

## OSMOSIS FOR NON-ELECTROLYTE SOLVENTS IN PERMEABLE PERIODIC POROUS MEDIA

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**ABSTRACT.** The paper gives a rigorous description, based on mathematical homogenization theory, for flows of solvents with not charged solute particles under osmotic pressure for periodic porous media permeable for solute particles. The effective Darcy type equations for the flow under osmotic pressure distributed within the porous media are derived. The effective Darcy law contains an additional flux term representing the osmotic pressure. Coefficients in the effective homogenized equations are related to the values of the phenomenological coefficients in the Kedem-Katchalsky formulae (2).

**1. Introduction.** The goal of the present paper is to give a rigorous description, based on mathematical homogenization theory, of flows of non-electrolyte solutions (that is electrically neutral solute particles) under osmotic pressure for periodic porous media permeable for solute particles.

Osmosis is historically the term for a phenomenon of spontaneous passage of water or other solvents through a membrane that is permeable to the solvent but is impermeable for solute particles. If a solution is separated by such a semipermeable membrane from the pure solvent, the pure solvent will move through the membrane making the solution at the other side of the membrane more dilute. This process can be stopped by applying external counter pressure that gives an idea of osmotic pressure.

Osmosis explains in particular how living cells as red blood cells or plant cells adapt their shape to the environment stress by changing concentration of solutes (sucrose in case of plants cells) inside them.

This phenomenon was discovered by French experimental physicist Jean-Antoine Nollet in 1748 in natural membranes but was first studied in detail by a German plant physiologist Wilhelm Pfeffer only in 1877. The term osmose or osmosis was introduced by a British chemist, Thomas Graham in 1854.

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The Dutch chemist van't Hoff showed in 1886 that, for dilute solutes the osmotic pressure varies with concentration and temperature similarly to an ideal gas. A classical formula by van't Hoff for osmotic pressure acting on solvent at the border of the membrane impermeable for solute particles reads [33]:

$$p_{\text{osm}} = \rho kT \quad (1)$$

where  $\rho$  is the concentration of solute particles,  $T$  is temperature and  $k$  is the Boltzmann constant. This relation led to practical methods for determining molecular weights of solutes.

Osmotic pressure plays important role in biological processes as transport in plants [19] and through cell membranes [21], [15] and also in several modern membrane technologies in particular for desalination and sustainable power generation [10], [38], [24].

There is a physical phenomenon called by chance similarly - electroosmosis. Electro-osmotic flows in micro channels are driven by external electric fields, acting on charged solute particles that initiate the solvent flow through the viscous interaction. This phenomenon was discovered by F.F. Reuss [28] in 1809. Electroosmosis of charged particles with corresponding electrokinetic models [25], [31] and osmosis of neutral particles have certain similarities from the mathematical point of view, but are rather different in physical nature.

In many situations porous membranes are not completely impermeable to solute particles, but depending on the size of pores, obstruct to some extent the passage of particles. The effect of osmotic pressure in this case is not concentrated on the surface of the membrane, but is distributed within the membrane's volume. A combination of several complicated phenomena define the joint transport of solute and solvent through the membrane in this case. The question about the nature of osmotic pressure in such intermediate regimes did not have a rigorous answer up to now.

Several phenomenological models based on general thermodynamical principles were suggested to extend formula (1) to the case when a porous membrane is partially permeable to neutral solute particles, as the Kedem-Katchalsky formulae

$$\begin{aligned} J_u &= L_p \Delta p - L_{pD} \Delta p_{\text{osm}} \\ J_D &= -L_{Dp} \Delta p + L_D \Delta p_{\text{osm}} \end{aligned} \quad (2)$$

that connect fluxes  $J_u$  and  $J_D$  of solvent and of solute particles through the slab of a porous material with the value of the pressure drop  $\Delta p$  in the solvent and the solute concentration jump  $\Delta \rho$  [20],[21]. Here the phenomenological coefficients  $L_p$ ,  $L_{pD}$ ,  $L_{Dp}$ ,  $L_D$  are called coefficients of filtration, osmotic transport, ultrafiltration and diffusion, respectively. The relation

$$\sigma = -L_{pD}/L_p \quad (3)$$

between the osmotic transport coefficient and the filtration coefficient is called membranes reflection coefficient.

The goal of the present paper is to derive using mathematical homogenization theory a consistent macroscopic model for transport of solvents and neutral solutes in porous media that are permeable for solute particles. We consider as a microscopic model a system of equations of Nernst-Planck-Stokes type describing a slow flow of viscous fluid solvent together with the advection-diffusion of the solute particles through a periodic porous solid microstructure with period  $\varepsilon \ll 1$  under

the effect of potential forces acting on the solute particles through a potential  $V$  concentrated along the surface of the porous structure.

Such kind of models for flows under osmotic pressure were considered in the case of one dimensional flows in thin channels by Anderson and his coauthors [8] [7] and were developed also in [35], [17], [16]. They were applied to simple geometries in [37], [19], [36]. Neither rigorous mathematical analysis nor numerical analysis of these models in case of general geometry has been done up to now.

Related mathematical problems for Nernst-Planck-Poisson and Nernst-Planck-Poisson-Stokes systems for non-stationary electrokinetic models were considered in papers [25], [30], [31], [6], [5]. In [25] the homogenization problem for periodic micro-structures for the stationary Nernst-Planck-Poisson-Stokes system is considered, and formal asymptotic expansions for solutions are constructed. Rigorous justification of convergence to homogenized solution is given for the non-stationary Nernst-Planck-Poisson-Stokes system in [31], [6]. Similar results for non-ideal transport when finite size of ions is taken into account, were obtained in [5]. A number of works on electro-osmosis in porous media is available in physical literature, see for example [12],[9],[29].

The main results of the present paper are following. Introduction and mathematical analysis of a new model for the microscopic picture of osmotic flow for non-electrolyte (not charged) solute transport at the pore level. Derivation using mathematical homogenization theory of new effective Darcy's type equations for the flow under osmotic pressure distributed within the porous media. The new formula (5.3) for the distribution of osmotic pressure inside the porous media gives a quantitative answer about the nature of the osmotic transport. Coefficients in the derived homogenized equations relate values of the phenomenological coefficients in (2) with properties of the osmotic flow at the pore level.

The present paper deals with the stationary transport of neutral solute particles where the potential of forces acting on the particles is given and can grow infinitely for points approaching the boundary. This leads to possible degeneracy of the diffusion equation in the vicinity of the boundary and to corresponding complications in mathematical analysis. In this respect the considered model is mathematically more complicated than models for electro-osmosis where the potential satisfies the Poisson equation and is regular. One of the new features of the studied problem is the choice of boundary conditions for the flow equations describing a flow through a reservoir with prescribed pressure drop between the inflow and outflow parts of the boundary.

We consider in the present paper an  $N$ -dimensional porous structure with  $N = 2, 3$ , that fills an open domain  $\Omega$  surrounded by solid lateral walls  $\Gamma_0$  and by flat inflow and outflow boundaries  $S_1$  and  $S_2$  in two planes orthogonal to one of the coordinate axes. It is assumed that  $\Gamma_0$  is a Lipschitz continuous surface.

Through this paper we suppose that the boundary of the porous structure is a Lipschitz continuous and periodic surface. The periodicity cell is denoted by  $Y$ . Without loss of generality we suppose that  $Y = [0, 1]^N$ . We denote by  $Y_F$  an open set on  $Y$  and assume that it is Lipschitz and its periodic extension to  $\mathbb{R}^N$  is a connected set. In what follows we refer to  $Y_F$  as the fluid part of the porous medium.  $Y_S = Y \setminus Y_F$  denotes the solid part of the structure in  $Y$ . The scaled periodicity cell is denoted by  $Y^\varepsilon$ . Cells including the structure match exactly the outflow and inflow boundaries  $S_1$  and  $S_2$  of  $\Omega$ .

$\Omega_\varepsilon$  denotes the fluid part of the domain  $\Omega$  together with the porous structure,

$$\Omega_\varepsilon = \Omega \cap \left( \bigcup_{i \in \mathbb{Z}^n} \varepsilon(Y_F + i) \right),$$

and  $\partial\Omega_\varepsilon$  is its boundary.  $\Gamma_\varepsilon$  is the solid part of the boundary  $\partial\Omega_\varepsilon$  of the flow domain including the structure boundary and the solid boundary  $\Gamma_0$  of  $\Omega$ . The inflow and outflow parts of  $\partial\Omega_\varepsilon$  are denoted by  $S_1^\varepsilon$  and  $S_2^\varepsilon$ .

We denote by  $C_\varepsilon$  the union of scaled periodicity cells that are completely included into the domain  $\Omega$ :

$$C_\varepsilon = \bigcup_{i \in K^\varepsilon} \varepsilon(Y + i), \quad K^\varepsilon = \{i \in \mathbb{Z}^N : \varepsilon(Y + i) \subset \Omega_\varepsilon\} \quad (4)$$

The fluid solvent is described by the Stokes equations for velocity  $u_\varepsilon$  and pressure  $p_\varepsilon$  with external forces coming from friction between the particles and the fluid. We impose non-slip boundary conditions for the velocity  $u_\varepsilon$  in the Stokes equations on the solid boundary  $\Gamma_\varepsilon$  and impose boundary conditions on the inflow and outflow boundaries  $S_1^\varepsilon$  and  $S_2^\varepsilon$  for pressure  $p_\varepsilon$  as constant values  $\bar{P}_1$  and  $\bar{P}_2$ , and for tangential component of velocity as  $u_{\varepsilon,\tau} = 0$ .

The solute concentration  $\rho_\varepsilon$  satisfies the advection diffusion equation with drift force defined in terms of the potential  $V_\varepsilon$  with support concentrated along the solid boundaries.  $V$  is a periodic function on  $Y$ , and we denote the scaled potential by  $V_\varepsilon(x) = V\left(\frac{x}{\varepsilon}\right)$ . We apply zero normal flux boundary condition for the solute concentration  $\rho_\varepsilon$  on the solid boundary  $\Gamma_\varepsilon$  and the Dirichlet boundary conditions for  $\rho_\varepsilon$  on inflow and outflow boundaries  $S_1^\varepsilon$  and  $S_2^\varepsilon$  defined as  $S_i^\varepsilon = S_i \cap \overline{\Omega_\varepsilon}$ ,  $i = 1, 2$ .

We consider a boundary value problem for the system of PDEs consisting of the Stokes equations for velocity  $u_\varepsilon$  and pressure  $p_\varepsilon$  of the solvent with the osmotic force  $\rho_\varepsilon \nabla V_\varepsilon$  and the advection-diffusion equation with advection velocity  $u_\varepsilon$  and drift term  $\text{div}(\kappa \rho_\varepsilon \nabla V_\varepsilon)$ .

The strong formulation of the boundary value problem reads:

$$\mu \Delta u_\varepsilon - \nabla p_\varepsilon - \rho_\varepsilon \nabla V_\varepsilon = 0, \quad x \in \Omega_\varepsilon; \quad (5a)$$

$$\text{div}(u_\varepsilon) = 0, \quad x \in \Omega_\varepsilon; \quad (5b)$$

$$u_\varepsilon = 0, \quad x \in \Gamma_\varepsilon; \quad (5c)$$

$$p_\varepsilon = \bar{P}_i, \quad u_{\varepsilon,\tau} = 0, \quad x \in S_i^\varepsilon, \quad i = 1, 2. \quad (5d)$$

for the Stokes equations and

$$\Delta \rho_\varepsilon + \frac{\kappa}{\lambda} \text{div}(\rho_\varepsilon \nabla V_\varepsilon) = \frac{1}{\lambda} \text{div}(\rho_\varepsilon u_\varepsilon), \quad x \in \Omega_\varepsilon; \quad (6a)$$

$$\left( \nabla \rho_\varepsilon + \frac{\kappa}{\lambda} (\rho_\varepsilon \nabla V_\varepsilon) - \frac{1}{\lambda} \rho_\varepsilon u_\varepsilon \right) \cdot n = 0, \quad x \in \Gamma_\varepsilon; \quad (6b)$$

$$\rho_\varepsilon = 0, \quad x \in S_1^\varepsilon, \quad \rho_\varepsilon = \theta_2 \beta_\varepsilon(x), \quad x \in S_2^\varepsilon, \quad (6c)$$

for the advection-diffusion equation. Here  $\mu$  is viscosity,  $\lambda$  is diffusion constant,  $\kappa$  is the mobility of solute particles,  $\theta_2 \geq 0$  is a constant, and  $\beta_\varepsilon(x) = \exp(-\frac{\kappa}{\lambda} V_\varepsilon(x))$ ;  $n$  is the exterior normal on  $\partial\Omega_\varepsilon$ .

The weak formulation of problem (5)-(6) and conditions for well posedness of this problem are given in Sections 2 and 3.

We notice that according to the Einstein-Smoluchowski relation [14], [34]

$$\frac{\lambda}{\kappa} = kT \quad (7)$$

where  $T$  is absolute temperature and  $k$  is the Boltzmann constant, and van't Hoff's formula (1) for osmotic pressure can in our notations be rewritten as

$$p_{\varepsilon, \text{osm}} = \rho_{\varepsilon} \left( \frac{\lambda}{\kappa} \right) \quad (8)$$

To illuminate the effects of osmosis in the Stokes equation we observe that

$$-\rho_{\varepsilon} \nabla V_{\varepsilon} = - \left( \frac{\lambda}{\kappa} \right) \beta_{\varepsilon} \nabla (\rho_{\varepsilon} \beta_{\varepsilon}^{-1}) + \left( \frac{\lambda}{\kappa} \right) \nabla \rho_{\varepsilon} \quad (9)$$

and rewrite the equation (5a) as

$$\mu \Delta u_{\varepsilon} - \nabla p_{\varepsilon} + \left( \frac{\lambda}{\kappa} \right) \nabla \rho_{\varepsilon} - \left( \frac{\lambda}{\kappa} \right) \beta_{\varepsilon} \nabla (\rho_{\varepsilon} \beta_{\varepsilon}^{-1}) = 0, \quad x \in \Omega_{\varepsilon}. \quad (10)$$

with the expression  $\left( \frac{\lambda}{\kappa} \right) \nabla \rho_{\varepsilon} = \nabla p_{\varepsilon, \text{osm}}$  for the osmotic pressure (8) included explicitly.

