2(1,t).\end{aligned}$$

Taking

$$e^{2\sum\limits_{i=1}^{n}\gamma_i}\le(\frac{1+\alpha}{1-\alpha})^2,$$

i.e.,

$$\sum_{i=1}^{n}\gamma_i\le\ln\left|\frac{1+\alpha}{1-\alpha}\right|,$$

we get $B(t)\le 0$. So

$$\dot V(t)\le-\sum_{i=1}^{n}\beta_iV_i(t)\le-\beta V(t),$$

and

$$V(t)\le V(0)e^{-\beta t}.$$

In particular, when $\gamma_i=\frac{\frac{1}{c_i}}{\sum\limits_{i=1}^{n}\frac{1}{c_i}}\ln\left|\frac{1+\alpha}{1-\alpha}\right|$, we have

$$\sum_{i=1}^{n}\gamma_i=\ln\left|\frac{1+\alpha}{1-\alpha}\right|,$$

and

$$\beta\ =\ \min_{i=1,2,\cdots,n}\beta_i=\min_{i=1,2,\cdots,n}\gamma_ic_i=\frac{1}{\sum\limits_{i=1}^{n}\frac{1}{c_i}}\ln\left|\frac{1+\alpha}{1-\alpha}\right|.$$

The desired result follows. □

Remark 4. By the spectral analysis, we can show that the asymptote of the spectrum is $\frac{1}{\sum\limits_{i=1}^{n}\frac{1}{c_i}}\ln\left|\frac{1+\alpha}{1-\alpha}\right|$.

3. **The decay rate of the generic tree of 1-d wave equations.** In this section we will extend the approach used in section 2 to more complex model. Here we mainly study the decay rate of the generic tree of 1-d wave equations. It is well known that if a network is of tree-shaped, and all boundary vertices (but one) are acted on control, then the system is exactly controllable, and under the feedback control laws, then the closed loop system is exponentially stable, see [13, 15, 1, 2, 50]. But there is no the estimate of decay rate. In this section we will estimate the decay rate of the generic tree wave networks.

3.1. **A simple tree-shaped network of 1-d wave equations.** In this subsection we consider a simple tree-shaped network of wave equations, which is governed by the following partial differential equations:

$$\begin{cases} w_{i,tt}(x,t) = c_i^2 w_{i,xx}(x,t), \ x \in (0,1), \ t \geq 0, \ i = 1,2,3, \\ w_1(0,t) = 0, \\ w_1(1,t) = w_2(0,t) = w_3(0,t), \\ c_1 w_{1,x}(1,t) = c_2 w_{2,x}(0,t) + c_3 w_{3,x}(0,t), \\ c_2 w_{2,x}(1,t) = -\alpha_2 u_2(t), \\ c_3 w_{3,x}(1,t) = -\alpha_3 u_3(t), \\ w_i(x,0) = w_{i,0}(x), \ w_{i,t}(x,0) = w_{i,1}(x). \end{cases} \tag{11}$$

where $c_i > 0$, $\alpha_i > 0$, $i = 1,2,3$. The functions $u_i(t)$, $i = 2,3$, are controls.

For the equations (11), we let

$$\begin{cases} \xi_i(x,t) = w_{i,t}(x,t) + c_i w_{i,x}(x,t), \\ \eta_i(x,t) = w_{i,t}(x,t) - c_i w_{i,x}(x,t). \end{cases}$$

Using (11) we can find out

$$\begin{cases} \xi_{i,t}(x,t) = c_i \xi_{i,x}(x,t), \\ \eta_{i,t}(x,t) = -c_i \eta_{i,x}(x,t), \ i = 1,2,3. \end{cases}$$

Similarly, we define the weighted energy functional by

$$V(t) = \sum_{i=1}^{3} V_i(t)$$

where

$$V_i(t) = \int_0^1 (p_i(x)\xi_i^2(x,t) + q_i(x)\eta_i^2(x,t))dx, \tag{12}$$

the weight functions $p_i(x)$ and $q_i(x)$ are defined as follows:

$$p_i(x) = p_i e^{\gamma_i x}, \quad q_i(x) = q_i e^{-\gamma_i x}, \tag{13}$$

with $p_i > 0$, $q_i > 0$, $\gamma_i > 0$, $i = 1,2,3$.

