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BV REGULARITY NEAR THE INTERFACE FOR NONUNIFORM CONVEX DISCONTINUOUS FLUX

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Abstract. In this paper, we discuss the total variation bound for the solution of scalar conservation laws with discontinuous flux. We prove the smoothing effect of the equation forcing the BV_{loc} solution near the interface for L^∞ initial data without the assumption on the uniform convexity of the fluxes made as in [1, 21]. The proof relies on the method of characteristics and the explicit formulas.

1. **Introduction.** Let us consider the following conservation laws with discontinuous flux,

$$\begin{aligned} u_t + F(x,u) &= 0 \\ u(x,0) &= u_0(x), \end{aligned} \tag{1}$$

the flux F is given by, $F(x,u) = H(x)f(u) + (1 - H(x))g(u)$, H is the Heaviside function. Throughout this paper we assume the fluxes f, g to be $C^2(\mathbb{R})$, strictly convex, superlinear growth (see definition 2.7) and $u_0 \in L^\infty$. We denote by f_+^{-1}, g_+^{-1} to be the inverses of the increasing part of f, g respectively and similarly f_-^{-1}, g_-^{-1}, the inverses of the decreasing part of f, g respectively. Let θ_f, θ_g be the unique minimums of the fluxes f and g respectively. By uniform convexity of the fluxes f, g, we mean $f, g \in C^2$ and there exists a positive constant α such that $f'' > \alpha,\ g'' > \alpha$.

The first order partial differential equation of type (1) has many applications namely, modeling gravity, continuous sedimentation in a clarifier-thickener unit, petroleum industry, traffic flow on a highway, semiconductor industry. For more details, one can see [13, 14, 15, 16, 17, 18, 24, 29] and the references therein.

It is well known that even if the flux and the initial data are sufficiently smooth the global classical solution of scalar conservation laws, does not exist always, which allows to define the weak notion of the solution. In general, there might be infinitely many weak solutions even for smooth flux. Moreover, the well-posedness theory for the Cauchy problem for scalar conservation laws with Lipschitz flux function was completely settled by Kruzkov [27].

In past few decades, the Cauchy problem for conservation laws with discontinuous flux of the type (1) has been well studied, it has been tackled in several ways. For convergence analysis of several important numerical schemes we refer to [5, 6, 8, 10, 11, 12, 14, 15, 32]. The solution of (1) can be achieved by vanishing viscosity

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limit [25, 9], front tracking ([13, 19, 20, 26]). On the other hand, one can obtain the solution of (1) also via Hamilton-Jacobi equation [4, 7, 30], which is the one we exploit in the present paper. Consider the following Hamilton-Jacobi equation

$$\begin{array}{lcl} v_t + g(v_x) & = & 0 \text{ if } x < 0, t > 0, \\ v_t + f(v_x) & = & 0 \text{ if } x > 0, t > 0, \\ v(x,0) & = & v_0 \text{ if } x \in \mathbb{R}. \end{array} \tag{2}$$

In [7], they dealt with the Hamilton-Jacobi equation (2) and proved the existence of the global Lipschitz solution of (2). Then $u = v_x$, solves the discontinuous conservation laws (1) with the initial data $v_{0_x} = u_0$. Also they proved the existence of infinitely many stable semigroups of entropy solutions based on (A, B) interface entropy condition (see definition 2.4). For any such choice of A, B, they have given an explicit representation of the entropy solution for the conservation law (1). Throughout our paper we use the (A, B) entropy solution (see definition 2.5) obtained as in [4, 7].

Despite the fact that the subject is well studied, the total variation bound near the interface remained unsolved for quite a long time. In [11], they observed that the solution stays in BV, away from the interface $x = 0$. Recently, in [1, 21] the question regarding the total variation bound near interface has been answered. In general, BV regularity of the solution is an extremely important phenomenon in the theory of conservation laws. For the case when $f = g$ (not necessarily convex), total variation diminishing (TVD) property holds, that is to say, $TV(u(\cdot,t)) \leq TV(u_0(\cdot))$, for all $t > 0$. Definitely, one cannot expect to have similar property in the case when $f \neq g$, because of the fact that constant initial data may lead to a non constant solution.

The first breakthrough results regarding the total variation bound have been obtained in [1]. First it was observed in [1] that the critical points plays a key role for the existence and nonexistence of total variation bound. It has been shown in [1] that if the connections (A, B) avoid the critical points (θ_g, θ_f) and $u_0 \in BV$, then the BV regularity holds. They observed in [1] that $u_0 \in BV$ is not enough, one needs to assume also $f_+^{-1}g(u_0), g_-^{-1}f(u_0) \in BV$ in order to prove that the solution is BV near the interface. Also they have constructed a counter example by rarefaction waves and shock waves separated by constant states near the interface to prove that total variation of the solution at time $t = 1$ may blow up even if $u_0 \in BV$. Hence the assumption $f_+^{-1}g(u_0), g_-^{-1}f(u_0) \in BV$ are important.

Later on in [21], it has been proved that if the connections (A, B) avoid the critical points (θ_g, θ_f) and $f'', g'' \geq \alpha > 0$, then $u \in BV$ for $u_0 \in L^\infty$. Also a very strong and surprising result has been obtained in [21] that if the lower heights of both the fluxes are same that is if $f(\theta_f) = g(\theta_g)$, then the solution stays in BV near the interface, even if the fluxes are not uniformly convex or even if $u_0 \notin BV$. On the other hand, if $f(\theta_f) \neq g(\theta_g)$, then the results does not hold in general (for instance the counter example in [1] at $t = 1$). If $f(\theta_f) \neq g(\theta_g)$, then one needs to put an extra assumption that Supp u_0 is compact, then the result holds true for uniformly convex flux, but for large time. One can not avoid to assume that the initial data u_0 is compactly supported due to the counter example (in [21]), which shows that even for uniformly convex fluxes, $u(x, T_n) \notin BV$ while $u_0 \in BV$, for all n and $\lim\limits_{n\to\infty} T_n = \infty$.

When $f = g$, we have the following Lax-Oleinik formula:

Theorem 1.1. *(Lax-Oleinik formula) Let the initial data $u_0 \in L^\infty(\mathbb{R})$. If the flux f is C^2, uniformly convex and of super linear growth. Then there exists a function $y(x,t)$ such that*

1. *$x \mapsto y(x,t)$ is non decreasing.*
2. *For a.e. $(x,t) \in \mathbb{R} \times (0,\infty)$, the solution u of (1) is given by*

$$u(x,t) = (f')^{-1}\left(\frac{x - y(x,t)}{t}\right). \tag{3}$$

An immediate observation from theorem 1.1 is the following:

If $f'' \geq \alpha > 0$ and then for all $t > 0$, $x \mapsto u(x,t)$ is in $BV_{loc}(\mathbb{R})$, even if $u_0 \notin BV$. When the flux is not uniformly convex, in the case $f = g$, BV regularity does not hold for $u_0 \in L^\infty$, even for large time (one can see a counter example in [2], though $f'(u) \in BV_{loc}$). For finer properties of characteristics in the case $f = g$, we refer to [2, 3, 22, 23, 28, 31].

