

**A COUPLING BETWEEN A NON-LINEAR 1D  
COMPRESSIBLE-INCOMPRESSIBLE LIMIT AND THE 1D  
 $p$ -SYSTEM IN THE NON SMOOTH CASE**

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ABSTRACT. We consider two compressible immiscible fluids in one space dimension and in the isentropic approximation. The first fluid is surrounded and in contact with the second one. As the sound speed of the first fluid diverges to infinity, we present the proof of rigorous convergence for the fully non-linear compressible to incompressible limit of the coupled dynamics of the two fluids. A linear example is considered in detail, where fully explicit computations are possible.

**1. Introduction.** The compressible to incompressible limit is widely studied in the literature, see for instance the well known results [7, 8, 9, 10], the more recent [12], the review [11] and the references therein. The usual setting considers regular solutions, whose existence is proved only for a finite time, to the compressible equations in 2 or 3 space dimensions. As the Mach number vanishes, these solutions are proved to converge to the solutions of the incompressible Euler equations.

Consider for instance the isentropic Euler equations in the three dimensional space:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \bar{P}(\rho) = 0, \end{cases} \quad \begin{cases} \bar{P}(\rho) > 0, & \bar{P}'(\rho) > 0, \\ (t, x) \in [0, +\infty[ \times \mathbb{R}^3. \end{cases}$$

For smooth solutions, this system is equivalent to

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla \bar{P}(\rho) = 0. \end{cases}$$

The Mach number is the ratio between the speed of the particles and the sound speed and it can be introduced into the equations in at least two different ways [11].

First, following [9], since the incompressible limit can be understood as the limit as the Mach number tends to zero, one begins by rescaling the fluid velocity  $u \rightarrow \kappa u$  where  $\kappa$  is a small parameter that eventually converges to zero. In order to capture the motion of the particles traveling with a small speed of order of  $\kappa$  one needs a

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space–time rescaling,  $\frac{x}{t} \rightarrow \kappa \frac{x}{t}$  which allows to obtain the system, in the rescaled variables,

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\rho} \frac{1}{\kappa^2} \nabla \bar{P}(\rho) = 0. \end{cases}$$

Alternatively, the same system is considered in [8], but motivated by the following approach, see [8, 10]. Consider fluids having equations of states  $\bar{P}_\kappa(\rho)$ , parametrized by  $\kappa$ , such that the speed of sound  $\sqrt{P'_\kappa(\rho)} \rightarrow +\infty$  as  $\kappa \rightarrow 0$ :

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla \bar{P}_\kappa(\rho) = 0. \end{cases}$$

The two approaches coincide if the one parameter family of pressure laws  $\bar{P}_\kappa(\rho)$  satisfies

$$\bar{P}'_\kappa(\rho) = \frac{1}{\kappa^2} \bar{P}'(\rho), \quad (1)$$

where  $\bar{P}(\rho)$  is a given fixed pressure law.

In the incompressible limit, the density is constant in time and space. Therefore, it is convenient to use the pressure instead of the density as unknown function. Since  $\bar{P}'_\kappa(\rho) > 0$ , we can take the inverse function  $R_\kappa(p) = (\bar{P}_\kappa)^{-1}(p)$  and rewrite the system in the unknown  $p$ :

$$\begin{cases} \frac{R'_\kappa(p)}{R_\kappa(p)} [\partial_t p + u \cdot \nabla p] + \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{R_\kappa(p)} \nabla p = 0. \end{cases}$$

As  $\kappa \rightarrow 0$ ,  $P'_\kappa(\rho) \rightarrow +\infty$ , therefore  $R'_\kappa(p) \rightarrow 0$ , and  $R_\kappa(p) \rightarrow \bar{\rho}$ , where  $\bar{\rho}$  is the constant density at the incompressible limit. Formally, we get the incompressible equations

$$\begin{cases} \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\bar{\rho}} \nabla p = 0. \end{cases}$$

In [7, 8] this limit is proved to hold for smooth solutions and small times. The heart of the matter is finding energy estimates independent of the small parameter  $\kappa$ .

Here, we want to recover similar convergence results, in a 1D setting, but within the framework of *weak entropy solutions*, proved to exist for *all times*. In particular, convergence is proved globally in time and solutions are assumed merely **BV** in the space variable.

**2. Two immiscible fluids.** In a 1D setting, an incompressible fluid behaves like a solid since its speed is constant in space. Therefore, we consider two compressible immiscible fluids and let one of the two become incompressible. Hence, we deal with a free boundary problem, with the boundary between the fluids converging to a solid wall. Below, we consider a volume of a compressible inviscid fluid, say the *liquid*, that fills the segment  $[a(t), b(t)]$  and which is surrounded by another compressible fluid, say the *gas*, filling the rest of the real line (see Figure 1). We assume that the gas follows a fixed pressure law  $\bar{P}(\rho)$ , while for the liquid we assume a one parameter family of pressure laws  $\bar{P}_\kappa(\rho)$  such that  $\bar{P}'_\kappa(\rho) \rightarrow +\infty$  as  $\kappa \rightarrow 0$ . The total mass of the liquid is fixed:  $\int_{a(t)}^{b(t)} \rho(t, x) dx = m$ . Since the two fluids are

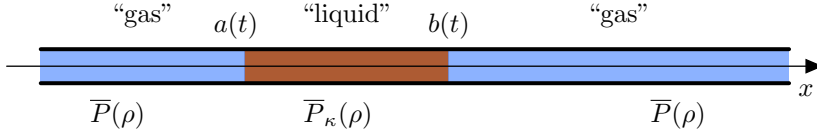


FIGURE 1. The two immiscible fluids: the liquid is in the middle, while the gas is on the two sides.

immiscible, Lagrangian coordinates are a natural choice:

$$z(t, x) = \int_{a(t)}^x \rho(t, \xi) d\xi, \quad \tau = \frac{1}{\rho}, \quad P(\tau) = \bar{P}\left(\frac{1}{\tau}\right), \quad (2)$$

with  $\tau$  being the specific volume. In these coordinates, the liquid occupies the fixed region  $[0, m]$  (see Figure 2).

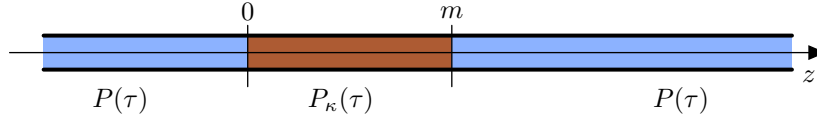


FIGURE 2. In Lagrangian coordinates, the boundaries separating the two fluids are fixed.

On  $P(\tau)$  and  $P_\kappa(\tau)$  we assume the usual hypotheses and the incompressible limit assumption:

$$P(\tau), P_\kappa(\tau) > 0; P'(\tau), P'_\kappa(\tau) < 0; P''(\tau), P''_\kappa(\tau) > 0; P'_\kappa(\tau) \xrightarrow{\kappa \rightarrow 0} -\infty. \quad (3)$$

In the isentropic approximation, the dynamics of the two fluids is described by the  $p$ -system

$$\begin{cases} \tau_t - v_z = 0 \\ v_t + P_\kappa(z, \tau)_z = 0, \end{cases} \quad P_\kappa(z, \tau) = \begin{cases} P_\kappa(\tau) & \text{for } z \in ]0, m[ \\ P(\tau) & \text{for } z \notin ]0, m[, \end{cases} \quad (4)$$

$v(t, z)$  being the speed of the fluids at the time  $t$  and at the Lagrangian coordinate  $z$ .

The Rankine–Hugoniot conditions for (4) applied at  $z = 0$  and  $z = m$  imply the following interface conditions for a.e.  $t \geq 0$ :

$$\begin{cases} v(t, 0-) = v(t, 0+) \\ P(\tau(t, 0-)) = P_\kappa(\tau(t, 0+)), \end{cases} \quad \begin{cases} v(t, m-) = v(t, m+) \\ P_\kappa(\tau(t, m-)) = P(\tau(t, m+)). \end{cases}$$

Remark that the pressure and the velocity have to be continuous across the interfaces. Hence, the pressure is a natural choice for the unknown, rather than the specific volume. Therefore, we introduce the inverse of the pressure laws:

$$\mathcal{T}(p) = P^{-1}(p), \quad \mathcal{T}_\kappa(p) = P_\kappa^{-1}(p), \quad \mathcal{T}'_\kappa(p) \xrightarrow{\kappa \rightarrow 0} 0 \quad (5)$$

and rewrite system (4) with  $(p, v)$  as unknowns

$$\begin{cases} \mathcal{T}_\kappa(z, p)_t - v_z = 0 \\ v_t + p_z = 0, \end{cases} \quad \mathcal{T}_\kappa(z, p) = \begin{cases} \mathcal{T}_\kappa(p) & \text{for } z \in ]0, m[ \\ \mathcal{T}(p) & \text{for } z \notin ]0, m[. \end{cases} \quad (6)$$

The conditions at the interfaces become continuity requirements on the unknown functions:

$$\begin{cases} v(t, 0-) = v(t, 0+) \\ p(t, 0-) = p(t, 0+) \end{cases} \quad \begin{cases} v(t, m-) = v(t, m+) \\ p(t, m-) = p(t, m+) \end{cases} \quad \text{for a.e. } t \geq 0. \quad (7)$$

This choice significantly simplifies the study of the Riemann problem at the interfaces.