A direct calculation gives

$$\begin{aligned} \dot{V}_i(t) &= 2\int_0^1 p_i(x)\xi_i(x,t)\xi_{i,t}(x,t)dx + 2\int_0^1 q_i(x)\eta_i(x,t)\eta_{i,t}(x,t)dx \\ &= 2c_i \int_0^1 p_i(x)\xi_i(x,t)\xi_{i,x}(x,t)dx - 2c_i \int_0^1 q_i(x)\eta_i(x,t)\eta_{i,x}(x,t))dx \\ &= c_i[p_i(x)\xi^2(x,t) - q_i(x)\eta^2(x,t)]|_0^1 - c_i \int_0^1 [p_i'(x)\xi_i^2(x,t) - q_i'(x)\eta_i^2(x,t)]dx \\ &= c_i[p_i(1)\xi_i^2(1,t) - q_i(1)\eta_i^2(1,t)] - c_i[p_i(0)\xi_i^2(0,t) - q_i(0)\eta_i^2(0,t)] - \beta_i V_i(t), \end{aligned}$$

where $\beta_i = \gamma_i c_i$. Thus, it holds that

$$\begin{aligned} \dot{V}(t) &= -\sum_{i=1}^{3} \beta_i V_i(t) + \sum_{i=1}^{3} c_i[p_i(1)\xi_i^2(1,t) - q_i(1)\eta_i^2(1,t)] \\ &\quad -\sum_{i=1}^{3} c_i[p_i(0)\xi_i^2(0,t) - q_i(0)\eta_i^2(0,t)] \end{aligned}$$

$$= -\sum_{i=1}^{3}\beta_i V_i(t)+\sum_{i=1}^{3}c_ip_ie^{\gamma_i}(w_{i,t}(1,t)+c_iw_{i,x}(1,t))^2$$
$$-\sum_{i=1}^{3}c_iq_ie^{-\gamma_i}(w_{i,t}(1,t)-c_iw_{i,x}(1,t))^2-\sum_{i=1}^{3}c_ip_i(w_{i,t}(0,t)+c_iw_{i,x}(0,t))^2$$
$$+\sum_{i=1}^{3}c_iq_i(w_{i,t}(0,t)-c_iw_{i,x}(0,t))^2.$$

For simplicity, we denote the boundary parts by

$$B(t)=\sum_{i=1}^{3}c_ip_ie^{\gamma_i}(w_{i,t}(1,t)+c_iw_{i,x}(1,t))^2-\sum_{i=1}^{3}c_iq_ie^{-\gamma_i}(w_{i,t}(1,t)-c_iw_{i,x}(1,t))^2$$
$$-\sum_{i=1}^{3}c_ip_i(w_{i,t}(0,t)+c_iw_{i,x}(0,t))^2+\sum_{i=1}^{3}c_iq_i(w_{i,t}(0,t)-c_iw_{i,x}(0,t))^2. \tag{14}$$

Using the boundary conditions in (11), we have

$$\begin{aligned} B(t) &= \sum_{i=2}^{3}c_ip_ie^{\gamma_i}(w_{i,t}(1,t)-\alpha_iu_i(t))^2-\sum_{i=2}^{3}c_iq_ie^{-\gamma_i}(w_{i,t}(1,t)+\alpha_iu_i(t))^2\\ &+ c_1p_1e^{\gamma_1}(w_{1,t}(1,t)+c_1w_{1,x}(1,t))^2-c_1q_1e^{-\gamma_1}(w_{1,t}(1,t)-c_1w_{1,x}(1,t))^2\\ &- \sum_{i=2}^{3}c_ip_i(w_{1,t}(1,t)+c_iw_{i,x}(0,t))^2+\sum_{i=2}^{3}c_iq_i(w_{1,t}(1,t)-c_iw_{i,x}(0,t))^2\\ &+ c_1^3(q_1-p_1)w_{1,x}^2(0,t). \end{aligned}$$

Our aim is to prove that we can select parameters p_i, q_i and γ_i and control $u_i(t)$ such that $B(t)\le 0$ for all $t\ge 0$.