When $f \neq g$, Lax-Oleinik type formula still holds:

Theorem 1.2. *(Adimurthi, Gowda [4], [7]) Let $u_0 \in L^\infty(\mathbb{R})$, then there exists an entropy solution u of (1) with $u \in L^\infty(\mathbb{R})$. Also there exist Lipschitz continuous functions $R(t) \geq 0, L(t) \leq 0$. and monotone functions $y_\pm(x,t)$, $t_\pm(x,t)$ such that*

For a fixed $t > 0$,

- *$t \geq t_+(x,t) \geq 0$ is a non-increasing function of x in $[0, R(t))$,*
- *$y_+(x,t) \geq 0$ is a non-decreasing function of x in $[R(t), \infty)$,*
- *$t \geq t_-(x,t) \geq 0$ is a non-decreasing function of x in $(L(t), 0]$,*
- *$y_+(x,t) \leq 0$ is a non-decreasing function of x in $(-\infty, L(t))$.*

$$u(x,t) = \begin{cases} (f')^{-1}\left(\frac{x - y_+(x,t)}{t}\right) & \text{if} \quad x \geq R(t), \\ (f')^{-1}\left(\frac{x}{t - t_+(x,t)}\right) & \text{if} \quad 0 \leq x < R(t). \end{cases} \tag{4}$$

$$u(x,t) = \begin{cases} (g')^{-1}\left(\frac{x - y_-(x,t)}{t}\right) & \text{if} \quad x \leq L(t), \\ (g')^{-1}\left(\frac{x}{t - t_-(x,t)}\right) & \text{if} \quad L(t) < x < 0. \end{cases} \tag{5}$$

One can ask the similar question as above that under which assumptions on f, g, u_0 can one expect to have $u \in BV_{loc}(-\infty, L(t))$, $u \in BV_{loc}(L(t), 0)$, $u \in BV_{loc}(0, R(t))$, $u \in BV_{loc}(R(t), \infty)$? In this present article we prove that $u \in BV_{loc}(L(t), 0)$, $u \in BV_{loc}(0, R(t))$ with nonuniform convex flux and $u_0 \in L^\infty$. that is to say we relax $f'', g'' \geq \alpha > 0$ and avoid $u_0 \in BV$ to prove the existence of the BV regularity near the interface. For the case when the connections avoid the critical points, we prove that the solution stays in BV near the interface, even if the flux is not uniformly convex (e.g. $f(u) = u^4, g(u) = u^6 + 1$) and even if the initial data is highly oscillatory ($u_0 \in L^\infty$). Also we prove that if one allows the connection to be the critical points, then similar results hold true without uniform convexity of fluxes and only with $u_0 \in L^\infty$, but for large time. The result in this paper is very surprising because, even if $f = g$, but not uniformly convex then in general there does not exist any region where the solution stays BV for L^∞ initial data, in other words, for $f = g$ and nonuniform convex fluxes one can always choose some $u_0 \in L^\infty$ such that $u(\cdot,t) \notin BV(K)$, where K is an interval in $\mathbb{R}$ and $t > 0$ (one can see the details in [2]). This happens due to the lack of Lipschitz continuity of $(f')^{-1}$. Here in this article we prove that near the interface i.e, in the region $(L(t), 0) \cup (0, R(t))$, either the solution does not oscillates too much or $(f')^{-1}, (g')^{-1}$ are Lipschitz continuous.

Note that for $x \in (L(t),0)\cup(0,R(t))$, the characteristic passing through the point (x,t), bended before time t if and only if $f \neq g$. The paper present a special case in which one can prove BV regularity for the solution near the interface (starting from L^∞ data) without enforcing the assumption on uniform convexity of the flux. However, the price to pay in order to relax the hypothesis on uniform convexity is not negligible. Our proof relies on the explicit Lax-Oleinik type formulas obtained in [4, 7] and the finer analysis of the characteristics curves [4, 7, 3].

In order to make the present article self contained, we describe enough prerequisites in section 2 before presenting the main results in section 3.

2. **Preliminaries.**

We recall some definitions and known results from [1, 3, 4, 7, 21].

Definition 2.1. Weak solution of (1): u is said to be weak solution of (1) if $u \in L^\infty_{\text{loc}}(\mathbb{R}\times\mathbb{R}_+)$ and it satisfies the following integral equality, for all $\phi \in C_0^\infty(\overline{\mathbb{R}\times\mathbb{R}_+})$

$$\int_0^\infty\int_{-\infty}^\infty \left(u\frac{\partial\phi}{\partial t} + (H(x)f(u) + (1-H(x)g(u))\frac{\partial\phi}{\partial x}\right)dxdt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0. \tag{6}$$

It is immediate to check that u satisfies (6) if and only if u satisfies the following in weak sense

$$\begin{array}{lll} u_t + g(u)_x & = 0 & \text{if } x<0, t>0,\\ u_t + f(u)_x & = 0 & \text{if } x>0, t>0,\\ u(x,0) & = u_0 & \text{if } x\in\mathbb{R}. \end{array} \tag{7}$$

Rankine-Hugoniot condition at interface: Let us denote $u(0+,t) = \lim_{x\to 0+} u(x,t)$ and $u(0-,t) = \lim_{x\to 0-} u(x,t)$. Then at $x=0$, u satisfies the following R-H condition

$$f(u(0+,t)) = g(u(0-,t)),\ \text{a.e. } t>0. \tag{8}$$

Definition 2.2. Interior entropy condition: A weak solution u of (7) is said to satisfy the interior entropy condition if

$$\begin{array}{l} \frac{\partial\phi_1(u)}{\partial t} + \frac{\partial\psi_1(u)}{\partial x} \le 0 \text{ for } x>0, t>0,\\ \frac{\partial\phi_2(u)}{\partial t} + \frac{\partial\psi_2(u)}{\partial x} \le 0 \text{ for } x<0, t>0, \end{array} \tag{9}$$

in the sense of distributions, where (ϕ_i,ψ_i) are the convex entropy pairs such that $(\psi_1'(u),\psi_2'(u)) = (\phi_1'(u)f'(u), \phi_2'(u)g'(u))$.

Definition 2.3. (**(A, B) Connection**): Let $(A,B)\in\mathbb{R}^2$, is called a connection if it satisfies the following
(i) $g(A) = f(B)$.
(ii) $g'(A)\le 0, f'(B)\ge 0$.

Definition 2.4. (**A, B**) **Interface entropy condition**: Let $u \in L^\infty_{\text{loc}}(\mathbb{R}\times\mathbb{R}_+)$ such that $u(0\pm,t)$ exist a.e. $t>0$. Define $I_{AB}(t)$ by

$$(g(u(0-,t))-g(A))\text{sign}(u(0-,t)-A)-(f(u(0+,t))-f(B))\text{sign}(u(0+,t)-B). \tag{10}$$

Then u is said to satisfy (A,B) Interface entropy condition if for a.e. $t>0$,

$$I_{AB}(t)\ge 0. \tag{11}$$

When $A = \theta_g$ or $B = \theta_f$, then (11) reduces to

$$\text{meas}\big\{t : f'(u(0+,t)) > 0, g'(u(0-,t)) < 0\big\} = 0. \tag{12}$$

Definition 2.5. $(\mathbf{A}, \mathbf{B})$ **Interface entropy solution**: $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ is said to be the (A, B) Interface entropy solution of (1), if it satisfy (6), (9) and (11).

Definition 2.6. Control curve: Let $0 \le t$ and $\gamma \in C([0,t], \mathbb{R})$. Then γ is called a control curve if the following holds.

(i). $\gamma(t) = x$.

(ii). γ consists of at most three linear curves and each segment lies completely in either $x \ge 0$ or $x \le 0$.

(iii). Let $0 = t_3 \le t_2 \le t_1 \le t_0 = t$ be such that for $i = 1, 2, 3, \quad \gamma_i = \gamma|_{[t_i, t_{i-1}]}$ be the linear parts of γ. If γ consists of three linear curves then $\gamma_2 = 0$ and $\gamma_1, \gamma_3 > 0$ or $\gamma_1, \gamma_3 < 0$.

