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MOVING RECURRENT PROPERTIES FOR THE DOUBLING MAP ON THE UNIT INTERVAL

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Abstract. Let $(X, T, \mathcal{B}, \mu)$ be a measure-theoretical dynamical system with a compatible metric d. Following Boshernitzan, call a point $x \in X$ is $\{n_k\}$-moving recurrent if

$$\inf_{k\geq 1} d\big(T^{n_k}x,\ T^{n_k+k}x\big) = 0,$$

where $\{n_k\}_{k\in\mathbb{N}}$ is a given sequence of integers. It was asked whether the set of $\{n_k\}$-moving recurrent points is of full μ-measure. In this paper, we restrict our attention to the doubling map and quantify the size of the set of $\{n_k\}$-moving recurrent points in the sense of measure (a class of 2-fold mixing measures) and Hausdorff dimension.

1. **Introduction.** For a measure-theoretical dynamical system $(X, T, \mathcal{B}, \mu)$ with a probability measure μ and a compatible metric d, Poincaré Recurrence theorem states a non-trivial recurrence to a measurable set with positive measure. Consequently, μ-almost surely,

$$\liminf_{n\to\infty} d(T^n x, x) = 0.$$

In other words, almost every orbit $\{T^n x\}_{n\geq 1}$ returns to a target with the fixed center x, the starting point, infinitely often. Instead of a target with fixed center or fixed target, Boshernitzan [4] asked how about the case when the target is also moving along the time? To state the question clearly, we cite some notation at first.

Problem ([4]). *Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence of integers. Define*

$$\psi_{\{n_k\}}(x) := \inf_{k\geq 1} d\big(T^{n_k}x, T^{n_k+k}x\big).$$

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If $\psi_{\{n_k\}}(x) = 0$, the point $x \in X$ is called $\{n_k\}$-moving recurrent. Is it true that μ-almost all points are $\{n_k\}$-moving recurrent?

In [4], Boshernitzan & Glasner showed that for a minimal dynamical system, the set of $\{n_k\}$-moving recurrent points is a dense G_δ set. In some weakly mixing systems, Grivaux & Roginskaya [9] and Grivaux [10] investigated the arithmetic properties of the sequence $\{n_k\}_{k\in\mathbb{N}}$ to ensure that almost all points are $\{n_k\}$-moving recurrent.

In this paper, we restrict our attention to the doubling map $Tx = 2x \pmod 1$ and quantify the size of the set consisting of $\{n_k\}$-moving recurrent points in the sense of measure and Hausdorff dimension. The results in the following can be generalized to β-expansions for all $\beta > 1$ with no more additional efforts.

Our first result concerns the size of the set of $\{n_k\}$-moving recurrent points in the sense of a class of measures.

Theorem 1.1. *Assume that $([0,1], T, \mu)$ is a 2-fold mixing system. Then for any sequence of integers $\{n_k\}_{k\geq 1}$ tending to infinity as $k \to \infty$, μ-almost all points are $\{n_k\}_{k\geq 1}$-moving recurrent.*

Remark 1. From its proof in Section 3, it can be seen that Theorem 1.1 is valid for any 2-fold mixing system. So, in this result, T is not necessarily the doubling map.

The next results concern a quantitative version of the set of $\{n_k\}$-moving recurrent points in the sense of Hausdorff dimension.

Let $\{n_k\}_{k\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ be two sequences of integers tending to infinity as $k \to \infty$. Define

$$R(\{n_k, r_k\}) = \Big\{x \in [0,1] : \big|T^{n_k}x - T^{n_k+k}x\big| < 2^{-r_k} \text{ for infinitely many } k \in \mathbb{N}\Big\},$$

which consists of $\{n_k\}$-moving recurrent points with additionally a quantitative convergent rate.

Theorem 1.2. *Let $\{n_k\}_{k\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ be two sequences of integers that tend to infinity. Then*

$$\dim_{\mathrm{H}} R(\{n_k, r_k\}) = \frac{1}{1+b} \quad \textit{with} \quad b = \liminf_{k\to\infty} \frac{r_k}{n_k + k},$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension of a set.

Theorem 1.3. *Let $\{n_k\}_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers that tends to infinity. Then*

$$\dim_{\mathrm{H}} \Big\{x \in [0,1] : \inf_{k\geq 1} \big|T^{n_k}x - T^{n_k+k}x\big| > 0\Big\} = 1,$$

i.e., the set of non $\{n_k\}$-moving recurrent points is full in the sense of Hausdorff dimension.

For more results of other kinds of recurrence properties in the sense of measure and dimension, one is referred to the papers of Fursternberg [6, 7], Hill & Velani [11], Boshernitzan [3], Akin [1], Barreira & Saussol [2], Glasner [8], Tan & Wang [13] and the references therein.