The simple selections for $u_i(t), i=2,3$ are

$$u_i(t)=w_{i,t}(1,t),\ i=2,3. \tag{15}$$

Hence the closed loop system associated with (11) is

$$\begin{cases} w_{i,tt}(x,t)=c_i^2w_{i,xx}(x,t),\ x\in(0,1),\ t>0,\ i=1,2,3,\\ w_1(0,t)=0,\\ w_1(1,t)=w_2(0,t)=w_3(0,t),\\ c_1w_{1,x}(1,t)=c_2w_{2,x}(0,t)+c_3w_{3,x}(0,t),\\ c_2w_{2,x}(1,t)=-\alpha_2w_{2,t}(1,t),\\ c_3w_{3,x}(1,t)=-\alpha_3w_{3,t}(1,t),\\ w_i(x,0)=w_{i,0}(x),\ w_{i,t}(x,0)=w_{i,1}(x). \end{cases} \tag{16}$$

The following theorem gives the selection condition of the parameters.

Theorem 3.1. *Let the boundary control laws u_2, u_3 be defined by (15). Suppose that $p_i,\ q_i,\ i=1,2,3$ satisfy the following conditions:*

$$\begin{cases} p_1=q_1,\\ p_2c_2=p_3c_3=\frac{3}{2}p_1c_1,\\ q_2c_2=q_3c_3=\frac{1}{2}p_1c_1,\\ \gamma_1\le \ln\frac{5}{4},\\ \gamma_2\le\frac{1}{2}\ln\frac{(1+\alpha_2)^2}{3(1-\alpha_2)^2},\\ \gamma_3\le\frac{1}{2}\ln\frac{(1+\alpha_3)^2}{3(1-\alpha_3)^2}. \end{cases} \tag{17}$$

then $B(t) \leq 0$, and hence for $\beta_i = \gamma_i c_i$, and

$$\beta = \min_{i=1,2,3} \{\beta_i\}, \tag{18}$$

it holds that $\dot{V}(t) \leq -\beta V(t)$, i.e.,the exponential decay rate of the closed loop systems (16) is at leat $\frac{\beta}{2}$.

Proof. Note that

$$\begin{aligned}
\dot{V}(t) &= -\sum_{i=1}^{3} \beta_i V_i(t) + B(t) \\
&\leq -\min_{i=1,2,3} \{\beta_i\} V(t) + B(t) \\
&= -\beta V(t) + B(t).
\end{aligned}$$

So, we only need to prove that under the conditions (17), it holds $B(t) \leq 0$.

Suppose that (15) and (17) hold, then we have

$$e^{2\gamma_i} - \frac{(1+\alpha_i)^2}{3(1-\alpha_i)^2} \leq 0, i = 2, 3,$$

and

$$\begin{aligned}
B(t) &= \frac{3}{2} p_1 c_1 \sum_{i=2}^{3} e^{\gamma_i} (1-\alpha_i)^2 w_{i,t}^2(1,t) - \frac{1}{2} p_1 c_1 \sum_{i=2}^{3} e^{-\gamma_i} (1+\alpha_i)^2 w_{i,t}^2(1,t) \\
&+ c_1 p_1 e^{\gamma_1} (w_{1,t}(1,t) + c_1 w_{1,x}(1,t))^2 - c_1 p_1 e^{-\gamma_1} (w_{1,t}(1,t) - c_1 w_{1,x}(1,t))^2 \\
&- \frac{3}{2} p_1 c_1 \sum_{i=2}^{3} (w_{1,t}(1,t) + c_i w_{i,x}(0,t))^2 + \frac{1}{2} p_1 c_1 \sum_{i=2}^{3} (w_{1,t}(1,t) - c_i w_{i,x}(0,t))^2 \\
&\leq c_1 p_1 e^{\gamma_1} (w_{1,t}(1,t) + c_1 w_{1,x}(1,t))^2 - c_1 p_1 e^{-\gamma_1} (w_{1,t}(1,t) - c_1 w_{1,x}(1,t))^2 \\
&- \frac{3}{2} p_1 c_1 \sum_{i=2}^{3} (w_{1,t}(1,t) + c_i w_{i,x}(0,t))^2 + \frac{1}{2} p_1 c_1 \sum_{i=2}^{3} (w_{1,t}(1,t) - c_i w_{i,x}(0,t))^2 \\
&=: c_1 p_1 B_1(t).
\end{aligned}$$