We formulate here also a boundary value problem for pressure that follows from (5)

$$\begin{aligned} \Delta p_{\varepsilon} &= -\operatorname{div}(\rho_{\varepsilon} \nabla V_{\varepsilon}), \quad x \in \Omega_{\varepsilon}; \\ \nabla p_{\varepsilon} \cdot n &= 0, \quad x \in \Gamma_{\varepsilon} \\ p_{\varepsilon} &= \bar{P}_i, \quad x \in S_i^{\varepsilon}, \quad i = 1, 2. \end{aligned} \quad (11)$$

Only the difference  $\delta \bar{P} = \bar{P}_1 - \bar{P}_2$  between pressure values at the inflow and outflow boundaries  $S_1$  and  $S_2$  has physical meaning. We will control only  $\delta \bar{P}$  and will normalize pressure by the condition

$$\int_{\Omega_{\varepsilon}} p_{\varepsilon} dx = 0 \quad (12)$$

The main result of the work is deriving a limit macroscopic system consisting of an effective diffusion equation (95) and a Darcy type equation (96) with additional flux representing the effect of the osmotic pressure distributed within the structure. In the case of a flat membrane, the corresponding effective matrices  $B_D$  and  $B_{\text{osm}}$  (90) in (96), are related to the filtration and osmotic transport coefficients  $L_p$  and  $L_{pD}$  in the Kedem-Katchalsky formula (2).

The paper is organized as follows. In Section 2 we provide the problem setup and obtain a priori estimates in weighted Sobolev spaces for solutions of the studied system. In the second part of this section we use contraction arguments to justify the well posedness of the system under consideration.

In Section 3 we pass to the two-scale limit in the advection-diffusion equation with potential forces. Here we use two-scale convergence in the variable spaces approach [39].

Section 4 is devoted to the homogenization of velocity and pressure satisfying the Stokes system with osmotic forces originated in potential forces acting on the solute and in the density gradient of the solute.

The goal of Section 5 is to derive the macroscopic Darcy's law with osmotic pressure distributed within the porous structure. The result is obtained by excluding the fast variable from the two-scaled effective system of equations.

Finally, in the Appendix we adapt results on the Friedrichs and Poincaré type inequalities from [22] and [27] to the weighted Sobolev spaces specific for our problems. Also we provide nontrivial examples of potentials and corresponding weights such that the desired Friedrichs and Poincaré inequalities hold true.

## 2. Weak formulation of the problems and a priori estimates.

**2.1. A priori estimates for the advection diffusion equation with drift by osmotic forces.** We derive in this section a weak formulation of problem (6) in terms of weighted spaces  $L_2(\Omega_\varepsilon, \beta_\varepsilon)$ ,  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  with scalar products

$$(\theta, \xi)_{L_2(\Omega_\varepsilon, \beta_\varepsilon)} = \int_{\Omega_\varepsilon} (\theta \xi) \beta_\varepsilon dx \quad (13)$$

$$(\theta, \xi)_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} = \int_{\Omega_\varepsilon} (\theta \xi + \nabla \theta \cdot \nabla \xi) \beta_\varepsilon dx \quad (14)$$

and weight

$$\beta_\varepsilon(x) = \exp\left(-\frac{\kappa}{\lambda} V(x/\varepsilon)\right). \quad (15)$$

The space  $L_p(\Omega_\varepsilon, \beta_\varepsilon)$  is defined by the norm

$$\|\theta\|_{L_p(\Omega_\varepsilon, \beta_\varepsilon)}^p = \int_{\Omega_\varepsilon} |\theta|^p \beta_\varepsilon dx. \quad (16)$$

Typical potentials  $V_\varepsilon(x)$  in our problems are nonnegative and bounded on compact subsets of  $\Omega_\varepsilon$ , and are rising, may be infinitely, for points tending to the solid part  $\Gamma_\varepsilon$  of the boundary of  $\Omega_\varepsilon$ .

By using the formula:

$$\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}) = \nabla \rho_\varepsilon + \frac{\kappa}{\lambda} (\rho_\varepsilon \nabla V_\varepsilon) \quad (17)$$

with  $\beta_\varepsilon^{-1} = 1/(\beta_\varepsilon)$ , the following symmetrization

$$\operatorname{div} \left[ \beta_\varepsilon \left( \frac{1}{\lambda} (\rho_\varepsilon \beta_\varepsilon^{-1} u_\varepsilon) - \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}) \right) \right] = 0 \quad (18)$$

of the advection diffusion equation with potential forces is achieved.

We multiply the advection-diffusion equation (6a) by an arbitrary function of the form  $\psi \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  and integrate the resulting relation by parts using (18). Boundary conditions imply that after integration by parts the sum of all fluxes on the solid boundary  $\Gamma_\varepsilon$  is zero.

$$\begin{aligned} & \int_{\Omega_\varepsilon} \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}) \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon dx - \frac{1}{\lambda} \int_{\Omega_\varepsilon} (\rho_\varepsilon \beta_\varepsilon^{-1}) u_\varepsilon \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon dx \quad (19) \\ &= \frac{1}{\lambda} \sum_i \int_{S_i} (\psi \beta_\varepsilon^{-1}) [u_{\varepsilon n} (\rho_\varepsilon \beta_\varepsilon^{-1})] \beta_\varepsilon d\sigma + \sum_i \int_{S_i} (\psi \beta_\varepsilon^{-1}) \left[ \frac{\partial}{\partial n} (\rho_\varepsilon \beta_\varepsilon^{-1}) \right] \beta_\varepsilon d\sigma. \end{aligned}$$

The last formulation motivates the introduction of the Hilbert space  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  defined in (14).

**Conditions on potential.** We consider the weighted space  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  with the weight  $\beta_\varepsilon = \exp(-\frac{\kappa}{\lambda} V_\varepsilon)$  and suppose that

- the Friedrichs inequality in  $\Omega_\varepsilon$  with zero boundary conditions  $f = 0$  on  $S_i^\varepsilon$ ,  $i = 1, 2$  is valid:

$$\int_{\Omega_\varepsilon} |f|^2 \beta_\varepsilon dx \leq C \left[ \sum_{i=1}^N \int_{\Omega_\varepsilon} \left| \frac{\partial}{\partial x_i} f \right|^2 \beta_\varepsilon dx \right] \quad (20)$$

- the spaces  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  and  $\overset{o}{W}_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  are compactly embedded into the space  $L_2(\Omega_\varepsilon, \beta_\varepsilon)$ . It implies that the Poincare inequality

$$\int_{\Omega_\varepsilon} |f|^2 \beta_\varepsilon dx \leq C_1 \left[ \left( \int_{\Omega_\varepsilon} \beta_\varepsilon dx \right)^{-1} \left| \int_{\Omega_\varepsilon} f \beta_\varepsilon dx \right|^2 + \sum_{i=1}^N \int_{\Omega_\varepsilon} \left| \frac{\partial}{\partial x_i} f \right|^2 \beta_\varepsilon dx \right] \quad (21)$$

is valid for all  $f \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$ .

- the space  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  is continuously embedded into  $L_6(\Omega_\varepsilon, (\beta_\varepsilon)^3)$ .  
If  $0 < c < \beta_\varepsilon < C < +\infty$  on  $\Omega_\varepsilon$  these conditions are satisfied in dimensions 2 and 3.

The connectedness of  $\Omega_\varepsilon$  and positivity of  $\beta_\varepsilon$  on  $\Omega_\varepsilon$  implies that the measure  $\beta_\varepsilon dx$  is ergodic in the sense of Zhikov [39]. By other words the equality  $\int_{\Omega_\varepsilon} |\nabla f|^2 \beta_\varepsilon dx = 0$  implies that  $f$  is constant almost everywhere with respect to the measure  $\beta_\varepsilon dx$ .

Potentials  $V_\varepsilon(x)$  appearing in the problems of interest are natural to interpret as functions of the distance  $d_\Gamma(x)$  from the solid boundary  $\Gamma_\varepsilon$ :  $V_\varepsilon(x) = V_\varepsilon(d_\Gamma(x))$ . If the potential  $V_\varepsilon(x)$  goes to infinity when  $x$  approaches the solid boundary  $\Gamma_\varepsilon$ , that can naturally happen in applications, the weight  $\beta_\varepsilon(x)$  degenerates at  $\Gamma_\varepsilon$ .

We provide in the Appendix a number of sufficient conditions for the Friedrichs inequality and the Poincare inequality in weighted Sobolev spaces from [27]. We also give examples of potentials  $V_\varepsilon(d_\Gamma(x))$  such that these conditions are satisfied for the weight  $\beta_\varepsilon(x) = \exp\{-\frac{\kappa}{\lambda} V_\varepsilon(d_\Gamma(x))\}$ .

For later analysis of the coupled advection-diffusion and Stokes equations we consider first two auxiliary problems for the equation (19): one with homogeneous boundary conditions on  $S_1^\varepsilon \cup S_2^\varepsilon$  with a given right hand side, and another one with inhomogeneous boundary conditions and zero right hand side.

The first problem in weak form reads: given  $G \in [L_2(\Omega_\varepsilon, \beta_\varepsilon)]^N$  find  $b_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  such that  $b_\varepsilon \beta_\varepsilon^{-1}$  satisfies the integral relation:

$$\int_{\Omega_\varepsilon} \nabla (b_\varepsilon \beta_\varepsilon^{-1}) \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon dx = \int_{\Omega_\varepsilon} G \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon dx. \quad (22)$$

and the boundary conditions

$$b_\varepsilon|_{S_1^\varepsilon \cup S_2^\varepsilon} = 0; \quad \left( -G_n + b_\varepsilon \frac{\kappa}{\lambda} \frac{\partial V_\varepsilon}{\partial n} + \frac{\partial b_\varepsilon}{\partial n} \right) \Big|_{\Gamma_\varepsilon} = 0.$$

for an arbitrary function  $\psi \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  that satisfies boundary conditions

$$\psi|_{S_1^\varepsilon \cup S_2^\varepsilon} = 0.$$

on the inflow and outflow parts  $S_1^\varepsilon$  and  $S_2^\varepsilon$  of the boundary. Here  $G_n$  stands for the normal component of the vector function  $G$ . For the coupled system of advection-diffusion and Stokes equations we will substitute  $G$  with  $G = \frac{1}{\lambda} (\rho_\varepsilon \beta_\varepsilon^{-1}) u_\varepsilon$ .

The Friedrichs inequality implies that problem (22) is coercive and the solution operator  $\mathcal{R}_1(G) = b_\varepsilon \beta_\varepsilon^{-1}$  is bounded from  $[L_2(\Omega_\varepsilon, \beta_\varepsilon)]^N$  to  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$ . Namely the following bound holds:

$$\|\mathcal{R}_1(G)\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq C_{\mathcal{R}_1} \|G\|_{[L_2(\Omega_\varepsilon, \beta_\varepsilon)]^N}. \quad (23)$$

Notice that according to (20) the constant  $C_{\mathcal{R}_1}$  does not depend on  $\varepsilon$ .

The second auxiliary problem in weak form reads: find  $a_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  such that  $a_\varepsilon$  satisfies the integral relation:

$$\int_{\Omega_\varepsilon} \nabla (a_\varepsilon \beta_\varepsilon^{-1}) \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon dx = 0 \quad (24)$$

and the boundary conditions

$$a_\varepsilon \beta_\varepsilon^{-1}|_{S_1^\varepsilon} = 0; \quad a_\varepsilon \beta_\varepsilon^{-1}|_{S_2^\varepsilon} = \theta_2; \quad \left( a_\varepsilon \frac{\kappa}{\lambda} \frac{\partial V_\varepsilon}{\partial n} + \frac{\partial a_\varepsilon}{\partial n} \right) \Big|_{\Gamma_\varepsilon} = 0 \quad (25)$$

for an arbitrary function  $\psi \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  that satisfies boundary conditions

$$\psi|_{S_1^\varepsilon \cup S_2^\varepsilon} = 0.$$

In order to construct a solution of this problem, we first introduce a function  $\tilde{a}_\varepsilon(x) \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  such that with constant  $\theta_2 > 0$

$$\tilde{a}_\varepsilon(x) \beta_\varepsilon^{-1}(x) = \theta_2, \quad x \in S_2^\varepsilon; \quad \tilde{a}_\varepsilon(x) \beta_\varepsilon^{-1}(x) = 0, \quad x \in S_1^\varepsilon; \quad \frac{\partial \tilde{a}_\varepsilon \beta_\varepsilon^{-1}}{\partial n} = 0, \quad x \in \Gamma_\varepsilon,$$

and  $\|\tilde{a}_\varepsilon\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq C\theta_2$ .

We represent  $a_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  as the sum  $a_\varepsilon = g_\varepsilon + \tilde{a}_\varepsilon$  with  $g_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  satisfying the following integral relation

$$\int_{\Omega_\varepsilon} \nabla (g_\varepsilon \beta_\varepsilon^{-1}) \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon dx = - \int_{\Omega_\varepsilon} \nabla (\tilde{a}_\varepsilon \beta_\varepsilon^{-1}) \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon dx \quad (26)$$

for an arbitrary function  $\psi \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  such that

$$\psi|_{S_1^\varepsilon \cup S_2^\varepsilon} = 0, \quad (27)$$

and the boundary conditions

$$g_\varepsilon(x) \beta_\varepsilon^{-1}(x) = 0, \quad x \in S_1^\varepsilon \cup S_2^\varepsilon; \quad \frac{\partial g_\varepsilon \beta_\varepsilon^{-1}}{\partial n} = 0, \quad x \in \Gamma_\varepsilon \quad (28)$$

Combining an energy estimate following from (26) with  $\psi \beta_\varepsilon^{-1} = g_\varepsilon \beta_\varepsilon^{-1}$  and the weighted Friedrichs inequality (20) we obtain by means of the Lax - Milgram lemma the existence and uniqueness of solutions to (26).

The corresponding solution operator  $\mathcal{R}_2(\theta_2) = a_\varepsilon \beta_\varepsilon^{-1}$  is bounded and satisfies the estimate:

$$\|\mathcal{R}_2(\theta_2)\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq C\mathcal{R}_2\theta_2 \quad (29)$$

**2.2. Weak formulation and a priori estimates for the Stokes equation.** We introduce the space

$$D^\#(\Omega_\varepsilon) = \left\{ \varphi \in [C^\infty(\Omega_\varepsilon)]^N : \operatorname{div}(\varphi) = 0, \varphi|_{\Gamma_\varepsilon} = 0, \varphi_\tau|_{S_1^\varepsilon \cup S_2^\varepsilon} = 0 \right\} \quad (30)$$

of smooth solenoidal vector valued functions equal to zero on the solid part  $\Gamma_\varepsilon$  of the boundary, having zero tangential component  $\varphi_\tau$  on the inflow and outflow parts of the boundary  $S_1^\varepsilon \cup S_2^\varepsilon$ , and possibly non zero normal component  $\varphi_n$  on  $S_1^\varepsilon \cup S_2^\varepsilon$ .

We will also use the space

$$J_1^\#(\Omega_\varepsilon) = \left\{ \varphi \in [W_2^1(\Omega_\varepsilon)]^N : \operatorname{div}(\varphi) = 0, \varphi|_{\Gamma_\varepsilon} = 0, \varphi_\tau|_{S_1^\varepsilon \cup S_2^\varepsilon} = 0 \right\}, \quad (31)$$

and the space

$$W_2^\#(\Omega_\varepsilon) = \left\{ \varphi \in [W_2^1(\Omega_\varepsilon)]^N : \varphi|_{\Gamma_\varepsilon} = 0, \varphi_\tau|_{S_1^\varepsilon \cup S_2^\varepsilon} = 0 \right\}. \quad (32)$$



A weak formulation of the Stokes boundary value problem with given constant pressure  $p_\varepsilon = \bar{P}_i$  and tangential velocity  $u_{\varepsilon,\tau} = 0$  on the inflow and outflow boundaries  $S_i^\varepsilon$ ,  $i = 1, 2$ , is formulated following ideas in [13] and [18].

