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ON A CONDITION OF STRONG PRECOMPACTNESS AND THE DECAY OF PERIODIC ENTROPY SOLUTIONS TO SCALAR CONSERVATION LAWS

Evgeny Yu. Panov

Novgorod State University
41 B. Sankt-Peterburgskaya
173003 Veliky Novgorod, Russia

Saint Petersburg State University
Universitetskaia nab. 7-9
199034 St. Petersburg, Russia

Abstract. We propose a new sufficient non-degeneracy condition for the strong precompactness of bounded sequences satisfying the nonlinear first-order differential constraints. This result is applied to establish the decay property for periodic entropy solutions to multidimensional scalar conservation laws.

1. **Introduction.** Let Ω be an open domain in $\mathbb{R}^n$. We consider the sequence $u_k(x)$, $k \in \mathbb{N}$, bounded in $L^\infty(\Omega)$, which converges weakly-$*$ in $L^\infty(\Omega)$ to some function $u(x)$: $u_k \underset{k\to\infty}{\rightharpoonup} u$. Now let $\varphi(x,u) \in L^2_{loc}(\Omega, C(\mathbb{R},\mathbb{R}^n))$ be a Caratheodory vector-function (i.e. it is continuous with respect to u and measurable with respect to x) such that the functions

$$\alpha_M(x) = \max_{|u|\le M} |\varphi(x,u)| \in L^2_{loc}(\Omega) \quad \forall M > 0 \tag{1}$$

(here and below $|\cdot|$ stands for the Euclidean norm of a finite-dimensional vector).

By $\theta(\lambda)$ we shall denote the Heaviside function:

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

Suppose that for every $p \in \mathbb{R}$ the sequence of distributions

$$\operatorname{div}_x \left[\theta(u_k - p)(\varphi(x,u_k) - \varphi(x,p))\right] \text{ is precompact in } W^{-1}_{d,loc}(\Omega) \tag{2}$$

for some $d > 1$. Recall that $W^{-1}_{d,loc}(\Omega)$ is a locally convex space of distributions $u(x)$ such that $uf(x)$ belongs to the Sobolev space W^{-1}_d for all $f(x) \in C_0^\infty(\Omega)$. The topology in $W^{-1}_{d,loc}(\Omega)$ is generated by the family of semi-norms $u \to \|uf\|_{W^{-1}_d}$, $f(x) \in C_0^\infty(\Omega)$.

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If the distributions $\operatorname{div}_x \varphi(x,k)$ are locally finite measures on Ω for all $k \in \mathbb{R}$, then the notion of entropy solutions (in Kruzhkov's sense) of the equation

$$\operatorname{div} \varphi(x,u) + \psi(x,u) = 0, \tag{3}$$

with a Caratheodory source function $\psi(x,u) \in L^1_{loc}(\Omega, C(\mathbb{R}))$, is defined, see [18] and [19] (in the latter paper the more general ultra-parabolic equations are studied). We underline that equations like (3) occur in various applications, for instance in heterogeneous media, and have been widely studied in recent years, see for example [12] and references therein.

As was shown in [19], assumption (2) is always satisfied for bounded sequences of entropy solutions of (3).

Our first result is the following strong precompactness property.

Theorem 1.1. *Suppose that for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the function $\lambda \to \xi \cdot \varphi(x,\lambda)$ is not constant in any vicinity of the point $u(x)$ (here and in the sequel "$\cdot$" denotes the inner product in $\mathbb{R}^n$). Then $u_k(x) \underset{k\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ (strongly).*

Theorem 1.1 extends the results of [18], where the strong precompactness property was established under the more restrictive non-degeneracy condition: *for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the function $\lambda \to \xi \cdot \varphi(x,\lambda)$ is not constant on nonempty intervals.*

The proof of Theorem 1.1 is based on a new localization principle for H-measure (with "continuous" indexes) corresponding to the sequence u_k, see Theorem 3.2 and its Corollary 2 below.

Using this theorem and results of [21], we will also derive the more precise criterion for the decay of periodic entropy solutions of scalar conservation laws

$$u_t + \operatorname{div}_x \varphi(u) = 0, \tag{4}$$

$u = u(t,x)$, $(t,x) \in \Pi = (0,+\infty) \times \mathbb{R}^n$. The flux vector $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u))$ is supposed to be merely continuous: $\varphi(u) \in C(\mathbb{R}, \mathbb{R}^n)$. Recall the definition of entropy solution to equation (4) in the Kruzhkov sense [9].

Definition 1.2. A bounded measurable function $u = u(t,x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (4) if for all $k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x [\operatorname{sign}(u-k)(\varphi(u) - \varphi(k))] \le 0 \tag{5}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$).

As usual, condition (5) means that for all non-negative test functions $f = f(t,x) \in C^1_0(\Pi)$

$$\int_\Pi [|u-k| f_t + \operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] dt dx \ge 0.$$

As was shown in [16] (see also [17]), an e.s. $u(t,x)$ always admits a strong trace $u_0 = u_0(x) \in L^\infty(\mathbb{R}^n)$ on the initial hyperspace $t = 0$ in the sense of relation

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \quad \text{in } L^1_{loc}(\mathbb{R}^n), \tag{6}$$

that is, $u(t,x)$ is an e.s. to the Cauchy problem for equation (4) with initial data

$$u(0,x) = u_0(x). \tag{7}$$

Remark 1. It was also established in [16, Corollary 7.1] that, after possible correction on a set of null measure, an e.s. $u(t,x)$ is continuous on $[0,+\infty)$ as a map $t \mapsto u(t,\cdot)$ into $L^1_{loc}(\mathbb{R}^n)$. In the sequel we will always assume that this property is satisfied.

Suppose that the initial function u_0 is periodic with a lattice of periods L, i.e., $u_0(x+e) = u_0(x)$ a.e. on $\mathbb{R}^n$ for every $e \in L$ (we will call such functions L-periodic). Denote by $\mathbb{T}^n = \mathbb{R}^n/L$ the corresponding n-dimensional torus, and by L' the dual lattice $L' = \{ \xi \in \mathbb{R}^n \mid \xi \cdot x \in \mathbb{Z} \ \forall x \in L \}$. In the case under consideration when the flux vector is merely continuous the property of finite speed of propagation for initial perturbation may be violated, which, in the multidimensional situation $n > 1$, may even lead to the nonuniqueness of e.s. to Cauchy problem (4), (7), see examples in [10, 11]. But for a periodic initial function $u_0(x)$, an e.s. $u(t,x)$ of (4), (7) is unique (in the class of all e.s., not necessarily periodic) and space-periodic, the proof can be found in [15]. It is also shown in [15] that the mean value of e.s. over the period does not depend on time:

$$\int_{\mathbb{T}^n} u(t,x)dx = I \doteq \int_{\mathbb{T}^n} u_0(x)dx, \tag{8}$$

where dx is the normalized Lebesgue measure on $\mathbb{T}^n$. The following theorem generalizes the previous results of [5, 21].

Theorem 1.3. *Suppose that*

$$\forall \xi \in L', \xi \neq 0 \quad \text{the function } u \to \xi \cdot \varphi(u) \\ \text{is not affine on any vicinity of } I. \tag{9}$$

Then

$$\lim_{t\to+\infty} u(t,\cdot) = I = \int_{\mathbb{T}^n} u_0(x)dx \quad \text{in } L^1(\mathbb{T}^n). \tag{10}$$

Moreover, condition (9) is necessary and sufficient for the decay property (10).

In the case $\varphi(u) \in C^2(\mathbb{R},\mathbb{R}^n)$ Theorem 1.3 was proved in [5]. As was noticed in [5, Remark 2.1], decay property (10) holds under the weaker regularity requirement $\varphi(u) \in C^1(\mathbb{R},\mathbb{R}^n)$ but under the more restrictive assumption that for each $\xi \in L'$ I is not an interior point of the closure of the union of all open intervals, over which the function $\xi \cdot \varphi'(u)$ is constant. Let us demonstrate that condition (9) is less restrictive than this assumption even in the case $\varphi(u) \in C^1(\mathbb{R},\mathbb{R}^n)$. Suppose that $n = 1$, $\varphi(u) \in C^1(\mathbb{R})$ is a primitive of the Cantor function, so that $\varphi'(u)$ is increasing, continuous, and maximal intervals, over which it remains constant, are exactly the connected component of the complement $\mathbb{R} \setminus K$ of the Cantor set $K \subset [0,1]$. Since K has the empty interior the assumption of [5] is never satisfied while (9) holds for each $I \in K$.

2. **Preliminaries.** We need the concept of measure valued functions (Young measures). Recall (see [6, 24]) that a measure-valued function on Ω is a weakly measurable map $x \mapsto \nu_x$ of Ω into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of ν_x means that for each continuous function $g(\lambda)$ the function $x \to \langle \nu_x, g(\lambda)\rangle \doteq \int g(\lambda)d\nu_x(\lambda)$ is measurable on Ω.