We expand above last terms as follows:

$$\begin{aligned}
B_1(t) &= e^{\gamma_1} (w_{1,t}(1,t) + c_1 w_{1,x}(1,t))^2 - e^{-\gamma_1} (w_{1,t}(1,t) - c_1 w_{1,x}(1,t))^2 \\
&- \frac{3}{2} \sum_{i=2}^{3} (w_{1,t}(1,t) + c_i w_{i,x}(0,t))^2 + \frac{1}{2} \sum_{i=2}^{3} (w_{1,t}(1,t) - c_i w_{i,x}(0,t))^2 \\
&= e^{\gamma_1} [w_{1,t}^2(1,t) + 2c_1 w_{1,t}(1,t) w_{1,x}(1,t) + c_1^2 w_{1,x}^2(1,t)] \\
&- e^{-\gamma_1} [w_{1,t}^2(1,t) - 2c_1 w_{1,t}(1,t) w_{1,x}(1,t) + c_1^2 w_{1,x}^2(1,t)] \\
&- \frac{3}{2} \sum_{i=2}^{3} [w_{1,t}^2(1,t) + 2c_i w_{1,t}(1,t) w_{i,x}(0,t) + c_i^2 w_{i,x}^2(0,t)] \\
&+ \frac{1}{2} \sum_{i=2}^{3} [w_{1,t}^2(1,t) - 2c_i w_{1,t}(1,t) w_{i,x}(0,t) + c_i^2 w_{i,x}^2(0,t)] \\
&= w_{1,t}^2(1,t)[(e^{\gamma_1} - e^{-\gamma_1}) - 2] + 2w_{1,t}(1,t)[(e^{\gamma_1} + e^{-\gamma_1}) c_1 w_{1,x}(1,t) \\
&- 2\sum_{i=2}^{3} c_i w_{i,x}(0,t)] + (e^{\gamma_1} - e^{-\gamma_1}) c_1^2 w_{1,x}^2(1,t) - \sum_{i=2}^{3} c_i^2 w_{i,x}^2(0,t)
\end{aligned}$$

$$\begin{aligned}
&= 2w_{1,t}^2(1,t)[\sinh\gamma_1 - 1] + 4w_{1,t}(1,t)c_1w_{1,x}(1,t)[\cosh\gamma_1 - 1] \\
&+ 2\sinh\gamma_1 c_1^2 w_{1,x}^2(1,t) - \sum_{i=2}^{3} c_i^2 w_{i,x}^2(0,t) \\
&\le 2w_{1,t}^2(1,t)[\sinh\gamma_1 - 1] + 2[\cosh\gamma_1 - 1][w_{1,t}^2(1,t) + c_1^2 w_{1,x}^2(1,t)] \\
&+ 2\sinh\gamma_1 c_1^2 w_{1,x}^2(1,t) - \sum_{i=2}^{3} c_i^2 w_{i,x}^2(0,t) \\
&= 2w_{1,t}^2(1,t)(e^{\gamma_1} - 2) + 2(e^{\gamma_1} - 1)c_1^2 w_{1,x}^2(1,t) - \sum_{i=2}^{3} c_i^2 w_{i,x}^2(0,t).
\end{aligned}$$

Using the condition

$$c_1 w_{1,x}(1,t) = c_2 w_{2,x}(0,t) + c_3 w_{3,x}(0,t),$$

$$c_1^2 w_{1,x}^2(1,t) = [c_2 w_{2,x}(0,t) + c_3 w_{3,x}(0,t)]^2 \le 2\sum_{i=2}^{3} c_i^2 w_{i,x}^2(0,t).$$

so $B(t) \le c_1 p_1 B_1(t)$ and

$$B_1(t) \le 2w_{1,t}^2(1,t)(e^{\gamma_1} - 2) + (4e^{\gamma_1} - 5)\sum_{i=2}^{3} c_i^2 w_{i,x}^2(0,t).$$