Particular care is due to select the one parameter family of pressure laws, the main constraint being the validity of (1) for all  $\kappa$ . Indeed, (1) ensures that we recover the same equations obtained through scaling and studied in [7, 8]. The family  $\bar{P}_\kappa(\rho) = \frac{1}{\kappa^2} \bar{P}(\rho)$  chosen in [7] diverges to  $+\infty$  as  $\kappa \rightarrow 0$ . This is not a problem if one studies only one fluid as in [7] because the pressure enters the equations only through its derivative. In our case, the value of the pressure is very relevant, since it enters the interface conditions. Therefore, we cannot allow the pressure to grow nonphysically to  $+\infty$ . So we fix the density  $\bar{\rho}$  of the incompressible fluid in the limit and impose that the pressure at that particular density  $\bar{p}$  is a constant, independent of  $\kappa$ :

$$\bar{P}_\kappa(\bar{\rho}) = \bar{p}, \quad \text{for all } \kappa \in [0, 1[. \quad (8)$$

For simplicity, we choose both for the gas and for the liquid the same pressure law  $\bar{P} = \bar{P}(\rho)$ . For instance, an admissible choice is the usual  $\gamma$ -law  $\bar{P}(\rho) = k \rho^\gamma$  with  $\gamma \geq 1$ ,  $\gamma = 1$  corresponding to the case of an isothermal gas.

Conditions (1) and (8) imply the following expression for  $\bar{P}_\kappa(\rho)$ , with  $\bar{P}(\bar{\rho}) = \bar{p}$ :

$$\bar{P}_\kappa(\rho) = \bar{p} + \frac{1}{\kappa^2} [\bar{P}(\rho) - \bar{p}],$$

which, with the substitution  $\rho = \frac{1}{\tau}$ , becomes:  $P_\kappa(\tau) = \bar{p} + \frac{1}{\kappa^2} [P(\tau) - \bar{p}]$ . Finally, in term of the inverse functions (5),  $\mathcal{T}_\kappa = P_\kappa^{-1}$ ,  $\mathcal{T} = P^{-1}$ :

$$\mathcal{T}_\kappa(p) = \mathcal{T}(\bar{p} + \kappa^2(p - \bar{p})), \quad \lim_{\kappa \rightarrow 0} \mathcal{T}_\kappa(p) = \mathcal{T}(\bar{p}) = \frac{1}{\bar{\rho}} \doteq \bar{\tau}. \quad (9)$$

In [5], the following linear approximation of (9) is used:

$$\mathcal{T}(\bar{p} + \kappa^2(p - \bar{p})) \approx \mathcal{T}(\bar{p}) + \kappa^2 \mathcal{T}'(\bar{p})(p - \bar{p}) = \bar{\tau} + \kappa^2 \mathcal{T}'(\bar{p})(p - \bar{p}), \quad (10)$$

so that the liquid phase turns out to be governed by a linear system. This approximation makes all the estimates simpler. Here we follow [4] and study the Cauchy problem for the fully non linear system:

$$\begin{cases} \mathcal{T}_\kappa(z, p)_t - v_z = 0 \\ v_t + p_z = 0, \end{cases} \quad \mathcal{T}_\kappa(z, p) = \begin{cases} \mathcal{T}(\bar{p} + \kappa^2(p - \bar{p})) & \text{for } z \in ]0, m[ \\ \mathcal{T}(p) & \text{for } z \notin ]0, m[ \end{cases} \quad (11)$$

where  $\mathcal{T} = \mathcal{T}(p)$  is the inverse function of a pressure law, see (5), and satisfy (3).

**3. Existence results for small fixed  $\kappa > 0$ .** Colombo and Schleper in [6, Theorem 2.5] proved that for any fixed small  $\kappa > 0$ , there exists a Lipschitz semigroup of solutions to (11), but their estimates are not uniform with respect to  $\kappa$ . Therefore, as  $\kappa \rightarrow 0$  the Lipschitz constant of the semigroup could blow up and its domain could shrink and become trivial. Here, we provide a full set of new estimates either uniform in  $\kappa$ , or with the dependence on  $\kappa$  made explicit. To this aim, we substantially improve the wave front tracking construction in [3, 6], devising and exploiting a different parametrization of the Lax curves. Hence, throughout the following, by

$\mathcal{O}(1)$  we denote a quantity that depends only on the pressure law  $\mathcal{T}$  in (11) and on uniform bounds on the initial data.

**3.1. Lax curves.** We collect below a few facts about the  $p$ -system in Lagrangian coordinates. In the gas phase, the Lax curves in the  $(p, v)$  plane, expressed with the velocity as a function of the pressure, are given by

$$\mathcal{V}_1(p; p_o, v_o) = \begin{cases} v_o - \int_{p_o}^p \sqrt{-\mathcal{T}'(\xi)} d\xi & \text{for } p \leq p_o, \\ v_o - \frac{\sqrt{(\mathcal{T}(p) - \mathcal{T}(p_o))(p_o - p)}}{p_o - p} & \text{for } p \geq p_o, \end{cases}$$

$$\mathcal{V}_2(p; p_o, v_o) = \begin{cases} v_o - \frac{\sqrt{(\mathcal{T}(p) - \mathcal{T}(p_o))(p_o - p)}}{p_o - p} & \text{for } p \leq p_o, \\ v_o + \int_{p_o}^p \sqrt{-\mathcal{T}'(\xi)} d\xi & \text{for } p \geq p_o, \end{cases}$$

with the characteristics speeds given by

$$\lambda_1(p) = -\sqrt{-P'(\mathcal{T}(p))} = -\sqrt{-\frac{1}{\mathcal{T}'(p)}}, \quad \lambda_2(p) = \sqrt{-\frac{1}{\mathcal{T}'(p)}}.$$

similarly, in the liquid phase we have:

$$\mathcal{V}_{1,\kappa}(p; p_o, v_o) = \begin{cases} v_o - \int_{p_o}^p \sqrt{-\mathcal{T}'_{\kappa}(\xi)} d\xi & \text{for } p \leq p_o, \\ v_o - \frac{\sqrt{(\mathcal{T}_{\kappa}(p) - \mathcal{T}_{\kappa}(p_o))(p_o - p)}}{p_o - p} & \text{for } p \geq p_o, \end{cases} \quad (12)$$

$$\mathcal{V}_{2,\kappa}(p; p_o, v_o) = \begin{cases} v_o - \frac{\sqrt{(\mathcal{T}_{\kappa}(p) - \mathcal{T}_{\kappa}(p_o))(p_o - p)}}{p_o - p} & \text{for } p \leq p_o, \\ v_o + \int_{p_o}^p \sqrt{-\mathcal{T}'_{\kappa}(\xi)} d\xi & \text{for } p \geq p_o, \end{cases}$$

$$\lambda_{1,\kappa}(p) = -\sqrt{-\frac{1}{\mathcal{T}'_{\kappa}(p)}} \xrightarrow{\kappa \rightarrow 0} -\infty, \quad \lambda_{2,\kappa}(p) = \sqrt{-\frac{1}{\mathcal{T}'_{\kappa}(p)}} \xrightarrow{\kappa \rightarrow 0} +\infty.$$

In Figure 3, Lax curves in the  $(p, v)$  plane for decreasing values of  $\kappa$  are drawn. As  $\kappa$  vanishes, the Lax curves tend to straight lines and their slopes converge to zero:

$$\frac{d}{dp} \mathcal{V}_{1,\kappa}(p; p_o, v_o) \Big|_{p=p_o} = -\sqrt{-\mathcal{T}'_{\kappa}(p_o)} \xrightarrow{\kappa \rightarrow 0} 0,$$

$$\frac{d}{dp} \mathcal{V}_{2,\kappa}(p; p_o, v_o) \Big|_{p=p_o} = \sqrt{-\mathcal{T}'_{\kappa}(p_o)} \xrightarrow{\kappa \rightarrow 0} 0.$$

Observe that  $\mathcal{T}_{\kappa}(p) \xrightarrow{\kappa \rightarrow 0} \bar{\tau}$ , where  $\bar{\tau}$  is the constant specific volume in the incompressible limit, see (9). Since at the interfaces the pressure and the velocity are continuous, it turns out that a useful way to measure the waves' strengths is the pressure difference. Therefore, we systematically use below the parametrization  $\sigma_i \rightarrow \mathcal{V}_i(p_o + \sigma_i; p_o, v_o)$  and  $\sigma_i \rightarrow \mathcal{V}_{i,\kappa}(p_o + \sigma_i; p_o, v_o)$  of the  $i$ -Lax curve. The size of the wave,  $\sigma_i$ , is the pressure difference. We call strength of the wave,  $|\sigma_i|$ , the absolute value of the size  $\sigma_i$ . Differently from the usual habit [3], we have that