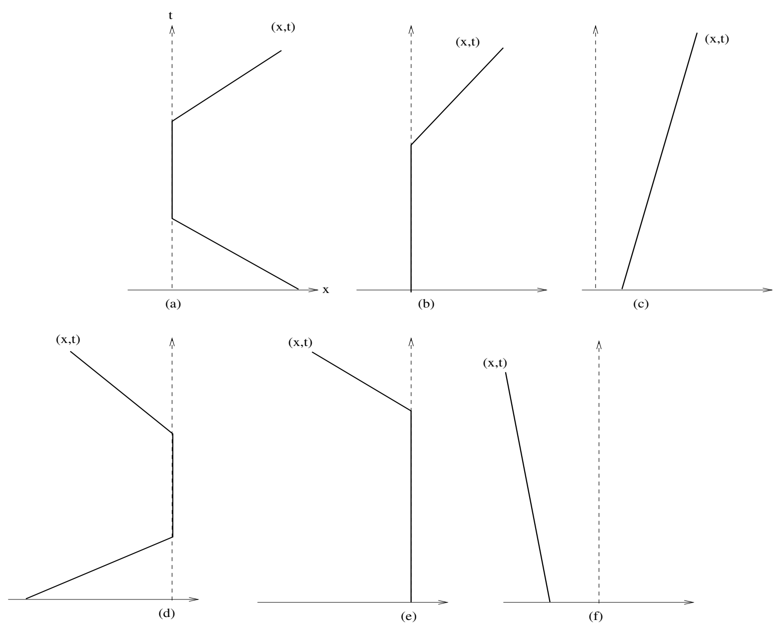

FIGURE 1. Control curves: fig (a), (b), (c) are positive control curves and fig (d), (e), (f) are negative control curves respectively.

Let us denote the set of control curves by $\Gamma(x,t)$.

Positive control curve: Let $x \ge 0$ then define the positive control curve $\Gamma_+(x,t)$ by

$$\Gamma_+(x,t) = \{\gamma \in \Gamma(x,t) \ : \ \gamma \ge 0\}.$$

Negative control curve: Let $x \le 0$ then define the negative control curve $\Gamma_-(x,t)$ by

$$\Gamma_-(x,t) = \{\gamma \in \Gamma(x,t) \ : \ \gamma \le 0\}.$$

Define $\bar{\Gamma}_\pm$ by

$$\bar{\Gamma}_\pm(x,t) = \big\{\gamma \in \Gamma_\pm(x,t) : \{s : \gamma(s) \neq 0\} = \text{the interval } (t_1, t), \text{ for some } t_1 \le t\big\}.$$

For simplicity, whenever we write the general flux h, it means that it is either f or g respectively in suitable sense.

Definition 2.7. Convexity, super linear growth, Legendre transformation and some useful facts: Let the flux h be a C^2 function and strictly convex, i.e., $\forall\ a, b \in \mathbb{R}$, and $\forall\ r \in [0,1]$, the following holds

$$h(ra + (1-r)b) < rh(a) + (1-r)h(b).$$

Also assume that the flux h satisfy superlinear growth property, i.e.,

$$\lim_{|u|\to\infty} \frac{h(u)}{|u|} = \infty.$$

Then one can define the Legendre transform h^* associated to h by

$$h^*(p) = \sup_q \{pq - h(q)\}.$$

It is easy to check that h^* enjoys the following properties

1. h^* is C^1 and strictly convex.
2. h^* has the superlinear growth property.
3. $h^{*\prime} = (h')^{-1}$.
4. $h = h^{**}$.
5. $h^*(h'(p)) = ph'(p) - h(p)$.
6. $h(h^{*'}(p)) = ph^{*'}(p) - h^*(p)$.
7. In addition, if $h'' > \alpha > 0$, for some α, then $(h')^{-1}$ is Lipschitz continuous with Lipschitz constant bounded by $\frac{1}{\alpha}$.

Definition 2.8. Cost functional: Let v_0 be the initial data for (2) and let h be the flux, then define the cost functionals as follows

$$J(\gamma, v_0, h) = v_0(\gamma(0)) + \int_0^t h^*(\gamma'(\theta))d\theta, \tag{13}$$

$$J_+(\gamma, v_0, h) = v_0(\gamma(0)) + \int_{\{t\,:\,\gamma(t)>0\}} h^*(\gamma'(\theta))d\theta, \tag{14}$$

$$J_-(\gamma, v_0, h) = v_0(\gamma(0)) + \int_{\{t\,:\,\gamma(t)<0\}} h^*(\gamma'(\theta))d\theta. \tag{15}$$

Let $v_0 \in C^1(\mathbb{R}\setminus\{0\}) \cup \mathrm{Lip}(\mathbb{R})$ and $v_0(0) = 0$. Define the following auxiliary functions $b_\pm$ by

$$b_+(t) := b_+(t, v_0, f) := \inf_{\{\gamma\in\Gamma_+(0,t)\}} J(\gamma, v_0, f) \tag{16}$$

$$b_-(t) := b_-(t, v_0, g) := \sup_{\{\gamma\in\Gamma_-(0,t)\}} J(\gamma, v_0, g). \tag{17}$$

Definition 2.9. Characteristics and other related functions: Let us define the set of characteristics $ch_\pm$ by

$$ch_\pm(t) \quad = \quad \{\gamma \in \Gamma_\pm(0,t)\ :\ b_\pm(t) = J(\gamma, u_0, h)\}. \tag{18}$$

Define $y_\pm, t_\pm$ by

$$
\begin{aligned}
&y_+(t,u_0,h) = \inf\{\gamma(0) \ : \ \gamma \in ch_+(t)\},\\
&y_-(t,u_0,h) = \sup\{\gamma(0) \ : \ \gamma \in ch_-(t)\},\\
&y_+(x,t) = \min\{\gamma(0) \ : \ \gamma \in ch_+(t)\},\\
&y_-(x,t) = \max\{\gamma(0) \ : \ \gamma \in ch_-(t)\},\\
&t_+(x,t) = \max\{t_1(\gamma) \ : \ \gamma \in ch_+(t)\},\\
&t_-(x,t) = \min\{t_1(\gamma) \ : \ \gamma \in ch_-(t)\},\\
&R(t) = \min\{x \ : \ t_+(x,t) = 0\},\\
&L(t) = \max\{x \ : \ t_-(x,t) = 0\},
\end{aligned}
\tag{19}
$$

where $t \mapsto y_-(t,u_0,h),\ t \mapsto y_+(t,u_0,h)$ are non-increasing and non-decreasing functions respectively. For a fix $t > 0,\ x \mapsto y_\pm(x,t),\ t_-(x,t)$ are non-decreasing functions and $x \mapsto t_+(x,t)$ is non-increasing function. The functions $t \mapsto R(t),\ t \mapsto L(t)$ are Lipschitz continuous functions and there exists some constant $C > 0$, such that $Ct \geq R(t) \geq 0,\ -Ct \leq L(t) \leq 0$.