We find a function  $u_\varepsilon \in J_1^\#(\Omega_\varepsilon)$  that for arbitrary  $\varphi \in J_1^\#(\Omega_\varepsilon)$  satisfies the integral relation:

$$\begin{aligned} & \mu (\nabla u_\varepsilon, \nabla \varphi) + \frac{\lambda}{\kappa} (\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}), \varphi) \\ &= - \int_{S_1^\varepsilon} \left( \bar{P}_1 - \bar{P}_2 + \frac{\lambda}{\kappa} \theta_2 \beta_\varepsilon \right) \varphi \cdot n d\sigma; \quad \varphi \in J_1^\#(\Omega_\varepsilon). \end{aligned} \tag{33}$$

To derive the weak formulation (33) from the strong one we multiply the Stokes equation (10) by a solenoidal test function  $\varphi \in D^\#(\Omega_\varepsilon)$  and integrate by parts taking into account boundary conditions:  $u_\varepsilon = \varphi = 0$  on the solid boundary  $\Gamma_\varepsilon$ ;  $p_\varepsilon = \bar{P}_i$  and  $u_{\varepsilon,\tau} = \varphi_\tau = 0$  on the inflow and outflow boundaries  $S_i^\varepsilon$ ,  $i = 1, 2$ . This yields

$$\int_\Omega \Delta u_\varepsilon \cdot \varphi dx = - \sum_{i,k} \int_\Omega \frac{\partial u_{\varepsilon,i}}{\partial x_k} \frac{\partial \varphi_i}{\partial x_k} dx + \int_{S_1^\varepsilon \cup S_2^\varepsilon} \frac{\partial u_{\varepsilon,n}}{\partial n} \varphi_n dS + \int_{S_1^\varepsilon \cup S_2^\varepsilon} \frac{\partial u_{\varepsilon,\tau}}{\partial n} \cdot \varphi_\tau dS.$$

We observe that in the case of dimension  $N = 3$ , for two orthogonal tangential directions  $\tau_1$  and  $\tau_2$  on  $S_1^\varepsilon \cup S_2^\varepsilon$

$$\frac{\partial u_{\varepsilon,n}}{\partial n} + \frac{\partial u_{\varepsilon,\tau_1}}{\partial x_{\tau_1}} + \frac{\partial u_{\varepsilon,\tau_2}}{\partial x_{\tau_2}} = \operatorname{div}(u_\varepsilon) = 0.$$

Condition  $u_{\varepsilon,\tau} = 0$  on  $S_1^\varepsilon \cup S_2^\varepsilon$  implies that  $\frac{\partial u_{\varepsilon,\tau_1}}{\partial x_{\tau_1}} + \frac{\partial u_{\varepsilon,\tau_2}}{\partial x_{\tau_2}} \equiv 0$  on  $S_1^\varepsilon \cup S_2^\varepsilon$ . Therefore  $\frac{\partial u_{\varepsilon,n}}{\partial n} \equiv 0$  on  $S_1^\varepsilon \cup S_2^\varepsilon$  and it leads to the following simplification:

$$\int_\Omega \Delta u_\varepsilon \cdot \varphi dx = - \int_\Omega \sum_{i,k} \frac{\partial u_{\varepsilon,i}}{\partial x_k} \frac{\partial \varphi_i}{\partial x_k} dx.$$

Similar formula evidently holds for dimension  $N = 2$ . Together with the relation

$$\begin{aligned} \int_\Omega \left( \nabla p_\varepsilon - \frac{\lambda}{\kappa} \nabla \rho_\varepsilon \right) \cdot \varphi dx &= \sum_{i=1,2} \int_{S_i^\varepsilon} \left( \bar{P}_i - \frac{\lambda}{\kappa} \rho_\varepsilon \beta_\varepsilon \right) \varphi \cdot n d\sigma \\ &= \int_{S_1^\varepsilon} \left( \bar{P}_1 - \bar{P}_2 + \frac{\lambda}{\kappa} \theta_2 \beta_\varepsilon \right) \varphi \cdot n d\sigma \end{aligned}$$

following from the constraint  $\operatorname{div}(\varphi) = 0$  and from the boundary conditions (6c), (5d) for  $\rho_\varepsilon$ ,  $p_\varepsilon$  it implies the equation (33).

Suppose that  $\beta_\varepsilon(x) \in W_2^{1/2}(S_i^\varepsilon)$  and  $\|\beta_\varepsilon\|_{W_2^{1/2}(S_i^\varepsilon)} \leq C_\beta$  with constant  $C_\beta$  independent of  $\varepsilon$ . It is valid for most reasonable potentials  $V$  for example for  $V(x) = [d_\Gamma(x)]^{-k}$ ,  $k > 0$ , because the tangential gradient of  $\beta_\varepsilon(x)$  on  $S_i^\varepsilon$  is  $\nabla_\tau \beta_\varepsilon(x) = -\left(\frac{\kappa}{\lambda}\right) \nabla_\tau V\left(\frac{x}{\varepsilon}\right) \exp\left(-\frac{\kappa}{\lambda} V\left(\frac{x}{\varepsilon}\right)\right)$  and the restriction of  $\beta_\varepsilon(x)$  on  $S_i^\varepsilon$  is a periodic function with period  $\varepsilon$  on cells of dimension  $N - 1$ . There is an auxiliary function  $\Pi_\varepsilon \in W_2^1(\Omega_\varepsilon)$  such that

$$\Pi_\varepsilon(x) = \bar{P}_2 - \frac{\lambda}{\kappa} \theta_2 \beta_\varepsilon(x/\varepsilon)$$

for  $x \in S_2^\varepsilon$ ,  $\Pi_\varepsilon(x) = \bar{P}_1$  for  $x \in S_1^\varepsilon$ , and

$$\|\Pi_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leq C(|\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2) \tag{34}$$

with the constant  $C$  independent of  $\varepsilon$ . We reformulate the equation (33) by subtracting  $\Pi_\varepsilon(x)$  from the pressure  $p_\varepsilon$ . Find function  $u_\varepsilon \in J_1^\#(\Omega_\varepsilon)$ , that for arbitrary  $\varphi \in J_1^\#(\Omega_\varepsilon)$ , satisfies the integral relation:

$$\mu(\nabla u_\varepsilon, \nabla \varphi) + \frac{\lambda}{\kappa} (\beta_\varepsilon \nabla(\rho_\varepsilon \beta_\varepsilon^{-1}), \varphi) - (\nabla \Pi_\varepsilon, \varphi) = 0; \quad \varphi \in J_1^\#(\Omega_\varepsilon) \quad (35)$$

similar to (33) but with zero boundary terms on  $S_i^\varepsilon$ .

For a fixed  $\rho_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  and for  $\Pi_\varepsilon \in W_2^1(\Omega_\varepsilon)$  this equation and the equivalent equation (33) have a unique solution in  $J_1^\#(\Omega_\varepsilon)$  by the Lax Milgram Lemma since the linear functional  $\mathcal{L}(\varphi) = (\beta_\varepsilon \nabla(\rho_\varepsilon \beta_\varepsilon^{-1}), \varphi) - (\nabla \Pi_\varepsilon, \varphi)$  is bounded in  $J_1^\#(\Omega_\varepsilon)$ . This argument is classical, see [23], [13], [18]. We consider corresponding estimates in more detail later.

We deal in this section with estimating solutions of the Stokes equation. This estimate is crucial for the homogenization analysis.

We consider first a general form of the Stokes equations with zero boundary terms on  $S_1^\varepsilon$  and  $S_2^\varepsilon$ :

$$\mu(\nabla u_\varepsilon, \nabla \varphi)_{[L_2(\Omega_\varepsilon)]^{N^2}} - (Q, \varphi)_{[L_2(\Omega_\varepsilon)]^N} = 0; \quad \varphi \in J_1^\#(\Omega_\varepsilon). \quad (36)$$

Consider this integral relation for  $\varphi = u_\varepsilon$ :

$$\mu \|\nabla u_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^{N^2}}^2 + (Q, u_\varepsilon)_{[L_2(\Omega_\varepsilon)]^N} = 0; \quad u_\varepsilon \in J_1^\#(\Omega_\varepsilon) \quad (37)$$

The scaling argument for Friedrichs inequality on the periodicity cell implies

$$\left| (Q, u_\varepsilon)_{[L_2(\Omega_\varepsilon)]^N} \right| \leq \|Q\|_{[L_2(\Omega_\varepsilon)]^N} \|u_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^N} \leq C\varepsilon \|Q\|_{[L_2(\Omega_\varepsilon)]^N} \|\nabla u_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^{N^2}};$$

and

$$\mu \|\nabla u_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^{N^2}} \leq C \left[ \varepsilon \|Q\|_{[L_2(\Omega_\varepsilon)]^N} \right].$$

and after one more similar argument an a priori estimate for the  $[L_2(\Omega_\varepsilon)]^N$  norm of  $u_\varepsilon$  follows:

$$\mu \|u_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^N} \leq C \left[ \varepsilon^2 \|Q\|_{[L_2(\Omega_\varepsilon)]^N} \right]. \quad (38)$$

Therefore the solution operator  $\mathcal{S}_1(\beta_\varepsilon \nabla(\rho_\varepsilon \beta_\varepsilon^{-1}))$  for the problem (36) with  $Q = \frac{\lambda}{\kappa} \beta_\varepsilon \nabla(\rho_\varepsilon \beta_\varepsilon^{-1})$  satisfies the estimates

$$\|\mathcal{S}_1(\beta_\varepsilon \nabla(\rho_\varepsilon \beta_\varepsilon^{-1}))\|_{[L_2(\Omega_\varepsilon)]^{N^2}} \leq \varepsilon^2 \frac{\lambda}{\kappa \mu} C \|\rho_\varepsilon \beta_\varepsilon^{-1}\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)}; \quad (39)$$

$$\|\mathcal{S}_1(\beta_\varepsilon \nabla(\rho_\varepsilon \beta_\varepsilon^{-1}))\|_{[W_2^1(\Omega_\varepsilon)]^N} \leq \varepsilon^1 \frac{\lambda}{\kappa \mu} C \|\rho_\varepsilon \beta_\varepsilon^{-1}\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)}.$$

The problem

$$\mu(\nabla u_\varepsilon, \nabla \varphi) - (\nabla \Pi_\varepsilon, \varphi) = 0; \quad \varphi \in J_1^\#(\Omega_\varepsilon). \quad (40)$$

with the potential  $\Pi_\varepsilon$  representing as above, the effect of the hydrostatic pressure drop  $\bar{P}_1 - \bar{P}_2$  between  $S_1^\varepsilon$  and  $S_2^\varepsilon$  together with the classical osmotic pressure  $\frac{\lambda}{\kappa} \rho_\varepsilon$ , has a solution operator  $\mathcal{S}_2$  satisfying estimates

$$\|\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)\|_{[L_2(\Omega_\varepsilon)]^N} \leq \varepsilon^2 \frac{1}{\mu} C_{\mathcal{S}_2} \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right); \quad (41)$$

$$\|\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)\|_{[W_2^1(\Omega_\varepsilon)]^N} \leq \varepsilon \frac{1}{\mu} C_{\mathcal{S}_2} \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right);$$

**Remark.** Notice that if the density  $\rho_\varepsilon$  of the solute has a constant value  $r_2$  on  $S_2^\varepsilon$  and is zero on  $S_1^\varepsilon$ , the last estimates depend just on a simple balance between the hydrostatic pressure drop  $\bar{P}_1 - \bar{P}_2$  and the osmotic pressure  $p_{\text{osm}} = \frac{\lambda}{\kappa} r_2$ :

$$\|\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)\|_{[L_2(\Omega_\varepsilon)]^N} \leq \varepsilon^2 \frac{1}{\mu} C_{\mathcal{S}_2} \left( \left| \bar{P}_1 - \bar{P}_2 + \frac{\lambda}{\kappa} r_2 \right| \right) \quad (42)$$

$$\|\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)\|_{[W_2^1(\Omega_\varepsilon)]^N} \leq \varepsilon \frac{1}{\mu} C_{\mathcal{S}_2} \left( \left| \bar{P}_1 - \bar{P}_2 + \frac{\lambda}{\kappa} r_2 \right| \right)$$

**3. Abstract contraction argument for quadratic non-linearity and a priori estimates for the coupled system.** We consider now the following joint system of equations for flow and advection-diffusion. Structure interacts with the solute through the potential  $V_\varepsilon$  and by that acts on the solvent. The Stokes equations and the advection-diffusion equation are coupled here through the first order terms. The joint system in weak form reads:

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla [(\rho_\varepsilon) \beta_\varepsilon^{-1}] \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon \, dx \\ = \frac{1}{\lambda} \int_{\Omega_\varepsilon} ((\rho_\varepsilon) \beta_\varepsilon^{-1}) u_\varepsilon \cdot \nabla (\psi \beta_\varepsilon^{-1}) \beta_\varepsilon \, dx \end{aligned} \quad (43a)$$

$$\begin{aligned} \mu (\nabla u_\varepsilon, \nabla \varphi)_{[L_2(\Omega_\varepsilon)]^{N^2}} + \frac{\lambda}{\kappa} (\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}), \varphi)_{[L_2(\Omega_\varepsilon)]^N} \\ - (\nabla \Pi_\varepsilon, \varphi)_{[L_2(\Omega_\varepsilon)]^N} = 0, \end{aligned} \quad (43b)$$

with velocity  $u_\varepsilon \in J_1^\#(\Omega_\varepsilon)$ , arbitrary  $\varphi \in J_1^\#(\Omega_\varepsilon)$ , scaled concentration  $\rho_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$ , arbitrary  $\psi \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$ , with  $\rho_\varepsilon(x) \beta_\varepsilon^{-1}(x) = \theta_2$  for  $x \in S_2^\varepsilon$ , and  $\rho_\varepsilon(x) \beta_\varepsilon^{-1}(x) = 0$  for  $x \in S_1^\varepsilon$ .

We reformulate this system of equations in abstract form using notations for solution operators of the decoupled auxiliary equations considered above:

$$\rho_\varepsilon \beta_\varepsilon^{-1} = \frac{1}{\lambda} \mathcal{R}_1 (\rho_\varepsilon \beta_\varepsilon^{-1} u_\varepsilon) + \mathcal{R}_2 (\theta_2) \quad (44a)$$

$$u_\varepsilon = \mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1})) + \mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2) \quad (44b)$$

Formally we can write down a non-linear operator equation for  $\rho_\varepsilon$  only:

$$\begin{aligned} \rho_\varepsilon \beta_\varepsilon^{-1} = \frac{1}{\lambda} \mathcal{R}_1 (\rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))]) \\ + \frac{1}{\lambda} \mathcal{R}_1 (\rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)]) + \mathcal{R}_2 (\theta_2) \end{aligned} \quad (45)$$

and want to show that for small  $\varepsilon$  the nonlinear operator

$$\begin{aligned} \mathcal{B}(\rho_\varepsilon \beta_\varepsilon^{-1}) = \frac{1}{\lambda} \mathcal{R}_1 (\rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))]) \\ + \frac{1}{\lambda} \mathcal{R}_1 (\rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)]) \end{aligned} \quad (46)$$

in the right hand side of (45) is a contraction in  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$ .