$i = 1$	$i = 2$
$\sigma_1 < 0 \Rightarrow$ rarefaction	$\sigma_2 < 0 \Rightarrow$ shock
$\sigma_1 > 0 \Rightarrow$ shock	$\sigma_2 > 0 \Rightarrow$ rarefaction

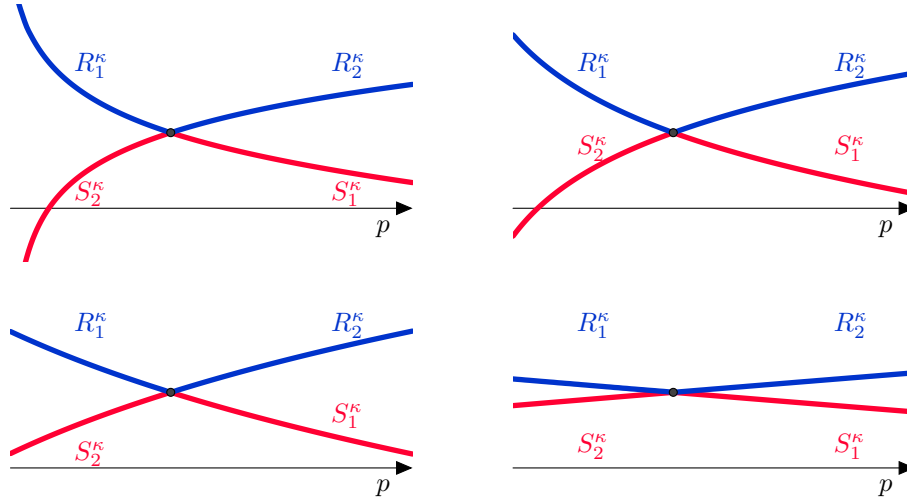


FIGURE 3. Lax curves (12) in the  $(p, v)$  plane for a non linear pressure law (9). As  $\kappa$  decreases, the Lax curves tend to horizontal straight lines.

**3.2. Wave front tracking.** The existence of solutions is shown through a wave front tracking algorithm [3, Chapter 7]: fix  $\varepsilon > 0$  and approximate the initial datum  $u_o = (p_o, v_o)$  by a sequence  $u_o^\varepsilon$  of piecewise constant initial data with a finite number of discontinuities such that

$$\begin{cases} u_o^\varepsilon(z) = u_o(0+) & \text{for all } z \in [-2\varepsilon^2, 2\varepsilon^2], \\ u_o^\varepsilon(z) = u_o(m-) & \text{for all } z \in [m - 2\varepsilon^2, m + 2\varepsilon^2], \end{cases}$$

$$\|u_o^\varepsilon - u_o\|_{\mathbf{L}^1} \leq \varepsilon, \quad \text{TV}(p_o^\varepsilon) \leq \text{TV}(p_o),$$

$$\text{TV}(v_o^\varepsilon, \mathbb{R} \setminus ]0, m]) \leq \text{TV}(v_o, \mathbb{R} \setminus ]0, m]) , \quad \text{TV}(v_o^\varepsilon, ]0, m]) \leq \text{TV}(v_o, ]0, m]) .$$

At each point of jump in the approximate initial datum, we solve the corresponding Riemann problem. As usual, see [3, Chapter 4], we approximate each rarefaction wave by a rarefaction fan consisting of  $\varepsilon$ -wavelets, each traveling with the characteristic speed of the state to its left. On the other hand, each shock wave is assigned its exact Rankine-Hugoniot speed. Similarly to what happens in the usual case, all the above Riemann problems have approximate solutions as long as  $\text{TV}(u_o)$  is sufficiently small. We introduce two strips around the two interfaces  $z = 0$  and  $z = m$ , where all 1-waves have speed  $-1$  and all 2-waves have speed  $1$ :

$$\mathcal{I}_\varepsilon^- = [-\varepsilon^2, \varepsilon^2] \times \mathbb{R}^+ \quad \text{and} \quad \mathcal{I}_\varepsilon^+ = [m - \varepsilon^2, m + \varepsilon^2] \times \mathbb{R}^+ .$$

This, together with [1, Lemma 2.5], allows us to avoid the introduction of non-physical waves, significantly simplifying the whole procedure. Hence, assign to all 1-waves entering  $\mathcal{I}_\varepsilon^- \cup \mathcal{I}_\varepsilon^+$  speed  $-1$ , while all 2-waves entering  $\mathcal{I}_\varepsilon^- \cup \mathcal{I}_\varepsilon^+$  are given speed  $+1$ , see Figure 4. Remark that the actual values attained by the approximate solution are not changed, only the wave speeds are modified. When exiting these strips, every wave is given back its correct speed. By this trick, no interaction among waves of the same family may take place in either of the two strips. This construction can be extended up to the first time  $t_1$  at which two waves interact, or a

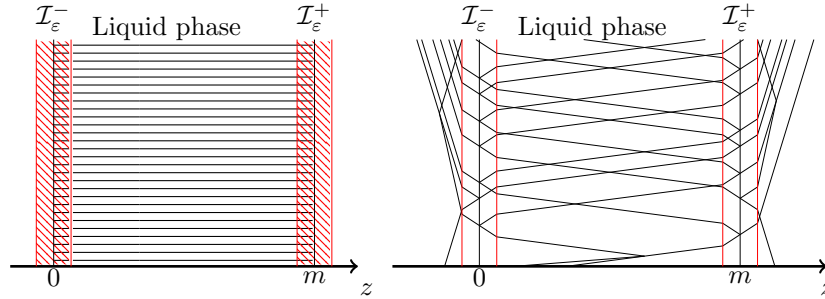


FIGURE 4. Modification to the usual wave front tracking algorithm: waves in the strips  $\mathcal{I}_\varepsilon^-$  and  $\mathcal{I}_\varepsilon^+$  are assign speed 1, if belonging to the first family, and  $-1$ , if of the second family.

wave hit one of the interfaces. At time  $t_1$ , the so constructed approximate solutions are piecewise constant with a finite number of discontinuities. Any such interaction gives rise to a new Riemann problem solved as at time  $t = 0$  if the interaction is in the interior of the two phases or as described in the following Lemma, proved in [4], whenever the interaction is along the left interface. A similar Lemma holds for interactions along the right interface.

**Lemma 3.1.** *Fix a state  $\bar{p}_o > 0$ . There exists positive  $\bar{\delta}$  such that for all  $\kappa \in ]0, 1]$  and for any couple of states  $(p^l, v^l)$ ,  $(p^r, v^r)$  with  $|p^l - \bar{p}_o| + |p^r - \bar{p}_o| < \bar{\delta}$  and  $|v^r - v^l| < \bar{\delta}$ , there exists a unique state  $(p^m, v^m)$  satisfying*

$$\mathcal{V}_1(p^m; p^l, v^l) = v^m \quad \text{and} \quad \mathcal{V}_{2,\kappa}(p^r; p^m, v^m) = v^r .$$

Moreover,  $|v^m - v^r| = \mathcal{O}(1) \kappa \bar{\delta}$ .

Any rarefaction wave, once arisen, is not further split even if its strength exceeds the threshold  $\varepsilon$  after subsequent interactions, with other waves or with the phase boundaries. The new rarefaction waves that may arise at the interfaces are split, if their strength exceeds  $\varepsilon$ , when they exit the strips  $\mathcal{I}_\varepsilon^\pm$ , since inside the strips they all travel with the same speed. We can thus iterate the previous construction at any subsequent interaction, provided suitable upper bounds on the total variation of the approximate solutions are available. As it is usual in this context, see [3, Chapter 7], we may assume that no more than 2 waves interact at any interaction point. A uniform bound on the total variation is obtained through a Glimm functional, as described in the following paragraph. Then, a bound on the total number of interaction points can be found through [1, Lemma 2.5], see [4] for details. Here, the strips  $\mathcal{I}_\varepsilon^\pm$  play a key role.

Specific to the present construction is our choice to measure waves' sizes through the pressure variation  $\sigma$  between the two states on the sides of a wave.