Definition 2.10. Boundary data: Let us define the boundary data $\lambda_\pm$ by:

$$
\lambda_+(t) = \begin{cases} f_-^{-1}(-b_+'(t)) & \text{if } -b_+'(t) > \max(-b_-'(t), f(B)) \\ f_+^{-1}(\max(-b_-'(t), f(B))) & \text{if } -b_+'(t) \leq \max(-b_-'(t), f(B)) \end{cases} \tag{20}
$$

$$
\lambda_-(t) = \begin{cases} g_+^{-1}(-b_-'(t)) & \text{if } -b_-'(t) \geq \max(-b_+'(t), g(A)) \\ g_-^{-1}(\max(-b_+'(t), g(A))) & \text{if } -b_-'(t) < \max(-b_+'(t), g(A)). \end{cases} \tag{21}
$$

Definition 2.11. Value functions: Let us define the value functions $v_\pm$ by:

$$
v_-(x,t) = \sup_{\gamma \in \Gamma_-(x,t)} \left[J_-(\gamma, v_0, g) - \int_{\gamma=0} g(\lambda_-(\theta))d\theta \right] \text{ if } x \leq 0, \tag{22}
$$

$$
v_+(x,t) = \sup_{\gamma \in \Gamma_+(x,t)} \left[J_+(\gamma, v_0, f) - \int_{\gamma=0} f(\lambda_+(\theta))d\theta \right] \text{ if } x \geq 0. \tag{23}
$$

Theorem 2.12. *(Adimurthi et. al. [7]) Let the fluxes f, g be strictly convex, smooth and of superlinear growth. Then*

1. $f(\lambda_+(t)) = g(\lambda_-(t))$.
2. *The value functions $v_\pm$ are Lipschitz continuous and $v_+(0,t) = v_-(0,t)$, for all $t > 0$. v_+, v_- are the solution of (2).*
3. *Define the following Lipschitz continuous function v by*

$$
v(x,t) = \begin{cases} v_-(x,t) \ \textit{if } x < 0,\ t > 0, \\ v_+(x,t) \ \textit{if } x > 0,\ t > 0. \end{cases} \tag{24}
$$

Then $u = v_x$ is the weak entropy solution of (1) and satisfies the interface entropy condition (11) and the interior entropy condition (9).

Some useful formulas: The case when $A = \theta_g$ or $B = \theta_f$. At the points of differentiability of $t_\pm, y_\pm$, we have

$$
u(x,t) = \begin{cases} (f')^{-1}\left(\frac{x - y_+(x,t)}{t}\right) = u_0(y_+(x,t)) & \text{if } \quad 0 \leq R(t) < x < \infty, \\ (f')^{-1}\left(\frac{x}{t - t_+(x,t)}\right) & \text{if } \quad 0 \leq x < R(t) \\ = u(0+, t_+(x,t)) & \\ = f_+^{-1} g(u(0-, t_+(x,t))) & \\ = f_+^{-1} g(g')^{-1}\left(\frac{-y_-(0-, t_+(x,t))}{t_+(x,t)}\right) & \\ = f_+^{-1} g(u_0(y_-(0-, t_+(x,t)))). & \end{cases} \tag{25}
$$

$$u(x,t)=\begin{cases}(g')^{-1}\left(\frac{x-y_-(x,t)}{t}\right)=u_0(y_-(x,t)) & \text{if}\quad -\infty<x\le L(t)\le 0,\\ (g')^{-1}\left(\frac{x}{t-t_-(x,t)}\right) & \text{if}\quad L(t)<x<0,\\ =u(0-,t_-(x,t)) & \\ =g_-^{-1}f(u(0+,t_-(x,t))) & \\ =g_-^{-1}f(f')^{-1}\left(\frac{-y_+(0+,t_-(x,t))}{t_-(x,t)}\right) & \\ =g_-^{-1}f(u_0(y_+(0+,t_-(x,t)))). & \end{cases} \tag{26}$$

The case when $A\neq\theta_g$ and $B\neq\theta_f$. At the points of differentiability of $t_\pm, y_\pm$, we have

$$u(x,t)=\begin{cases}(f')^{-1}\left(\frac{x-y_+(x,t)}{t}\right)=u_0(y_+(x,t) & \text{if}\quad 0\le R(t)<x<\infty,\\ (f')^{-1}\left(\frac{x}{t-t_+(x,t)}\right) & \text{if}\quad 0\le x<R(t)\\ =u(0-,t_+(x,t)) & \\ =\lambda_+(t_+(x,t)) & \\ =f_+^{-1}(\max(-b'_-(t_+(x,t)),f(B))). & \end{cases} \tag{27}$$

$$u(x,t)=\begin{cases}(g')^{-1}\left(\frac{x-y_-(x,t)}{t}\right)=u_0(y_-(x,t)) & \text{if}\quad -\infty<x\le L(t)\le 0,\\ (g')^{-1}\left(\frac{x}{t-t_-(x,t)}\right) & \text{if}\quad L(t)<x<0,\\ =u(0-,t_-(x,t)) & \\ =\lambda_-(t_-(x,t)) & \\ =g_-^{-1}(\max(-b'_+(t_-(x,t)),g(A))), & \end{cases} \tag{28}$$

where $b'_\pm$ satisfies the following

$$b'_-(t)=\begin{cases}-g\left((g')^{-1}\left(-\frac{y_-(t,u_0,g)}{t}\right)\right) & \text{if}\quad y_-(t,u_0,g)<0,\\ -g(\theta_g) & \text{if}\quad y_-(t,u_0,g)=0,\end{cases} \tag{29}$$

$$b'_+(t)=\begin{cases}-f\left((f')^{-1}\left(-\frac{y_+(t,u_0,g)}{t}\right)\right) & \text{if}\quad y_+(t,u_0,f)>0,\\ -f(\theta_f) & \text{if}\quad y_+(t,u_0,f)=0.\end{cases} \tag{30}$$

Theorem 2.13. *(Adimurthi et.al [1]). Let $u_0\in L^\infty(\mathbb{R})$ and u be the solution of (1). Let $t>0,\ \epsilon>0, M>\epsilon,\ I(M,\epsilon)=\{x:\epsilon\le|x|\le M\}$. Then*

(1). Suppose there exists an $\alpha>0$ such that $f''\ge\alpha,\ g''\ge\alpha$, then there exist $C=C(\epsilon,M,\alpha)$ such that

$$TV\big(u(\cdot,t),I(M,\epsilon)\big)\le C(\epsilon,M,t).$$

(2). Suppose $u_0\in BV$, and $T>0$. Then there exists $C(\epsilon,T)$ such that for all $0<t\le T$

$$TV\big(u(\cdot,t),|x|>\epsilon\big)\le C(\epsilon,t)\,TV(u_0)+4||u_0||_\infty.$$

(3). Let $u_0\in BV$, $T>0$ and $A\neq\theta_g$ and $B\neq\theta_f$. Then there exists $C>0$ such that for all $0<t\le T$,

$$TV\big(u(\cdot,t)\big)\le C\ \ TV(u_0)+6||u_0||_\infty.$$

(4). Let $u_0,\ f_+^{-1}(g(u_0)),\ g_-^{-1}(f(u_0))\in BV$, $T>0$ and $A=\theta_g$. Then for all $0<t\le T$,

$$TV\big(u(\cdot,t)\big)\le TV(u_0)+\max\big(TV(f_+^{-1}(g(u_0))),TV(g_-^{-1}(f(u_0)))\big)+6||u_0||_\infty.$$

(5). For a certain choice of fluxes f and g there exists $u_0 \in BV \cap L^\infty$ such that $TV(u(\cdot,1)) = \infty$ if $A = \theta_g$ or $B = \theta_f$.