2. **Preliminaries.**

In this section, we fix some notation about the dyadic expansions, k-fold mixing systems and a dimensional result about Cantor set for later use.

2.1. **Notation.** Let T be the doubling map defined on $[0,1)$ by

$$Tx = 2x - \lfloor 2x \rfloor,$$

where $\lfloor \xi \rfloor$ denotes the integer part of ξ. Then every $x \in [0,1)$ can be expressed uniquely as a finite or infinite series

$$x = \frac{\epsilon_1(x)}{2} + \cdots + \frac{\epsilon_n(x)}{2^n} + \cdots, \tag{1}$$

where, for $n \geq 1$, $\epsilon_n(x) = \lfloor 2T^{n-1}x \rfloor$ is called the n-th digit of x. Sometimes we rewrite (1) as

$$x = \big(\epsilon_1(x), \cdots, \epsilon_n(x), \cdots\big).$$

Let $\Sigma = \{0,1\}$ and denote

$$\Sigma^n = \Big\{(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) : \epsilon_j \in \Sigma\Big\} \quad (\Sigma^0 = \emptyset),$$

the collection of all words with length n and Σ^∞ the infinite words.

For any finite word $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in \Sigma^n$ of length $n \geq 1$, call

$$I_n\big(\epsilon_1, \epsilon_2, \cdots, \epsilon_n\big) = cl\big\{x \in [0,1) : \epsilon_j(x) = \epsilon_j, 1 \leq j \leq n\big\}$$

a *dyadic interval of generation n,* where clA denotes the closure of A. It is clear that a dyadic interval of generation n is an interval of length 2^{-n}.

The concatenation of two finite words $\sigma = (\sigma_1, \cdots, \sigma_k) \in \Sigma^k$ and $\tau = (\tau_1, \cdots, \tau_m) \in \Sigma^m$ is given as

$$(\sigma, \tau) = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m).$$

2.2. **k-fold mixing system.** In this short part, we cite the definition of the k-fold mixing system.

Definition 2.1 ([12])**.** A measure preserving system $(X, \mathcal{B}, \mu, T)$ is k-fold mixing, mixing of order k or mixing on $k+1$ sets if

$$\mu\Big(A_0 \bigcap T^{-n_1} A_1 \bigcap \cdots \bigcap T^{-n_k} A_k\Big) \to \mu(A_0) \cdots \mu(A_k)$$

as

$$n_1, n_2 - n_1, n_3 - n_2, \cdots, n_k - n_{k-1} \to \infty$$

for any sets $A_0 \cdots A_k \in \mathcal{B}$.

2.3. **Cantor set.** To estimate the Hausdorff dimension of a fractal set from below, it is a very classic way to construct a suitable Cantor subset inside and then apply the following widely used dimensional result.

Proposition 1 (Falconer [5])**.** *Let $[0,1] = \mathbb{F}_0 \supset \mathbb{F}_1 \supset \cdots$ be a decreasing sequence of sets, with each $\mathbb{F}_i$ a union of a finite number of disjoint closed intervals. Assume that each interval of $\mathbb{F}_i$ contains at least m_{i+1} intervals of $\mathbb{F}_{i+1}$ which are separated by gaps of lengths at least η_{i+1}. If $m_i \geq 2$ and $\eta_i > \eta_{i+1} > 0$, then*

$$\dim_{\mathrm{H}} \bigcap_{i \geq 1} \mathbb{F}_i \geq \liminf_{i \to \infty} \frac{\log(m_1 m_2 \cdots m_i)}{-\log(m_{i+1}\eta_{i+1})}.$$

3. **Proof of Theorem 1.1.** It should be noted that the proof presented here only works for systems with 2-fold mixing property. Recall that what we need to show is that

$$\mu\Big(\big\{x \in [0,1) : \inf_{k\geq 1} \big|T^{n_k}x - T^{n_k+k}x\big| = 0\big\}\Big) = 1.$$

For any $\epsilon > 0$ and $k \geq 1$, let

$$F_k(\epsilon) = \Big\{x \in [0,1) : \big|T^{n_k}x - T^{n_k+k}x\big| < \epsilon\Big\}.$$

Then set

$$A(\epsilon) = \limsup_{k\to\infty} F_k(\epsilon).$$

It is clear that $A(\epsilon)$ is decreasing with respect to ϵ and

$$\bigcap_{\epsilon>0} A(\epsilon) \subset \Big\{x \in [0,1) : \inf_{k\geq 1} \big|T^{n_k}x - T^{n_k+k}x\big| = 0\Big\}.$$

Now we show that

$$\mu\Big(\bigcap_{\epsilon>0} A(\epsilon)\Big) = 1.$$

Suppose this is not true, then there exists some $\epsilon > 0$ such that $\mu\big(A(\epsilon)\big) < 1$. Let $B = [0,1] \setminus A(\epsilon)$. Since $\mu(B) > 0$, there exists a subset $B_\epsilon \subset B$ such that $\mu(B_\epsilon) > 0$ and $|B_\epsilon| < \epsilon$. This can be done by just partitioning $[0,1]$ into intervals of length less than ϵ.