**3.3. A Glimm functional.** The necessary uniform bounds on the total variation of the approximate solutions are obtained following the classical techniques based on a Glimm functional. If  $x_\alpha(t)$ ,  $\alpha = 1, \dots, N$  are the locations of the discontinuities in the wave front tracking approximate solution at time  $t$  and  $\sigma_\alpha(t)$  is the size of the corresponding waves, we introduce, for any time  $t$  at which there are no interactions,

the potentials

$$\begin{aligned}
V_{\mathcal{G}_{in}} &= \sum_{\alpha \in \mathcal{G}_{in}} |\sigma_\alpha| & V_{\mathcal{G}_{out}} &= \sum_{\alpha \in \mathcal{G}_{out}} |\sigma_\alpha| & V_{\mathcal{L}} &= \sum_{\alpha \in \mathcal{L}} |\sigma_\alpha| \\
Q_{\mathcal{G}} &= \sum_{(\alpha, \beta) \in \mathcal{A}_{\mathcal{G}}} |\sigma_\alpha \sigma_\beta| & Q_{\mathcal{L}} &= \sum_{(\alpha, \beta) \in \mathcal{A}_{\mathcal{L}}} |\sigma_\alpha \sigma_\beta| & & (13) \\
\Upsilon &= K_{in} V_{\mathcal{G}_{in}} + V_{\mathcal{G}_{out}} + K_{\mathcal{L}} V_{\mathcal{L}} + H_{\mathcal{G}} Q_{\mathcal{G}} + \kappa^2 H_{\mathcal{L}} Q_{\mathcal{L}},
\end{aligned}$$

where  $K_{in}, K_{\mathcal{L}}, H_{\mathcal{G}}$  and  $H_{\mathcal{L}}$  are constants independent of  $\kappa$  to be precisely defined below. Above, we denoted

- $\mathcal{G}_{in}$  2-waves supported in  $]-\infty, 0[$  and 1-waves supported in  $]m, +\infty[$ .
- $\mathcal{G}_{out}$  1-waves supported in  $]-\infty, 0[$  and 2-waves supported in  $]m, +\infty[$ .
- $\mathcal{L}$  all waves supported in the liquid phase  $]0, m[$ .
- $\mathcal{A}_{\mathcal{G}}$  pairs of approaching waves supported in the gas phase.
- $\mathcal{A}_{\mathcal{L}}$  pairs of approaching waves supported in the liquid phase.

Here, we define as *approaching* two waves both supported in the same phase  $]-\infty, 0[$ ,  $[0, m]$  or  $]m, +\infty[$ , either of the same family and when one of the two is a shock, or of different families with the one of the first family on the right. In [4], the following Lemma is proved.

**Lemma 3.2.** *There exist weights  $K_{in}, K_{\mathcal{L}}, H_{\mathcal{G}}$  and  $H_{\mathcal{L}}$  and a positive  $\bar{\delta}$ , all independent of  $\kappa \in ]0, 1[$ , such that for all piecewise constant initial data  $u_o$  with  $\Upsilon(0+) < \bar{\delta}$ , the function  $t \rightarrow \Upsilon(t)$  is non increasing. Moreover, calling  $\sigma_\alpha, \sigma_\beta$  the sizes of two waves interacting at a point  $(\bar{t}, \bar{z})$ , with  $\sigma_\alpha$  coming from the left, the following estimates hold:*

$$\begin{aligned}
\bar{z} \in \mathcal{G}_{in} \cup \mathcal{G}_{out} & \quad \Delta \Upsilon(\bar{t}) \leq -|\sigma_\alpha \sigma_\beta| \\
\bar{z} = 0 & \quad \Delta \Upsilon(\bar{t}) \leq -|\sigma_\alpha| - \kappa |\sigma_\beta| \\
\bar{z} \in \mathcal{L} & \quad \Delta \Upsilon(\bar{t}) \leq -\kappa^2 |\sigma_\alpha \sigma_\beta| \\
\bar{z} = m & \quad \Delta \Upsilon(\bar{t}) \leq -\kappa |\sigma_\alpha| - |\sigma_\beta|.
\end{aligned} \tag{14}$$

Here, we give the main ideas in the proof of Lemma 3.2 and refer to [4] for the technical details.

First, we consider a wave hitting the interface from the liquid (see Figure 5, left). Since  $R_2^\kappa$  and  $S_1^\kappa$  have slopes of order of  $\kappa$ , and since we measure the waves' sizes through the pressure difference, we have (see Figure 5, right) for a suitable  $c > 0$ ,

$$|\sigma_1^+| \leq \mathcal{O}(1) \kappa |\sigma_1^-| \quad \text{and} \quad |\sigma_2^+| \leq (1 - c\kappa) |\sigma_1^-|, \tag{15}$$

uniformly in  $\kappa$ . By (15),

$$\begin{aligned}
\Delta \mathcal{V}_{\mathcal{L}} &\leq -c\kappa |\sigma_1^-|, & \Delta \mathcal{V}_{\mathcal{G}_{out}} &\leq \mathcal{O}(1) \kappa |\sigma_1^-|, & \Delta \mathcal{V}_{\mathcal{G}_{in}} &= 0 \\
\Delta Q_{\mathcal{G}} &\leq \mathcal{O}(1) \kappa \delta |\sigma_1^-|, & \Delta Q_{\mathcal{L}} &\leq \delta |\sigma_1^-|,
\end{aligned}$$

where  $\delta$  is an upper bound on  $\Upsilon$ . Therefore, if we choose  $K_{\mathcal{L}}$  sufficiently large, we can compensate the increase in the other terms, provided  $\delta$  is sufficiently small (see [4] for details). Observe that, due to (15), all the quantities that increase along this kind of interaction, increase by a quantity of order  $\kappa |\sigma_1^-|$  (thanks also to the coefficient  $\kappa^2$  in front of the potential  $Q_{\mathcal{L}}$ ). This is a crucial point in our estimates since we eventually let  $\kappa$  tend to zero.

Consider now the case of a wave hitting the interface from the gas (see Figure 6). Since the difference in the pressure along the wave  $S_2^\kappa$  is bounded by a constant



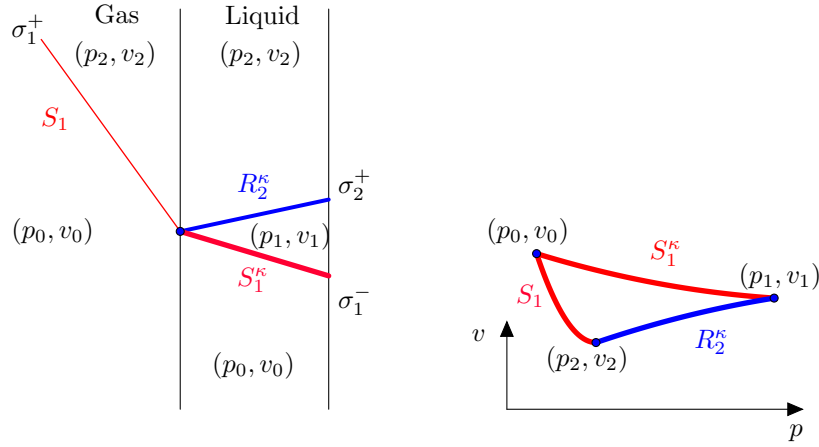


FIGURE 5. A shock wave hits the interface from the liquid phase. Left, in the  $(t, z)$  plane and, right, the corresponding Lax curves in the  $(p, v)$  plane.

uniform in  $\kappa$ , depending only on  $|\sigma_2^-|$  and the slopes of the curves  $S_1$  and  $S_2$ , the uniform estimate

$$|\sigma_1^+| \leq |\sigma_2^+| \leq \mathcal{O}(1) |\sigma_2^-| \tag{16}$$

holds. By (16),

$$\begin{aligned} \Delta \mathcal{V}_\mathcal{L} &\leq \mathcal{O}(1) |\sigma_2^-|, & \Delta \mathcal{V}_{\mathcal{G}_{out}} &\leq \mathcal{O}(1) |\sigma_2^-|, & \Delta \mathcal{V}_{\mathcal{G}_{in}} &\leq -|\sigma_2^-| \\ \Delta Q_\mathcal{G} &\leq \mathcal{O}(1) \delta |\sigma_2^-|, & \Delta Q_\mathcal{L} &\leq \mathcal{O}(1) \delta |\sigma_2^-|, \end{aligned}$$

If we choose a suitably large constant  $K_{in}$ , we can use the decrease in  $V_{\mathcal{G}_{in}}$  to compensate the increase in the other terms, provided  $\delta$  is sufficiently small (see [4] for details).