To reach this goal we estimate first  $\mathcal{R}_1(\rho_\varepsilon \beta_\varepsilon^{-1} u_\varepsilon)$  and  $(\rho_\varepsilon \beta_\varepsilon^{-1}) u_\varepsilon$  that appear in the weak form of the advection-diffusion equation. The Hölder inequality, the

estimate  $\beta_\varepsilon \leq 1$  and the Sobolev imbedding theorems  $W_2^{1/2}(\Omega_\varepsilon) \hookrightarrow L_3(\Omega_\varepsilon)$  and  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon) \hookrightarrow L_6(\Omega_\varepsilon, (\beta_\varepsilon)^3)$  for weighted Sobolev spaces yield

$$\begin{aligned} \|(\rho_\varepsilon \beta_\varepsilon^{-1}) u_\varepsilon\|_{[L_2(\Omega_\varepsilon, \beta_\varepsilon)]^{[N]}} &\leq \|u_\varepsilon\|_{[L_3(\Omega_\varepsilon)]^N} \|(\rho_\varepsilon \beta_\varepsilon^{-1})\|_{L_6(\Omega_\varepsilon, (\beta_\varepsilon)^3)} \\ &\leq C \|u_\varepsilon\|_{[W_2^{1/2}(\Omega_\varepsilon)]^N} \|(\rho_\varepsilon \beta_\varepsilon^{-1})\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)}. \end{aligned} \quad (47)$$

We point out that despite the fact that  $\Omega_\varepsilon$  depends on  $\varepsilon$ , the last estimates are uniform with respect to  $\varepsilon \rightarrow 0$  because the porous structure boundary  $\Gamma_\varepsilon$  is Lipschitz on the periodicity cell  $Y$  and therefore admits a uniformly bounded extension within the same Sobolev class.

The interpolation inequality

$$\|u_\varepsilon\|_{[W_2^{1/2}(\Omega_\varepsilon)]^N} \leq C \|u_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^N}^{1/2} \|u_\varepsilon\|_{[W_2^1(\Omega_\varepsilon)]^N}^{1/2} \quad (48)$$

together with the earlier estimates (41) for the Stokes solution operators implies that

$$\begin{aligned} \|\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))\|_{[W_2^{1/2}(\Omega_\varepsilon)]^N} &\leq \varepsilon^{3/2} \frac{\lambda}{\kappa \mu} C \left( \|[\rho_\varepsilon \beta_\varepsilon^{-1}]\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \right), \\ \|\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)\|_{[W_2^{1/2}(\Omega_\varepsilon)]^N} &\leq \varepsilon^{3/2} \mu^{-1} C \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right). \end{aligned} \quad (49)$$

Using estimates (49) and (47), for the operators  $\mathcal{R}_1((\theta \beta_\varepsilon^{-1}) [\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))])$  and  $\mathcal{R}_1((\rho_\varepsilon \beta_\varepsilon^{-1}) [\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)])$  we obtain the following inequalities :

$$\begin{aligned} &\|\mathcal{R}_1((\theta \beta_\varepsilon^{-1}) [\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))])\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq \\ &C \|(\theta \beta_\varepsilon^{-1}) \mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))\|_{[L_2(\Omega_\varepsilon, \beta_\varepsilon)]^N} \leq \\ &C \|\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))\|_{[W_2^{1/2}(\Omega_\varepsilon)]^N} \|(\theta \beta_\varepsilon^{-1})\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq \\ &C \|\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))\|_{[L_2(\Omega_\varepsilon)]^N}^{1/2} \|\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))\|_{[W_2^1(\Omega_\varepsilon)]^N}^{1/2} \|(\theta \beta_\varepsilon^{-1})\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq \\ &C \frac{\lambda}{\kappa \mu} \left( \varepsilon^2 \|[\rho_\varepsilon \beta_\varepsilon^{-1}]\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \right)^{1/2} \left( \varepsilon \|[\rho_\varepsilon \beta_\varepsilon^{-1}]\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \right)^{1/2} \|(\theta \beta_\varepsilon^{-1})\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} = \\ &\varepsilon^{3/2} \frac{\lambda}{\kappa \mu} C \|[\rho_\varepsilon \beta_\varepsilon^{-1}]\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \|(\theta \beta_\varepsilon^{-1})\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \end{aligned} \quad (50)$$

and similarly

$$\begin{aligned} &\|\mathcal{R}_1((\rho_\varepsilon \beta_\varepsilon^{-1}) [\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)])\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq \\ &C \|\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)\|_{[W_2^{1/2}(\Omega_\varepsilon)]^N} \|(\rho_\varepsilon \beta_\varepsilon^{-1})\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq \\ &\varepsilon^{3/2} \mu^{-1} C \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right) \|(\rho_\varepsilon \beta_\varepsilon^{-1})\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \end{aligned} \quad (51)$$

The last two estimates imply that in any ball of radius  $R_0$  in  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  for a sufficiently small  $\varepsilon$  the operator  $\mathcal{B}(\rho_\varepsilon \beta_\varepsilon^{-1})$  is a contraction. Indeed,

$$\begin{aligned} &\lambda [\mathcal{B}(\rho_\varepsilon \beta_\varepsilon^{-1}) - \mathcal{B}(\theta_\varepsilon \beta_\varepsilon^{-1})] = \\ &\mathcal{R}_1 \{ \rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))] - \theta_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\beta_\varepsilon \nabla (\theta_\varepsilon \beta_\varepsilon^{-1}))] \} + \\ &\mathcal{R}_1 \{ [(\rho_\varepsilon - \theta_\varepsilon)] \beta_\varepsilon^{-1} [\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)] \} = \end{aligned}$$

$$\begin{aligned}
 &\mathcal{R}_1 \{ \rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}))] - \rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\beta_\varepsilon \nabla (\theta_\varepsilon \beta_\varepsilon^{-1}))] \} + \\
 &\quad \mathcal{R}_1 \{ \rho_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\theta_\varepsilon \nabla V_\varepsilon)] - \theta_\varepsilon \beta_\varepsilon^{-1} [\mathcal{S}_1(\theta_\varepsilon \nabla V_\varepsilon)] \} \\
 &\quad + \mathcal{R}_1 \{ [(\rho_\varepsilon - \theta_\varepsilon)] \beta_\varepsilon^{-1} [\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)] \} = \\
 &\quad \mathcal{R}_1 \{ \rho_\varepsilon \beta_\varepsilon^{-1} \mathcal{S}_1((\beta_\varepsilon \nabla ([\rho_\varepsilon - \theta_\varepsilon] \beta_\varepsilon^{-1}))) \} + \\
 &\quad \mathcal{R}_1 \{ [\rho_\varepsilon - \theta_\varepsilon] \beta_\varepsilon^{-1} \mathcal{S}_1(\beta_\varepsilon \nabla (\theta_\varepsilon \beta_\varepsilon^{-1})) \} \\
 &\quad + \mathcal{R}_1 \{ [\rho_\varepsilon - \theta_\varepsilon] \beta_\varepsilon^{-1} [\mathcal{S}_2(\bar{P}_1, \bar{P}_2, \theta_2)] \}
 \end{aligned} \tag{52}$$

and finally

$$\begin{aligned}
 &\lambda \| \mathcal{B}(\rho_\varepsilon \beta_\varepsilon^{-1}) - \mathcal{B}(\theta_\varepsilon \beta_\varepsilon^{-1}) \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq \\
 &\varepsilon^{3/2} \frac{\lambda}{\kappa \mu} C \| [\rho_\varepsilon \beta_\varepsilon^{-1}] \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \| ([\rho_\varepsilon \beta_\varepsilon^{-1} - \theta_\varepsilon \beta_\varepsilon^{-1}]) \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \\
 &+ \varepsilon^{3/2} \frac{\lambda}{\kappa \mu} C \| [[\rho_\varepsilon \beta_\varepsilon^{-1} - \theta_\varepsilon \beta_\varepsilon^{-1}]] \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \| (\theta_\varepsilon \beta_\varepsilon^{-1}) \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \\
 &\varepsilon^{3/2} \frac{C}{\mu} \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right) \| ([\rho_\varepsilon \beta_\varepsilon^{-1} - \theta_\varepsilon \beta_\varepsilon^{-1}]) \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq \\
 &\varepsilon^{3/2} \frac{C}{\mu} \left[ \frac{\lambda}{\kappa} \left( \| [\rho_\varepsilon \beta_\varepsilon^{-1}] \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} + \| (\theta_\varepsilon \beta_\varepsilon^{-1}) \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \right) + \right. \\
 &\quad \left. \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right) \right] \| ([\rho_\varepsilon - \theta_\varepsilon] \beta_\varepsilon^{-1}) \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)}
 \end{aligned} \tag{53}$$

Choosing  $\varepsilon_0$  so that

$$(\varepsilon_0)^{3/2} \frac{C}{\lambda \mu} \left[ \frac{\lambda}{\kappa} 2R_0 + \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right) \right] < 1/2 \tag{54}$$

we conclude that for any  $\varepsilon \leq \varepsilon_0$  the operator  $\mathcal{B}$  is a contraction in the ball of radius  $R_0$  in  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  and maps this ball into itself.

**Theorem 3.1.** *For  $0 < \varepsilon \leq \varepsilon_0$  with  $\varepsilon_0$  satisfying (54), the nonlinear operator equation (45) corresponding to the system of equations (43) has a unique solution  $\rho_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$ . The system of equations (43) has a unique solution  $(\rho_\varepsilon \beta_\varepsilon^{-1}, u_\varepsilon)$  with  $\rho_\varepsilon \beta_\varepsilon^{-1} \in W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  and  $u_\varepsilon \in J_1^\#(\Omega_\varepsilon)$  satisfying the following estimates with constants  $C$  independent of  $\varepsilon$ :*

$$\| \rho_\varepsilon \beta_\varepsilon^{-1} \|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \leq C \theta_2 \tag{55}$$

$$\| u_\varepsilon \|_{[W_2^1(\Omega_\varepsilon)]^N} \leq \frac{\varepsilon}{\mu} C \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right) \tag{56}$$

$$\| u_\varepsilon \|_{[L_2(\Omega_\varepsilon)]^N} \leq \frac{\varepsilon^2}{\mu} C \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right) \tag{57}$$

We notice that  $\bar{P}_1 - \bar{P}_2$  is the pressure drop between inflow and outflow parts of the boundary  $S_1^\varepsilon$  and  $S_2^\varepsilon$ , and the expression  $\frac{\lambda}{\kappa} \theta_2$  is similar to the classical formula (8) for the osmotic pressure in the vicinity of an impermeable membrane.

**4. Homogenization by two-scale convergence for the concentration of solute particles.** This and the next section are devoted to passing to the two-scale limit [3] [26] in the system of equations (43) and obtaining a homogenized limit problem. Uniformity of the obtained estimates lets extend solutions  $u_\varepsilon, (\rho_\varepsilon \beta_\varepsilon^{-1})$  to the whole domain  $\Omega$  in such a way that estimates (55)-(57) hold for the extended

functions with a constant  $C$  that does not depend on  $\varepsilon$ , see [1]. We also extend  $\beta_\varepsilon$  by zero outside  $\Omega_\varepsilon$  for convenience. We keep here the same notations for the extended functions.

We rewrite the weak formulation of our problem (5)-(6) introducing an indicator function  $\mathbf{1}_{\Omega_\varepsilon}$ , test functions  $\psi\beta_\varepsilon^{-1} \in W_2^1(\Omega, \beta_\varepsilon)$  vanishing on  $S_1^\varepsilon \cup S_2^\varepsilon$ , and test functions  $\varphi \in J_1^\#(\Omega_\varepsilon)$ :

$$\int_{\Omega} \nabla(\rho_\varepsilon\beta_\varepsilon^{-1}) \cdot \nabla(\psi\beta_\varepsilon^{-1}) \beta_\varepsilon \mathbf{1}_{\Omega_\varepsilon} dx = \frac{1}{\lambda} \int_{\Omega} (\rho_\varepsilon\beta_\varepsilon^{-1}) u_\varepsilon \cdot \nabla(\psi\beta_\varepsilon^{-1}) \beta_\varepsilon \mathbf{1}_{\Omega_\varepsilon} dx \tag{58}$$

$$\int_{\Omega} \mu \nabla u_\varepsilon \cdot (\mathbf{1}_{\Omega_\varepsilon} \nabla \varphi) dx + \int_{\Omega} \beta_\varepsilon \nabla(\rho_\varepsilon\beta_\varepsilon^{-1}) \cdot (\mathbf{1}_{\Omega_\varepsilon} \varphi) dx - \int_{\Omega} \nabla \Pi_\varepsilon \cdot (\mathbf{1}_{\Omega_\varepsilon} \varphi) dx = 0, \tag{59}$$

Estimate (55) for concentration is formulated in terms of weighted Sobolev space depending on the parameter  $\varepsilon$ . It makes two-scale convergence of functions  $\rho_\varepsilon\beta_\varepsilon^{-1}$  and  $\nabla(\rho_\varepsilon\beta_\varepsilon^{-1})$  in weighted Sobolev spaces  $W_2^1(\mathbb{R}^N, \mathbf{1}_{\Omega_\varepsilon}\beta_\varepsilon dx)$  depending on the parameter  $\varepsilon$  an appropriate tool for deriving a homogenized model. We refer here to several relevant definitions and results from [39].

Let  $\mu$  be a periodic Borel measure normalized on the periodicity cell  $Y$ :  $\mu(Y) = 1$  and  $\mu_\varepsilon$  be the scaled measure defined by

$$\mu_\varepsilon(B) = \varepsilon^N \mu(\varepsilon^{-1}B)$$

for each Borel set  $B$ . The measure  $\mu_\varepsilon$  converges weakly to the Lebesgue measure  $dx$  in the sense that  $\int_{\mathbb{R}^N} \varphi d\mu_\varepsilon \rightarrow \int_{\mathbb{R}^N} \varphi dx$  for any  $\varphi \in C_0(\mathbb{R}^N)$ .

We consider a sequence of measures  $\mu_\varepsilon$  and a sequence of functions  $z_\varepsilon \in L_2(\Omega, d\mu_\varepsilon)$  and test functions  $\Phi(x, y) = \varphi(x)\psi(y)$  with  $\varphi \in C_0^\infty(\Omega)$  and  $\psi \in C_{\text{per}}^\infty(Y)$ , where  $C_{\text{per}}^\infty(Y)$  stands for the space of smooth periodic functions on  $Y$ .