Obviously, when $e^{\gamma_1} \le \frac{5}{4}$, i.e., $0 \le \gamma_1 \le \ln\frac{5}{4}$, we have $B(t) \le 0$. The desired result follows. □

Remark 5. Since the decay rate is determined by

$$\beta = \min_{j=1,2,3} \beta_i = \min_{j=1,2,3} \gamma_j c_j,$$

we can choose

$$\gamma_1 = \ln\frac{5}{4}, \gamma_j = \frac{1}{2}\ln\left|\frac{1+\alpha_j}{1-\alpha_j}\right|, j = 2, 3,$$

thus the decay rate estimate is given by

$$\frac{\beta}{2} = \min\left\{\frac{c_1}{2}\ln\frac{5}{4}, \frac{c_2}{4}\ln\left|\frac{1+\alpha_2}{1-\alpha_2}\right|, \frac{c_3}{4}\ln\left|\frac{1+\alpha_3}{1-\alpha_3}\right|\right\}.$$

From above we see that even if $\alpha_2 = \alpha_3 = 1$, the decay rate of the system is $\frac{\beta}{2} = \frac{c_1}{2}\ln\frac{5}{4}$. So the system is not super-stable.

3.2. **The decay rate of the generic tree of 1-d wave equation.** In this subsection, we extend the previous model to more extensive case. We consider 1-d wave networks which are governed by the following partial differential equations:

$$\begin{cases}
w_{i,tt}(x,t) = c_i^2 w_{i,xx}(x,t), i = 1,2,\cdots,m, x \in (0,1),\ t > 0, \\
w_1(0,t) = 0, \\
w_1(1,t) = w_j(0,t), j = 2,3,\cdots,m, \\
c_1 w_{1,x}(1,t) = \sum_{j=2}^{m} c_j w_{j,x}(0,t), \\
c_j w_{j,x}(1,t) = -\alpha_j w_{j,t}(1,t), j = 2,\cdots,m, \\
w_i(x,0) = w_{i,0}(x),\ w_{i,t}(x,0) = w_{i,1}(x).
\end{cases} \tag{19}$$

where $c_i > 0$, $\alpha_i > 0$, $i = 2, \cdots, m$.

As before, we set

$$\begin{cases} \xi_i(x,t) = w_{i,t}(x,t) + c_i w_{i,x}(x,t), \\ \eta_i(x,t) = w_{i,t}(x,t) - c_i w_{i,x}(x,t). \end{cases}$$

then

$$\begin{cases} \xi_{i,t}(x,t) = c_i \xi_{i,x}(x,t), \\ \eta_{i,t}(x,t) = -c_i \eta_{i,x}(x,t),\ i = 1,2,,\cdots,m. \end{cases}$$

We define the weighted energy functional by $V(t)=\sum_{i=1}^m V_i(t)$, where

$$V_i(t) = \int_0^1 (p_i(x)\xi_i^2(x,t) + q_i(x)\eta_i^2(x,t))dx,$$

the weight functions $p_i(x)$ and $q_i(x)$ are defined as follows:

$$p_i(x) = p_i e^{\gamma_i x}, \quad q_i(x) = q_i e^{-\gamma_i x},$$

with $p_i > 0$, $q_i > 0$, $i = 1,2,\cdots,m$.

A direct calculation gives

$$\begin{aligned} \dot{V}(t) \quad = \quad & -\sum_{j=1}^m \beta_j V_j(t) + c_1 q_1 [w_{1,t}(0,t) - c_1 w_{1,x}(0,t)]^2 \\ & -c_1 p_1 [w_{1,t}(0,t) + c_1 w_{1,x}(0,t)]^2 + c_1 p_1 e^{\gamma_1} [w_{1,t}(1,t) + c_1 w_{1,x}(1,t)]^2 \\ & -c_1 q_1 e^{-\gamma_1} [w_{1,t}(1,t) - c_1 w_{1,x}(1,t)]^2 - \sum_{j=2}^m c_j p_j [w_{j,t}(0,t) + c_j w_{j,x}(0,t)]^2 \\ & +\sum_{j=2}^m c_j q_j [w_{j,t}(0,t) - c_j w_{j,x}(0,t)]^2 + \sum_{j=2}^m c_j p_j e^{\gamma_j} [w_{j,t}(1,t) + c_j w_{j,x}(1,t)]^2 \\ & -\sum_{j=2}^m c_j q_j e^{-\gamma_j} [w_{j,t}(1,t) - c_j w_{j,x}(1,t)]^2, \end{aligned}$$

where $\beta_i = \gamma_i c_i$.