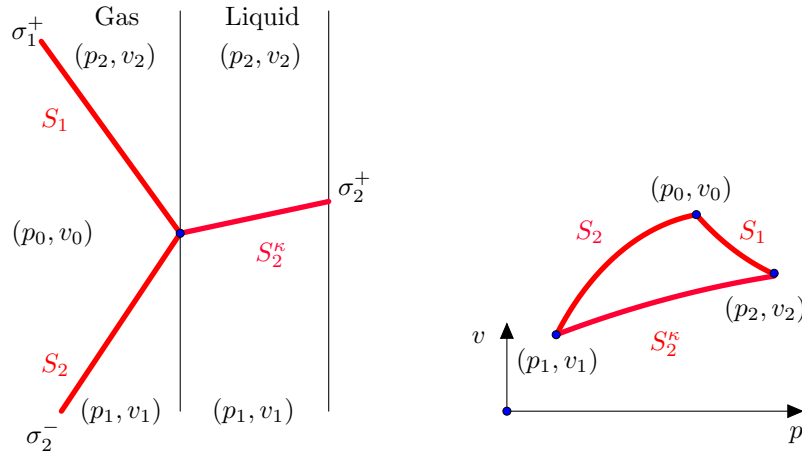


FIGURE 6. A shock wave hits the interface from the gas

When two waves interacts in the gas (see Figure 7) one has the usual interaction estimates:

$$\begin{aligned} \left| \sigma_1^+ - \sigma_1^- \right| + \left| \sigma_2^+ - \sigma_2^- \right| &\leq \mathcal{O}(1) \left| \sigma_1^- \sigma_2^- \right| && \text{(Figure 7, left),} \\ \left| \sigma_1^+ \right| + \left| \sigma_2^+ - (\sigma' + \sigma'') \right| &\leq \mathcal{O}(1) \left| \sigma' \sigma'' \right| && \text{(Figure 7, middle),} \\ \left| \sigma_1^+ - (\sigma' + \sigma'') \right| + \left| \sigma_2^+ \right| &\leq \mathcal{O}(1) \left| \sigma' \sigma'' \right| && \text{(Figure 7, right),} \end{aligned} \tag{17}$$

Therefore, as usual [3], choosing  $H_G$  sufficiently large we can compensate the in-

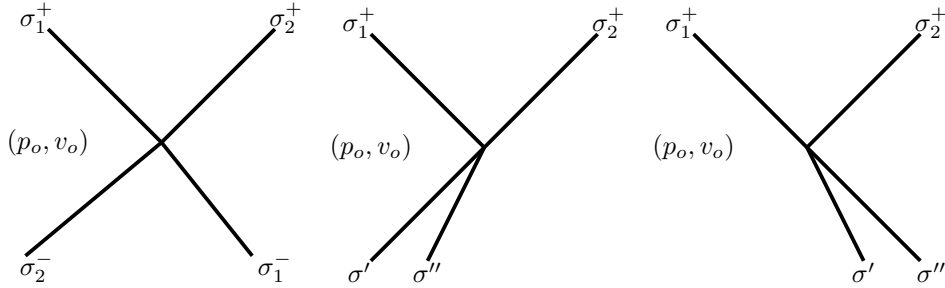


FIGURE 7. Left, an interaction between waves of different families. Middle, an interaction between waves of the second family. Right, an interaction between waves of the first family.

crease in the linear potential  $V_G$ .

Finally, when two waves interacts in the liquid (see Figure 7), in [4] the following refined interaction estimates are proved:

$$\begin{aligned} \left| \sigma_1^+ - \sigma_1^- \right| + \left| \sigma_2^+ - \sigma_2^- \right| &\leq \mathcal{O}(1) \kappa^2 \left| \sigma_1^- \sigma_2^- \right| && \text{(Figure 7, left),} \\ \left| \sigma_1^+ \right| + \left| \sigma_2^+ - (\sigma' + \sigma'') \right| &\leq \mathcal{O}(1) \kappa^2 \left| \sigma' \sigma'' \right| && \text{(Figure 7, middle),} \\ \left| \sigma_1^+ - (\sigma' + \sigma'') \right| + \left| \sigma_2^+ \right| &\leq \mathcal{O}(1) \kappa^2 \left| \sigma' \sigma'' \right| && \text{(Figure 7, right).} \end{aligned} \tag{18}$$

As the systems tends to be incompressible, i.e.,  $\kappa \rightarrow 0$ , the system also gets more and more similar to a linear one. This heuristically justifies the presence of  $\kappa^2$  in (18). Therefore, at these interactions, the small term  $\kappa^2 H_{\mathcal{L}} Q_{\mathcal{L}}$  in  $\Upsilon$  is sufficient to compensate the increase in  $K_{\mathcal{L}} V_{\mathcal{L}}$ , provided  $H_{\mathcal{L}}$  is sufficiently large.

**3.4. Initial data.** To ensure that the value of the functional at  $t = 0+$  is sufficiently small in order that all the above interaction estimates hold true, we need some conditions on the total variation of the initial data. The standard estimates on the solution of the Riemann problem (see [3]) imply that, in the gas, it is sufficient that the initial data have sufficiently small total variation. On the other hand, in the liquid, the estimates on the Riemann problem depend on the small parameter  $\kappa$ , as shown in Figure 8. Indeed, in the liquid region, the Lax curves have a slope of the order of  $\kappa$ , hence a jump  $\Delta v$  in the initial velocity, generates waves of the order of  $\frac{1}{\kappa}$ . With reference to Figure 8, we have  $|\sigma_1| + |\sigma_2| = \mathcal{O}(1) [|p_r - p_l| + \frac{1}{\kappa} |v_r - v_l|]$ . This estimate suggests the introduction of the weighted total variation

$$\text{WTV}_{\kappa}(p, v) = \text{TV}(p, \mathbb{R}) + \text{TV}(v, \mathbb{R} \setminus ]0, m[) + \frac{1}{\kappa} \text{TV}(v, ]0, m[).$$

It is not difficult to see [4] that, if the initial piecewise constant approximate data  $u_o^\varepsilon$  satisfies  $\text{WTV}_{\kappa}(u_o^\varepsilon) \leq \delta$ , then  $\Upsilon(0+) \leq \mathcal{O}(1) \delta$ . Hence for  $\delta$  sufficiently small, the wave front tracking procedure defines an approximate solution for all times.

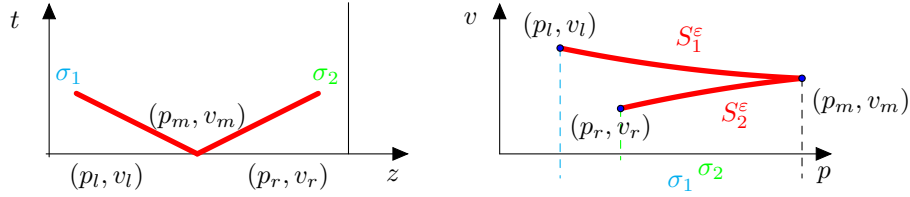


FIGURE 8. Riemann problem in the liquid. Left, in the  $(t, z)$  plane and, right, in the  $(p, v)$  plane.

The following theorem is proved in [4]. Observe that, in the liquid region, waves travel with a speed of the order of  $\frac{1}{\kappa}$ .

**Theorem 3.3.** *Given a positive constant  $c > 0$ , there exist  $\delta, \Delta, L > 0$  independent of  $\kappa$  such that if  $\text{WTV}_\kappa(p_o^\kappa, v_o^\kappa) < \delta$  and  $p_o^\kappa \geq c$  hold, then the wave front tracking approximate solution  $u^{\kappa, \varepsilon}$  to the Cauchy problem for (11) with  $u_o^\kappa = (p_o^\kappa, v_o^\kappa)$  as initial data can be constructed for all times  $t \geq 0$ . Moreover, if we recover the specific volume as  $\tau^{\kappa, \varepsilon}(t, z) = \mathcal{T}_\kappa(z, p^{\kappa, \varepsilon}(t, z))$ , the following estimates hold:*

For any  $t, t_1, t_2 \geq 0$

$$\text{WTV}_\kappa((p^{\kappa, \varepsilon}, v^{\kappa, \varepsilon})(t, \cdot)) \leq \Delta,$$

$$\text{TV}(p^{\kappa, \varepsilon}(t, \cdot), ]0, m[) \leq \Delta, \quad \int_0^m |p^{\kappa, \varepsilon}(t_2, z) - p^{\kappa, \varepsilon}(t_1, z)| dz \leq \frac{1}{\kappa} L |t_2 - t_1|,$$

$$\text{TV}(v^{\kappa, \varepsilon}(t, \cdot), ]0, m[) \leq \kappa \Delta, \quad \int_0^m |v^{\kappa, \varepsilon}(t_2, z) - v^{\kappa, \varepsilon}(t_1, z)| dz \leq L |t_2 - t_1|,$$

$$\text{TV}(\tau^{\kappa, \varepsilon}(t, \cdot), ]0, m[) \leq \kappa^2 \Delta, \quad \int_0^m |\tau^{\kappa, \varepsilon}(t_2, z) - \tau^{\kappa, \varepsilon}(t_1, z)| dz \leq \kappa L |t_2 - t_1|,$$

$$\text{TV}(p^{\kappa, \varepsilon}(t, \cdot), \mathbb{R} \setminus ]0, m[) \leq \Delta, \quad \int_{\mathbb{R} \setminus ]0, m[} |p^{\kappa, \varepsilon}(t_2, z) - p^{\kappa, \varepsilon}(t_1, z)| dz \leq L |t_2 - t_1|,$$

$$\text{TV}(v^{\kappa, \varepsilon}(t, \cdot), \mathbb{R} \setminus ]0, m[) \leq \Delta, \quad \int_{\mathbb{R} \setminus ]0, m[} |v^{\kappa, \varepsilon}(t_2, z) - v^{\kappa, \varepsilon}(t_1, z)| dz \leq L |t_2 - t_1|,$$

$$\text{TV}(\tau^{\kappa, \varepsilon}(t, \cdot), \mathbb{R} \setminus ]0, m[) \leq \Delta, \quad \int_{\mathbb{R} \setminus ]0, m[} |\tau^{\kappa, \varepsilon}(t_2, z) - \tau^{\kappa, \varepsilon}(t_1, z)| dz \leq L |t_2 - t_1|.$$