Remind that here μ is of 2-*fold mixing* and n_k tends to infinity as $k \to \infty$. Then by Definition 2.1 of k-*fold mixing*, we have

$$\mu\Big(B_\epsilon \bigcap T^{-n_k}B_\epsilon \bigcap T^{-n_k-k}B_\epsilon\Big) \to \mu(B_\epsilon)^3 > 0$$

as $k \to \infty$. Then it gives that

$$\limsup_{k\to\infty} \left(B_\epsilon \bigcap T^{-n_k}B_\epsilon \bigcap T^{-(n_k+k)}B_\epsilon\right) \neq \emptyset.$$

Write $G_k = B_\epsilon \bigcap T^{-n_k}B_\epsilon \bigcap T^{-n_k-k}B_\epsilon$, for every $k \geq 1$. By the continuity of a finite measure from above, we obtain that

$$\mu\Big(\limsup_{k\to\infty} G_k\Big) \geq \limsup_{k\to\infty} \mu\big(G_k\big) = \mu(B_\epsilon)^3 > 0.$$

Hence there exists a point $x \in B_\epsilon$ such that $\big|T^{n_k}x - T^{n_k+k}x\big| < \epsilon$ holds for infinitely many k, then $x \in A(\epsilon)$. This leads to a contradiction!

4. **Proof of Theorem 1.2.** The proof of the dimension of $R(\{n_k, r_k\})$ is divided into two parts: upper bound and lower bound.

4.1. **Upper bound.** The upper bound can be obtained by considering the natural coverings of $R(\{n_k, r_k\})$. Evidently,

$$R(\{n_k, r_k\}) = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \bigcup_{(\epsilon_1,\cdots,\epsilon_{n_k+k})\in\sum^{n_k+k}} R_k(\epsilon_1, \cdots, \epsilon_{n_k+k})$$

where

$$R_k(\epsilon_1, \cdots, \epsilon_{n_k+k}) = \Big\{x : \epsilon_j(x) = \epsilon_j, 1 \leq j \leq n_k + k, \big|T^{n_k}x - T^{n_k+k}x\big| < 2^{-r_k}\Big\}.$$

Now we estimate the length of $R_k(\epsilon_1, \cdots, \epsilon_{n_k+k})$: for $x \in R_k(\epsilon_1, \cdots, \epsilon_{n_k+k})$,

$$\left|T^{n_k}x - T^{n_k+k}x\right| = \left|\frac{\epsilon_{n_k+1}}{2} + \cdots + \frac{\epsilon_{n_k+k} + T^{n_k+k}x}{2^k} - T^{n_k+k}x\right| < 2^{-r_k}.$$

This gives that

$$\Big(1 - \frac{1}{2^k}\Big)T^{n_k+k}x \in \Big(\frac{\epsilon_{n_k+1}}{2} + \cdots + \frac{\epsilon_{n_k+k}}{2^k} - 2^{-r_k}, \frac{\epsilon_{n_k+1}}{2} + \cdots + \frac{\epsilon_{n_k+k}}{2^k} + 2^{-r_k}\Big).$$

We infer that, for every $k \geq 1$,

$$\left|R_k(\epsilon_1, \cdots, \epsilon_{n_k+k})\right| \leq \frac{4}{2^{r_k}}.$$

As a result, the s-dimensional Hausdorff measure of $R(\{n_k, r_k\})$ can be estimated as

$$\begin{aligned}\mathcal{H}^s\big(R(\{n_k, r_k\})\big) &\leq \liminf_{N\to\infty} \sum_{k=N}^{\infty} \sum_{(\epsilon_1,\cdots,\epsilon_{n_k+k})\in\sum^{n_k+k}} \left|R_k(\epsilon_1, \cdots, \epsilon_{n_k+k})\right|^s \\ &\leq \liminf_{N\to\infty} \sum_{k=N}^{\infty} 2^{n_k+k}\Big(\frac{4}{2^{r_k}}\Big)^s,\end{aligned}$$

which is finite for any $s > \frac{1}{1+b}$. The arbitrariness of s yields that

$$\dim_{\mathrm{H}} R(\{n_k, r_k\}) \leq \frac{1}{1+b}.$$

4.2. **Lower bound.** We will use Proposition 1 to give the lower bound of the Hausdorff dimension of $R(\{n_k, r_k\})$.