A measure-valued function ν_x is said to be bounded if there exists $M > 0$ such that $\operatorname{supp} \nu_x \subset [-M, M]$ for almost all $x \in \Omega$.

Measure-valued functions of the kind $\nu_x(\lambda)=\delta(\lambda-u(x))$, where $u(x)\in L^\infty(\Omega)$ and $\delta(\lambda-u^*)$ is the Dirac measure at $u^*\in\mathbb{R}$, are called *regular*. We identify these measure-valued functions and the corresponding functions $u(x)$, so that there is a natural embedding of $L^\infty(\Omega)$ into the set $\mathrm{MV}(\Omega)$ of bounded measure-valued functions on Ω.

Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem by L. Tartar [24].

Theorem 2.1. *Let $u_k(x)\in L^\infty(\Omega)$, $k\in\mathbb{N}$, be a bounded sequence. Then there exist a subsequence (we keep the notation $u_k(x)$ for this subsequence) and a bounded measure valued function $\nu_x\in\mathrm{MV}(\Omega)$ such that*

$$\forall g(\lambda)\in C(\mathbb{R})\quad g(u_k)\underset{k\to\infty}{\rightharpoonup}\langle\nu_x,g(\lambda)\rangle\quad \textit{weakly-}*\textit{ in } L^\infty(\Omega). \tag{11}$$

Besides, ν_x is regular, i.e., $\nu_x(\lambda)=\delta(\lambda-u(x))$ if and only if $u_k(x)\underset{k\to\infty}{\to}u(x)$ in $L^1_{loc}(\Omega)$ (strongly).

We will essentially use in the sequel the variant of H-measures with “continuous indexes” introduced in [13]. This variant extends the original concept of H-measure invented by L. Tartar [25] and P. Gerárd [7] and it appears to be an efficient tool in nonlinear analysis.

Suppose $u_k(x)$ is a bounded sequence in $L^\infty(\Omega)$. Passing to a subsequence if necessary, we can suppose that this sequence converges to a bounded measure valued function $\nu_x\in\mathrm{MV}(\Omega)$ in the sense of relation (11). We introduce the measures $\gamma_x^k(\lambda)=\delta(\lambda-u_k(x))-\nu_x(\lambda)$ and the corresponding distribution functions $U_k(x,p)=\gamma_x^k((p,+\infty))$, $u_0(x,p)=\nu_x((p,+\infty))$ on $\Omega\times\mathbb{R}$. Observe that $U_k(x,p),u_0(x,p)\in L^\infty(\Omega)$ for all $p\in\mathbb{R}$, see [13, Lemma 2]. We define the set

$$E=E(\nu_x)=\left\{\ p_0\in\mathbb{R}\ \mid\ u_0(x,p)\underset{p\to p_0}{\to}u_0(x,p_0)\ \text{ in } L^1_{loc}(\Omega)\ \right\}.$$

As was shown in [13, Lemma 4], the complement $\mathbb{R}\setminus E$ is at most countable and if $p\in E$ then $U_k(x,p)\underset{k\to\infty}{\rightharpoonup}0$ weakly-$*$ in $L^\infty(\Omega)$.

Let $F(u)(\xi)$, $\xi\in\mathbb{R}^n$, be the Fourier transform of a function $u(x)\in L^2(\mathbb{R}^n)$, $S=S^{n-1}=\{\ \xi\in\mathbb{R}^n\ \mid\ |\xi|=1\ \}$ be the unit sphere in $\mathbb{R}^n$. Denote by $u\to\overline{u}$, $u\in\mathbb{C}$ the complex conjugation.

The next result was established in [13, Theorem 3], [14, Proposition 2, Lemma 2].

Proposition 1. *(i) There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Omega\times S$ and a subsequence $U_r(x,p)=U_{k_r}(x,p)$ such that for all $\Phi_1(x),\Phi_2(x)\in C_0(\Omega)$ and $\psi(\xi)\in C(S)$*

$$\langle\mu^{pq},\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle=\lim_{r\to\infty}\int_{\mathbb{R}^n}F(\Phi_1U_r(\cdot,p))(\xi)\overline{F(\Phi_2U_r(\cdot,q))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi; \tag{12}$$

(ii) For any $p_1,\dots,p_l\in E$ the matrix $\{\mu^{p_ip_j}\}_{i,j=1}^l$ is Hermitian and nonnegative definite, that is, for all $\zeta_1,\dots,\zeta_l\in\mathbb{C}$ the measure

$$\sum_{i,j=1}^l\zeta_i\overline{\zeta_j}\mu^{p_ip_j}\ge 0.$$

We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the H-measure corresponding to the subsequence $u_r(x)=u_{k_r}(x)$.

As was demonstrated in [13], the H-measure $\mu^{pq} = 0$ for all $p, q \in E$ if and only if the subsequence $u_r(x)$ converges as $r \to \infty$ strongly (in $L^1_{loc}(\Omega)$).

We denote by $|\mu|$ the variation of a Borel measure μ (this is the minimal of nonnegative Borel measures ν such that $|\mu(A)| \le \nu(A)$ for all Borel sets A). Let $\mathrm{pr}_\Omega|\mu^{pq}|$ be the projection of the measure $|\mu^{pq}|$ on the domain Ω. By definition it is a nonnegative Borel measure determined by the relation $\mathrm{pr}_\Omega|\mu^{pq}|(A) = |\mu^{pq}|(A \times S)$ for any Borel set $A \subset \Omega$. Since $|U_k(x,p)| \le 1$, it readily follows from (12) and Plancherel's equality that $\mathrm{pr}_\Omega|\mu^{pq}| \le \mathrm{meas}$ for $p, q \in E$, where meas is the Lebesgue measure on Ω. This implies the representation $\mu^{pq} = \mu^{pq}_x dx$ (the disintegration of H-measures). More exactly, choose a countable dense subset $D \subset E$. The following statement was proved in [14, Proposition 3], see also [18, Proposition 3].

Proposition 2. *There exists a family of complex finite Borel measures $\mu^{pq}_x \in \mathrm{M}(S)$ in the sphere S with $p, q \in D$, $x \in \Omega'$, where Ω' is a subset of Ω of full measure, such that $\mu^{pq} = \mu^{pq}_x dx$, that is, for all $\Phi(x,\xi) \in C_0(\Omega \times S)$ the function*

$$x \to \langle \mu^{pq}_x(\xi), \Phi(x,\xi)\rangle = \int_S \Phi(x,\xi) d\mu^{pq}_x(\xi)$$

is Lebesgue-measurable on Ω, bounded, and

$$\langle \mu^{pq}, \Phi(x,\xi)\rangle = \int_\Omega \langle \mu^{pq}_x(\xi), \Phi(x,\xi)\rangle dx.$$

Moreover, for $p, p', q \in D$, $p' > p$

$$\mathrm{Var}\, \mu^{pq}_x \doteq |\mu^{pq}_x|(S) \le 1 \quad \textit{and} \quad \mathrm{Var}\,(\mu^{p'q}_x - \mu^{pq}_x) \le 2\left(\nu_x((p,p'))\right)^{1/2}. \tag{13}$$

We choose a non-negative function $K(x) \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\int K(x)dx = 1$ and set $K_m(x) = m^n K(mx)$ for $m \in \mathbb{N}$. Clearly, the sequence K_m converges in $\mathcal{D}'(\mathbb{R}^n)$ to the Dirac δ-function (that is, this sequence is an approximate unity). We define $\Phi_m(x) = (K_m(x))^{1/2}$. As was shown in [14, Remark 4] (see also [18, Remark 2(b)]), the measures μ^{pq}_x can be explicitly represented by the relation

$$\Phi(x)\langle \mu^{pq}_x, \psi(\xi)\rangle = \lim_{m\to\infty} \langle \mu^{pq}_x(y,\xi), \Phi(y)K_m(x-y)\psi(\xi)\rangle =$$

$$\lim_{m\to\infty}\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi\Phi_m U_r(\cdot,p))(\xi)\overline{F(\Phi_m U_r(\cdot,q))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi \tag{14}$$

for all $\psi(\xi) \in C(S)$, where $\Phi\Phi_m U_r(\cdot,p) = \Phi(y)\Phi_m(x-y)U_r(y,p)$, $\Phi_m U_r(\cdot,q) = \Phi_m(x-y)U_r(y,q)$, and $\Phi(y) \in L^2_{loc}(\Omega)$ be an arbitrary function such that x is its Lebesgue point.