(19)

For any  $z, z_1, z_2 \in ]0, m[$

$$\text{TV}(p^{\kappa, \varepsilon}(\cdot, z), \mathbb{R}^+) \leq \frac{\Delta}{\kappa}, \quad \int_0^{+\infty} |p^{\kappa, \varepsilon}(t, z_2) - p^{\kappa, \varepsilon}(t, z_1)| dt \leq L |z_2 - z_1|,$$

$$\text{TV}(v^{\kappa, \varepsilon}(\cdot, z), \mathbb{R}^+) \leq \Delta, \quad \int_0^{+\infty} |v^{\kappa, \varepsilon}(t, z_2) - v^{\kappa, \varepsilon}(t, z_1)| dt \leq \kappa L |z_2 - z_1|, \quad (20)$$

$$\text{TV}(\tau^{\kappa, \varepsilon}(\cdot, z), \mathbb{R}^+) \leq \kappa \Delta, \quad \int_0^{+\infty} |\tau^{\kappa, \varepsilon}(t, z_2) - \tau^{\kappa, \varepsilon}(t, z_1)| dt \leq \kappa^2 L |z_2 - z_1|.$$

For any  $z, z_1, z_2 \in \mathbb{R} \setminus ]0, m[$

$$\text{TV}(p^{\kappa, \varepsilon}(\cdot, z), \mathbb{R}^+) \leq \Delta, \quad \int_0^{+\infty} |p^{\kappa, \varepsilon}(t, z_2) - p^{\kappa, \varepsilon}(t, z_1)| dt \leq L |z_2 - z_1|,$$

$$\text{TV}(v^{\kappa, \varepsilon}(\cdot, z), \mathbb{R}^+) \leq \Delta, \quad \int_0^{+\infty} |v^{\kappa, \varepsilon}(t, z_2) - v^{\kappa, \varepsilon}(t, z_1)| dt \leq L |z_2 - z_1|, \quad (21)$$

$$\text{TV}(\tau^{\kappa, \varepsilon}(\cdot, z), \mathbb{R}^+) \leq \Delta, \quad \int_0^{+\infty} |\tau^{\kappa, \varepsilon}(t, z_2) - \tau^{\kappa, \varepsilon}(t, z_1)| dt \leq L |z_2 - z_1|.$$

Since all the estimates in Theorem 3.3 do not depend on the parameter  $\varepsilon$ , the following theorem is a straightforward application of Helly’s Compactness Theorem, see [3, Section 7.4]. See also [3, Chapter 7] for the definition of entropy solutions.

**Theorem 3.4.** *Given a positive constant  $c > 0$ , there exist  $\delta, \Delta, L > 0$  independent of  $\kappa$  such that if  $WTV_\kappa(p_o^\kappa, v_o^\kappa) < \delta$  and  $p_o^\kappa \geq c$ , then the Cauchy problem for (11) with  $u_o^\kappa = (p_o^\kappa, v_o^\kappa)$  as initial data has an entropy solution  $u^\kappa = (p^\kappa, v^\kappa)$  defined for all times  $t \geq 0$ . Moreover, if we recover the specific volume as  $\tau^\kappa(t, z) = \mathcal{T}_\kappa(z, p^\kappa(t, z))$ , the estimates (19), (20), (21) in Theorem 3.3 hold with  $(p^\kappa, v^\kappa, \tau^\kappa)$  in place of their wave front tracking approximations.*

**4. A linear example.** In the linear case, explicit computations are possible and useful to gain a better insight in the limiting procedure. Set  $m = 1$  and choose the following linear pressure laws:

$$\mathcal{T}_\kappa(p) = \bar{\tau} - \kappa^2(p - \bar{p}) \quad \text{and} \quad \mathcal{T}(p) = \bar{\tau} - (p - \bar{p}),$$

so that our system (7)–(11) becomes

$$\begin{cases} -\kappa^2 \partial_t p - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0, \end{cases} \quad z \in ]0, 1[, \quad \begin{cases} -\partial_t p - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0, \end{cases} \quad z \in \mathbb{R} \setminus [0, 1], \tag{22}$$

$$\begin{cases} p(t, 0-) = p(t, 0+), \quad p(t, 1-) = p(t, 1+), & \text{for a.e. } t \geq 0, \\ v(t, 0-) = v(t, 0+), \quad v(t, 1-) = v(t, 1+), & \text{for a.e. } t \geq 0. \end{cases}$$

All waves in the liquid travel with speed  $\pm \frac{1}{\kappa}$  while all waves in the gas travel with speed  $\pm 1$ . Fix constant states  $p_o, v_o > 0$  and consider the following initial datum for  $t = -1$ :

$$(p, v)(-1, z) = \begin{cases} (p_o + v_o, v_o) & \text{for } z \leq -1 \\ (p_o, 0) & \text{for } z > -1. \end{cases} \tag{23}$$

The two states  $(p_o + v_o, v_o)$  and  $(p_o, 0)$  in the gas are separated by a wave of the second family. This wave hits the left interface at time  $t = 0$ . The resulting wave front pattern can be explicitly computed by induction, see Figure 9 and (24). For  $i = 0, 1, 2, 3, \dots$  we have:

$$p_0^l = p_o + v_o, \quad v_0^l = v_o; \quad p_0^r = p_o, \quad v_0^r = 0$$

$$\begin{cases} p_{i+1}^l = \frac{v_i^l - v_i^r + p_i^l + \kappa p_i^r}{1 + \kappa}, \\ v_{i+1}^l = \frac{\kappa v_i^l + v_i^r + \kappa (p_i^l - p_i^r)}{1 + \kappa}, \end{cases} \quad \begin{cases} p_{i+1}^r = \frac{v_{i+1}^l - v_i^r + \kappa p_{i+1}^l + p_i^r}{1 + \kappa}, \\ v_{i+1}^r = \frac{v_{i+1}^l + \kappa v_i^r + \kappa (p_{i+1}^l - p_i^r)}{1 + \kappa}. \end{cases} \tag{24}$$

Here, as  $\kappa$  tends to zero (see Figure 10):

- The speed of the waves in the liquid increases towards  $+\infty$ , therefore they bounce back and forth between the interfaces at an increasing rate.
- The waves refracted from the liquid into the gas have sizes of order  $\kappa$ , but as  $\kappa$  tends to zero, their number increases, so that their total size does not vanish.
- The waves in the liquid that bounce back and forth have a non vanishing size, therefore the total variation in time of the liquid pressure at a fixed  $z$  blows up.

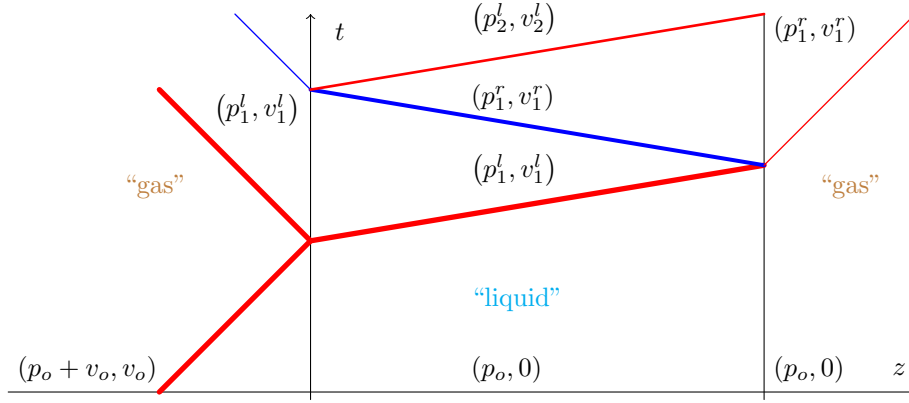


FIGURE 9. Initial wave front tracking pattern for the system (22) with initial data (23)

- Since, in the liquid, along a wave in the  $(p, v)$  plane we have  $|\Delta v| = \kappa |\Delta p|$ , the size of any single jump in the velocity vanishes. Therefore, the total variation in time of the velocity at any fixed  $z \in [0, m]$  stays bounded, even if the number of jumps diverges to  $+\infty$ . On the other hand, the total variation in space of the velocity in the liquid phase at any  $t > 0$  vanishes.
- For  $\kappa$  small, the first discontinuity hits the left interface at time  $t = 0$ . A big wave is reflected back. Two wave fans composed of many very small waves arise from the interfaces into the gas. In the liquid, the pressure widely oscillates.