4.2.1. *Construction of a Cantor subset.* We construct a Cantor set $\mathbb{F}(\infty)$ such that for each $x \in \mathbb{F}(\infty)$

$$\left|T^{n_k}x - T^{n_k+k}x\right| < 2^{-r_k} \tag{2}$$

holds for infinitely many $k \in \mathbb{N}$.

Let

$$x = \big(\epsilon_1, \cdots, \epsilon_n, \cdots\big)$$

be its dyadic expansion. We pose some restrictions on the digit sequence of x to ensure the validity of (2). Two cases are distinguished according as $r_k \leq k$ or not.

Case (i). $r_k \leq k$. We demand

$$\epsilon_{n_k+i} = \epsilon_{n_k+k+i},\ \ 1 \leq i \leq r_k,$$

i.e., the block $(\epsilon_{n_k+1}, \cdots, \epsilon_{n_k+r_k})$ is repeated after the position $n_k + k$. Thus the $(n_k + k + r_k)$-dyadic interval containing x can be expressed as:

$$I_{n_k+k+r_k}\big(\epsilon_1, \cdots, \epsilon_{n_k}, \underbrace{\epsilon_{n_k+1}, \cdots, \epsilon_{n_k+r_k}}_{r_k}, \epsilon_{n_k+r_k+1}, \cdots, \epsilon_{n_k+k}, \underbrace{\epsilon_{n_k+1}, \cdots, \epsilon_{n_k+r_k}}_{r_k}\big).$$

Case (ii). $r_k > k$. We repeat the word $(\epsilon_{n_k+1}, \cdots, \epsilon_{n_k+k})$. More precisely, write $r_k = (q_k - 1)k + t_k$ for some $q_k \geq 2$ and $0 \leq t_k < k$ and then demand

$$\begin{aligned}\epsilon_{n_k+i} &= \epsilon_{n_k+jk+i},\ \ 1 \leq i \leq k, 1 \leq j \leq q_k - 1;\\ \epsilon_{n_k+i} &= \epsilon_{n_k+q_kk+i},\ \ 1 \leq i \leq t_k.\end{aligned}$$

In other words, after the position n_k, a periodic word with period $(\epsilon_{n_k+1}, \cdots, \epsilon_{n_k+k})$ is followed up to the (n_k+k+r_k)th position. Thus the (n_k+k+r_k)-dyadic interval containing x can be expressed as:

$$I_{n_k+k+r_k}\left(\epsilon_1, \cdots, \epsilon_{n_k}, \left(\epsilon_{n_k+1}, \cdots, \epsilon_{n_k+k}\right)^{q_k}, \epsilon_{n_k+q_k k+1}, \cdots, \epsilon_{n_k+q_k k+t_k}\right),$$

where w^q means the concatenation of q many words w.

For any $k \in \mathbb{N}$, either $r_k \le k$ or $r_k > k$ occurs. So we must meet one case for infinitely many k. In the sequel, we assume $r_k \le k$ appears for infinitely many k. The other case can be treated similarly.

Choose a largely sparse subsequence $\{k_i\}_{i\ge 1}$ of $\mathbb{N}$ such that

$$\lim_{i\to\infty} \frac{r_{k_i}}{n_{k_i}+k_i} = b$$

with $k_i \gg n_{k_{i-1}} + k_{i-1} + r_{k_{i-1}}$, and $k_i \ge r_{k_i}$ for all $i \ge 1$.

For each $\left(\epsilon_1, \cdots, \epsilon_{n_{k_i}+k_i}\right) \in \Sigma^{n_{k_i}+k_i}$, define

$$\begin{aligned} &J_i\big(\epsilon_1, \cdots, \epsilon_{n_{k_i}+k_i}\big) \\ =&I_{n_{k_i}+k_i+r_{k_i}}\big(\epsilon_1, \cdots, \epsilon_{n_{k_i}}, \epsilon_{n_{k_i}+1}, \cdots, \epsilon_{n_{k_i}+k_i}, \underbrace{\epsilon_{n_{k_i}+1}, \cdots, \epsilon_{n_{k_i}+r_{k_i}}}_{r_{k_i}}\big), \end{aligned}$$

and call it an i-th *basic cylinder*. This is just a cylinder realizing (2) for $k = k_i$. It should be reminded that J_i is a dyadic interval of order $n_{k_i} + k_i + r_{k_i}$, but it depends only on the first $n_{k_i} + k_i$ coordinates.

The first generation of the Cantor set.

Denote by Ξ_1 half of the words in $\Sigma^{n_{k_1}+k_1}$ such that any two dyadic intervals $I_{n_{k_1}+k_1}(w_1)$ and $I_{n_{k_1}+k_1}(w_2)$, with $w_1, w_2 \in \Xi_1$, are separated by a gap of length at least $2^{-n_{k_1}-k_1}$.