From this representation (with $\Phi \equiv 1$) and Proposition 1(ii) it follows that for all $p_1, \ldots, p_l \in D$, $x \in \Omega'$, $\zeta_1, \ldots, \zeta_l \in \mathbb{C}$ the measure

$$\mu = \sum_{i,j=1}^{l} \zeta_i \overline{\zeta_j} \mu^{p_i p_j}_x \ge 0. \tag{15}$$

Indeed, for every nonnegative $\psi(\xi) \in C(S)$

$$< \mu(\xi), \psi(\xi) >= \lim_{m\to\infty} \left\langle \sum_{i,j=1}^{l} \zeta_i\overline{\zeta_j}\mu^{p_i p_j}(y,\xi), K_m(x-y)\psi(\xi)\right\rangle \ge 0.$$

This, in particular implies, that $\mu^{pp}_x \ge 0$, $\mu^{qp}_x = \overline{\mu^{pq}_x}$, and for every Borel set $A \subset S$

$$|\mu^{pq}_x|(A) \le (\mu^{pp}_x(A)\mu^{qq}_x(A))^{1/2} \tag{16}$$

(see [14, 18]). For completeness we provide below the simple proof of (16). In view of (15) (with l=2) the matrix $M = \begin{pmatrix} \mu_x^{pp}(A) & \mu_x^{pq}(A) \\ \mu_x^{qp}(A) & \mu_x^{qq}(A) \end{pmatrix}$ is Hermitian and nonnegative definite. Therefore,

$$\mu_x^{pp}(A)\mu_x^{qq}(A) - |\mu_x^{pq}(A)|^2 = \mu_x^{pp}(A)\mu_x^{qq}(A) - \mu_x^{pq}(A)\mu_x^{qp}(A) = \det M \geq 0.$$

By Young's inequality for any positive constant c and all Borel sets $A \subset S$

$$|\mu_x^{pq}(A)| \leq (\mu_x^{pp}(A)\mu_x^{qq}(A))^{1/2} \leq \frac{c}{2}\mu_x^{pp}(A) + \frac{1}{2c}\mu_x^{qq}(A).$$

Since $\mu = \frac{c}{2}\mu_x^{pp} + \frac{1}{2c}\mu_x^{qq}$ is nonnegative Borel measure, it follows from this inequality that the variation $|\mu_x^{pq}| \leq \mu$. This implies that

$$|\mu_x^{pq}|(A) \leq \frac{c}{2}\mu_x^{pp}(A) + \frac{1}{2c}\mu_x^{qq}(A) \quad \forall c > 0. \tag{17}$$

It is easily computed that

$$\inf_{c>0}\left(\frac{c}{2}\mu_x^{pp}(A) + \frac{1}{2c}\mu_x^{qq}(A)\right) = (\mu_x^{pp}(A)\mu_x^{qq}(A))^{1/2}$$

and (16) follows from (17).

3. **Localization principles and the strong precompactness property.**

Lemma 3.1. *For each $p, q \in \mathbb{R}$, $x \in \Omega'$ there exist one-sided limits in the space* $\mathrm{M}(S)$ *of finite Borel measures on S (with the standard norm* $\mathrm{Var}\,\mu$*):*

$$\mu_x^{p'q'} \to \mu_x^{pq+} \quad \text{as } (p', q') \to (p, q), \quad p', q' \in D, p' > p, q' > q,$$
$$\mu_x^{p'q'} \to \mu_x^{pq-} \quad \text{as } (p', q') \to (p, q), \quad p', q' \in D, p' < p, q' < q.$$

Moreover, $\mathrm{Var}\,\mu^{pq\pm} \leq 1$ *and for every Borel set $A \subset S$ and each $p_i \in \mathbb{R}$, $i = 1, \dots, l$ the matrices $\{\mu_x^{p_i p_j \pm}(A)\}_{i,j=1}^l$ are Hermitian and nonnegative definite, that is, the measures*

$$\sum_{i,j=1}^{l} \zeta_i \overline{\zeta_j} \mu_x^{p_i p_j \pm} \geq 0 \tag{18}$$

for all complex $\zeta_i \in \mathbb{C}$, $i = 1, \dots, l$.

Proof. Let $x \in \Omega'$, $p, q \in \mathbb{R}$, $p_1, q_1, p_2, q_2 \in D$, $p_2 > p_1 > p$, $q_2 > q_1 > q$. Then, in view of (13) and the equality $\mu_x^{qp} = \overline{\mu_x^{pq}}$,

$$\mathrm{Var}\,(\mu_x^{p_2q_2} - \mu_x^{p_1q_1}) \leq \mathrm{Var}\,(\mu_x^{p_2q_2} - \mu_x^{p_1q_2}) + \mathrm{Var}\,(\mu_x^{q_2p_1} - \mu_x^{q_1p_1}) \leq$$
$$2\nu_x((p_1,p_2)) + 2\nu_x((q_1,q_2)) \leq 2\nu_x((p,p_2)) + 2\nu_x((q,q_2)) \underset{(p_2,q_2)\to(p,q)}{\to} 0.$$

By the Cauchy criterion, this implies that there exists a limit μ_x^{pq+} in $\mathrm{M}(S)$ of the measures $\mu_x^{p'q'}$ as $(p', q') \to (p, q)$, $p', q' \in D$, $p' > p$, $q' > q$. Similarly, for each $p_1, q_1, p_2, q_2 \in D$ such that $p_2 < p_1 < p$, $q_2 < q_1 < q$

$$\mathrm{Var}\,(\mu_x^{p_2q_2} - \mu_x^{p_1q_1}) \leq 2\nu_x((p_2,p_1)) + 2\nu_x((q_2,q_1)) \leq$$
$$2\nu_x((p_2,p)) + 2\nu_x((q_2,q)) \underset{(p_2,q_2)\to(p,q)}{\to} 0,$$

which implies existence of a left-sided limit μ_x^{pq-} in $\mathrm{M}(S)$ of $\mu_x^{p'q'}$ as $(p', q') \to (p, q)$, $p', q' \in D$, $p' < p$, $q' < q$. By Proposition 2 $\mathrm{Var}\,\mu_x^{p'q'} \leq 1$, which implies in the limits as $p' \to p\pm$, $q' \to q\pm$ that $\operatorname{Var}\mu_x^{pq\pm} \le 1$. Finally, for every $p'_i \in D$, $\zeta_i \in \mathbb{C}$, $i = 1, \dots, l$ the measures

$$\sum_{i,j=1}^{l} \zeta_i \overline{\zeta_j} \mu_x^{p'_i p'_j \pm} \ge 0.$$

In the limits as $p'_i \to p_i\pm$ this implies (18). □

Corollary 1. *Let $p, q \in \mathbb{R}$, $x \in \Omega'$. Then for every Borel set $A \subset S$*

$$|\mu_x^{pq+}|(A) \le \left(\mu_x^{pp+}(A)\mu_x^{qq+}(A)\right)^{1/2}, \ |\mu_x^{pq-}|(A) \le \left(\mu_x^{pp-}(A)\mu_x^{qq-}(A)\right)^{1/2}. \quad (19)$$

Proof. Relations (19) follow from (18) in the same way as in the proof of inequality (16) above. □

Remark 2. By continuity of μ_x^{pq} with respect to variables $p, q \in D$, we see that for $p \in D$

$$\mu_x^{pq\pm} = \lim_{q' \to q\pm} \lim_{p' \to p\pm} \mu_x^{p'q'} = \lim_{q' \to q\pm} \mu_x^{pq'} \quad \text{in } \mathrm{M}(S).$$

Analogously, if $q \in D$, then

$$\mu_x^{pq\pm} = \lim_{p' \to p\pm} \mu_x^{p'q} \quad \text{in } \mathrm{M}(S).$$

If the both indices $p, q \in D$, then evidently $\mu_x^{pq\pm} = \mu_x^{pq}$.

Now we suppose that $f(x, \lambda) \in L^2_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$ is a Caratheodory vector-function on $\Omega \times \mathbb{R}$. In particular,

$$\forall M > 0 \quad \|f(x, \cdot)\|_{M,\infty} = \max_{|\lambda| \le M} |f(x, \lambda)| = \alpha_M(x) \in L^2_{loc}(\Omega). \quad (20)$$

Since the space $C(\mathbb{R}, \mathbb{R}^n)$ is separable with respect to the standard locally convex topology generated by seminorms $\|\cdot\|_{M,\infty}$, then, by the Pettis theorem (see [8, Chapter 3]), the map $x \to F(x) = f(x, \cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ is strongly measurable and in view of estimate (20) we see that $|F(x)|^2 \in L^1_{loc}(\Omega, C(\mathbb{R}))$. In particular (see again [8, Chapter 3]), the set Ω_f of common Lebesgue points of the maps $F(x), |F(x)|^2$ has full measure. As was demonstrated in [18], for $x \in \Omega_f$

$$\lim_{m \to \infty} \int K_m(x-y)\|F(x) - F(y)\|^2_{M,\infty} dy = 0 \quad \forall M > 0. \quad (21)$$

Clearly, each $x \in \Omega_f$ is a common Lebesgue point of all functions $y \to f(y, \lambda)$, $\lambda \in \mathbb{R}$. Let $\Omega'' = \Omega' \cap \Omega_f$, $\gamma_y^r(\lambda) = \delta(\lambda - u_r(y)) - \nu_y(\lambda)$, where $u_r(y) = u_{k_r}(y)$.