Theorem 2.14. *(Ghoshal [21]). Let $u_0 \in L^\infty(\mathbb{R})$ and u be a solution of (1). Let $t > 0,\ \epsilon > 0,\ M > \epsilon$ and*

$$\begin{aligned} I(M) &= \{x : |x| < M\}, \\ I(R_1(t)) &= \{x > 0 : x < R_1(t)\},\ I(L_1(t)) = \{x < 0 : x > L_1(t)\}. \end{aligned}$$

(i). Let $f(\theta_f) \neq g(\theta_g)$ and $f'' \geq \alpha,\ g'' \geq \alpha$, for some $\alpha > 0$, also assuming the fact that Supp $u_0 \subset [-K, K]$, for some $K > 0$, then there exists a $T_0 > 0$ such that for all $t > T_0$,

$$TV\,(u(\cdot\,,t), I(M)) \leq C(M,t). \tag{31}$$

As a consequence we have, for all $t > T_0$,

$$TV(u(\cdot\,,t), \mathbb{R}) \leq C(t), \tag{32}$$

where $C(t), C(M,t) > 0$ are some constants.

(ii). Let $f(\theta_f) = g(\theta_g)$ then for all $t > 0$,

$$TV\,(u(\cdot\,,t), I(R_1(t)) \cup I(L_1(t))) \leq C(t). \tag{33}$$

In addition if $f'' \geq \alpha,\ g'' \geq \alpha$, for some $\alpha > 0$, then for all $t > 0$,

$$TV\,(u(\cdot\,,t), I(M)) \leq C(M,t). \tag{34}$$

As a consequence, if Supp $u_0 \subset [-K, K]$, for some $K > 0$, then for all $t > 0$,

$$TV(u(\cdot\,,t), \mathbb{R}) \leq C(t). \tag{35}$$

(iii). Let $f(\theta_f) = g(\theta_g)$ and $u_0 \in BV(\mathbb{R})$ then for all $t > 0$,

$$TV\,(u(\cdot\,,t)) \leq C(t)(TV(u_0) + 1) + 4\|u_0\|\infty. \tag{36}$$

(iv). Let $A \neq \theta_g$ and $B \neq \theta_f$. If $f'', g'' \geq \alpha > 0, u_0 \in L^\infty(\mathbb{R})$ then for all $t > 0$,

$$TV\,(u(\cdot\,,t), I(\epsilon) \cup I(M,\epsilon)) \leq C_1(\epsilon) + C_2(\epsilon, M, t). \tag{37}$$

As a consequence, if $u_0 \in BV(\mathbb{R})$ then for all $t > 0$,

$$TV\,(u(\cdot\,,t)) \leq C(\epsilon,t)(TV(u_0) + 1) + 4\|u_0\|\infty. \tag{38}$$

Counter example *(Ghoshal [21]): Let $f(u) = (u-1)^2 - 1,\ g(u) = u^2$. Then there exists an initial data $u_0 \in BV$ and a sequence T_n, such that $\lim\limits_{n\to\infty} T_n = \infty$ and*

$$TV\,(u(\cdot\,,T_n)) = \infty,\ \text{for all } n. \tag{39}$$

3. **Main results.**

In the following Theorems we have relaxed the assumptions of uniform convexity of the fluxes made in [1, 21] and allowed $u_0 \in L^\infty$. Let us denote $I(R(t)) = \{x > 0 : x < R(t)\},\ I(L(t)) = \{x < 0 : x > L(t)\}$.

Hypothesis on the fluxes $\boldsymbol{f}, \boldsymbol{g}$: Let the fluxes satisfy the following

***H*1**. f, g be C^2, strictly convex and of superlinear growth (see definition 2.7).

***H*2**. Either $f'' > \alpha > 0$, for some α or the zero of f' and f'' are the same, i.e., if there exists $p \in \mathbb{R}$ such that $f''(p) = 0$, then $f'(p) = 0$.

***H*3**. Either $g'' > \alpha > 0$, for some α or the zero of g' and g'' are the same.

Theorem 3.1. *Let $u_0 \in L^\infty$. Let the fluxes satisfy **H1**, **H2** and **H3** as above. Let the connection satisfies $A \neq \theta_g, B \neq \theta_f$, then*

$$TV(u(\cdot\,,t), I(R(t)) \cup I(L(t))) \leq C(t). \tag{40}$$

Theorem 3.2. *Let $u_0 \in L^\infty$. Let the fluxes satisfy **H1**, **H2** and **H3** as above. Let the connection satisfies $A = \theta_g$ or $B = \theta_f$. If $f(\theta_f) \neq g(\theta_g), f(\theta_f) \neq g(0), f(0) \neq g(\theta_g), f(0) \neq g(0)$ and Supp $u_0 \subset [-M, M]$, for some $M > 0$, then there exists $T > 0$ such that for all $t > T$,*

$$TV(u(\cdot, t), I(R(t)) \cup I(L(t))) \leq C(t). \tag{41}$$

Proof. (Proof of Theorem 3.1). We consider the following three cases.
Case 1: $R(t) > 0, L(t) = 0$.
Case 2: $R(t) = 0, L(t) < 0$.
Case 3: $R(t) > 0, L(t) < 0$.

Case 1. Since the characteristics speed is positive, for $x \in (0, R(t))$, we have

$$u(x,t) = (f')^{-1}\left(\frac{x}{t - t_+(x,t)}\right) \tag{42}$$

$$= \lambda_+(t_+(x,t)) \tag{43}$$

$$= f_+^{-1}(\max(-b'_-(t_+(x,t)), f(B))). \tag{44}$$

Now b'_- satisfies the following relations

$$b'_-(t_+(x,t)) = \begin{cases} -g\left((g')^{-1}\left(-\dfrac{y_-(t_+(x,t), u_0, g)}{t_+(x,t)}\right)\right) & \text{if} \quad y_-(t_+(x,t), u_0, g) < 0, \\ -g(\theta_g) & \text{if} \quad y_-(t_+(x,t), u_0, g) = 0, \end{cases} \tag{45}$$

where $t \mapsto y_-(t, u_0, g)$ is a non-increasing function.

Let us define $\epsilon(t) = \sup\{x > 0 \ : y_-(t_+(x,t), u_0, g) < 0\}$. If $\epsilon(t) = 0$, then from (43), (44) and (45), we have $u(x,t) = f_+^{-1}(\max(g(\theta_g), f(B))) =$ constant, for all $x \in (0, R(t))$, hence in $u \in BV(0, R(t))$. Now we assume that $\epsilon(t) > 0$. Due to the monotonicity of y_-, for all $x \in (0, \epsilon(t))$

$$y_-(t_+(0+,t), u_0, g) \leq y_-(t_+(x,t), t), u_0, g) \leq y_-(t_+(\epsilon(t), t), u_0, g) < 0. \tag{46}$$

First we prove the result in $(0, \epsilon(t))$ then in $(\epsilon(t), R(t))$. From the monotonicity of y_-, t_+, we conclude

$$\begin{aligned} \frac{-y_-(t_+(0+,t), u_0, g)}{t_+(\epsilon(t), t)} &\geq \frac{-y_-(t_+(x,t), u_0, g)}{t_+(\epsilon(t), t)} \\ &\geq \frac{-y_-(t_+(x,t), u_0, g)}{t_+(x,t)} \\ &\geq \frac{-y_-(t_+(\epsilon(t), t), u_0, g)}{t_+(x,t)} \\ &\geq \frac{-y_-(t_+(\epsilon(t), t), u_0, g)}{t} > 0. \end{aligned} \tag{47}$$