The first generation $\mathbb{F}(1)$ is defined as

$$\mathbb{F}(1) = \bigcup J_1\big(\epsilon_1, \cdots, \epsilon_{n_{k_1}+k_1}\big),$$

where the union is taken over all $(\epsilon_1, \cdots, \epsilon_{n_{k_1}+k_1}) \in \Xi_1$.

Then the first generation consists of $m_1 = \frac{1}{2}2^{n_{k_1}+k_1}$ many intervals each of which are separated by a gap at least $\eta_1 = 2^{-n_{k_1}-k_1}$.

The second generation of the Cantor set. The second generation is composed of collections of subintervals of each 1-th *basic cylinders* $J_1 \in \mathbb{F}(1)$:

$$\mathbb{F}(2) = \bigcup_{J_1\in\mathbb{F}(1)} \mathbb{F}(2, J_1).$$

Now we give the definition of $\mathbb{F}(2, J_1)$ for each $J_1 \in \mathbb{F}(1)$.

Fix an element $J_1 = J_1\big(\epsilon_1, \cdots, \epsilon_{n_{k_1}+k_1}\big) \in \mathbb{F}(1)$ which is a dyadic interval of order $n_{k_1} + k_1 + r_{k_1}$. Denote by Ξ_2 half of the words in $\Sigma^{n_{k_2}+k_2-(n_{k_1}+k_1+r_{k_1})}$ such that any two dyadic intervals

$$I_{n_{k_2}+k_2}(\epsilon_1, \cdots, \epsilon_{n_{k_1}+k_1+r_{k_1}}, w_1), \quad I_{n_{k_2}+k_2}(\epsilon_1, \cdots, \epsilon_{n_{k_1}+k_1+r_{k_1}}, w_2),$$

with $w_1, w_2 \in \Xi_2$, are separated by a gap of length at least $2^{-n_{k_2}-k_2}$.

Then the collection of intervals $\mathbb{F}(2, J_1)$ is defined as

$$\mathbb{F}(2, J_1) = \bigcup J_2\big(\epsilon_1, \cdots, \epsilon_{n_{k_1}+k_1+r_{k_1}}, \epsilon_{n_{k_1}+k_1+r_{k_1}+1}, \cdots, \epsilon_{n_{k_2}+k_2}\big),$$

where the union is taken over $(\epsilon_{n_{k_1}+k_1+r_{k_1}+1}, \cdots, \epsilon_{n_{k_2}+k_2}) \in \Xi_2$.

As a result, each element $J_1 \in \mathbb{F}(1)$ contains $m_2 = \frac{1}{2} \cdot 2^{n_{k_2}+k_2-(n_{k_1}+k_1+r_{k_1})}$ elements in $\mathbb{F}_2$, which are separated by a gap of length at least $\eta_2 = 2^{-n_{k_2}-k_2}$.

The inductive step of the construction: Suppose that the i-th generation $\mathbb{F}(i)$ is already defined. With the same construction as $\mathbb{F}(2)$, for each $J_i \in \mathbb{F}(i)$, we construct $\mathbb{F}(i+1, J_i)$ and then define

$$\mathbb{F}(i+1) = \bigcup_{J_i \in \mathbb{F}(i)} \mathbb{F}(i+1, J_i).$$

Still we have that each element $J_i \in \mathbb{F}(i)$ contains $m_{i+1} = \frac{1}{2} \cdot 2^{n_{k_{i+1}}+k_{i+1}-(n_{k_i}+k_i+r_{k_i})}$ elements in $\mathbb{F}(i+1)$, which are separated by a gap at least $\eta_{i+1} = 2^{-n_{k_{i+1}}-k_{i+1}}$.

The Cantor set. Finally, we obtain a nested sequence $\{\mathbb{F}(i)\}_{i\ge 1}$ composed of *basic cylinders*. Then the desired Cantor set is defined as

$$\mathbb{F}(\infty) = \bigcap_{i=1}^{\infty} \mathbb{F}(i).$$

It is clear that

$$\mathbb{F}(\infty) \subset R(\{n_k, r_k\}).$$

The dimension of $\mathbb{F}(\infty)$ is estimated by applying Proposition 1:

$$\dim_{\mathrm{H}} \mathbb{F}(\infty) \ge \liminf_{i\to\infty} \frac{\log m_1 \cdots m_i}{-\log m_{i+1}\eta_{i+1}} \ge \liminf_{i\to\infty} \frac{n_{k_i}+k_i}{n_{k_i}+k_i+r_{k_i}} = \frac{1}{1+b},$$

where, for the second inequality, we used the fact that $\sum_{j=1}^{i-1}(n_{k_j}+k_j+r_{k_j})$ is negligible compared with n_{k_i}.

5. **Proof of Theorem 1.3.** Recall

$$E = \left\{x \in [0,1] : \inf_{k\ge 1} \left|T^{n_k}x - T^{n_k+k}x\right| > 0\right\}$$

is the set in question. We will try to construct a Cantor subset $\mathbb{F}_\infty$ inside E. Then Proposition 1 applies.