**Definition 4.1.** The sequence  $z_\varepsilon$  such that  $\|z_\varepsilon\|_{L_2(\Omega, d\mu_\varepsilon)} \leq \text{const}$  is weakly two-scale convergent to a periodic in  $y \in Y$  function  $z = z(x, y) \in L_2(\Omega \times Y, dx d\mu) = L_2(\Omega \times Y)$ , or  $z_\varepsilon(x) \xrightarrow{2s} z(x, y)$ , if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \Phi(x, \varepsilon^{-1}x) z_\varepsilon(x) d\mu_\varepsilon = \int_{\Omega} \int_Y \Phi(x, y) z(x, y) dx d\mu$$

for each test function  $\Phi(x, y)$ .

**Proposition 1.** *If the sequence  $z_\varepsilon$  is bounded in  $L_2(\Omega, d\mu_\varepsilon)$ , then there is a subsequence that converges weakly two-scale to some  $z = z(x, y) \in L_2(\Omega \times Y, dx d\mu)$  periodic in  $y \in Y$ .*

**Definition 4.2.** The sequence  $z_\varepsilon$  is strongly two-scale convergent to a periodic in  $y \in Y$  function  $z = z(x, y) \in L_2(\Omega \times Y, dx d\mu) = L_2(\Omega \times Y)$ , or  $z_\varepsilon(x) \xrightarrow{2s} z(x, y)$ , if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_\varepsilon(x) z_\varepsilon(x) d\mu_\varepsilon = \int_{\Omega} \int_Y v(x, y) z(x, y) dx d\mu$$

for any two-scale weakly convergent  $v_\varepsilon(x) \xrightarrow{2s} v(x, y)$ .

Taking  $v_\varepsilon(x) = z_\varepsilon(x)$  gives

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} (z_\varepsilon(x))^2 d\mu_\varepsilon = \int_{\Omega} \int_Y z^2(x, y) dx d\mu. \tag{60}$$

**Proposition 2.** *The following properties of weak two-scale convergence are useful.*

1. *If  $z_\varepsilon(x) \xrightarrow{2s} z(x, y)$  and  $a \in L^\infty(Y, \mu)$  is a periodic function on  $Y$ , then  $a(\varepsilon^{-1}x) z_\varepsilon(x) \xrightarrow{2s} a(y) z(x, y)$*

- 2. If  $z_\varepsilon(x) \xrightarrow{2s} z(x, y)$ , then  $z_\varepsilon(x) \rightharpoonup \int_Y z(x, y) d\mu = \bar{z}(x)$
- 3. Weak two-scale convergence  $z_\varepsilon(x) \xrightarrow{2s} z(x, y)$  together with (60) implies strong two-scale convergence.

**Theorem 4.3.** [39]. Let  $\mu$  be an ergodic measure, and assume that the following conditions hold:

$$z_\varepsilon(x) \xrightarrow{2s} z(x, y)$$

$$\varepsilon \|\nabla z_\varepsilon(x)\|_{[L_2(\Omega, d\mu_\varepsilon)]^N} \rightarrow 0$$

Then the two-scale limit  $z(x, y)$  is independent of  $y$ :  $z(x, y) = z(x)$ .

**Theorem 4.4.** [39]. Let  $\mu$  be an ergodic measure,  $z_\varepsilon \in \overset{o}{W}_2^1(\Omega, \mu_\varepsilon)$ ,  $z_\varepsilon$  and  $\nabla z_\varepsilon$  be bounded in  $[L_2(\Omega, d\mu_\varepsilon)]^N$  and

$$z_\varepsilon(x) \xrightarrow{2s} z(x)$$

$$\nabla z_\varepsilon(x) \xrightarrow{2s} p(x, y)$$

Then  $z(x) \in \overset{o}{W}_2^1(\Omega)$  and

$$\nabla z_\varepsilon(x) \xrightarrow{2s} \nabla z + v(x, y)$$

where  $v \in L^2(\Omega, V_{\text{pot}})$ , and  $V_{\text{pot}}$  is the closure of gradients of smooth periodic functions on  $Y$  in norm  $L_2(Y, \beta dy)$ . Poincare inequality implies that in our case any such function is a gradient of a periodic function from  $W_2^1(Y, \beta dy)$ .

Turning to our problem notice that the measure  $d\mu_\varepsilon = \beta_\varepsilon \mathbf{1}_{\Omega_\varepsilon} dx$  converges weakly to the measure  $\mathbf{1}_\Omega \bar{\beta} dx$  with

$$\bar{\beta} = \int_{Y_F} \beta(y) dy$$

in the sense that  $\int_{\mathbb{R}^N} \beta_\varepsilon \mathbf{1}_{\Omega_\varepsilon} \varphi dx \rightarrow \int_{\mathbb{R}^N} \mathbf{1}_\Omega \bar{\beta} \varphi dx$  for any  $\varphi \in C_0(\mathbb{R}^N)$ .

**Theorem 4.5.** The diffusion component  $\rho_\varepsilon \beta_\varepsilon^{-1}$  of the solution  $(u_\varepsilon, \rho_\varepsilon \beta_\varepsilon^{-1})$  to system (43) converges strongly in  $L_2(\Omega, d\mu_\varepsilon)$  to a solution  $\Theta^0$  of the boundary value problem:

$$\text{div}(A_{\text{eff}} \nabla \Theta^0) = 0 \tag{61}$$

with boundary conditions for  $\Theta^0(x)$  the same as in the original problem:

$$\begin{aligned} \Theta^0|_{S_1} &= 0, & \Theta^0|_{S_2} &= \theta_2 \\ A_{\text{eff}} \nabla \Theta^0 \cdot n|_{\Gamma_0} &= 0. \end{aligned} \tag{62}$$

with a positive definite matrix  $A_{\text{eff}}$  defined by

$$A_{\text{eff}} = \int_Y (I + [\nabla_y \chi(y)]) \beta(y) \mathbf{1}_{Y_F}(y) dy \tag{63}$$

Here  $\chi(y)$  is the periodic solution to the cell problem

$$\begin{aligned} \text{div}(\beta(y) (\nabla_y \chi + I)) &= 0 \\ \frac{\partial}{\partial y_n}(\chi) &= -n(y), \quad y \in \partial Y_S; \end{aligned} \tag{64}$$

*Proof.* The  $L_2(\Omega, d\mu_\varepsilon)$  uniform estimates for  $\rho_\varepsilon\beta_\varepsilon^{-1}$  and  $\nabla(\rho_\varepsilon\beta_\varepsilon^{-1})$  and the ergodicity of the measure  $\mathbf{1}_{\Omega_\varepsilon}\beta_\varepsilon dx$  imply according to the properties of the two-scale convergence above that for a subsequence  $\varepsilon \rightarrow 0$  it holds

$$\rho_\varepsilon\beta_\varepsilon^{-1} \rightharpoonup \Theta^0(x), \tag{65}$$

$$\nabla(\rho_\varepsilon\beta_\varepsilon^{-1}) \xrightarrow{2s} \nabla_x\Theta^0(x) + \nabla_y\Theta^1(x, y), \tag{66}$$

where  $\Theta^0 \in W_2^1(\Omega, \bar{\beta}dx)$ ,  $\Theta^1(x, y) \in L_2(\Omega, W_2^1(Y, \beta))$ .

Choosing in (58) a test function  $\psi\beta_\varepsilon^{-1} = \varepsilon \varphi_1(\frac{x}{\varepsilon})\varphi_2(x)$  with smooth  $\varphi_1(y)$  periodic in  $y \in Y$ , and  $\varphi_2(x) \in C_0^\infty(\Omega)$ , and passing to the two-scale limit we obtain the following equation:

$$\int_Y \int_\Omega (\nabla_x\Theta^0(x) + \nabla_y\Theta^1(x, y)) \cdot \nabla_y\varphi_1(y)\varphi_2(x)\beta(y)\mathbf{1}_{Y_F}(y)dx dy = 0, \tag{67}$$

where  $Y_F$  is the fluid part of the periodic cell  $Y$  and  $\mathbf{1}_{Y_F}(y)$  is its characteristic function. Zero limit for the right hand side is an immediate consequence of the estimates (55)-(57) for solutions. This yields that

$$\int_Y (\nabla_x\Theta^0(x) + \nabla_y\Theta^1(x, y)) \cdot \nabla_y\varphi_1(y)\beta(y)\mathbf{1}_{Y_F}(y)dy = 0$$

for almost all  $x \in \Omega$ . Therefore

$$\Theta^1(x, y) = \chi(y) \cdot \nabla_x\Theta^0(x) \tag{68}$$

with  $\chi(y)$  being a periodic solution to the cell problem (64). The cell problem is well posed since apriori estimates on  $W_2^1(Y, d\mu)$  are fulfilled.

Choosing now an arbitrary test function  $\varphi \in C^\infty(\Omega)$ ,  $\varphi = 0$  in the vicinity of  $S_1^\varepsilon \cup S_2^\varepsilon$ , in the weak form of the advection-diffusion equation

$$\int_\Omega \nabla(\rho_\varepsilon\beta_\varepsilon^{-1}) \cdot \nabla(\varphi)\beta_\varepsilon\mathbf{1}_{\Omega_\varepsilon}dx = \int_\Omega (\rho_\varepsilon\beta_\varepsilon^{-1})u_\varepsilon \cdot \nabla(\varphi)\beta_\varepsilon\mathbf{1}_{\Omega_\varepsilon}dx$$

we in a similar way get

$$\int_\Omega \int_Y (\nabla_x\Theta^0(x) + [\nabla_y\chi(y)]\nabla_x\Theta^0(x))\beta(y)\mathbf{1}_{Y_F}(y)dy \cdot \nabla\varphi(x)dx = 0$$

and

$$\int_\Omega \left\{ \int_Y (I + [\nabla_y\chi(y)])\beta(y)\mathbf{1}_{Y_F}(y)dy \right\} \nabla_x\Theta^0(x) \cdot \nabla\varphi(x)dx = 0$$

Integration with respect to  $y$  yields

$$\int_\Omega A_{\text{eff}}\nabla_x\Theta^0(x) \cdot \nabla\varphi(x)dx = 0 \tag{69}$$

which is the weak formulation of (61). The boundary conditions (62) are evidently inherited from the original system. Strong convergence of  $\rho_\varepsilon\beta_\varepsilon^{-1}$  to  $\Theta^0$  follows from the apriori estimates (55) and the compactness of the embedding from  $W_2^1(\Omega_\varepsilon, \beta_\varepsilon)$  to  $L_2(\Omega_\varepsilon, \beta_\varepsilon)$ .  $\square$

We notice that the limit equation for the scaled concentration  $\rho_\varepsilon\beta_\varepsilon^{-1}$  is decoupled from the flow equation. But  $\rho_\varepsilon\beta_\varepsilon^{-1}$  plays a role in the Stokes part of the system and its limit  $\Theta^0$  enters a homogenized Darcy type equation for flow.



**5. Homogenization by two-scale convergence for the velocity and pressure of the solvent.** Now we consider the two-scale limit for the Stokes equations (33). We need to extend the velocity field and pressure to the whole domain  $\Omega$  to consider two-scale limits of solutions in a fixed domain. Velocity  $u_\varepsilon$  is extended in a trivial way by zero with apriori estimates preserved for the extended function:

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}$$

The extension of the pressure  $p_\varepsilon$  is more tricky and needs sophisticated estimates uniform with respect to  $\varepsilon$  to carry out a limit when  $\varepsilon \rightarrow 0$ .

**5.1. Extension of pressure.** The homogenization of the Stokes equations relies on an extension of pressure and on uniform with respect to  $\varepsilon$  estimates for pressure: [32], [2]. The following technical lemma from [11] is used here in the construction.

**Lemma 5.1.** *Let  $g \in L_2(\Omega_\varepsilon)$  and  $\int_{\Omega_\varepsilon} g dx = 0$ , and assume that the cell domain  $Y_F$  can be represented as a finite union of domains with Lipschitz boundaries. Then there is a vector valued function  $w_\varepsilon \in \left[ \overset{o}{W}_2^1(\Omega_\varepsilon) \right]^N$ , such that  $\operatorname{div}(w_\varepsilon)(x) = g(x)$  and the following estimates are satisfied:*

$$\|w_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^N} \leq C \|g\|_{L_2(\Omega_\varepsilon)}; \quad \|\nabla w_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^{N^2}} \leq \frac{1}{\varepsilon} C \|g\|_{L_2(\Omega_\varepsilon)} \quad (70)$$

with  $C > 0$  independent of  $\varepsilon$  and  $g$ .

**Lemma 5.2.** *For the pressure  $p_\varepsilon$  normalized by  $\int_{\Omega_\varepsilon} p_\varepsilon dx = 0$  and satisfying the equations (5) the following estimate holds:*

$$\|p_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leq C \left[ (|\bar{P}_1 - \bar{P}_2|) + \frac{\lambda}{\kappa} \theta_2 \right] \quad (71)$$

with  $C > 0$  independent of  $\varepsilon$ .