Suppose that $x \in \Omega''$, $p \in \mathbb{R}$, H_+, H_- are the minimal linear subspaces of $\mathbb{R}^n$, containing supports of the measures μ_x^{pp+}, μ_x^{pp-}, respectively. We fix $q \in D$ and introduce for $p' \in D$ the function

$$I_r(y, p') = \int f(y, \lambda)(\theta(\lambda - p') - \theta(\lambda - q)) d\gamma_y^r(\lambda) \in L^2_{loc}(\Omega, \mathbb{R}^n). \quad (22)$$

Proposition 3. *Assume that $q > p$ and $f(x, \lambda) \in H_+^\perp$ for all $\lambda \in \mathbb{R}$. Then*

$$\lim_{p' \to p+} \overline{\lim_{m \to \infty}} \, \overline{\lim_{r \to \infty}} \left| \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi \right| = 0 \quad (23)$$

for all $\psi(\xi)\in C(S)$. *Analogously, if* $q<p$ *and* $f(x,\lambda)\in H_-^\perp$ $\forall\lambda\in\mathbb{R}$, *then* $\forall\psi(\xi)\in C(S)$

$$\lim_{p'\to p-}\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right|=0. \tag{24}$$

Here $\Phi_m=\Phi_m(x-y)=\sqrt{K_m(x-y)}$ *and* $I_r(y,p'),U_r(y,p')$ *are functions of the variable* $y\in\Omega$.

Proof. Note that starting from some index m the supports of the functions $\Phi_m(x-y)$ lie in some compact subset B of Ω. Without loss of generality we can assume that $\operatorname{supp}\Phi_m\subset B$ for all $m\in\mathbb{N}$. Let

$$\tilde I_r(y,p')=\int f(x,\lambda)(\theta(\lambda-p')-\theta(\lambda-q))d\gamma_y^r(\lambda)\in L^2_{loc}(\Omega,\mathbb{R}^n),$$

$M=\sup\limits_{r\in\mathbb{N}}\|u_r\|_\infty$. Then $\operatorname{supp}\gamma_y^r\subset[-M,M]$, and

$$|I_r(y,p')-\tilde I_r(y,p')|\le\int|f(y,\lambda)-f(x,\lambda)|d|\gamma_y^r|(\lambda)\le 2\|F(y)-F(x)\|_{M,\infty}.$$

By Plancherel's identity

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi-\right.\\ &\left.\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m \tilde I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right|=\\ &\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m (I_r(\cdot,p')-\tilde I_r(\cdot,p')))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right|\le\\ &\|\psi\|_\infty\|\Phi_m(I_r(\cdot,p')-\tilde I_r(\cdot,p'))\|_2\|\Phi_m U_r(\cdot,p')\|_2\le\\ &\|\psi\|_\infty\|\Phi_m(I_r(\cdot,p')-\tilde I_r(\cdot,p'))\|_2\le\\ &2\|\psi\|_\infty\left(\int K_m(x-y)\|F(y)-F(x)\|^2_{M,\infty}dy\right)^{1/2}.\end{aligned}$$

Here we take account of the equality

$$\|\Phi_m\|_2=\left(\int_\Omega K_m(x-y)dy\right)^{1/2}=1.$$

From the above estimate and (21) it follows that

$$\begin{aligned}\lim_{m\to\infty}&\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi-\right.\\ &\left.\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m \tilde I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right|=0\end{aligned} \tag{25}$$

uniformly in r,p'. Observe that the function $\tilde f(\lambda)=f(x,\lambda)\in C(\mathbb{R},H_+^\perp)$ is continuous and does not depend on y. Therefore for any $\varepsilon>0$ there exists a piece-wise constant vector-valued function $g(\lambda)$ of the form $g(\lambda)=\sum\limits_{i=1}^k v_i\theta(\lambda-p_i)$, where $v_i\in H_+^\perp$, $p=p_1<p_2<\cdots<p_k=q$ such that $\|\tilde f\chi-g\|_\infty\le\varepsilon$ on $\mathbb{R}$. Here

$\chi(\lambda) = \theta(\lambda - p) - \theta(\lambda - q)$. Moreover, by the density of D, we may suppose that $p_i \in D$ for $i > 1$. We define for $p' \in D \cap (p, p_2)$

$$J_r(y, p') = \int g(\lambda)\theta(\lambda - p')d\gamma_y^r(\lambda).$$

Using again Plancherel's identity and the fact that

$$|\tilde{I}_r(y, p') - J_r(y, p')| = \left|\int (\tilde{f}\chi - g)(\lambda)\theta(\lambda - p')d\gamma_y^r(\lambda)\right| \le \int |(\tilde{f} \cdot \chi - g)(\lambda)|d|\gamma_y^r|(\lambda) \le 2\varepsilon,$$

we obtain

$$\left|\int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m \tilde{I}_r(\cdot, p'))(\xi)\overline{F(\Phi_m U_r(\cdot, p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi - \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m J_r(\cdot, p'))(\xi)\overline{F(\Phi_m U_r(\cdot, p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi\right| =$$

$$\left|\int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m(\tilde{I}_r(\cdot, p') - J_r(\cdot, p')))(\xi)\overline{F(\Phi_m U_r(\cdot, p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi\right| \le$$

$$\|\Phi_m(\tilde{I}_r(\cdot, p') - J_r(\cdot, p'))\|_2 \cdot \|\Phi_m U_r(\cdot, p')\|_2 \cdot \|\psi\|_\infty \le 2\|\psi\|_\infty\varepsilon \tag{26}$$

for all $\psi(\xi) \in C(S)$. Since

$$J_r(y, p') = \int \left(\sum_{i=1}^k v_i\theta(\lambda - p_i')\right) d\gamma_y^r(\lambda) = \sum_{i=1}^k v_i U_r(y, p_i'),$$

where $p_i' = \max(p_i, p') \in D$, it follows from (14) with account of Remark 2 that

$$\lim_{p'\to p+} \lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m J_r(\cdot, p'))(\xi)\overline{F(\Phi_m U_r(\cdot, p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi =$$

$$\lim_{p'\to p+} \sum_{i=1}^k \langle \mu_x^{p_i'p'}, (v_i \cdot \xi)\psi(\xi)\rangle = \sum_{i=1}^k \langle \mu_x^{p_ip+}, (v_i \cdot \xi)\psi(\xi)\rangle = 0. \tag{27}$$

The last equality is a consequence of the inclusion $\operatorname{supp} \mu_x^{p_ip+} \subset \operatorname{supp} \mu_x^{pp+} \subset H_+$ (because of Corollary 1) combined with the relation $v_i \perp H_+$. By (25), (26) and (27), we have

$$\overline{\lim_{p'\to p+}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}} \left|\int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot, p'))(\xi)\overline{F(\Phi_m U_r(\cdot, p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi\right| \le \text{const}\cdot\varepsilon,$$

and it suffices to observe that $\varepsilon > 0$ can be arbitrary to complete the proof of (23). The proof of relation (24) is similar to the proof of (23) and is omitted. □

Now we assume that the sequence u_k satisfies constraints (2). We choose a subsequence u_r and the corresponding H-measure $\mu^{pq} = \mu_x^{pq}dx$. Assume that $x \in \Omega'' = \Omega' \cap \Omega_\varphi$, $p_0 \in \mathbb{R}$. As above, let H_+, H_- be the minimal linear subspaces of $\mathbb{R}^n$ containing $\operatorname{supp} \mu_x^{p_0p_0+}$, $\operatorname{supp} \mu_x^{p_0p_0-}$, respectively.