5.1. **A subset E_m of E.** Fix an integer $m \ge 3$ and define

$$E_m = \{x = (\epsilon_1 \cdots \epsilon_n \cdots) : \text{satisfies (3) for any } k \ge 1 \text{ and (4) for any } i \ge 0,\}$$

where (3) and (4) are some restrictions on the digit sequence of the dyadic expansion of x:

$$(\epsilon_{n_k+k+1}, \cdots, \epsilon_{n_k+k+m}) \ne (\epsilon_{n_k+1}, \cdots, \epsilon_{n_k+m}); \tag{3}$$

$$(\epsilon_{n_m+im+1}, \cdots, \epsilon_{n_m+im+m}) \ne 0^m. \tag{4}$$

We claim that $E_m \subset E$. More precisely, given a point $x \in E_m$, we assume without loss of generality that $T^{n_k}x \ge T^{n_k+k}x$. One one hand, by (3), it follows that

$$T^{n_k+k}x = \frac{\epsilon_{n_k+k+1}}{2} + \cdots + \frac{\epsilon_{n_k+k+m}}{2^m} + \cdots \le \frac{\epsilon_{n_k+k+1}}{2} + \cdots + \frac{\epsilon_{n_k+k+m}}{2^m} + \frac{1}{2^m}$$
$$\le \frac{\epsilon_{n_k+1}}{2} + \cdots + \frac{\epsilon_{n_k+m}}{2^m} \quad \text{(by (3))}.$$

On the other hand, the formula (4) implies that every $2m$ consecutive digits of x after the n_mth digit contains at least one non-zero terms. Thus

$$\begin{aligned} T^{n_k}x &= \frac{\epsilon_{n_k+1}}{2} + \cdots + \frac{\epsilon_{n_k+m}}{2^m} + \frac{\epsilon_{n_k+m+1}}{2^{m+1}} + \cdots + \frac{\epsilon_{n_m+3m}}{2^{3m}} + \cdots \\ &\geq \frac{\epsilon_{n_k+1}}{2} + \cdots + \frac{\epsilon_{n_k+m}}{2^m} + \frac{1}{2^{3m}}. \end{aligned}$$

Combining the above two inequalities together, we conclude that for any $x \in E_m$ and any $k \geq n_m$

$$\left|T^{n_k}x - T^{n_k+k}x\right| \geq 2^{-3m} > 0.$$

5.2. **Cantor subset of E_m.** We construct a Cantor subset $\mathbb{F}_\infty$ of E_m. The restriction (4) can be realized easily, so we only pay close attention on (3).

We partition the natural numbers $\mathbb{N}$ into a countable union of disjoint sets according to the positions of $\{n_k + k + 1\}$: $\mathbb{N} = \bigsqcup_{i\geq 0} \mathbb{N}_i$, where

$$\begin{aligned} \mathbb{N}_0 &= \Big\{k \in \mathbb{N} : 1 < n_k + k + 1 \leq n_m + m\Big\} = \big\{1, \cdots, m-1\big\} \\ \mathbb{N}_\ell &= \Big\{k \in \mathbb{N} : n_m + \ell m < n_k + k + 1 \leq n_m + (\ell+1)m\Big\},\ (\ell \geq 1). \end{aligned} \tag{5}$$

Note that $m \in \mathbb{N}_1$ and that $\mathbb{N}_\ell$ may be empty for some $\ell \geq 1$.

I. ***Realize (3) for $k \in \mathbb{N}_0$ and the 0th level of the Cantor subset.***
Choose one word $(\epsilon_1, \cdots, \epsilon_{n_m+m}) \in \Sigma^{n_m+m}$ such that

$$\epsilon_{n_k+k+1} \neq \epsilon_{n_k+1} \text{ for all } k \in \mathbb{N}_0,$$

which realizes (3) for all $k \in \mathbb{N}_0$. The digits of such a word can be chosen one by one in the following way.

The first step. Denote $s_1 = \max\big\{n_1 + 1 + 1, n_2 + 1\big\}$. Fix the first s_1 digits such that

$$\epsilon_{n_1+1+1} \neq \epsilon_{n_1+1}.$$

This ensures the validity of (3) for $k = 1$.

The second step. Note that $n_2 + 2 + 1 > s_1$, which means that the position $n_2 + 2 + 1$ is after the s_1th position. This enables us to choose the next digits up to the position $s_2 = \max\big\{n_2 + 2 + 1, n_3 + 1\big\}$ such that

$$\epsilon_{n_2+2+1} \neq \epsilon_{n_2+1}.$$

This ensures the validity of (3) for $k = 2$.