*Proof.* Using lemma 5.1 and recalling our normalization for pressure we construct a function  $w_\varepsilon \in \left[ \overset{o}{W}_2^1(\Omega_\varepsilon) \right]^N$  such that  $\operatorname{div}(w_\varepsilon) = p_\varepsilon$  in  $\Omega_\varepsilon$ , and

$$\begin{aligned} \|\nabla w_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^N} &\leq \frac{1}{\varepsilon} C \|p_\varepsilon\|_{L_2(\Omega_\varepsilon)} \\ \|w_\varepsilon\|_{[L_2(\Omega_\varepsilon)]^N} &\leq C \|p_\varepsilon\|_{L_2(\Omega_\varepsilon)} \end{aligned} \quad (72)$$

Multiplying the Stokes equation repeated here from (10)

$$\mu \Delta u_\varepsilon - \nabla p_\varepsilon + \left( \frac{\lambda}{\kappa} \right) \nabla \rho_\varepsilon - \left( \frac{\lambda}{\kappa} \right) \beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}) = 0, \quad x \in \Omega_\varepsilon$$

by  $w_\varepsilon$ , integrating the resulting relation by parts over  $\Omega_\varepsilon$  we obtain

$$\begin{aligned} (p_\varepsilon, \operatorname{div}(w_\varepsilon))_{L_2(\Omega_\varepsilon)} &= \mu (\nabla u_\varepsilon, \nabla w_\varepsilon)_{[L_2(\Omega_\varepsilon)]^{N^2}} + \left( \left( \frac{\lambda}{\kappa} \right) \rho_\varepsilon, \operatorname{div}(w_\varepsilon) \right)_{L_2(\Omega_\varepsilon)} \\ &\quad + \left( \left( \frac{\lambda}{\kappa} \right) \beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}), w_\varepsilon \right)_{[L_2(\Omega_\varepsilon)]^N} \end{aligned}$$

and pointing out that  $\int_{\Omega_\varepsilon} \operatorname{div}(w_\varepsilon) dx = 0$ , we get

$$\begin{aligned} \|p_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2 &= \mu (\nabla u_\varepsilon, \nabla w_\varepsilon)_{[L_2(\Omega_\varepsilon)]^{N^2}} + \left( \left( \frac{\lambda}{\kappa} \right) \rho_\varepsilon, p_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \\ &\quad + \left( \left( \frac{\lambda}{\kappa} \right) \beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}), w_\varepsilon \right)_{[L_2(\Omega_\varepsilon)]^N} \end{aligned} \tag{73}$$

Combining estimates

$$\begin{aligned} \left| \left( \frac{\lambda}{\kappa} \beta_\varepsilon \nabla (\rho_\varepsilon \beta_\varepsilon^{-1}), w_\varepsilon \right)_{[L_2(\Omega_\varepsilon)]^N} \right| &\leq \frac{\lambda}{\kappa} \|\rho_\varepsilon \beta_\varepsilon^{-1}\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \|p_\varepsilon\|_{L_2(\Omega_\varepsilon)} \\ \left| \left( \left( \frac{\lambda}{\kappa} \right) \rho_\varepsilon, p_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \right| &\leq \frac{\lambda}{\kappa} \|\rho_\varepsilon \beta_\varepsilon^{-1}\|_{L_2(\Omega_\varepsilon, \beta_\varepsilon)} \|p_\varepsilon\|_{L_2(\Omega_\varepsilon)} \end{aligned}$$

with estimates (55), (56), (72) for  $\rho_\varepsilon, u_\varepsilon, \nabla w_\varepsilon, w_\varepsilon$  we receive the following estimate for pressure  $p_\varepsilon$ :

$$\begin{aligned} \|p_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2 &\leq C \left( |\bar{P}_1 - \bar{P}_2| + \frac{\lambda}{\kappa} \theta_2 \right) \|p_\varepsilon\|_{L_2(\Omega_\varepsilon)} \\ &\quad + \frac{\lambda}{\kappa} \|\rho_\varepsilon \beta_\varepsilon^{-1}\|_{W_2^1(\Omega_\varepsilon, \beta_\varepsilon)} \|p_\varepsilon\|_{L_2(\Omega_\varepsilon, \beta_\varepsilon)}, \end{aligned}$$

which yields (71).

Denoting  $Y_{i,F}^\varepsilon = \varepsilon(Y_F + i), Y_{i,S}^\varepsilon = \varepsilon(Y_S + i), i \in \mathbb{Z}^N$ , and using the estimate (71) we can extend pressure from  $\Omega_\varepsilon$  to  $\Omega$  by

$$P^\varepsilon = \begin{cases} \frac{1}{|Y_{i,F}^\varepsilon|} \int_{Y_{i,F}^\varepsilon} p_\varepsilon dx, & \text{in } Y_{i,S}^\varepsilon \\ p_\varepsilon, & \text{in } Y_{i,F}^\varepsilon \end{cases}. \tag{74}$$

as in [2] for  $Y_{i,S}^\varepsilon \subset C_\varepsilon$ . If the porous structure crosses the lateral boundary  $\Gamma_0$  of  $\Omega$  one can [2] complete this definition by extending  $p_\varepsilon$  by zero on  $\Omega \setminus C_\varepsilon$ :

$$P^\varepsilon = p_\varepsilon \text{ in } (\Omega \setminus C_\varepsilon) \cap \Omega_\varepsilon, \quad P^\varepsilon = 0 \text{ in } (\Omega \setminus C_\varepsilon) \setminus \Omega_\varepsilon$$

□

**5.2. Homogenization for velocity and pressure in the Stokes equations with osmotic forces.** In this section we deal with the Stokes part of the system (58)-(59) and consider properties of the two-scale limits of the extended velocity  $\tilde{u}_\varepsilon$  and pressure  $P^\varepsilon$ . The estimates (56), (57), (71) for velocity and pressure imply that there are functions  $u_0(x, y) \in L_2(\Omega; [W_2^1(Y)]^N), \xi_0(x, y) \in L_2(\Omega; [L_2(Y)]^{N^2}),$  and  $p_0(x, y) \in L_2(\Omega \times Y)$  periodic with respect to  $y \in Y$ , such that extensions  $\mu\varepsilon^{-2}\tilde{u}_\varepsilon, \mu\varepsilon^{-1}\nabla\tilde{u}_\varepsilon, P^\varepsilon$  converge two-scale to these functions:

$$\begin{aligned} \mu\varepsilon^{-2}\tilde{u}_\varepsilon &\xrightarrow{2s} u_0(x, y) \\ \mu\varepsilon^{-1}\nabla\tilde{u}_\varepsilon &\xrightarrow{2s} \xi_0(x, y) \\ P^\varepsilon &\xrightarrow{2s} p_0(x, y) \end{aligned} \tag{75}$$

It means that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mu\varepsilon^{-2}\tilde{u}_\varepsilon \Psi \left( x, \frac{x}{\varepsilon} \right) dx = \int_{\Omega} \int_Y u_0(x, y) \Psi(x, y) dy dx \tag{76}$$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mu \varepsilon^{-1} \nabla \tilde{u}_{\varepsilon} \Xi \left( x, \frac{x}{\varepsilon} \right) dx &= \int_{\Omega} \int_Y \xi_0(x, y) \Xi(x, y) dy dx \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} P^{\varepsilon} \Phi \left( x, \frac{x}{\varepsilon} \right) dx &= \int_{\Omega} \int_Y p_0(x, y) \Phi(x, y) dy dx \end{aligned}$$

for any  $\Psi \in C_0^{\infty} \left[ \Omega; [C_{\text{per}}^{\infty}(Y)]^N \right]$ , any  $\Xi \in C_0^{\infty} \left[ \Omega; [C_{\text{per}}^{\infty}(Y)]^{N^2} \right]$  and any  $\Phi \in C_0^{\infty} \left[ \Omega; C_{\text{per}}^{\infty}(Y) \right]$ .

Integrating by parts in the second of equations (76) and passing to the two-scale limit leads in a standard way to

$$\begin{aligned} \int_{\Omega} \int_Y \xi_0(x, y) \Xi(x, y) dy dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mu \varepsilon^{-1} \nabla \tilde{u}_{\varepsilon} \Xi \left( x, \frac{x}{\varepsilon} \right) dx = \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mu \varepsilon^{-2} \tilde{u}_{\varepsilon} \left( \varepsilon \operatorname{div}_x \left( \Xi \left( x, \frac{x}{\varepsilon} \right) \right) + \operatorname{div}_y \left( \Xi \left( x, \frac{x}{\varepsilon} \right) \right) \right) dx &= \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mu \varepsilon^{-2} \tilde{u}_{\varepsilon} \operatorname{div}_y \left( \Xi \left( x, \frac{x}{\varepsilon} \right) \right) dx &= \int_{\Omega} \int_Y u_0 \operatorname{div}_y \left( \Xi(x, y) \right) dx dy \end{aligned}$$

and after integration by parts with respect to  $y$  over the periodicity cell  $Y$  to the relation

$$\xi_0(x, y) = \nabla_y u_0. \tag{77}$$

The two-scale limit  $p_0(x, y)$  has a specific structure that is one of the main results of the present paper. We express it in the following lemma.

**Lemma 5.3.** *The two scale limit  $p_0(x, y)$  of  $P^{\varepsilon}$  is the sum of a function  $p(x)$  that can be interpreted as hydrodynamic pressure, and a term expressing local osmotic pressure:*

$$p_0(x, y) = p(x) - \frac{\lambda}{\kappa} \Theta^0(x) \beta(y) \tag{78}$$

*Proof.* Multiplying the Stokes equation (10) by a test function  $\varepsilon \psi(x, \frac{x}{\varepsilon})$  where  $\psi(x, y) \in C^{\infty} \left[ \Omega; [C^{\infty}(Y)]^N \right]$  and has finite support in  $\Omega \times Y_f$ , and integrating the resulting relation by parts we get

$$\begin{aligned} \varepsilon \mu \left( \nabla u_{\varepsilon}, \nabla_x \psi + \frac{1}{\varepsilon} \nabla_y \psi \right)_{[L_2(\Omega_{\varepsilon})]^N} + \varepsilon \left( \beta_{\varepsilon} \nabla \left( \rho_{\varepsilon} \beta_{\varepsilon}^{-1} \right), \psi \right)_{[L_2(\Omega_{\varepsilon})]^N} + \\ \varepsilon \left( P_{\varepsilon} - \beta_{\varepsilon} \frac{\lambda}{\kappa} \left( \rho_{\varepsilon} \beta_{\varepsilon}^{-1} \right), \operatorname{div}_x \psi + \frac{1}{\varepsilon} \operatorname{div}_y \psi \right)_{L_2(\Omega_{\varepsilon})} = 0 \end{aligned}$$

Passing to the two-scale limit in

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \left[ \mu \left( \nabla u_{\varepsilon}, \nabla_x \psi \right)_{[L_2(\Omega_{\varepsilon})]^{N^2}} + \varepsilon^{-1} \mu \left( \nabla u_{\varepsilon}, \nabla_y \psi \right)_{[L_2(\Omega_{\varepsilon})]^{N^2}} \right] - \tag{79}$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \left[ \left( \frac{\lambda}{\kappa} \beta_{\varepsilon} \left( \rho_{\varepsilon} \beta_{\varepsilon}^{-1} \right), \operatorname{div}_x \psi \right)_{[L_2(\Omega_{\varepsilon})]^N} + \varepsilon^{-1} \left( \frac{\lambda}{\kappa} \beta_{\varepsilon} \left( \rho_{\varepsilon} \beta_{\varepsilon}^{-1} \right), \operatorname{div}_y \psi \right)_{L_2(\Omega_{\varepsilon})} \right] \tag{80}$$

$$\begin{aligned} + \lim_{\varepsilon \rightarrow 0} \varepsilon \left[ \left( P_{\varepsilon}, \operatorname{div}_x \psi \right)_{L_2(\Omega_{\varepsilon})} + \varepsilon^{-1} \left( P_{\varepsilon}, \operatorname{div}_y \psi \right)_{L_2(\Omega_{\varepsilon})} \right] \\ + \lim_{\varepsilon \rightarrow 0} \varepsilon \left[ \left( \beta_{\varepsilon} \nabla \left( \rho_{\varepsilon} \beta_{\varepsilon}^{-1} \right), \psi \right)_{[L_2(\Omega_{\varepsilon})]^N} \right] = 0 \end{aligned} \tag{81}$$

and using (66) and (68) implies a relation between the two scale limit  $p_0(x, y)$  of pressure  $P^\varepsilon$  and the two scale limit  $\Theta^0$  of the scaled concentration  $\rho_\varepsilon \beta_\varepsilon^{-1}$ :

$$\int_{\Omega} \int_Y \left[ p_0(x, y) + \frac{\lambda}{\kappa} \Theta^0(x) \beta(y) \right] \operatorname{div}_y \psi(x, y) dx dy = 0$$

Taking into account that  $\psi(x, y)$  is arbitrary we yield the desired formula (78).  $\square$

**Remark 1.** We point out that a similar formula was derived in [8] in one dimensional case for an infinitely long cylindric channel.

The same argument in situation without osmotic forces leads to the conclusion that the two scale limit  $p_0(x, y) = p(x)$  is independent of  $y$ .

We proceed with clarifying properties of the two-scale limit  $u_0(x, y)$  of velocity. The incompressibility conditions for  $u_0(x, y)$  and  $u(x) = \int_Y u_0(x, y) dy$  and boundary conditions for  $u_0(x, y)$  and  $u(x)$  are formulated in the following lemma.

**Lemma 5.4.**

$$\begin{aligned} \operatorname{div}_y u_0(x, y) &= 0 \text{ in } \Omega \times Y & (82) \\ \operatorname{div}_x \left[ \int_Y u_0(x, y) dy \right] &= 0 \text{ in } \Omega \\ u_0(x, y) &= 0 \text{ in } \Omega \times Y_S \\ \left[ \int_Y u_0(x, y) dy \right] \cdot n &= 0 \text{ on } \Gamma_0 \end{aligned}$$

*Proof.* Integrating by parts the equation  $\operatorname{div}(\tilde{u}_\varepsilon) = 0$  with a test function  $\lambda(x)$  that is zero on  $\partial\Omega$ , passing to the two-scale limit and integrating by parts again we obtain

$$0 = \int_{\Omega} \operatorname{div}(\tilde{u}_\varepsilon) \lambda(x) dx = - \int_{\Omega} \tilde{u}_\varepsilon \cdot \nabla \lambda(x) dx \rightarrow - \int_{\Omega} u \cdot \nabla \lambda(x) dx = \int_{\Omega} \operatorname{div}(u) \lambda(x) dx$$

and conclude that  $\operatorname{div}_x(u) = \operatorname{div}_x \left( \int_Y u_0(x, y) dy \right) = 0$ .

Integrating by parts the equation  $\operatorname{div}(\tilde{u}_\varepsilon) = 0$  with a test function  $\lambda(x)$  that is zero only on the inflow and outflow part  $S_1 \cup S_2$  of the boundary  $\partial\Omega$ , taking into account the boundary condition  $\tilde{u}_\varepsilon = 0$  on  $\Gamma_0$ , passing to the two-scale limit, and integrating by parts again

$$\begin{aligned} 0 &= \int_{\Omega} \operatorname{div}(\tilde{u}_\varepsilon) \lambda(x) dx = \int_{S_1 \cup S_2} \tilde{u}_\varepsilon \cdot n \lambda(x) d\sigma + \int_{\Gamma_0} \tilde{u}_\varepsilon \cdot n \lambda(x) d\sigma - \int_{\Omega} \tilde{u}_\varepsilon \cdot \nabla \lambda(x) dx \\ &\xrightarrow{\varepsilon \rightarrow 0} - \int_{\Omega} u \cdot \nabla \lambda(x) dx = \int_{\Gamma_0} u \cdot n \lambda(x) d\sigma + \int_{\Omega} \operatorname{div}(u) \lambda(x) dx = \int_{\Gamma_0} u \cdot n \lambda(x) d\sigma \end{aligned}$$

we conclude that  $u(x) \cdot n = \left( \int_Y u_0(x, y) dy \right) \cdot n = 0$  for  $x \in \Gamma_0$ .