Theorem 3.2 (localization principle)**.** *There exists a positive δ such that $(\varphi(x, \lambda) - \varphi(x, p)) \cdot \xi = 0$ for all $\xi \in H_+$, $\lambda \in [p_0, p_0 + \delta]$ and all $\xi \in H_-$, $\lambda \in [p_0 - \delta, p_0]$.*

Proof. The proof is analogous to the proof of [18, Theorem 4] (if $d = 2$), for arbitrary $d > 1$ see the proof of [19, Theorem 4] (where the more general case of ultra-parabolic constraints was treated). For completeness we provide the details. Observe firstly that in view of (2) the sequence of distributions

$$\mathcal{L}_p^r(y) = \mathrm{div}_y \left(\int \theta(\lambda - p)(\varphi(y,\lambda) - \varphi(y,p)) d\gamma_y^r(\lambda) \right) \underset{r\to\infty}{\to} 0 \text{ in } W^{-1}_{d,loc}(\Omega). \quad (28)$$

For $p, q \in D$, $q > p > p_0$ we consider the sequence of distributions

$$\mathcal{L}_q^r - \mathcal{L}_p^r = \mathrm{div}_y \, (Q_r^p(y)), \quad r \in \mathbb{N},$$

where the vector-valued functions $Q_r^p(y)$ (for fixed $q \in D$) are as follows:

$$\begin{aligned} Q_r^p(y) = \int (\varphi(y,\lambda) - \varphi(y,q))\theta(\lambda - q) d\gamma_y^r(\lambda) - \\ \int (\varphi(y,\lambda) - \varphi(y,p))\theta(\lambda - p) d\gamma_y^r(\lambda) = \\ \int (\varphi(y,q) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda) - \\ \int (\varphi(y,q) - \varphi(y,p))\theta(\lambda - p) d\gamma_y^r(\lambda) = \\ \int (\varphi(y,q) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda) - (\varphi(y,q) - \varphi(y,p)) U_r(y,p); \end{aligned} \quad (29)$$

here $\chi(\lambda) = \theta(\lambda - p) - \theta(\lambda - q)$ is the indicator function of the interval $(p, q]$. As was already noted, $\mathrm{div}_y \, (Q_r^p(y)) \underset{r\to\infty}{\to} 0$ in $W^{-1}_{d,loc}(\Omega)$ and if $\Phi(y) \in C_0^\infty(\Omega)$ then

$$\mathrm{div}_y \, (Q_r^p \Phi(y)) \underset{r\to\infty}{\to} 0 \text{ in } W_d^{-1}. \quad (30)$$

Using the Fourier transformation, from (30) we obtain

$$|\xi|^{-1}\xi \cdot F(Q_r^p \Phi)(\xi) = F(g_r), \quad g_r \underset{r\to\infty}{\to} 0 \text{ in } L^d(\mathbb{R}^n) \quad (31)$$

(see [18, 19] for details).

Let $\psi(\xi) \in C^\infty(S)$. By the known Marcinkiewicz multiplier theorem (cf. [23, Chapter 4]) $\psi(\xi/|\xi|)$ is a Fourier multiplier in L^s for all $s > 1$. This implies that

$$\overline{F(U_r(\cdot,p)\Phi)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) = \overline{F(h_r)}(\xi), \quad (32)$$

where the sequence h_r is bounded in $L^{d'}$, $d' = d/(d-1)$.

By (31), (32) we obtain

$$\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(Q_r^p \Phi)(\xi) \overline{F(U_r(\cdot,p)\Phi)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) = \int_{\mathbb{R}^n} g_r(x) \overline{h_r(x)} dx \to 0$$

as $r \to \infty$, or in view of (29),

$$\begin{aligned} \lim_{r\to\infty} \Big\{ \int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(U(\cdot,p) f \Phi)(\xi) \overline{F(U_r(\cdot,p)\Phi)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi - \\ \int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(V_r(\cdot,p)\Phi)(\xi) \overline{F(U_r(\cdot,p)\Phi)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi \Big\} = 0, \end{aligned} \quad (33)$$

where

$$f(y) = \varphi(y,q) - \varphi(y,p) \text{ and } V_r(y,p) = \int (\varphi(y,q) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda).$$

Obviously, (33) remains valid for merely continuous $\psi(\xi)$. We set in (33) $\Phi(y) = \Phi_m(x - y)$, where the functions Φ_m were defined in section 2, and pass to the limit as $m \to \infty$, $p \to p_0+$. By (14) with $\Phi(y) = \varphi(y,q) - \varphi(y,p)$ and Lemma 3.1, we obtain

$$\lim_{p\to p_0+}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(U_r(\cdot,p)f\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = \lim_{p\to p_0+}(\varphi(x,q)-\varphi(x,p))\cdot\langle\mu_x^{pp},\xi\psi(\xi)\rangle = (\varphi(x,q)-\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle,$$

therefore

$$(\varphi(x,q)-\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle = \lim_{p\to p_0+}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(V_r(\cdot,p)\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi. \quad (34)$$

Let π_1 and π_2 be the orthogonal projections of $\mathbb{R}^n$ onto the subspaces H_+ and $H_+^\perp$, respectively; let $\tilde\varphi(y,\lambda) = \pi_1(\varphi(y,\lambda))$, $\bar\varphi(y,\lambda) = \pi_2(\varphi(y,\lambda))$. Recall that H_+ is the smallest subspace containing $\operatorname{supp}\mu_x^{p_0p_0+}$. This readily implies that $\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle \in H_+$. Hence

$$(\varphi(x,q)-\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle = (\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle. \quad (35)$$

Further, $V_r(y,p) = \pi_1(V_r(y,p)) + \pi_2(V_r(y,p))$ and

$$\pi_1(V_r(y,p)) = \int(\tilde\varphi(y,q)-\tilde\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda),$$
$$\pi_2(V_r(y,p)) = \int(\bar\varphi(y,q)-\bar\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda).$$

Observe that

$$\pi_2(V_r(y,p)) = I_r(y,p),$$

where the function $I_r(y,p)$ is defined in (22) (with p' replaced by p) for a vector-function $f(y,\lambda) = \bar\varphi(y,q) - \bar\varphi(y,\lambda) \in H_+^\perp$. By Proposition 3 we obtain

$$\lim_{p\to p_0+}\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\left|\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\pi_2(V_r(y,p))\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| = 0. \quad (36)$$

Let $\tilde V_r(y,p) = \pi_1(V_r(y,p))$. From (34), in view of (35) and (36), we see that

$$|(\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle| \le \overline{\lim_{p\to p_0+}}\,\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\left|\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\tilde V_r(\cdot,p)\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right|,$$

which in turn, by Bunyakovskii inequality and Plancherel's equality, gives us the estimate

$$\left|(\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle\right| \le \overline{\lim_{p\to p_0+}}\,\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\|\tilde V_r(\cdot,p)\Phi_m\|_2\cdot\|U_r(\cdot,p)\Phi_m\|_2\cdot\|\psi\|_\infty \le \overline{\lim_{p\to p_0+}}\,\overline{\lim_{m\to\infty}}\,\overline{\lim_{r\to\infty}}\|\tilde V_r(\cdot,p)\Phi_m\|_2\cdot\|\psi\|_\infty. \quad (37)$$

Next, for $M_q(y) = \max_{\lambda\in[p_0,q]}|\tilde\varphi(y,q)-\tilde\varphi(y,\lambda)|$

$$|\tilde V_r(y,p)| \le M_q(y)\left|\int\chi(\lambda)d\left(\delta(\lambda-u_r(y))+\nu_y(\lambda)\right)\right| = M_q(y)(u_r(y,p)-u_r(y,q)+u_0(y,p)-u_0(y,q)),$$

where $u_r(y,\lambda) = \theta(u_r(y) - \lambda)$. In view of the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ and the relation $0 \le u_r(y,p) - u_r(y,q) \le 1$, $r \in \mathbb{N}\cup\{0\}$, we have

$$\|\tilde V_r(\cdot,p)\Phi_m\|_2^2 \le 2\int_\Omega (M_q(y))^2\big((u_r(y,p)-u_r(y,q))^2 + (u_0(y,p)-u_0(y,q))^2\big)K_m(x-y)dy \le 2\int_\Omega (M_q(y))^2(u_r(y,p)-u_r(y,q) + u_0(y,p)-u_0(y,q))K_m(x-y)dy. \quad (38)$$

Since $p,q \in D \subset E$, then

$$u_r(y,p)-u_r(y,q) = U_r(y,p)-U_r(y,q)+u_0(y,p)-u_0(y,q) \rightharpoonup u_0(y,p)-u_0(y,q)$$

as $r \to \infty$ in the weak-$*$ topology of $L^\infty(\Omega)$ and from (38) we now obtain the estimate

$$\overline{\lim_{r\to\infty}} \|\tilde V_r(\cdot,p)\Phi_m\|_2^2 \le 4\int_\Omega (M_q(y))^2(u_0(y,p)-u_0(y,q))K_m(x-y)dy,$$

from which, passing to the limit as $m \to \infty$, we obtain

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}} \|\tilde V_r(\cdot,p)\Phi_m\|_2^2 \le 4(M_q(x))^2(u_0(x,p)-u_0(x,q)). \quad (39)$$

Here we bear in mind that by the definition of Ω' (see, for instance, [18, Proposition 3]) x is a Lebesgue point of the functions $u_0(y,p)$, $u_0(y,q)$. It is also used that $x \in \Omega_\varphi$ is a Lebesgue point of the function $(M_q(y))^2$ as well (this easily follows from the fact that x is a Lebesgue point of the maps $y \to \varphi(y,\cdot)$, $y \to |\varphi(y,\cdot)|^2$ into the spaces $C(\mathbb{R},\mathbb{R}^n)$, $C(\mathbb{R})$, respectively). From (39) in the limit as $p \to p_0$ it follows that

$$\overline{\lim_{p\to p_0}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{r\to\infty}} \|\tilde V_r(\cdot,p)\Phi_m\|_2^2 \le 4(M_q(x))^2(u_0(x,p_0)-u_0(x,q)). \quad (40)$$

In view of (37) and (40),

$$|(\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle| \le 2\|\psi\|_\infty M_q(x)\omega(q), \quad (41)$$
$$\omega(q) = (u_0(x,p_0)-u_0(x,q))^{1/2} = (\nu_x(p_0,q])^{1/2} \underset{q\to p_0}{\to} 0.$$

It is clear that the set of vectors of the form $\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle$, with real $\psi(\xi) \in C(S)$ spans the subspace H_+. Hence we can choose functions $\psi_i(\xi) \in C(S)$, $i = 1,\dots,l$ such that the vectors $v_i = \langle\mu_x^{p_0p_0+},\xi\psi_i(\xi)\rangle$ make up an algebraic basis in H_+.