For simplicity, we denote the connection parts by

$$\begin{aligned} N(t) \quad = \quad & c_1 p_1 e^{\gamma_1} [w_{1,t}(1,t) + c_1 w_{1,x}(1,t)]^2 - c_1 q_1 e^{-\gamma_1} [w_{1,t}(1,t) - c_1 w_{1,x}(1,t)]^2 \\ & -\sum_{j=2}^m c_j p_j [w_{j,t}(0,t) + c_j w_{j,x}(0,t)]^2 + \sum_{j=2}^m c_j q_j [w_{j,t}(0,t) - c_j w_{j,x}(0,t)]^2 \end{aligned} \tag{20}$$

and the boundary parts by

$$\begin{aligned} B(t) \quad = \quad & c_1 q_1 [w_{1,t}(0,t) - c_1 w_{1,x}(0,t)]^2 - c_1 p_1 [w_{1,t}(0,t) + c_1 w_{1,x}(0,t)]^2 \\ & +\sum_{j=2}^m c_j p_j e^{\gamma_j} [w_{j,t}(1,t) + c_j w_{j,x}(1,t)]^2 - \sum_{j=2}^m c_j q_j e^{-\gamma_j} [w_{j,t}(1,t) - c_j w_{j,x}(1,t)]^2 \end{aligned} \tag{21}$$

Using the boundary conditions in (19), taking $p_1 = q_1$, $c_i p_i = \mu_p$ and $c_i q_i = \mu_q$,

$$B(t) \quad = \quad \sum_{j=2}^m w_{j,t}^2(1,t)[\mu_p e^{\gamma_j}(1-\alpha_j)^2 - \mu_q e^{-\gamma_j}(1+\alpha_j)^2].$$

Using the connection conditions in (19), we get

$$\begin{aligned} N(t) &= [2c_1p_1\sinh\gamma_1 + (m-1)(\mu_q-\mu_p)]w_{1,t}^2(1,t) + 2c_1p_1\sinh\gamma_1 c_1^2 w_{1,x}^2(1,t) \\ &\quad + 2w_{1,t}(1,t)c_1w_{1,x}(1,t)[2c_1p_1\cosh\gamma_1 - (\mu_p+\mu_q)] + (\mu_q-\mu_p)\sum_{j=2}^m c_j^2 w_{j,x}^2(0,t). \end{aligned}$$

When $2c_1p_1\cosh\gamma_1 - (\mu_p+\mu_q) \geq 0$, it holds that

$$\begin{aligned} N(t) &\leq w_{1,t}^2(1,t)[2c_1p_1e^{\gamma_1} + (m-1)(\mu_q-\mu_p) - (\mu_p+\mu_q)] \\ &\quad + [2c_1p_1e^{\gamma_1} - (\mu_p+\mu_q)]c_1^2w_{1,x}^2(1,t) + (\mu_q-\mu_p)\sum_{j=2}^m c_j^2 w_{j,x}^2(0,t) \\ &\leq w_{1,t}^2(1,t)[2c_1p_1e^{\gamma_1} + (m-1)(\mu_q-\mu_p) - (\mu_p+\mu_q)] \\ &\quad + (m-1)\sum_{j=2}^m c_j^2 w_{j,x}^2(0,t)[2c_1p_1e^{\gamma_1} - (\mu_p+\mu_q) + \frac{(\mu_q-\mu_p)}{m-1}]. \end{aligned}$$