Integrating by parts the equation  $\operatorname{div}(\tilde{u}_\varepsilon) = 0$  with the test function  $\varepsilon \lambda(x, x/\varepsilon)$  that is zero on  $\partial\Omega$ , passing to the two-scale limit and integrating by parts again

$$\begin{aligned} 0 &= \varepsilon \int_{\Omega} \operatorname{div}(\tilde{u}_\varepsilon) \lambda(x, x/\varepsilon) dx = -\varepsilon \int_{\Omega} \tilde{u}_\varepsilon \cdot \nabla_x \lambda(x, x/\varepsilon) dx - \int_{\Omega} \tilde{u}_\varepsilon \cdot \nabla_y \lambda(x, x/\varepsilon) dx \\ &\xrightarrow{\varepsilon \rightarrow 0} - \int_Y \int_{\Omega} u_0(x, y) \cdot \nabla_y \lambda(x, y) dx dy = \int_Y \int_{\Omega} \operatorname{div}_y (u_0(x, y)) \lambda(x, y) dx dy \end{aligned}$$

we conclude that  $\operatorname{div}_y(u_0(x, y)) = 0$ . □

**6. Darcy’s law with distributed osmotic forces.**

**Theorem 6.1.** *The extension  $((\mu\varepsilon^{-2})\tilde{u}_\varepsilon, P_\varepsilon)$  of the weak solution (35) to the Stokes equations (5) with osmotic forces defined by the solution  $\rho_\varepsilon$  to the advection-diffusion equation (6) and with fixed pressure drop  $\delta\bar{P}$  between  $S_1^\varepsilon$  and  $S_2^\varepsilon$  two-scale converges to  $(u_0(x, y), p_0(x, y))$  with  $p_0(x, y) = p(x) - \frac{\lambda}{\kappa}\Theta^0(x)\beta(y)$ , where  $(u_0(x, y), p(x))$  is the unique solution of the two-scale homogenized problem*

$$\begin{aligned} -\Delta_{yy}u_0(x, y) &= -\nabla_y p_1(x, y) - \nabla_x p(x) \\ &\quad + \frac{\lambda}{\kappa} [I - \nabla_y \chi(y)] \beta(y) \nabla_x \Theta^0(x) \quad \text{in } \Omega \times Y_S \\ \operatorname{div}_y u_0(x, y) &= 0 \quad \text{in } \Omega \times Y; \quad \operatorname{div}_x \left[ \int_Y u_0(x, y) dy \right] = 0 \quad \text{in } \Omega \quad (83) \\ u_0(x, y) &= 0 \quad \text{in } \Omega \times Y_S \\ \left[ \int_Y u_0(x, y) dy \right] \cdot n &= 0 \quad \text{in } \Gamma_0, \quad \int_\Omega p dx = 0 \\ \bar{P}_1 - \bar{P}_2 &= \delta\bar{P}; \quad p(x) = \bar{P}_i \quad \text{in } S_i; \quad u_0(x, \cdot) \text{ is periodic in } Y, \end{aligned}$$

$\chi(y)$  is the solution to the cell diffusion problem (64) and  $\Theta^0$  is the solution of the homogenized problem (61), (62).

*Proof.* We follow the way of reasoning from [4]. Choose a test function  $\psi(x, y) \in C_0^\infty(\Omega; [C^\infty(Y)]^N)$  with  $\psi(x, y) \equiv 0$  in  $\Omega \times Y_S$ , so that  $\psi(x, x/\varepsilon) \in \left[ \overset{\circ}{W}_2^1(\Omega_\varepsilon) \right]^N$ . We suppose also that  $\psi(x, y)$  satisfies incompressibility conditions  $\operatorname{div}_y \psi(x, y) = 0$ ,  $\operatorname{div}_x \left[ \int_Y \psi(x, y) dy \right] = 0$ . Multiplication of the Stokes equation in form (10) by the test function  $\psi(x, \frac{x}{\varepsilon})$ , taking into account the incompressibility condition for  $\psi$  in  $y$ , and integration by parts yields

$$\begin{aligned} &\int_{\Omega_\varepsilon} p_\varepsilon(x) \operatorname{div}_x \left( \psi(x, \frac{x}{\varepsilon}) \right) dx - \int_{\Omega_\varepsilon} \left[ \frac{\lambda}{\kappa} \rho_\varepsilon(x) \right] \operatorname{div}_x \left( \psi(x, \frac{x}{\varepsilon}) \right) dx \\ &\quad - \int_{\Omega_\varepsilon} \left[ \frac{\lambda}{\kappa} \beta_\varepsilon \nabla(\rho_\varepsilon \beta_\varepsilon^{-1}) \right] \psi(x, \frac{x}{\varepsilon}) dx \quad (84) \\ &= \int_{\Omega_\varepsilon} \mu \varepsilon^{-1} \nabla u_\varepsilon(x) \cdot \nabla_y \psi(x, \frac{x}{\varepsilon}) dx + \int_{\Omega_\varepsilon} \mu \nabla u_\varepsilon(x) \cdot \nabla_x \psi(x, \frac{x}{\varepsilon}) dx. \end{aligned}$$

We can replace the integration domain in the last equation with  $\Omega$  and  $p_\varepsilon$  with  $P_\varepsilon$  since the test function  $\psi(x, \frac{x}{\varepsilon})$  is zero outside  $\Omega_\varepsilon$ .

Passing to the two-scale limit in the first term in (84) gives the expression

$$- \int_{\Omega \times Y} \frac{\lambda}{\kappa} \Theta^0(x) \beta(y) \operatorname{div}_x (\psi(x, y)) dx dy$$

because the first term in the two-scale limit  $p_0(x, y) = p(x) - \frac{\lambda}{\kappa} \Theta^0(x) \beta(y)$  of  $P_\varepsilon$  does not depend on  $y$  and  $\psi$  satisfies  $\operatorname{div}_x \left[ \int_Y \psi(x, y) dy \right] = 0$ . Passing to the two-scale limit in other terms in (84) and using (66) and (68) gives

$$\int_{\Omega \times Y} \left[ -\frac{\lambda}{\kappa} \Theta^0(x) \beta(y) \right] \operatorname{div}_x \psi(x, y) dx dy$$

$$\begin{aligned}
& - \int_{\Omega \times Y} \frac{\lambda}{\kappa} \Theta^0(x) \beta(y) \operatorname{div}_x \psi(x, y) dx dy \\
& - \int_{\Omega \times Y} \left[ \frac{\lambda}{\kappa} [I + \nabla_y \chi(y)] \nabla_x \Theta^0(x) \beta(y) \right] \cdot \psi(x, y) dx dy \\
& = \int_{\Omega \times Y} \nabla_y u_0(x, y) \cdot \nabla_y \psi(x, y) dx dy
\end{aligned} \tag{85}$$

The boundary term disappears because of the boundedness of  $\psi(x, \frac{x}{\varepsilon})$  and its support. The last term in the right hand side of (84) disappears by the estimates for  $\nabla u_\varepsilon(x)$ . Finally after integration by parts, taking into account boundary conditions for  $p(x)$  and  $\Theta^0(x)$  and cancelling two integrals with  $\frac{\lambda}{\kappa} \Theta^0(x) \beta(y)$ , the variational form of the homogenized equation with osmotic forces reads

$$\begin{aligned}
& \int_{\Omega \times Y} \frac{\lambda}{\kappa} [I - \nabla_y \chi(y)] \beta(y) \nabla_x \Theta^0(x) \psi(x, y) dx dy \\
& = \int_{\Omega \times Y} \nabla_y u_0(x, y) \cdot \nabla_y \psi(x, y) dx dy
\end{aligned} \tag{86}$$

By density the last equation holds for  $\psi(x, y)$  in the Hilbert space  $\mathcal{V}$  of functions periodic in  $y \in Y$ , defined by

$$\begin{aligned}
\psi(x, y) & \in L^2 \left[ \Omega; W_2^1(Y)^N \right] \\
\operatorname{div}_y \psi(x, y) & = 0 \text{ in } \Omega \times Y, \quad \operatorname{div}_x \left[ \int_Y \psi(x, y) dy \right] = 0 \text{ in } \Omega \\
\psi(x, y) & = 0 \text{ in } \Omega \times Y_S \\
\left[ \int_Y \psi(x, y) dy \right] \cdot n & = 0 \text{ in } \Gamma_0
\end{aligned} \tag{87}$$

One can check that the Lax-Milgram lemma holds for the problem (86) and that it has therefore a unique solution  $u_0(x, y) \in \mathcal{V}$ . Let  $L_{2, \text{per}}(Y)$  be the space of periodic on  $Y$ , square integrable functions with standard scalar product.

By a variant of the Weyl decomposition, see [4] we conclude that the orthogonal complement  $\mathcal{V}^\perp$  of  $\mathcal{V}$  with respect to the scalar product in  $L^2 \left[ \Omega; [L_{2, \text{per}}(Y)]^N \right]$  coincides with vector fields of the form  $\nabla_x q(x) + \nabla_y q_1(x, y)$  with  $q(x) \in W_2^1(\Omega)$  and  $q_1(x, y) \in L^2 \left[ \Omega; L_{2, \text{per}}(Y_F) \right]$  having zero mean values over  $\Omega$  and  $Y_F$  correspondingly. Using this statement and integrating by parts in (86) we get the strong form (83) of the two-scale homogenized limit for our problem. We must show that the pressure like expression  $p(x) - \frac{\lambda}{\kappa} \Theta^0(x) \beta(y)$  arising from the incompressibility constraint  $\operatorname{div}_x \left[ \int_Y u_0(x, y) dy \right] = 0$  is the same as the two-scale limit  $p_0(x, y)$  of the pressure  $P^\varepsilon$ . We multiply the Stokes equation (10) by a test function  $\psi(x, y)$  that is divergence free only in  $y$ :  $\operatorname{div}_y \psi(x, y) = 0$ , integrate the resulting expression by parts as in (85) and identify two-scale limits:

$$\begin{aligned}
& \int_{\Omega \times Y} p(x) \operatorname{div}_x \psi(x, y) dx dy \\
& + \int_{\Omega \times Y} \left[ \frac{\lambda}{\kappa} [I - \nabla_y \chi(y)] \nabla_x \Theta^0(x) \beta(y) \right] \cdot \psi(x, y) dx dy \\
& = \int_{\Omega \times Y} \nabla_y u_0(x, y) \cdot \nabla_y \psi(x, y) dx dy
\end{aligned} \tag{88}$$

Since the system (83) has a unique solution  $(u_0(x, y), p(x))$ , the entire sequence  $(\tilde{u}_\varepsilon, P^\varepsilon)$  converges to  $(u_0(x, y), p(x) - \frac{\lambda}{\kappa}\Theta^0(x)\beta_\varepsilon(y))$ .  $\square$

We are in the position to separate variables in the two-scale homogenized system (83) and reduce it to a periodic cell problem of  $y$  variable on  $Y$  and a homogenized problem of  $x$  variable only in the domain  $\Omega$ .

**Theorem 6.2.** *The extension  $(\tilde{u}_\varepsilon, P^\varepsilon)$  of the velocity and pressure  $(u_\varepsilon, p_\varepsilon)$  satisfying the system (5)-(6) converges weakly in  $[L_2(\Omega)]^N \times [L^2(\Omega)]$  to the unique solution  $(u, p)$  of the homogenized problem*

$$\begin{cases} u(x) = B_D(-\nabla p) + B_{\text{osm}}(\nabla\Theta^0) & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{in } \Gamma_0; \\ p = \bar{P}_i & \text{in } S_i, \quad \bar{P}_1 - \bar{P}_2 = \delta\bar{P} \end{cases} \tag{89}$$

where  $u(x) = \int_{Y_F} u_0(x, y)dy$ , the values  $P_1$  and  $P_2$  are uniquely defined by the normalization  $\int_\Omega p dx = 0$  and the pressure drop  $\delta\bar{P}$ .  $B_D$  and  $B_{\text{osm}}$  are constant symmetric matrices with entries defined by

$$\begin{aligned} B_D e_i &= \int_Y w_i(y) dy \\ B_{\text{osm}} e_i &= \int_Y W_i(y) dy \end{aligned} \tag{90}$$

where for  $1 \leq i \leq N$ ,  $w_i(y)$  and  $W_i(y)$  are unique periodic solutions to the cell Stokes problems

$$\begin{aligned} \nabla_y q_i - \Delta_{yy} w_i &= e_i, \quad \operatorname{div}(w_i) = 0 & \text{in } Y_F \\ w_i &= 0 & \text{in } Y_S \end{aligned} \tag{91}$$

and

$$\begin{aligned} \nabla_y Q_i - \Delta_{yy} W_i &= \frac{\lambda}{\kappa} [I - \nabla_y \chi(y)] \beta(y) e_i, \quad \operatorname{div}(W_i) = 0 & \text{in } Y_F \\ W_i &= 0 & \text{in } Y_S \end{aligned} \tag{92}$$

*Proof.* The two-scale homogenized problem (83) is equivalent to (89) through the relation

$$u_0(x, y) = \sum_{i=1}^3 w_i(y) \left(-\frac{\partial}{\partial x_i} p\right) + \sum_{i=1}^3 W_i(y) \left(\frac{\partial}{\partial x_i} \Theta^0(x)\right) \tag{93}$$

$$p_1(x, y) = \sum_{i=1}^3 q_i(y) \left(-\frac{\partial}{\partial x_i} p\right) + \sum_{i=1}^3 Q_i(y) \left(\frac{\partial}{\partial x_i} \Theta^0(x)\right) \tag{94}$$

The incompressibility condition for  $u_0(x, y)$  implies

$$\int_Y \operatorname{div}_x \left\{ \sum_{i=1}^3 w_i(y) \left(-\frac{\partial}{\partial x_i} p\right) + \sum_{i=1}^3 W_i(y) \left(\frac{\partial}{\partial x_i} \Theta^0(x)\right) \right\} dy = 0$$

After integrating the last expression over  $Y$  and recalling the problem (61), (62) for  $\Theta^0$  we arrive at the following macroscopic system of equations for  $p(x)$  and  $\Theta^0$ :

$$\operatorname{div}(A_{\text{eff}} \nabla \Theta^0) = 0 \tag{95a}$$

$$\operatorname{div}_x (B_D (\nabla_x p)) - \operatorname{div}_x (B_{\text{osm}} \nabla_x \Theta^0(x)) = 0 \tag{95b}$$

with boundary conditions

$$\Theta^0|_{S_1} = 0, \quad \Theta^0|_{S_2} = \theta_2 \quad (96a)$$

$$A_{\text{eff}} \nabla \Theta^0 \cdot n|_{\Gamma_0} = 0. \quad (96b)$$

$$p = \bar{P}_i \text{ in } S_i, \quad \bar{P}_1 - \bar{P}_2 = \delta \bar{P} \quad (96c)$$

$$(B_D (\nabla_x p) - B_{\text{osm}} \nabla_x \Theta^0(x)) \cdot n = 0 \text{ in } \Gamma_0; \quad \int_{\Omega} p dx = 0. \quad (96d)$$

where the values  $P_1$  and  $P_2$  are uniquely defined by the normalization  $\int_{\Omega} p dx = 0$  and the pressure drop  $\delta \bar{P}$ . An expression for  $u(x)$  follows:

$$u(x) = B_D (-\nabla p) + (B_{\text{osm}} \nabla \Theta^0). \quad (97)$$

□

**Remark.** We notice that the limit macroscopic system (95) consists of a decoupled effective diffusion equation and a Darcy type equation with an additional flux term  $B_{\text{osm}} \nabla_x \Theta^0(x)$  representing the osmotic pressure effect well known in physical chemistry. The input of the present paper is a rigorous description of flows of non-electrolytic solutions under osmotic pressure in intermediate regimes when a porous media is permeable for solvent particles. Diagonal elements in matrices  $B_D$  and  $B_{\text{osm}}$  are related to the filtration and osmotic transport coefficients  $L_p$  and  $L_{pD}$  in the Kedem-Katchalsky formula (2) in the case of a flat porous membrane.