4.1. **Passing to the limit.** In the present linear case, the solution to (22)-(23) can be explicitly computed as follows. The states  $(p_i^l, v_i^l)$  and  $(p_i^r, v_i^r)$  in the solution (24) to (22) are depicted in Figure 9 and can be written as  $U_{i+1} = A_\kappa U_i$  where

$$U_i = \begin{bmatrix} p_i^l \\ v_i^l \\ p_i^r \\ v_i^r \end{bmatrix}, A_\kappa = \frac{1}{1 + \kappa} \begin{bmatrix} 1 & 1 & \kappa & -1 \\ \kappa & \kappa & -\kappa & 1 \\ \frac{2\kappa}{1+\kappa} & \frac{2\kappa}{1+\kappa} & \frac{1+\kappa^2}{1+\kappa} & -\frac{2\kappa}{1+\kappa} \\ \frac{2\kappa}{1+\kappa} & \frac{2\kappa}{1+\kappa} & -\frac{2\kappa}{1+\kappa} & \frac{1+\kappa^2}{1+\kappa} \end{bmatrix} \text{ and } U_0 = \begin{bmatrix} p_0 + v_0 \\ v_0 \\ p_0 \\ 0 \end{bmatrix}.$$

Note that  $A_\kappa = R_\kappa D_\kappa R_\kappa^{-1}$ , where

$$R_\kappa = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ -\frac{1-\kappa}{1+\kappa} & 1 & 0 & 0 \\ -\frac{1-\kappa}{1+\kappa} & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D_\kappa = \begin{bmatrix} \left(\frac{1-\kappa}{1+\kappa}\right)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$U_i = (A_\kappa)^i U_0 = R_\kappa (D_\kappa)^i R_\kappa^{-1} U_0.$$

It is therefore possible to compute explicitly the limit as  $\kappa \rightarrow 0$  of the solution to (22)-(23). Indeed, the wave speed in the liquid is  $1/\kappa$  and, as Figure 9 shows,

$$\begin{aligned} u^\kappa(t, 0-) &= (p_i^l, v_i^l) + o(\kappa) \\ u^\kappa(t, 1+) &= (p_i^r, v_i^r) + o(\kappa) \end{aligned} \quad \text{as } \kappa \rightarrow 0, \quad \text{where } i = \left\lfloor \frac{t}{2\kappa} \right\rfloor$$

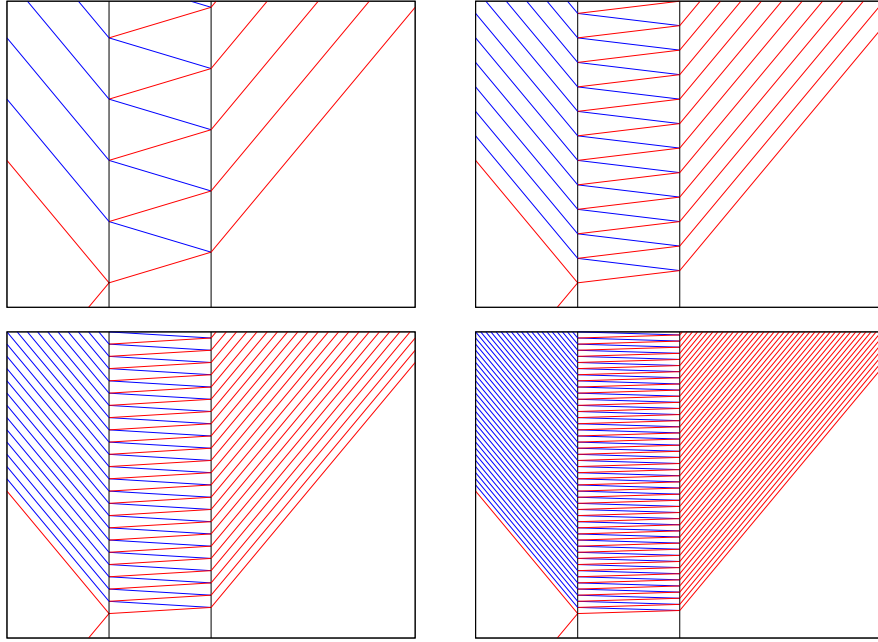


FIGURE 10. Wave patterns of the solution to (22) with initial data (23), in the  $(t, z)$  plane, for decreasing values of  $\kappa$ .

and  $[\xi]$  denotes the integer part of  $\xi$ . Moreover, define

$$\mathcal{D}_t = \lim_{\kappa \rightarrow 0} D_\kappa^{\lfloor t/(2\kappa) \rfloor} = \begin{bmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{R}_0 = \lim_{\kappa \rightarrow 0} R_\kappa = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

leading to

$$\begin{aligned} \lim_{\kappa \rightarrow 0} \begin{bmatrix} u^\kappa(t, 0-) \\ u^\kappa(t, 1+) \end{bmatrix} &= \lim_{\kappa \rightarrow 0} U_{\lfloor t/(2\kappa) \rfloor} \\ &= \lim_{\kappa \rightarrow 0} (A_\kappa)^{\lfloor t/(2\kappa) \rfloor} U_o \\ &= \lim_{\kappa \rightarrow 0} R_\kappa (D_\kappa)^{\lfloor t/(2\kappa) \rfloor} R_\kappa^{-1} U_o \\ &= \mathcal{R}_0 \mathcal{D}_t \mathcal{R}_0^{-1} U_o \\ &= \begin{bmatrix} (1 + e^{-2t})v_o + p_o \\ (1 - e^{-2t})v_o \\ (1 - e^{-2t})v_o + p_o \\ (1 - e^{-2t})v_o \end{bmatrix} \end{aligned} \quad (25)$$

As proved in Theorem 5.1, the same result can be obtained passing first in (22)–(23) to the limit  $\kappa \rightarrow 0$  and then solving the resulting problem. Indeed, (22)–(23)

as  $\kappa \rightarrow 0$  becomes a coupled system of ODE–PDE, see [2]:

$$\begin{cases} \begin{cases} -\partial_t p - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0, \end{cases} & z \in \mathbb{R} \setminus [0, 1], & \begin{cases} \dot{v}_l = p(t, 0-) - p(t, 1+) \\ \partial_z p = -\dot{v}_l & z \in ]0, 1[, \end{cases} \\ \begin{cases} v(t, z) = v_l(t) & \text{for } 0 < z < 1, t \geq 0, \\ p(t, 0-) = p(t, 0+), p(t, 1-) = p(t, 1+), & \text{for a.e. } t \geq 0, \\ v(t, 0-) = v(t, 0+), v(t, 1-) = v(t, 1+), & \text{for a.e. } t \geq 0. \end{cases} \end{cases} \quad (26)$$

Its solution is given by (see also Figure 11, left):

$$p(t, z) = \begin{cases} p_o + v_o & \text{for } z < -|t| \\ p_o & \text{for } z > t + \chi_{[0, +\infty[}(t) \\ p_o + v_o \cdot (1 + e^{-2(z+t)}) & \text{for } -t < z < 0 \\ p_o + v_o \cdot (1 - e^{-2(-z+t+1)}) & \text{for } 1 < z < t + 1 \\ p_o + v_o + v_o \cdot (1 - 2z) e^{-2t} & \text{for } 0 < z < 1, \end{cases}$$

$$v(t, z) = \begin{cases} v_o & \text{for } z < -|t| \\ 0 & \text{for } z > t + \chi_{[0, +\infty[}(t) \\ v_o \cdot (1 - e^{-2(z+t)}) & \text{for } -t < z < 0 \\ v_o \cdot (1 - e^{-2(-z+t+1)}) & \text{for } 1 < z < t + 1 \\ v_o \cdot (1 - e^{-2t}) & \text{for } 0 < z < 1, \end{cases}$$

which is consistent with (25).

In Eulerian coordinate, the displacement of the interfaces  $a(t)$  and  $b(t)$  can be obtained through the integration of the speed. Assume that the liquid phase is initially located to the right of the origin, so that  $a(t) = 0$  for  $t \leq 0$ . Then,

$$a(t) = \int_0^t v_l(\xi) d\xi = \int_0^t v_o \cdot (1 - e^{-2\xi}) d\xi = v_o \cdot \left( t - \frac{1}{2} (1 - e^{-2t}) \right)$$

$$b(t) = a(t) + 1 = v_o \cdot \left( t - \frac{1}{2} (1 - e^{-2t}) \right) + 1.$$

The structure of the exact solution to (22)–(23) in Eulerian coordinate follows from (2) and is depicted in Figure 11, right. When the discontinuity hits the liquid at  $t = 0$  a big wave (which would be a shock in the non linear case) is reflected backward. This causes a difference in the pressure between the two sides of the liquid. Hence, the liquid accelerates, moving to the right. A wave fan (which would be a rarefaction wave in the non linear case) propagates backward while another wave fan (a compression wave in the non linear case) is generated by the movement of the liquid and moves to the right.