From (47), $\dfrac{-y_-(t_+(x,t), u_0, g)}{t_+(x,t)}$ is away from 0, hence

$$(g')^{-1} \text{ is Lipschitz continuous in } \left(\frac{-y_-(t_+(\epsilon(t), t), u_0, g)}{t}, \frac{-y_-(t_+(0+,t), u_0, g)}{t_+(\epsilon(t), t)}\right). \tag{48}$$

Let us choose a partition $\{x_i\}_{i=1}^N$ in $(0,\epsilon(t))$. In view of the fact that t_+, y_- are monotone and using (46), (47) to obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\left|\frac{-y_-(t_+(x_{i+1},t),u_0,g)}{t_+(x_{i+1},t)}+\frac{y_-(t_+(x_i,t),u_0,g)}{t_+(x_i,t)}\right| \\
&\leq \sum_{i=1}^{N}\frac{|y_-(t_+(x_{i+1},t),u_0,g)||t_+(x_i,t)-t_+(x_{i+1},t)|}{|t_+(x_i,t)t_+(x_{i+1},t)|} \\
&+\sum_{i=1}^{N}\frac{|t_+(x_{i+1},t)||y_-(t_+(x_{i+1},t),u_0,g)-y_-(t_+(x_i,t),u_0,g)|}{|t_+(x_i,t)t_+(x_{i+1},t)|} \\
&\leq \frac{|y_-(t_+(0+,t),u_0,g)||t-t_+(\epsilon(t),t)|}{|t_+(\epsilon(t),t)|^2}+\frac{t|y_-(t_+(0+,t),u_0,g)|}{|t_+(\epsilon(t),t)|^2} \\
&\leq \frac{2t|y_-(t_+(0+,t),u_0,g)|}{|t_+(\epsilon(t),t)|^2}.
\end{aligned} \tag{49}$$

Since $B\neq\theta_f$, we conclude $f_+(\lambda_+(t_+(x,t)))=(\max(-b'_-(t_+(x,t)),f(B)))\geq f(B)$ $>f(\theta_f)$, therefore

$$f_+^{-1} \text{ is Lipschitz continuous in the interval } \big(f(B),\sup_{x\in(0,R(t))}-b'_-(t_+(x,t))\big). \tag{50}$$

Since $f,g\in C^2$ using (42), (43), (44), (50), (49), we get

$$\begin{aligned}
&\sum_{i=1}^{N}|u(x_{i+1},t)-u(x_i,t)| \\
&=\sum_{i=1}^{N}\left|f_+^{-1}(\max(-b'_-(t_+(x_{i+1},t)),f(B)))-f_+^{-1}(\max(-b'_-(t_+(x_i,t)),f(B)))\right| \\
&\leq C(t)\sum_{i=1}^{N}\left|\max(-b'_-(t_+(x_{i+1},t)),f(B))-\max(-b'_-(t_+(x_i,t)),f(B))\right| \\
&\leq C(t)\sum_{i=1}^{N}\left|b'_-(t_+(x_{i+1},t))-b'_-(t_+(x_i,t))\right| \\
&=C(t)\sum_{i=1}^{N}\left|\frac{-y_-(t_+(x_{i+1},t),u_0,g)}{t_+(x_{i+1},t)}+\frac{y_-(t_+(x_i,t),u_0,g)}{t_+(x_i,t)}\right| \\
&\leq C(t)\frac{2t|y_-(t_+(0+,t),u_0,g)|}{|t_+(\epsilon(t),t)|^2}.
\end{aligned} \tag{51}$$

Hence $u\in BV(0,\epsilon(t))$. Next, we prove the results in $(\epsilon(t),R(t))$. Due to the monotonicity of t_+, for $x\in(\epsilon(t),R(t))$, we have

$$\begin{aligned}
&t_+(\epsilon(t),t)\geq t_+(x,t)\geq t_+(R(t)-,t) \\
&t-t_+(\epsilon(t),t)\leq t-t_+(x,t)\leq t-t_+(R(t)-,t)<t,
\end{aligned} \tag{52}$$

hence for some $C>0$,

$$\begin{aligned}
0<\frac{\epsilon(t)}{t}\leq\frac{\epsilon(t)}{t-t_+(R(t)-,t)}\leq\frac{x}{t-t_+(x,t)}\leq\frac{R(t)}{t-t_+(x,t)} &\leq \frac{R(t)}{t-t_+(\epsilon(t),t)} \\
&\leq \frac{Ct}{t-t_+(\epsilon(t),t)}.
\end{aligned} \tag{53}$$

Whence from (53), $\dfrac{x}{t-t_+(x,t)}$ is away from 0, which allows

$$(f')^{-1} \text{ is Lipschitz continuous in } \left[\frac{\epsilon(t)}{t}, \frac{Ct}{t-t_+(\epsilon(t),t)}\right] \tag{54}$$

and the Lipschitz constant depends on t. Let us choose a partition $\{x_i\}_{i=1}^N$ in $(\epsilon(t), R(t))$. Because of the monotonicity of t_+, (52) and (53) we observe

$$\begin{aligned}
&\sum_{i=1}^{N}\left|\frac{x_{i+1}}{t-t_+(x_{i+1},t)} - \frac{x_i}{t-t_+(x_i,t)}\right| \\
&\leq \sum_{i=1}^{N} \frac{t|x_{i+1}-x_i| + |x_{i+1}t_+(x_i,t) - x_i t_+(x_{i+1},t)|}{(t-t_+(x_{i+1},t))(t-t_+(x_i,t))} \\
&\leq \sum_{i=1}^{N} \frac{t|x_{i+1}-x_i| + |x_{i+1}t_+(x_i,t) - x_i t_+(x_{i+1},t)|}{(t-t_+(\epsilon(t),t))^2} \\
&\leq \sum_{i=1}^{N} \frac{t|x_{i+1}-x_i| + |x_{i+1}||t_+(x_i,t) - t_+(x_{i+1},t)| + t_+(x_{i+1},t)|x_{i+1}-x_i|}{(t-t_+(\epsilon(t),t))^2} \\
&\leq \frac{3R(t)t}{(t-t_+(\epsilon(t),t))^2} \\
&\leq \frac{3Ct^2}{(t-t_+(\epsilon(t),t))^2}.
\end{aligned} \tag{55}$$

(42), (54) and (55) yields

$$\begin{aligned}
&\sum_{i=1}^{N} |u(x_{i+1},t) - u(x_i,t)| \\
&= \sum_{i=1}^{N}\left|(f')^{-1}\left(\frac{x_{i+1}}{t-t_+(x_{i+1},t)}\right) - (f')^{-1}\left(\frac{x_i}{t-t_+(x_i,t)}\right)\right| \\
&\leq \sum_{i=1}^{N} C(t)\left|\frac{x_{i+1}}{t-t_+(x_{i+1},t)} - \frac{x_i}{t-t_+(x_i,t)}\right| \\
&\leq \frac{3CC(t)t^2}{(t-t_+(\epsilon(t),t))^2}.
\end{aligned} \tag{56}$$

Hence for the Case 1, $u(\cdot,t) \in BV(0,R(t))$.

Case 2 and Case 3. When $L(t) < 0$, one can prove the result exactly as $R(t) > 0$. Therefore both the cases follows exactly like as Case 1. Hence the Theorem. □

Proof. (Proof of Theorem 3.2). Let us assume that $g(\theta_g) > f(\theta_f)$. Denote $0 < \delta_1 = g(\theta_g) - f(\theta_f)$. We have to consider the following three cases.
Case 1: $R(t) > 0, L(t) = 0$.
Case 2: $R(t) = 0, L(t) < 0$.
Case 3: $R(t) > 0, L(t) < 0$.