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**Appendix. Poincare and Friedrichs inequalities in weighted Sobolev spaces.** Potentials  $V_{\varepsilon}(x)$  acting in the problems of interest are natural to interpret as functions of the distance  $d_{\Gamma}(x)$  from the solid boundary  $\Gamma_{\varepsilon}$ :  $V_{\varepsilon}(x) = V_{\varepsilon}(d_{\Gamma}(x))$ . The weight that appears in our problems is  $\beta_{\varepsilon}(x) = \exp \{-V_{\varepsilon}(d_{\Gamma}(x))\}$  depends on the point  $x$  through  $d_{\Gamma}(x)$ . If the potential  $V_{\varepsilon}(x)$  goes to infinity when  $x$  approaches the solid boundary  $\Gamma_{\varepsilon}$ , that can naturally happen in applications, the weight  $\beta_{\varepsilon}(x)$  degenerates at  $\Gamma_{\varepsilon}$ .

We provide below some specific results about conditions implying Friedrichs inequality (20) and the Poincare inequality (21) as well as embedding of  $W_2^1(\Omega_{\varepsilon}, \beta_{\varepsilon})$  into  $L_6(\Omega_{\varepsilon}, [\beta_{\varepsilon}]^6)$  in weighted Sobolev spaces and give nontrivial examples of potentials  $V_{\varepsilon}(d_{\Gamma}(x))$  such that these conditions are satisfied for the weight  $\beta_{\varepsilon}(x) = \exp(-\frac{\kappa}{\lambda} V_{\varepsilon}(x))$ .

**General Hardy inequalities in one dimension.** The most flexible and practical results for embedding, and Poincare and Friedrichs inequalities in weighted Sobolev spaces with weights degenerate only on the boundary follow from one dimensional Hardy inequalities on a finite interval and estimates on a thin stripe along the boundary. For Lipschitz domains and weights depending on the distance from the boundary corresponding estimates are similar to ones for the interval because the distance to the graph of a Lipschitz function along the corresponding coordinate direction and the usual distance  $d(x)$  are equivalent for small distances.



The following results on one-dimensional Hardy inequalities from [27] are useful both for estimates and for embedding results in weighted Sobolev spaces with particular choice of weights degenerating on the boundary of Lipschitz domains. Global results (20), (21) can be reduced in this case to local one-dimensional estimates by considering a thin stripe along the boundary similarly as in [22].

In domains with periodic perforated structure these results can be gained by combining the Friedrichs and Poincare inequalities in reference domains that do not depend on the small parameter, and by scaling arguments.

Let  $W(a, b)$  be the set of measurable positive functions finite almost everywhere on  $(a, b)$ .

**Theorem 6.3.** [27] *Let  $1 \leq p \leq q \leq \infty$ ,  $v, w \in W(a, b)$ . Define*

$$F_R(x) = F_R(x; a, b, w, v, q, p) = \left[ \int_a^x w(s) ds \right]^{1/q} \left[ \int_x^b v^{-1/(p-1)}(s) ds \right]^{(p-1)/p}, \tag{98}$$

and

$$B_R = B_R(a, b, w, v, q, p) = \sup_{a < x < b} F_R(x). \tag{99}$$

Then the Hardy inequality

$$\left[ \int_a^b u^q(s) w(s) ds \right]^{1/q} \leq C_R \left[ \int_a^b [u'(s)]^p v(s) ds \right]^{1/p},$$

or

$$\|u\|_{L_q((a,b),w)} \leq C_R \|u'\|_{L_p((a,b),v)} \tag{100}$$

is valid if and only if  $B_R = B_R(a, b, w, v, q, p) < \infty$ . The best possible constant  $C_R$  satisfies

$$B_R \leq C_R \leq B_R \left( \left(1 + \frac{q}{p}\right)^{1/q} \left(1 + \frac{p'}{q}\right)^{1/p'} \right).$$

This result can be also formulated as the boundedness of the Hardy operator  $H_R$

$$(H_R f)(x) = \int_x^b f(s) ds \tag{101}$$

as acting from  $L_p(a, b; v)$  to  $L_q(a, b; w)$ .

**Theorem 6.4.** [27] *Let  $1 \leq p \leq q \leq \infty$ ,  $v, w \in W(a, b)$ . The Hardy operator  $H_R : L_p(a, b; v) \rightarrow L_q(a, b; w)$  defined by (101) is compact if and only if  $B_R = B_R(a, b, w, v, q, p) < \infty$  and  $\lim_{x \rightarrow b^-} F_R(x) = \lim_{x \rightarrow a^+} F_R(x) = 0$ .*

**Example.** We check that conditions of Theorem 6.4 are fulfilled for weights  $w(x) = \exp(-\beta/x^n)$  and  $v(x) = \exp(-\alpha/x^n)$  with  $\alpha \leq \beta$  and  $p = q = 2$  on the interval  $(0, b)$  for functions equal to zero on the right endpoint. It implies as in [22] that Poincare and Friedrichs inequalities are valid in the Lipschitz domain  $\Omega_\varepsilon$  for weights  $w(x) = \exp(-\beta/(d_\Gamma(x))^n)$  and  $v(x) = \exp(-\alpha/(d_\Gamma(x))^n)$  corresponding to potentials with singularity  $V(d) \sim 1/d^n$ ,  $n > 0$ , at the boundary.

It is sufficient to verify compactness of the operator  $H_R$  (101) acting from  $L_2(0, b; v)$  to  $L_2(0, b; w)$ . Corresponding value of the function  $F_R(x)$  is:

$$\begin{aligned} F_R(x) &= \left[ \int_0^x \exp(-\beta/s^n) ds \right]^{1/2} \left[ \int_x^b \exp(\alpha/s^n) ds \right]^{1/2} \\ &= [F_{1,n}(x)]^{1/2} [F_{2,n}(x)]^{1/2}. \end{aligned}$$

We estimate integral  $F_{1,n}(x) = \int_0^x \exp(-\beta/s^n) ds$  by introducing variable  $y = \beta/s^n$ ,  $s = (\beta/y)^{1/n}$ .

$$\begin{aligned} F_{1,n}(x) &= \int_0^x \exp(-\beta/s^n) ds \\ &= \beta^{\frac{1}{n}} \int_{+\infty}^{\beta/x^n} \exp(-y) \frac{d}{dy} (y)^{-\frac{1}{n}} dy = \frac{1}{n} \beta^{\frac{1}{n}} \int_{\beta/x^n}^{+\infty} \exp(-y) (y)^{-\left(\frac{1}{n}+1\right)} dy = \\ &= \frac{1}{n} \beta^{\frac{1}{n}} \exp\left(-\frac{\beta}{x^n}\right) \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+1\right)} - \beta^{\frac{1}{n}} \frac{1}{n} \left(\frac{1}{n}+1\right) \int_{\beta/x^n}^{+\infty} \exp(-y) (y)^{-\left(\frac{1}{n}+2\right)} dy \\ &= \frac{1}{n} \beta^{\frac{1}{n}} \left\{ e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+1\right)} - \left(\frac{1}{n}+1\right) e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+2\right)} \right\} \\ &\quad + \frac{1}{n} \beta^{\frac{1}{n}} \left\{ \left(\frac{1}{n}+1\right) \left(\frac{1}{n}+2\right) \int_{\left(\frac{\beta}{x^n}\right)}^{\infty} e^{-y} y^{-\left(\frac{1}{n}+3\right)} dy \right\} \\ &\leq \frac{1}{n} \beta^{\frac{1}{n}} \left\{ e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+1\right)} - \left(\frac{1}{n}+1\right) e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+2\right)} \right\} \\ &\quad + \frac{1}{n} \beta^{\frac{1}{n}} \left(\frac{\beta}{x^n}\right)^{-2} \left(\frac{1}{n}+1\right) \left(\frac{1}{n}+2\right) \int_{\left(\frac{\beta}{x^n}\right)}^{\infty} e^{-y} y^{-\left(\frac{1}{n}+1\right)} dy \\ &= \frac{1}{n} \beta^{\frac{1}{n}} \left\{ e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+1\right)} - \left(\frac{1}{n}+1\right) e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+2\right)} \right\} \\ &\quad + \left(\frac{\beta}{x^n}\right)^{-2} \left(\frac{1}{n}+1\right) \left(\frac{1}{n}+2\right) F_{1,n}(x). \end{aligned}$$

This yields

$$\begin{aligned} F_{1,n}(x) &\leq \frac{1}{n} \beta^{\frac{1}{n}} \left\{ e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+1\right)} - \left(\frac{1}{n}+1\right) e^{-\frac{\beta}{x^n}} \left(\frac{\beta}{x^n}\right)^{-\left(\frac{1}{n}+2\right)} \right\} \\ &\quad \times \left[ 1 - \left(\frac{\beta}{x^n}\right)^{-2} \left(\frac{1}{n}+1\right) \left(\frac{1}{n}+2\right) \right]^{-1}. \end{aligned}$$

We estimate integral  $F_{2,n}(x) = \int_x^b \exp(\alpha/s^n) ds$  by introducing variable  $y = \alpha/s^n$ ,  $s = (\alpha/y)^{1/n}$ ,  $\frac{d}{dy} s = \frac{d}{dy} (\alpha/y)^{1/n} = -(\alpha)^{\frac{1}{n}} \frac{1}{n} (y)^{-\left(\frac{1}{n}+1\right)}$ . Then

$$F_{2,n}(x) = -(\alpha)^{\frac{1}{n}} \frac{1}{n} \int_{\alpha/x^n}^{\alpha/b^n} \exp(y) y^{-\left(\frac{1}{n}+1\right)} dy$$

$$\begin{aligned}
&= \alpha^{\frac{1}{n}} \frac{1}{n} \left\{ e^{\frac{\alpha}{x^n}} \left( \frac{\alpha}{x^n} \right)^{-\left(\frac{1}{n}+1\right)} - e^{\frac{\alpha}{b^n}} \left( \frac{\alpha}{b^n} \right)^{-\left(\frac{1}{n}+1\right)} \right. \\
&\quad \left. + \left( \frac{1}{n} + 1 \right) \int_{\frac{\alpha}{x^n}}^{\frac{\alpha}{b^n}} e^y y^{-\left(\frac{1}{n}+2\right)} dy \right\} \\
&\leq (\alpha)^{\frac{1}{n}} \frac{1}{n} \left\{ e^{\frac{\alpha}{x^n}} \left( \frac{\alpha}{x^n} \right)^{-\left(\frac{1}{n}+1\right)} - e^{\frac{\alpha}{b^n}} \left( \frac{\alpha}{b^n} \right)^{-\left(\frac{1}{n}+1\right)} \right\} \\
&\quad + \left( \frac{1}{n} + 1 \right) \left( \frac{\alpha}{x^n} \right)^{-1} F_{2,n}(x) \\
F_{2,n}(x) &\leq (\alpha)^{\frac{1}{n}} \frac{1}{n} \left\{ e^{\frac{\alpha}{x^n}} \left( \frac{\alpha}{x^n} \right)^{-\left(\frac{1}{n}+1\right)} - e^{\frac{\alpha}{b^n}} \left( \frac{\alpha}{b^n} \right)^{-\left(\frac{1}{n}+1\right)} \right\} \\
&\quad \times \left[ 1 - \left( \frac{1}{n} + 1 \right) \left( \frac{\alpha}{x^n} \right)^{-1} \right]^{-1}.
\end{aligned}$$

It is easy to observe that we have the following relation for  $x \rightarrow 0+$

$$\begin{aligned}
[F_{1,n}(x)] [F_{2,n}(x)] &= \frac{1}{n} \beta^{\frac{1}{n}} \exp(-\beta/x^n) (\beta/x^n)^{-\left(\frac{1}{n}+1\right)} \\
&\quad \times \exp(\alpha/x^n) (\alpha/x^n)^{-\left(\frac{1}{n}+1\right)} (\alpha)^{\frac{1}{n}} \frac{1}{n} [1 + O(x)].
\end{aligned}$$

Therefore

$$\lim_{x \rightarrow 0+} F_R(x) = 0. \quad (102)$$

We also observe that  $F_R(x)$  is bounded, and

$$\lim_{x \rightarrow b-} F_R(x) = 0. \quad (103)$$

It implies that the operator  $H_R$  is compact from  $L_2(0, b; \exp(-\beta/x^n))$  to itself.

Checking boundedness of  $H_R$  acting from the space  $L_p(0, b; \exp(-\alpha/x^n))$  to the space  $L_q(0, b; \exp(-\beta/x^n))$  we observe that

$$\begin{aligned}
F_R(x) &= \left[ \int_a^x w(s) ds \right]^{1/q} \left[ \int_x^b v^{-1/(p-1)}(s) ds \right]^{(p-1)/p} \\
&\leq e^{\frac{\alpha}{x^{n/p}}} e^{-\frac{\beta}{x^{n/q}}} \left\{ \frac{1}{n} \beta^{\frac{1}{n}} \left\{ \left( \frac{\beta}{x^n} \right)^{-\left(\frac{1}{n}+1\right)} - \left( \frac{1}{n} + 1 \right) \left( \frac{\beta}{x^n} \right)^{-\left(\frac{1}{n}+2\right)} \right\} \right\}^{\frac{1}{q}} \\
&\quad \times \left\{ \left[ 1 - \left( \frac{\beta}{x^n} \right)^{-2} \left( \frac{1}{n} + 1 \right) \left( \frac{1}{n} + 2 \right) \right]^{-1} \right\}^{1/q} \\
&\times \left\{ \left( \frac{\alpha}{p-1} \right)^{\frac{1}{n}} \frac{1}{n} \left\{ \left( \frac{\alpha}{x^n(p-1)} \right)^{-\left(\frac{1}{n}+1\right)} - \left( \frac{\alpha}{b^n(p-1)} \right)^{-\left(\frac{1}{n}+1\right)} \right\} \right\}^{\frac{p-1}{p}} \\
&\quad \times \left\{ \left[ 1 - \left( \frac{1}{n} + 1 \right) \left( \frac{\alpha}{x^n(p-1)} \right)^{-1} \right]^{-1} \right\}^{\frac{p-1}{p}}.
\end{aligned}$$

The main term to estimate is  $e^{-\frac{\beta}{x^{n/q}}} e^{\frac{\alpha}{x^{n/p}}}$ . It is bounded in the case  $\alpha/p \leq \beta/q$ . Similarly as above  $\lim_{x \rightarrow 0+} F_R(x) = 0$  and  $\lim_{x \rightarrow b-} F_R(x) = 0$  in this case. It

means that  $H_R$  will be both bounded and compact from  $L_p(0, b; \exp(-\alpha/x^n))$  to  $L_q(0, b; \exp(-\beta/x^n))$  if  $\alpha/p \leq \beta/q$ .

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