The $(m-1)$-th step. Continue this process: provided that $(\epsilon_1, \cdots, \epsilon_{s_{m-2}})$ has been determined, where $s_{m-2} = \max\big\{n_{m-2} + (m-2) + 1, n_{m-1} + 1\big\}$. Similarly $n_{m-1} + (m-1) + 1 > s_{m-2}$ which enables us to choose the next digits up to $s_{m-1} = \max\big\{n_{m-1} + (m-1) + 1, n_m + 1\big\}$ satisfying that

$$\epsilon_{n_{m-1}+(m-1)+1} \neq \epsilon_{n_{m-1}+1}.$$

This ensures the validity of (3) for the last integer in $\mathbb{N}_0$, i.e. $k = m - 1$.

Now we get a word $(\epsilon_1, \cdots, \epsilon_{s_{m-1}}) \in \Sigma^{s_{m-1}}$ satisfying (3) for all $k \in \mathbb{N}_0$. Choose the next digits up to $(n_m + m)$th position satisfying that

$$\big(\epsilon_{n_m+1}, \cdots, \epsilon_{n_m+m}\big) \neq 0^m,$$

which realizes (4) for $i = 0$. The word $w_0 = (\epsilon_1, \cdots, \epsilon_{n_m+m})$ is a desired one.

Define the 0th level of the Cantor subset as

$$\mathbb{F}_0 = \bigcup I_{n_m+m}(\epsilon_1, \cdots, \epsilon_{n_m+m}),$$

where the union is taken over all elements in Λ_0:

$$\Lambda_0 = \{w_0\}.$$

II. ***Realize (3) for*** $k \in \mathbb{N}_1$ ***and the 1th level of the Cantor subset.***
It is clear that $m \in \mathbb{N}_1$, so we write $\mathbb{N}_1 = \{m, m+1, \cdots, m+\ell\}$.

For each $w_0 \in \Lambda_0$, we will choose the next m digits $(\epsilon_{n_m+m+1}, \cdots, \epsilon_{n_m+2m})$ such that (3) is valid for $k \in \mathbb{N}_1$. (In fact, Λ_0 is a singleton. The only reason we said like this is to illustrate the ideas in constructing $\mathbb{F}_\infty$.)

Recall the definition of $\mathbb{N}_1$: for each $k \in \mathbb{N}_1$,

$$n_m + m < n_k + k + 1 \le n_m + 2m. \tag{6}$$

Write

$$t_k = (n_m + 2m) - (n_k + k), \quad (\text{note: } 1 \le t_k \le m) \tag{7}$$

which is the distance from the position n_k+k+1 to n_m+2m. Then the inequalities (6), together with $k \ge m$, implies that

$$n_k + 1 \le n_m + m, \text{ moreover } n_k + t_k \le n_m + m.$$

This means that for each $k \in \mathbb{N}_1$, the first $n_k + t_k$ digits has already been chosen in the 0th step. So, to ensure (3), we choose $(\epsilon_{n_m+m+1}, \cdots, \epsilon_{n_m+2m})$ such that for all $k \in \mathbb{N}_1$,

$$(\overbrace{\epsilon_{n_k+k+1}, \cdots, \epsilon_{n_m+2m}}^{t_k}) \ne (\overbrace{\epsilon_{n_k+1} \cdots \epsilon_{n_k+t_k}}^{t_k}). \tag{8}$$

This is stronger than (3) since $t_k \le m$.

The restriction (4) can be realized by avoiding one more case that $(\epsilon_{n_m+m+1}, \cdots, \epsilon_{n_m+2m}) = 0^m$.

As a summary, we have proposed a method to realize (3) for all $k \in \mathbb{N}_1$ and (4) for $i = 1$. Now we arrive at giving the first level of the Cantor set. Let

$$\Lambda_1 = \Big\{ \xi = (\epsilon_{n_m+m+1}, \cdots, \epsilon_{n_m+2m}) : \xi \ne 0^m \text{ and (8) is satisfied for all } k \in \mathbb{N}_1 \Big\}.$$

Then we get a collection of dyadic intervals

$$\mathcal{F}_1 := \Big\{ I_{n_m+2m}(w_0, \xi) : \xi \in \Lambda_1 \Big\}.$$

Leave out half of the intervals in $\mathcal{F}_1$, or equivalently choose half of the elements in Λ_1 (denoted by Ξ_1), such that the remaining intervals are separated by a gap at least 2^{-n_m-2m}. Then the first level of the Cantor set is defined as

$$\mathbb{F}_1 = \bigcup_{(\epsilon_{n_m+m+1}, \cdots, \epsilon_{n_m+2m}) \in \Xi_1} I_{n_m+2m}\big(w_0, (\overbrace{\epsilon_{n_m+m+1}, \cdots, \epsilon_{n_m+2m}}^{m})\big),$$

It should be noted that Ξ_1 depends on the word w_0, but this dependence will not play a role in the following argument, so will not be addressed explicitly.