Case 1. In this case,

$$u(x,t) = (f')^{-1}\left(\frac{x}{t-t_+(x,t)}\right) \text{ for } x \in (0,R(t)). \tag{57}$$

Let $\epsilon(t) > 0$ be a small number such that $\epsilon(t) < R(t)$. First we prove the result in $(0,\epsilon(t))$ then in $(\epsilon(t), R(t))$. Suppose there exist constants $T_0 > 0, C > 0$ such that

$$|t_+(x,t)| < C, \text{ for } t > T_0, x \in (0, R(t)). \tag{58}$$

Since for $t > T_0$, $\epsilon(t), t_+(x,t)$ are bounded, hence from (57), (58) and the fact that the characteristics speed is positive for $x \in (0, R(t))$ there exists a small $\delta_2 > 0$, such that

$$u(x,t) \in [\theta_f, \theta_f + \delta_2], \text{ for } t > T_0, x \in (0, \epsilon(t)). \tag{59}$$

One can re-choose T_0 large, $\delta_2 > 0$ small such that $f(\theta_f + \delta_2) < g(\theta_g)$ and (59) still holds. From (59), it is clear that for $t > T_0$,

$$u(0+,t) \in [\theta_f, \theta_f + \delta] \text{ and so } f(u(0+,t)) \in [f(\theta_f), g(\theta_g)) \tag{60}$$

Therefore from (60), it is easy to see that R-H condition $f(u(0+,t) = g(u(0-,t))$ does not hold for $t > T_0$, which is a contradiction. Hence (58) is false and so for $x \in (0, \epsilon(t))$,

$$\overline{\lim_{t\to\infty}}\, t_+(x,t) = \infty. \tag{61}$$

From the R-H condition and the explicit formulas we have

$$f(u(x,t)) = f(u(0+, t_+(x,t))) = g(u(0-, t_+(x,t))) \tag{62}$$

$$= g\left((g')^{-1}\left(-\frac{y_-(0-, t_+(x,t))}{t_+(x,t)}\right)\right) \tag{63}$$

$$= g(u_0(y_-(0-, t_+(x,t)))). \tag{64}$$

Now if for some $x_0 \in (0, \epsilon(t))$, $y_-(0-, t_+(x,t)) < -M$, then by the monotonicity of y_-, $y_-(0-, t_+(x,t)) < -M$ for all $x \in (0, x_0)$. Since Supp $u_0 \subset [-M, M]$ and (64), we obtain $u(x,t) = f^{-1}g(0)$. Hence choosing $\epsilon(t) = x_0$, one has $u(\cdot, t) \in BV(0, \epsilon(t))$.

Let us assume the case when

$$y_-(0-, t_+(x,t)) \in [-M, 0] \text{ for all } x \in (0, \epsilon(t)). \tag{65}$$

Using (61), (63) and (65), it is immediate that $-\dfrac{y_-(0-, t_+(x,t))}{t_+(x,t)} \to 0$ as $t \to \infty$. In view of the fact that the characteristic speed in positive in $(0, R(t))$, there exists a small $\delta_3 > 0$ and a large T_0 (re-choosing the previous one), such that for all $x \in (0, \epsilon(t)), t > T_0$

$$g\left((g')^{-1}\left(-\frac{y_-(0-, t_+(x,t))}{t_+(x,t)}\right)\right) \in [g(\theta_g), g(\theta_g) + \delta_3]. \tag{66}$$

Thanks to $g(\theta_g) > f(\theta_f)$ and (66), we get

$$f^{-1} \text{ is Lipschitz continuous in } [g(\theta_g), g(\theta_g) + \delta_3] \tag{67}$$

and the Lipschitz constant depends on t. Since y_+, t_+ are monotone functions we have the following relations

$$t_+(\epsilon(t), t) \le t_+(x,t) < t, \text{ for all } x \in (0, \epsilon(t)) \tag{68}$$

$$-M \le y_-(0-, t_+(x,t)) \le y_-(0-, t_+(\epsilon(t), t)) \le 0, \text{ for all } x \in (0, \epsilon(t)) \tag{69}$$

$$0 < -\frac{y_-(0-, t_+(\epsilon(t), t))}{t} < -\frac{y_-(0-, t_+(\epsilon(t), t))}{t_+(x,t)} \tag{70}$$

$$\le -\frac{y_-(0-, t_+(x,t))}{t_+(x,t)} \le -\frac{y_-(0-, t_+(x,t))}{t_+(\epsilon(t), t)} \le \frac{M}{t_+(\epsilon(t), t)}, \tag{71}$$

for all $x \in (0, \epsilon(t))$.

Therefore from (71), it is clear that for a fixed $t > T_0$, $-\dfrac{y_-(0-, t_+(x,t))}{t_+(x,t)}$ is away from 0, hence

$$(g')^{-1} \text{ is Lipschitz continuous in } \left(-\frac{y_-(0-, t_+(\epsilon(t),t))}{t}, \frac{M}{t_+(\epsilon(t),t)}\right) \tag{72}$$

and the Lipschitz constant depends on t. Let us choose a partition $\{x_i\}_{i=1}^N$ in $(0, \epsilon(t))$. In view of the fact that t_+, y_- are monotone and using (68), (69), (71), to obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \left| -\frac{y_-(0-, t_+(x_{i+1},t))}{t_+(x_{i+1},t)} + \frac{y_-(0-, t_+(x_i,t))}{t_+(x_i,t)} \right| \\
&\leq \sum_{i=1}^{N} \frac{|y_-(0-, t_+(x_{i+1},t))||t_+(x_i,t) - t_+(x_{i+1},t)|}{|t_+(x_i,t)t_+(x_{i+1},t)|} \\
&+ \sum_{i=1}^{N} \frac{|t_+(x_{i+1},t)||y_-(0-, t_+(x_{i+1},t)) - y_-(0-, t_+(x_i,t))|}{|t_+(x_i,t)t_+(x_{i+1},t)|} \\
&\leq \frac{M|t - t_+(\epsilon(t),t)|}{|t_+(\epsilon(t),t)|^2} + \frac{Mt}{|t_+(\epsilon(t),t)|^2} \\
&\leq \frac{2Mt}{|t_+(\epsilon(t),t)|^2}
\end{aligned} \tag{73}$$

As a results of (62), (63), (67), (72) and (73), we get

$$\begin{aligned}
&\sum_{i=1}^{N} |u(x_{i+1},t) - u(x_i,t)| \\
&= \sum_{i=1}^{N} \left| f^{-1}g\left((g')^{-1}\left(-\tfrac{y_-(0-,t_+(x_{i+1},t))}{t_+(x_{i+1},t)}\right)\right) - f^{-1}g\left((g')^{-1}\left(-\tfrac{y_-(0-,t_+(x_i,t))}{t_+(x_i,t)}\right)\right)\right| \\
&\leq C_1(t) \sum_{i=1}^{N} \left| g\left((g')^{-1}\left(-\tfrac{y_-(0-,t_+(x_{i+1},t))}{t_+(x_{i+1},t)}\right)\right) - g\left((g')^{-1}\left(-\tfrac{y_-(0-,t_+(x_i,t))}{t_+(x_i,t)}\right)\right)\right| \\
&\leq C_2 C_1(t) \sum_{i=1}^{N} \left| (g')^{-1}\left(-\tfrac{y_-(0-,t_+(x_{i+1},t))}{t_+(x_{i+1},t)}\right) - (g')^{-1}\left(-\tfrac{y_-(0-,t_+(x_i,t))}{t_+(x_i,t)}\right)\right| \\
&\leq C_3(t) C_2 C_1(t) \sum_{i=1}^{N} \left| \left(-\tfrac{y_-(0-,t_+(x_{i+1},t))}{t_+(x_{i+1},t)}\right) - \left(-\tfrac{y_-(0-,t_+(x_i,t))}{t_+(x_i,t)}\right)\right| \\
&\leq C_3(t) C_2 C_1(t) \tfrac{2M|t - t_+(\epsilon(t),t)|}{|t_+(\epsilon(t),t)|^2}.
\end{aligned} \tag{74}$$