By (41), for $\psi(\xi) = \psi_i(\xi)$, $i = 1,\dots,l$, we obtain

$$|(\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot v_i| \le c_i\omega(q)M_q(x), \quad c_i = \text{const},$$

and since v_i, $i = 1,\dots,l$ is a basis in H_+, these estimates show that

$$|\tilde\varphi(x,q)-\tilde\varphi(x,p_0)| \le c\omega(q)M_q(x) = c\omega(q)\max_{\lambda\in[p_0,q]}|\tilde\varphi(x,q)-\tilde\varphi(x,\lambda)|, \quad c = \text{const}. \quad (42)$$

We take $q = p_0+\delta$, where $\delta > 0$ is so small that $2c\omega(q) = \varepsilon < 1$. Then, in view of (42),

$$|\tilde\varphi(x,q)-\tilde\varphi(x,p_0)| \le \frac{\varepsilon}{2}\max_{\lambda\in[p_0,q]}|\tilde\varphi(x,q)-\tilde\varphi(x,\lambda)|, \quad (43)$$

and since $\varphi(x,q)$ is continuous with respect to q and the set D is dense, the estimate (43) holds for all $q \in [p_0, p_0+\delta]$.

We claim that $\tilde{\varphi}(x,p) = \tilde{\varphi}(x,p_0)$ for $p \in [p_0, p_0+\delta]$. Indeed, assume that for $p' \in [p_0, p_0+\delta]$

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)|.$$

Then for $\lambda \in [p_0, p']$ we have

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \le |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| + \\ |\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \le 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|$$

and

$$\max_{\lambda\in[p_0,p']} |\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \le 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|.$$

We now derive from (43) with $q = p'$ that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \le \varepsilon|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|,$$

and since $\varepsilon < 1$, this implies that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| = 0.$$

We conclude that $\varphi(x,\lambda) - \varphi(x,p_0) \in H_+^\perp$ for all $\lambda \in [p_0, p_0+\delta]$, i.e., $(\varphi(x,\lambda) - \varphi(x,p_0)) \cdot \xi = 0$ on the segment $[p_0, p_0+\delta]$ for all $\xi \in H_+$.

To prove that for some sufficiently small $\delta > 0$ $(\varphi(x,\lambda) - \varphi(x,p_0)) \cdot \xi = 0$ on the segment $[p_0-\delta, p_0]$ for all $\xi \in H_-$, we take $p, q \in D$, $q < p < p_0$ and repeat the reasonings used in the first part of the proof. As a result, we obtain the relation similar to (41)

$$|(\tilde{\varphi}(x,q) - \tilde{\varphi}(x,p_0)) \cdot \langle \mu_x^{p_0p_0-}, \xi\psi(\xi)\rangle| \le 2\|\psi\|_\infty M_q(x)\omega(q),$$

where

$$M_q(x) = \max_{\lambda\in[q,p_0]} |\tilde{\varphi}(y,q) - \tilde{\varphi}(y,\lambda)|,$$

$$\omega(q) = \lim_{p\to p_0-} (u_0(x,q) - u_0(x,p))^{1/2} = (\nu_x(q,p_0))^{1/2} \underset{q\to p_0}{\to} 0.$$

This relation readily implies the desired statement $(\varphi(x,\lambda) - \varphi(x,p_0)) \cdot \xi = 0$ on the segment $[p_0-\delta, p_0]$ for all $\xi \in H_-$, where δ is sufficiently small.

The proof is complete. □

Corollary 2. *Let $x \in \Omega''$, $[a,b]$ be the minimal segment, containing* $\operatorname{supp} \nu_x$ *and $p_0 \in (a,b)$. Assume that $S_+, S_- \subset S$ are Borel sets such that*

$$\mu_x^{p_0p_0+}(S \setminus S_+) = \mu_x^{p_0p_0-}(S \setminus S_-) = 0. \tag{44}$$

Then $S_+ \cap S_- \ne \emptyset$. In particular, $\operatorname{supp} \mu_x^{p_0p_0+} \cap \operatorname{supp} \mu_x^{p_0p_0-} \ne \emptyset$ *and, in the notations of Theorem 3.2, for all $\xi \in H_+ \cap H_-$, $\xi \ne 0$ the function $\xi \cdot \varphi(x,\lambda)$ is constant in a vicinity of p_0.*

Proof. First, note that since $x \in \Omega'' \subset \Omega'$ is a Lebesgue point of the functions $u_0(\cdot,p)$ for all $p \in D$ while D is dense, the distribution function $u_0(x,\lambda) = \nu_x((\lambda,+\infty))$ is uniquely defined by the relation $u_0(x,\lambda) = \sup\limits_{p\in D, p>\lambda} u_0(x,p)$. In particular, the measure ν_x is well-defined at the point x.

The statement that the function $\lambda \to \xi \cdot \varphi(x,\lambda)$ is constant in a vicinity of p_0 for all $\xi \in H_+ \cap H_-$, $\xi \ne 0$ readily follows from the assertion of Theorem 3.2. Hence, we only need to show that $S_+ \cap S_- \ne \emptyset$ whenever Borel sets S_+, S_- satisfy condition (44). We assume to the contrary that $S_+ \cap S_- = \emptyset$. Denote $C_+ = S \setminus S_+$,

$C_- = S \setminus S_-$, Then $S = C_+ \cup C_-$, $\mu_x^{p_0p_0+}(C_+) = \mu_x^{p_0p_0-}(C_-) = 0$. Therefore, by relation (16), for all $p, q \in D$, $p < p_0 < q$

$$\operatorname{Var}\mu_x^{pq} = |\mu_x^{pq}|(S) \le |\mu_x^{pq}|(C_+) + |\mu_x^{pq}|(C_-) \le$$
$$(\mu_x^{pp}(C_+)\mu_x^{qq}(C_+))^{1/2} + (\mu_x^{pp}(C_-)\mu_x^{qq}(C_-))^{1/2} \le (\mu_x^{qq}(C_+))^{1/2} + (\mu_x^{pp}(C_-))^{1/2},$$

where we use that $\mu_x^{pp}(A) \le \mu_x^{pp}(S) \le 1$ for all $p \in D$ and every Borel set $A \subset S$, see (13). It follows from the obtained estimate and Lemma 3.1 that

$$\lim_{p\to p_0-}\lim_{q\to p_0+} \operatorname{Var}\mu_x^{pq} \le \left(\mu_x^{p_0p_0+}(C_+)\right)^{1/2} + \left(\mu_x^{p_0p_0-}(C_-)\right)^{1/2} = 0.$$

Thus,

$$\mu_x^{pq} \to 0 \quad \text{in } \mathrm{M}(S) \quad \text{as } p \to p_0-, q \to p_0+. \tag{45}$$

On the other hand, by (14)

$$\mu_x^{pq}(S) = \lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi_m U_r(\cdot,p))(\xi)\overline{F(\Phi_m U_r(\cdot,q))(\xi)}d\xi =$$
$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} U_r(y,p)U_r(y,q)K_m(x-y)dy. \tag{46}$$