Now we count the number of the elements in Ξ_1. For each $k \in \mathbb{N}_1$, to realize (8), one needs to avoid 2^{m-t_k} possible choices of $(\epsilon_{n_m+m+1}, \cdots, \epsilon_{n_m+2m})$. Hence

$$m_1 := \sharp \Xi_1 = \frac{1}{2} \sharp \Lambda_1 \ge \frac{1}{2} \cdot (2^m - \sum_{k \in \mathbb{N}_1} 2^{m-t_k} - 1).$$

Recall the definition of t_k (7) and note that $t_k \geq 1$. Moveover since n_k is strictly increasing, $t_{k+1} - t_k \geq 2$. As a result,

$$m_1 \geq 2^{m-1} - (2^{m-2} + 2^{m-4} + \cdots) - \frac{1}{2} \geq \frac{2^{m-1}}{3} - 1 \geq 2^{-c} \cdot 2^m, \tag{9}$$

for some $c > 0$. Remember also that all the elements in $\mathbb{F}_1$ are separated by a gap of length at least $\eta_1 = 2^{-n_m - 2m}$..

The following is an inductive step.

III. ***Realize (3) for $k \in \mathbb{N}_\ell$ and the ℓth level of the Cantor subset.***

One can use completely the same argument as in the first step to construct $\mathbb{F}_\ell$. Assume that $\mathbb{F}_{\ell-1}$ has been defined which is a collection of cylinders of order $n_m + \ell m$. For each $I_{n_m+\ell m}(w_0, \cdots, w_{\ell-1}) \in \mathbb{F}_\ell$, we choose $\Lambda_\ell \subset \Sigma^m$ to ensure the validity of (3) for all $k \in \mathbb{N}_\ell$ and (4) for $i = \ell$.

If $\mathbb{N}_\ell = \emptyset$, choose $\Lambda_\ell = \Sigma^m \setminus \{0^m\}$.

If $\mathbb{N}_\ell \neq \emptyset$, choose Λ_ℓ and Ξ_ℓ by the same method as we did in the first step with $(w_0, \cdots, w_{\ell-1})$ taking place the role of w_0.

Then define

$$\mathbb{F}_\ell\Big(I_{n_m+\ell m}(w_0, \cdots, w_{\ell-1})\Big)$$
$$= \bigcup I_{n_m+(\ell+1)m}(w_0, \cdots, w_{\ell-1}, \epsilon_{n_m+\ell m+1}, \cdots, \epsilon_{n_m+(\ell+1)m})$$

where the union is taken over all $(\epsilon_{n_m+\ell m+1}, \cdots, \epsilon_{n_m+(\ell+1)m}) \in \Xi_\ell$. Still with the same argument, we have that

$$m_\ell := \Xi_\ell = \frac{1}{2}\sharp\Lambda_\ell \geq 2^{-c} \cdot 2^m$$

Also the elements in $\mathbb{F}_\ell$ are separated by a gap at least $\eta_\ell = 2^{-n_m-(\ell+1)m}$.

The ℓth level of the Cantor is then defined as

$$\mathbb{F}_\ell = \bigcup_{I_{n_m+\ell m}(w_0,\cdots,w_{\ell-1}) \in \mathbb{F}_{\ell-1}} \mathbb{F}_\ell\Big(I_{n_m+\ell m}(w_0, \cdots, w_{\ell-1})\Big)$$

The Cantor subset. Finally, we obtain a nested sequence $\{\mathbb{F}_\ell\}_{\ell \geq 0}$ composed of cylinders. The desired Cantor set is obtained as

$$\mathbb{F}_\infty = \bigcap_{\ell \geq 0}^{\infty} \mathbb{F}_\ell,$$

which is a subset of E_m.

5.3. **Hausdorff dimension of $\mathbb{F}_\infty$.** In fact, m_ℓ is just the number of the elements in $\mathbb{F}_\ell$ which are contained in a common element in $\mathbb{F}_{\ell-1}$. Then gaps between the elements $I_{n_m+(\ell+1)m}(w_0, \cdots, w_\ell)$ in $\mathbb{F}_\ell$ are at least

$$2^{-(n_m+(\ell+1)m+m)}.$$

So by Proposition 1,

$$\dim_{\mathrm{H}} E_m \geq \dim_{\mathrm{H}} \mathbb{F}_\infty \geq \liminf_{\ell \to \infty} \frac{\ell m - \ell c}{(\ell+1)m + c + n_m} = 1 - \frac{c}{m}.$$

Hence

$$\dim_{\mathrm{H}} E \geq \dim_{\mathrm{H}} \bigcup_{m \geq 3} E_m \geq 1.$$

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