Set $\mu_p > \mu_q$ and

$$2c_1p_1e^{\gamma_1} - (\mu_p+\mu_q) + \frac{(\mu_q-\mu_p)}{m-1} \leq 0,$$

that implies

$$2c_1p_1e^{\gamma_1} + (m-1)(\mu_q-\mu_p) - (\mu_p+\mu_q) \leq 0.$$

Thus we need solve the following inequality

$$\begin{cases} \mu_p > \mu_q, \quad \gamma_1 > 0, \\ 2c_1p_1\cosh\gamma_1 - (\mu_p+\mu_q) \geq 0, \\ 2c_1p_1e^{\gamma_1} - (\mu_p+\mu_q) - \frac{(\mu_p-\mu_q)}{m-1} \leq 0. \end{cases} \tag{22}$$

We can set

$$\mu_p = (1+(m-2)\varepsilon)c_1p_1,$$
$$\mu_q = (1-(m-2)\varepsilon)c_1p_1,$$

then

$$\mu_p + \mu_q = 2c_1p_1, \quad (\mu_p-\mu_q) = 2(m-2)\varepsilon c_1p_1,$$

and

$$2c_1p_1e^{\gamma_1} - (\mu_p+\mu_q) = 2c_1p_1(e^{\gamma_1}-1) \geq 0, \forall\gamma_1 > 0.$$

Substituting the last inequality we get

$$e^{\gamma_1} - 1 - \frac{m-2}{m-1}\varepsilon \leq 0.$$

If we take

$$\varepsilon = \frac{m-1}{m-2}(e^{\gamma_1}-1), \quad (1-(m-2)\varepsilon) > 0,$$

we have

$$\varepsilon = \frac{m-1}{m-2}(e^{\gamma_1}-1) = \frac{1}{m-1} < \frac{1}{m-2}.$$

From above we get

$$\gamma_1 = \ln\frac{m^2-m-1}{(m-1)^2}, \quad \varepsilon = \frac{1}{m-1}.$$

The following theorem gives the selection conditions of the parameters.

Theorem 3.2. *Let the boundary control laws $u_i, i = 2, \cdots, m$, be defined by (15). Suppose that p_i, q_i, $i = 1, \cdots, m$, satisfy the following conditions:*

$$\begin{cases} p_1 = q_1, \\ p_j c_j = p_1 c_1 (1 + \frac{m-2}{m-1}), \\ q_j c_j = p_1 c_1 (1 - \frac{m-2}{m-1}), \\ \gamma_1 \le \ln \frac{m^2-m-1}{(m-1)^2}, \\ \gamma_j \le \frac{1}{2} \ln \frac{(1+\alpha_j)^2}{(2m-3)(1-\alpha_j)^2}, j = 2, 3, \cdots, m. \end{cases} \tag{23}$$

it holds that $\dot{V}(t) \le -\beta V(t)$, i.e., the exponential decay rate of the closed loop systems (19) is at leat $\frac{\beta}{2}$,and

$$\frac{\beta}{2} = \min_{j=1,\cdots m} \frac{\beta_i}{2} = \min_{j=1,\cdots m} \frac{\gamma_j c_j}{2}$$

$$= \min \left\{ \frac{c_j}{2} \ln \frac{m^2-m-1}{(m-1)^2}, \frac{c_j}{4} \ln \frac{(1+\alpha_j)^2}{(2m-3)(1-\alpha_j)^2}, j = 2, 3, \cdots, m \right\}$$

4. **Conclusions.** In this paper, we use the weighted energy functional of the systems to estimate the decay rate of the wave networks. Under the boundary velocity feedback control laws, we calculate the decay rate β of the weighted energy functional, and hence the system has decay rate estimate $\frac{\beta}{2}$. The key point of this approach is to select the parameters such that boundary parts and the connection parts are less or equal to zero. This will lead to solve an inequality group for the parameters. By a suitable choices of parameters, we can get the estimate of decay rate of the systems. Usually, if the parameters γ_j are selected suitable large, we can get more accurate estimate for the decay rate of the system. But according to the current method, we notice that the parameters γ_j are more and more small, and may tend to zero. Because we can't get accurate decay rate for complicated tree-shaped wave networks by spectral methods, in next step, we will analyze whether the parameters γ_j are smaller and smaller with the increase of m. This problem is worth thinking.

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