Observe that $U_r(y,\lambda) = \theta(u_r(y) - \lambda) - u_0(y,\lambda)$. Since $U_r(\cdot,p) \underset{r\to\infty}{\rightharpoonup} 0$ for all $p \in D$ and $(\theta(u_r(y) - p) - 1)\theta(u_r(y) - q) \equiv 0$, we find

$$\lim_{r\to\infty}\int_{\mathbb{R}^n} U_r(y,p)U_r(y,q)K_m(x-y)dy =$$
$$\lim_{r\to\infty}\int_{\mathbb{R}^n} (U_r(y,p) - 1)U_r(y,q)K_m(x-y)dy =$$
$$\lim_{r\to\infty}\int_{\mathbb{R}^n} (\theta(u_r(y) - p) - 1 - u_0(y,p))(\theta(u_r(y) - q) - u_0(y,q))K_m(x-y)dy =$$
$$\lim_{r\to\infty}\int_{\mathbb{R}^n} [(1 - \theta(u_r(y) - p))u_0(y,q) - u_0(y,p)(\theta(u_r(y) - q) - u_0(y,q))]K_m(x-y)dy$$
$$= \int_{\mathbb{R}^n} (1 - u_0(y,p))u_0(y,q)K_m(x-y)dy.$$

In the limit as $m \to \infty$ this yields

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} U_r(y,p)U_r(y,q)K_m(x-y)dy =$$
$$\lim_{m\to\infty}\int_{\mathbb{R}^n} (1 - u_0(y,p))u_0(y,q)K_m(x-y)dy = (1 - u_0(x,p))u_0(x,q).$$

Here we take into account that x is a Lebesgue point of the functions $u_0(y,p)$, $u_0(y,q)$. By (45), (46) we find

$$0 = \lim_{p\to p_0-}\lim_{q\to p_0+} \mu_x^{pq}(S) =$$
$$\lim_{p\to p_0-}\lim_{q\to p_0+} (1 - u_0(x,p))u_0(x,q) = \nu_x((-\infty,p_0))\nu_x((p_0,+\infty)) > 0,$$

since $a < p_0 < b$ and $[a,b]$ is the minimal segment containing $\operatorname{supp}\nu_x$. The obtained contradiction implies that $S_+ \cap S_- \ne \emptyset$ and completes the proof. □

Remark 3. Let us consider the particular case $n = 2$, $\varphi(u) = (u, f(u))$ with $f(u) \in C(\mathbb{R})$. Let $\Omega \subset \mathbb{R}^2$ be an open plain domain. Suppose that a sequence

$u_k = u_k(t,x)$ converges weakly-$*$ in $L^\infty(\Omega)$ to a function $u = u(t,x)$ and satisfies restrictions (2), that is, for $p \in \mathbb{R}$ the sequences

$$((u_k - p)^+)_t + [\theta(u_k - p)(f(u_k) - f(p))]_x$$

are precompact in $W^{-1}_{d,loc}(\Omega)$ for some $d > 1$. Here we use the standard notation $v^+ = \theta(v)v = \max(v, 0)$.

Further, we assume that a subsequence $u_r = u_{k_r}$ of the sequence u_k converges as $r \to \infty$ to a bounded measure valued function $\nu_{t,x}$ in the sense of relation (11). Let $[a(t,x), b(t,x)]$ be the minimal segment containing $\operatorname{supp} \nu_{t,x}$. By Corollary 2, for a.e. $(t,x) \in \Omega$ and each $p \in (a(t,x), b(t,x))$ there is a nonzero vector $\xi = (\xi_1, \xi_2)$ such that the function $\xi_1 u + \xi_2 f(u) = \text{const}$ in a vicinity of p. This simply means that the function $f(u)$ is affine in a neighborhood of any point $p \in (a(t,x), b(t,x))$. Obviously, this implies that $f(u)$ is affine on the segment $[a(t,x), b(t,x)]$. Therefore, for a.e. $(t,x) \in \Omega$

$$\int f(\lambda) d\nu_{t,x}(\lambda) = f\left(\int \lambda d\nu_{t,x}(\lambda)\right) = f(u(t,x)).$$

In view of (11) we obtain that $f(u_r) \rightharpoonup f(u)$ as $r \to \infty$ weakly-$*$ in $L^\infty(\Omega)$. Since the limit function $f(u)$ does not depend on a choice of the subsequence u_r, we claim that the obtained limit relation actually holds for the original sequence as well:

$$f(u_k) \underset{k\to\infty}{\rightharpoonup} f(u) \quad \text{weakly-}* \text{ in } L^\infty(\Omega).$$

It follows from this relation that

$$(u_k)_t + f(u_k)_x \rightharpoonup u_t + f(u)_x \quad \text{in } \mathcal{D}'(\Omega)$$

as $k \to \infty$. For instance, if the sequence $(u_k)_t + f(u_k)_x$ weakly converges to zero in $\mathcal{D}'(\Omega)$, then the limit function $u(t,x)$ is a weak solution of equation

$$u_t + f(u)_x = 0. \tag{47}$$

In particular, this implies the known result that the weak limit of a sequence of entropy solutions of equation (47) is a weak solution of this equation. This result is usually proved with the help of compensated compactness theory, see [3, 4, 24]. Here we obtain it again relying only on the localization properties of H-measures. Observe also that, as it was established in [20], actually a weak limit of the sequence of entropy solutions of (47) is not only weak but also an entropy solution of this equation (in the case $\Omega = (0,T) \times \mathbb{R}$).

Now we are ready to prove Theorem 1.1.

Proof. Let $u_r = u_{k_r}$ be a subsequence of u_k chosen in accordance with Proposition 1. In particular, this subsequence converges to a measure-valued function $\nu_x \in \mathrm{MV}(\Omega)$. In view of (11) for a.e. $x \in \Omega$

$$u(x) = \int \lambda d\nu_x(\lambda). \tag{48}$$

We define the set of full measure $\Omega'' \subset \Omega$ and the minimal segment $[a(x), b(x)]$, containing $\operatorname{supp} \nu_x$, $x \in \Omega''$, as is required in Corollary 2. In view of (48) $u(x) \in (a(x), b(x))$ whenever $a(x) < b(x)$. By Corollary 2 the function $\xi \cdot \varphi(x, \cdot)$ is constant in a vicinity of $u(x)$ for some vector $\xi \neq 0$. But this contradicts to the assumption of Theorem 1.1. Therefore, $a(x) = b(x) = u(x)$ for a.e. $x \in \Omega$. This means that $\nu_x(\lambda) = \delta(\lambda - u(x))$. By Theorem 2.1 the subsequence $u_r \to u$ as $r \to \infty$ in $L^1_{loc}(\Omega)$. Finally, since the limit function $u(x)$ does not depend on the choice of a subsequence

u_r, we conclude that the original sequence $u_k \to u$ in $L^1_{loc}(\Omega)$ as $k \to \infty$. The proof is complete. □

4. **Decay property.** This section is devoted to the proof of Theorem 1.3. Suppose that $u(t,x)$ is a unique e.s. to problem (4), (7) with the periodic initial data $u_0(x)$. By Remark 1 we can assume that $u(t,x) \in C([0,+\infty), L^1(\mathbb{T}^n))$ (after possible correction on a set of null measure). We consider the sequence $u_k(t,x) = u(kt,kx)$, $k \in \mathbb{N}$, consisting of e.s. of (4). As was firstly shown in [2], the decay property (10) is equivalent to the strong convergence $u_r(t,x) \underset{r\to\infty}{\to} I = \text{const}$ in $L^1_{loc}(\Pi)$ of a subsequence $u_r = u_{k_r}(t,x)$. As follows from [21, Lemma 3.2(i)], $u_r \rightharpoonup u^*$, where $u^* = u^*(t)$ is a weak-$*$ limit of the sequence $a_0(k_r t)$, where $a_0(t) = \int_{\mathbb{T}^n} u(t,x)dx$. Since $u(t,x)$ is an e.s. of (4), this function is constant: $a_0(t) \equiv I = \int_{\mathbb{T}^n} u_0(x)dx$, in view of (8). Therefore, $u_r \rightharpoonup I$ as $r \to \infty$ (actually, the original sequence $u_k \rightharpoonup I$ as $k \to \infty$).

Let μ^{pq}, $p,q \in E$, be the H-measure corresponding to a subsequence $u_r = u_{k_r}(t,x)$. Recall that $\mu^{pq} = \mu^{pq}(t,x,\tau,\xi) \in \mathrm{M}_{loc}(\Pi \times S)$, where

$$S = \{\ \hat{\xi} = (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid |\hat{\xi}|^2 = \tau^2 + |\xi|^2 = 1\ \}$$

is a unit sphere in the dual space $\mathbb{R}^{n+1}$ (the variable τ corresponds to the time variable t).