In the non linear case, this structure is qualitatively similar, see [6]. However, a rigorous treatment is technically far more intricate. Due to the interactions among the non linear waves on the two sides of the liquid, new waves are generated and interact again with the interfaces.

In the next section, the rigorous compressible to incompressible limit for the fully non linear case is presented.

**5. The general incompressible limit.** Choose an initial data with constant speed in the region  $]0, m[$ . Then, the weighted total variation remains small, uniformly with respect to the parameter  $\kappa$ . Indeed, let  $c > 0$  be a positive lower bound

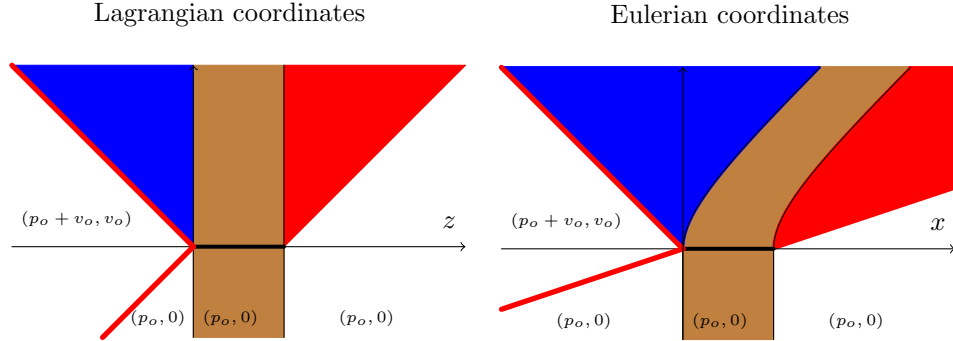


FIGURE 11. Solution for the limit system (26) with initial data (23) in Lagrangian and Eulerian coordinates

for the initial pressure and fix an initial data  $(p_o, v_o)$  such that

$$\begin{aligned} \text{TV}(p_o) + \text{TV}(v_o) &\leq \delta, \quad p_o \geq c, \\ v_o(z) &= \tilde{v} \text{ for all } z \in ]0, m[. \end{aligned}$$

Then, one has  $\text{WTV}_\kappa(p_o, v_o) \leq \delta$  for all  $\kappa \in ]0, 1[$ . The following theorem is proved in [4], using the uniform (with respect to  $\kappa$ ) estimate in Theorem 3.4.

**Theorem 5.1.** *Given a positive constant  $c$ , there exists a positive  $\delta$  such that, for any  $\tilde{v}$  and any initial data  $(p_o, v_o)$  satisfying  $\text{TV}(p_o) + \text{TV}(v_o) \leq \delta$ ,  $p_o \geq c$ ,  $v_o(z) = \tilde{v}$  for all  $z \in ]0, m[$ , there exists, for any  $\kappa \in ]0, 1[$ , an entropy solution  $u_o^\kappa = (p_o^\kappa, v_o^\kappa)$  to the Cauchy problem for (11) with initial datum  $(p_o, v_o)$ . Define the specific volume as  $\tau^\kappa(t, z) = \mathcal{T}_\kappa(z, p^\kappa(t, z))$ , then as  $\kappa \rightarrow 0$ , up to subsequences, we have the following convergence results.*

For all  $t \geq 0$ ,

$$\begin{aligned} \tau^\kappa(t, \cdot) &\rightarrow \bar{\tau} & \text{in } \mathbf{L}^1(]0, m[) & \quad \text{and} & \quad \tau^\kappa(t, \cdot) &\rightarrow \tau^*(t, \cdot) \\ v^\kappa(t, \cdot) &\rightarrow v_l(t) & \text{in } \mathbf{L}^1(]0, m[) & \quad \text{and} & \quad v^\kappa(t, \cdot) &\rightarrow v^*(t, \cdot) \text{ in } \mathbf{L}^1(\mathbb{R} \setminus [0, m]) . \\ p^\kappa(t, \cdot) & & & & \quad p^\kappa(t, \cdot) &\rightarrow p^*(t, \cdot) \end{aligned}$$

For all  $z \in \mathbb{R} \setminus [0, m]$

$$\begin{aligned} \tau^\kappa(\cdot, z) &\rightarrow \tau^*(\cdot, z) \\ v^\kappa(\cdot, z) &\rightarrow v^*(\cdot, z) \text{ in } \mathbf{L}^1(\mathbb{R}^+). \\ p^\kappa(\cdot, z) &\rightarrow p^*(\cdot, z) \end{aligned}$$

For all  $z \in ]0, m[$

$$\begin{aligned} \tau^\kappa(\cdot, z) &\rightarrow \bar{\tau} & \text{in } \mathbf{L}^1(\mathbb{R}^+) & \quad \text{and} & \quad p^\kappa(\cdot, \cdot) &\overset{*}{\rightharpoonup} p_l(\cdot, \cdot) \text{ in } \mathbf{L}^\infty(]0, m[ \times \mathbb{R}^+) . \\ v^\kappa(\cdot, z) &\rightarrow v_l(\cdot) & & & & \end{aligned}$$

The limit functions  $p^*$ ,  $v^*$ ,  $\tau^*$  satisfy the Lipschitz conditions (21) and, together with  $v_l$ , they are entropy solutions [2] to

$$\begin{aligned} \begin{cases} \partial_t \mathcal{T}(p^*) - \partial_z v^* = 0 \\ \partial_t v^* + \partial_z p^* = 0, \end{cases} & \quad \begin{cases} p^*(0, z) = p_o(z) & t \geq 0 \\ v^*(0, z) = v_o(z) & z \in \mathbb{R} \setminus [0, m], \end{cases} \\ m \frac{d}{dt} v_l(t) = p^*(t, 0-) - p^*(t, 0+), & \quad v_l(0) = \tilde{v}, \quad \text{a.e. } t \geq 0, \\ v_l(t) = v^*(t, 0-) = v^*(t, m+), & \quad \text{a.e. } t \geq 0. \end{aligned} \tag{27}$$



Moreover,

$$p_l(t, z) = \left(1 - \frac{z}{m}\right) p^*(t, 0-) + \frac{z}{m} p^*(t, 0+) \text{ a.e. } t \geq 0, z \in [0, m].$$

**Remark 1.** System (27) is a coupled system of PDE and ODE whose well posedness was proved in [2].

Observe that from the Eulerian coordinates’ point of view, the locations of the boundaries of the liquid phase can be recovered through a time integration, as in the linear example. We fix the initial location of the left interface at a given point  $x = a_o$ . In Theorem 5.1 the initial pressure is chosen independent of  $\kappa$ . The initial specific volume in the liquid is then  $\tau_o^\kappa(z) = \mathcal{T}(\bar{p} + \kappa^2(p_o(z) - \bar{p}))$ , which depends on  $\kappa$ . The total mass  $m$  of the liquid is fixed. Hence, the initial location of the right interface also depends on  $\kappa$  and we call it  $x = b_o^\kappa$ . Since  $\tau_o^\kappa(z) \rightarrow \bar{\tau}$  as  $\kappa \rightarrow 0$ , we have  $b_o^\kappa \rightarrow b_o = a_o + m\bar{\tau}$ . Note however that in the particular case of the constant initial pressure  $p_o = \bar{p}$  in the liquid,  $b_o^\kappa$  turns out to be independent of  $\kappa$ .

Let  $a^\kappa(t)$  and  $b^\kappa(t)$  be the locations of the interfaces (in Eulerian coordinates) at time  $t$  for positive  $\kappa$ , while  $a(t)$  and  $b(t)$  are the corresponding limits as  $\kappa \rightarrow 0$ . Then, if  $v^\kappa$  is the velocity in Lagrangian coordinates, we have:

$$\begin{aligned} a^\kappa(t) &= a_o + \int_0^t v^\kappa(\xi, 0-) d\xi & a(t) &= a_o + \int_0^t v_l(\xi) d\xi \\ b^\kappa(t) &= b_o^\kappa + \int_0^t v^\kappa(\xi, m+) d\xi & b(t) &= b_o + \int_0^t v_l(\xi) d\xi. \end{aligned} \tag{28}$$

Using Theorem 5.1 in the explicit expressions of the locations of the interfaces (28), one can see that the boundaries of the two phases are Lipschitz continuous functions and moreover  $a^\kappa \rightarrow a$ ,  $b^\kappa \rightarrow b$  uniformly on bounded time intervals, as  $\kappa \rightarrow 0$ .

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