Hence $u \in BV(0, \epsilon(t))$. Next, we prove the results in $(\epsilon(t), R(t))$. Due to the monotonicity of t_+, for $x \in (\epsilon(t), R(t))$, we have

$$\begin{aligned}
&t_+(\epsilon(t),t) \geq t_+(x,t) \geq t_+(R(t)-,t) \\
&t - t_+(\epsilon(t),t) \leq t - t_+(x,t) \leq t - t_+(R(t)-,t) < t.
\end{aligned} \tag{75}$$

Whence

$$0 < \frac{\epsilon(t)}{t} < \frac{x}{t - t_+(x,t)} < \frac{C_4 t}{t - t_+(\epsilon(t),t)}, \tag{76}$$

for some constant $C_4 > 0$. For a fix $t > T_0$, it is clear from (76) that $\dfrac{x}{t - t_+(x,t)}$ is away from 0, which allows

$$(f')^{-1} \text{ is Lipschitz continuous in } \left[\frac{\epsilon(t)}{t}, \frac{C_4 t}{t - t_+(\epsilon(t),t)}\right] \tag{77}$$

and the Lipschitz constant depends on t. Let us choose a partition $\{x_i\}_{i=1}^N$ in $(\epsilon(t), R(t))$. Because of the monotonicity of t_+, (75) and (76) we observe

$$\begin{aligned}
&\sum_{i=1}^{N}\left|\frac{x_{i+1}}{t-t_+(x_{i+1},t)}-\frac{x_i}{t-t_+(x_i,t)}\right| \\
&\leq \sum_{i=1}^{N}\frac{t|x_{i+1}-x_i|+|x_{i+1}t_+(x_i,t)-x_it_+(x_{i+1},t)|}{(t-t_+(\epsilon(t),t))^2} \\
&\leq \sum_{i=1}^{N}\frac{t|x_{i+1}-x_i|+|x_{i+1}||t_+(x_i,t)-t_+(x_{i+1},t)|+t_+(x_{i+1},t)|x_{i+1}-x_i|}{(t-t_+(\epsilon(t),t))^2} \\
&\leq \frac{3R(t)t}{(t-t_+(\epsilon(t),t))^2} \\
&\leq \frac{3C_4t^2}{(t-t_+(\epsilon(t),t))^2}
\end{aligned} \tag{78}$$

(57), (77) and (78) yields

$$\begin{aligned}
&\sum_{i=1}^{N}|u(x_{i+1},t)-u(x_i,t)| \\
&=\sum_{i=1}^{N}\left|(f')^{-1}\left(\frac{x_{i+1}}{t-t_+(x_{i+1},t)}\right)-(f')^{-1}\left(\frac{x_i}{t-t_+(x_i,t)}\right)\right| \\
&\leq \sum_{i=1}^{N}C_5(t)\left|\frac{x_{i+1}}{t-t_+(x_{i+1},t)}-\frac{x_i}{t-t_+(x_i,t)}\right| \\
&\leq \frac{3CC_5(t)t^2}{(t-t_+(\epsilon(t),t))^2}.
\end{aligned} \tag{79}$$

Hence for the Case 1, $u(\cdot,t)\in BV(0,R(t))$.

Case 2. For $x\in(L(t),0)$, we have the following

$$u(x,t) = (g')^{-1}\left(\frac{x}{t-t_-(x,t)}\right) \tag{80}$$

$$= u(0-,t_-(x,t)) \tag{81}$$

$$= g^{-1}f\left((f')^{-1}\left(-\frac{y_+(0+,t_-(x,t))}{t_-(x,t)}\right)\right) \tag{82}$$

$$= g^{-1}f\left((u_0(y_+(x,t))\right). \tag{83}$$

We consider the following four subcases.

Subcase I. For all $x\in(L(t),0)$, and for all $t>T_0$, $y_+(0+,t_-(x,t))\leq M$, $t_-(x,t)<C$, for some constant $C>0$.

Subcase II. For some $x_0\in(L(t),0)$, $y_+(0+,t_-(x_0,t))>M$ and for all $t>T_0$,for all $x\in(L(t),0)$, $t_-(x,t)<C$, for some constant $C>0$.

Subcase III. For some $x_0\in(L(t),0)$, for all $t>T_0$, $y_+(0+,t_-(x_0,t))>M$ and for $x\in(L(t),0)$, $\overline{\lim\limits_{t\to\infty}}\, t_-(0-,t)=\infty$

Subcase IV. For all $x\in(L(t),0)$, for all $t>T_0$, $y_+(0+,t_-(x,t))\leq M$ and $\overline{\lim\limits_{t\to\infty}}\, t_-(0-,t)=\infty$.

Subcase I. Since $t_-(x,t) < C$, whence for $t > T_0$, we obtain

$$u(0-,t) = \lim_{x\to 0-} u(x,t) = \lim_{x\to 0-} (g')^{-1}\left(\frac{x}{t - t_-(x,t)}\right) = (g')^{-1}(0) = \theta_g. \qquad (84)$$

In this Case 2, $R(t) = 0$. For $x > 0$, we have $u(x,t) = (f')^{-1}\left(\frac{x - y_+(x,t)}{t}\right)$.

If $y_+(0+,t) > M$ then due to the compact support of u_0, and monotonicity of y_+,

$$u(0+,t) = \lim_{x\to 0+} u(x,t) = \lim_{x\to 0+} 0 = 0. \qquad (85)$$

As a result of R-H condition, (84) and (85), we obtain $f(0) = g(\theta_g)$, which contradicts the assumption $f(0) \neq g(\theta_g)$.

If $y_+(0+,t) < M$, then there exists a large $T_0 > 0$ and a small $\delta > 0$ such that for $x \in (0, \epsilon(t))$

$$u(0+,t) \in [\theta_f - \delta, \theta_f]. \qquad (86)$$

Since $\delta > 0$ is small and $g(\theta_g) > f(\theta_f)$, (86) violets the R-H condition $f(u(0+,t)) = g(u(0-,t))$. Therefore the Subcase I, can never occur.

SubCase II. Since for some $x_0 \in (L(t), 0)$, $y_+(0+, t_-(x_0,t)) > M$. From the monotonicity of y_+ and Supp $u_0 \subset [-M, M]$, we deduce $u(x,t) = g^{-1}f(0)$, for $x \in (x_0, 0)$, therefore $u \in BV(x_0, 0)$. Then one can repeat the same argument as in (79) to prove $u \in BV(L(t), x_0)$.

Subcase III. Follows as in Subcase II.

Subcase IV. Since $y_+(0+, t_-(x,t)) \leq M$, for $x \in (L(t), 0)$, therefore from (82) and similarly as in (86), it contradicts the R-H condition. Therefore the Subcase IV, can never occur.

This proves Case 2.

Case 3. This case is not allowed due to the interface entropy condition (12).

Similarly one can repeat the arguments above for the case when $f(\theta_f) > g(\theta_g)$.

Hence the Theorem. □

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