By [21, Theorem 3.1] the following localization principle holds

$$\operatorname{supp} \mu^{pq} \subset \Pi \times S_0,$$

where

$$S_0 = \{\ \hat{\xi}/|\hat{\xi}| \mid \hat{\xi} = (\tau,\xi) \neq 0, \tau \in \mathbb{R},\ \xi \in L'\ \}.$$

As was demonstrated in Proposition 2, $\mu^{pq} = \mu^{pq}_{t,x} dtdx$ for all $p,q \in D$, where $D \subset E$ is a countable dense subset and measures $\mu^{pq}_{t,x} \in \mathrm{M}(S)$, are defined for all (t,x) belonging to a set of full measure $\Pi' \subset \Pi$. Obviously, the identity

$$\langle \mu^{pp}, \Phi(t,x,\hat{\xi})\rangle = \int_\Pi \langle \mu^{pp}_{t,x}(\hat{\xi}), \Phi(t,x,\hat{\xi})\rangle dtdx, \tag{49}$$

$\Phi(t,x,\hat{\xi}) \in C_0(\Pi \times S)$, remains valid also for compactly supported Borel functions Φ. Taking $\Phi = \phi(t,x)h(\hat{\xi})$, where $\phi(t,x) \in C_0(\Pi)$, $\phi(t,x) \geq 0$ while $h(\hat{\xi})$ is an indicator function of the set $S \setminus S_0$, we derive from (49) that

$$\int_\Pi \mu^{pp}_{t,x}(S \setminus S_0)\phi(t,x)dtdx = 0$$

and since $\mu^{pp}_{t,x} \geq 0$ and $\phi(t,x) \in C_0(\Pi)$ is arbitrary nonnegative function, it follows from this identity that $\mu^{pp}_{t,x}(S \setminus S_0) = 0$ for all $p \in D$, $(t,x) \in \Pi_0$, where $\Pi_0 \subset \Pi'$ is a subset of full measure. By relation (16) we claim that, more generally, $|\mu^{pq}_{t,x}|(S \setminus S_0) = 0$ for all $p,q \in D$, $(t,x) \in \Pi_0$. Finally, in view of Lemma 3.1, we find that

$$|\mu^{pq\pm}_{t,x}|(S \setminus S_0) = 0\ \forall p,q \in \mathbb{R}, (t,x) \in \Pi_0. \tag{50}$$

Further, $u_r(t,x)$ is a sequence of entropy solutions of (4). Therefore (see for instance [19]) the sequences

$$\operatorname{div}\,[\theta(u_r - p)(\hat{\varphi}(u_r) - \hat{\varphi}(p))] = ((u_r - p)^+)_t + \operatorname{div}_x\,[\theta(u_r - p)(\varphi(u_r) - \varphi(p))]$$

are compact in $H^{-1}_{d,loc}(\Pi)$ for some $d > 1$ and all $p \in \mathbb{R}$, where $\hat{\varphi}(u) = (u, \varphi(u)) \in C(\mathbb{R}, \mathbb{R}^{n+1})$.

Denote by $\nu_{t,x} \in \mathrm{MV}(\Pi)$ the limit measure valued function for a sequence u_r, and by $[a(t,x), b(t,x)]$ the minimal segment containing $\operatorname{supp}\nu_{t,x}$.

Suppose that $(t,x) \in \Pi_0$, $a(t,x) < b(t,x)$. Then $I = \int \lambda d\nu_{t,x}(\lambda) \in (a(t,x), b(t,x))$. Taking (50) into account, we see that the Borel sets $S_\pm = \operatorname{supp}\mu^{II\pm}_{t,x} \cap S_0$ satisfy requirement (44). By Corollary 2 we find that $\operatorname{supp}\mu^{II+}_{t,x} \cap \operatorname{supp}\mu^{II-}_{t,x} \cap S_0 \neq \emptyset$. Therefore, there exist $\hat{\xi} = (\tau, \xi) \in \operatorname{supp}\mu^{II+}_{t,x} \cap \operatorname{supp}\mu^{II-}_{t,x} \cap S_0$ and $\delta > 0$ such that the function

$$\lambda \to \hat{\xi} \cdot \hat{\varphi}(\lambda) = \tau\lambda + \xi \cdot \varphi(\lambda) = c = \mathrm{const} \tag{51}$$

on the interval $V = \{\lambda \mid |\lambda - I| < \delta\}$. Since $\hat{\xi} \in S_0$, we can assume that $\xi \in L'$ in (51). Evidently, $\xi \neq 0$ (otherwise, $\tau\lambda \equiv c$ on V for $\tau \neq 0$). Hence the function $\xi \cdot \varphi(\lambda) = c - \tau\lambda$ is affine, which contradicts (9). Thus, $a(t,x) = b(t,x) = I$ for a.e. $(t,x) \in \Pi$. We conclude that $\nu_{t,x}(\lambda) = \delta(\lambda - I)$ and by Theorem 2.1 the sequence $u_r \to I$ as $r \to \infty$ strongly (in $L^1_{loc}(\Pi)$). As was mentioned above (one can simply repeat the conclusive part of the proof of Theorem 1.1 in [21]), this implies (10).

Conversely, if the assumption (9) fails, we can find $\xi \in L'$, $\xi \neq 0$, and constants $a, b \in \mathbb{R}$ such that $\xi \cdot \varphi(\lambda) \equiv a\lambda + b$ on a segment $[I - \delta, I + \delta]$, $\delta > 0$. Then, as is easily verified, the function

$$u(t,x) = I + \delta \sin(2\pi(\xi \cdot x - at))$$

is the e.s. of (4), (7) with initial data $u_0(x) = I + \delta\sin(2\pi(\xi \cdot x))$. It is clear that $u_0(x)$ is L-periodic and $\int_{\mathbb{T}^n} u_0(x)dx = I$, but the e.s. $u(t,x)$ does not satisfy the decay property.

Example. Let $n = 1$, $\varphi(u) = |u|$. Let $u = u(t,x)$ be an e.s. of the problem

$$u_t + (|u|)_x = 0, \quad u(0,x) = u_0(x), \tag{52}$$

where $u_0(x) \in L^\infty(\mathbb{R})$ is a nonconstant periodic function with a period l (for a constant $u_0 \equiv c$ the e.s. $u \equiv c$ and the decay property is evident). Notice that no previous results [2, 5, 21] can help to answer the question whether the decay property is satisfied. However, as follows from Theorem 1.3, if $I = \frac{1}{l}\int_0^l u_0(x)dx = 0$, then the decay property holds: $\int_0^l |u(t,x)|dx \to 0$ as $t \to \infty$. Actually, the condition $\int_0^l u_0(x)dx = 0$ is also necessary for the decay property (10). Indeed, $u(t,x) = u_0(x \mp t)$ if $\pm u_0(x) \geq 0$ (then $\pm I > 0$), and the decay property is evidently violated. In the remaining case when u_0 changes sign we define the functions $u_+(t,x) = v_+(x-t)$, $u_-(t,x) = v_-(x+t)$, where $v_+(x) = \max(u_0(x), 0) \geq 0$, $v_-(x) = \min(u_0(x), 0) \leq 0$. Note that these functions take zero values on sets of positive measures. By the construction, $v_-(x) \leq u_0(x) \leq v_+(x)$ and $u_\pm(t,x)$ are e.s. of (52) with initial data $v_\pm(x)$. In view of the known property of monotone dependence of e.s. on initial data $u_-(t,x) \leq u(t,x) \leq u_+(t,x)$ a.e. on Π. These inequality can be written in the form

$$u(t, x-t) \geq v_-(x), \quad u(t, x+t) \leq v_+(x). \tag{53}$$

Assuming that $u(t,x)$ satisfies the decay property, we find, with the help of x-periodicity of $u(t,\cdot)$, that

$$\int_0^l |u(t, x \pm t) - I|dx = \int_0^l |u(t,x) - I|dx \to 0 \text{ as } t \to +\infty,$$

that is, the functions $u(t, x \pm t) \underset{t\to+\infty}{\to} I$ in $L^1([0, l])$. Passing to the limit as $t \to +\infty$ in (53), we find that $v_-(x) \le I \le v_+(x)$ for a.e. $x \in \mathbb{R}$. The latter is possible only if $I = 0$. We conclude that the decay property holds only in the case $I = 0$.

Remark 4. Theorem 1.3 can be extended to more general case of almost periodic initial data (in the Besicovitch sense [1]). Repeating the arguments of [22], we arrive at the following analogue of Theorem 1.3.

Theorem 4.1. *Let M_0 be the additive subgroup of $\mathbb{R}^n$ generated by the spectrum of u_0. Assume that for all $\xi \in M_0$, $\xi \neq 0$ the function $\xi \cdot \varphi(\lambda)$ is not affine in any vicinity of $I = ⨍_{\mathbb{R}^n} u_0(x)$. Then the e.s. $u(t, x)$ of (4), (7) satisfies the decay property*

$$\lim_{t\to+\infty} ⨍_{\mathbb{R}^n} |u(t, x) - I| dx = 0.$$

Here $⨍_{\mathbb{R}^n} v(x)dx$ denotes the mean value of an almost periodic function $v(x)$ (see [1]).

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E-mail address: Eugeny.Panov@novsu.ru