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NON-CRITICAL FRACTIONAL CONSERVATION LAWS IN DOMAINS WITH BOUNDARY

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Abstract. We study bounded solutions for a multidimensional conservation law coupled with a power $s \in (0,1)$ of the Dirichlet laplacian acting in a domain. If $s \leq 1/2$ then the study centers on the concept of entropy solutions for which existence and uniqueness are proved to hold. If $s > 1/2$ then the focus is rather on the C^∞-regularity of weak solutions. This kind of results is known in $\mathbb{R}^N$ but perhaps not so much in domains. The extension given here relies on an abstract spectral approach, which would also allow many other types of nonlocal operators.

1. **Introduction.** This short note reports on some joint work with Nathael Alibaud on the perturbation of scalar conservation laws by nonlocal operators *in domains*. Complete proofs and technical details will appear in two forthcoming papers [3] [4]. The purpose of this note is rather to give an overview of the general program of study, which covers problems of both hyperbolic and parabolic nature.

In an open bounded subset $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, we consider the Cauchy problem for a scalar conservation law perturbed by a fractional power $(0 < s < 1)$ of the Dirichlet laplacian in Ω, i.e.

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}_x\big(f(u(t,x))\big) + (-\Delta_x)^s u(t,x) = 0 & (t,x) \in I \times \Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \\ u(t,x) = 0 & (t,x) \in I \times \partial\Omega. \end{cases} \tag{1}$$

Here the time interval $I = [0,T]$ is finite $(0 < T < \infty)$ and the flux function $f : \mathbb{R} \to \mathbb{R}^N$ is assumed smooth (say $f \in C^\infty$ like in Burgers's equation). The problem is set in the framework of bounded real-valued solutions $u = u(t,x)$ of time and space. The Dirichlet condition $u = 0$ applied on the boundary $\partial\Omega$ is for the time being only formal.

To the best of our knowledge, (1) has mainly been investigated so far for homogeneous spaces $\Omega = \mathbb{R}^N$ only. Our present goal is to extend some existing results on $\mathbb{R}^N$ to domains by means of abstract functional analysis on the power, rather than through an explicit representation of $(-\Delta)^s$ as a space integral as is used in most works on the subject.

In terms of differential order, $0 < s < 1/2$ resp. $1/2 < s < 1$ corresponds to a hyperbolic resp. parabolic problem, while the value $s = 1/2$ of the power is critical in the sense that it is related to a change of PDE class.

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- When $s > 1/2$ the problem may be placed into 'parabolic semigroup theory', in the sense that the evolution governed by the generator $-(-\Delta)^s$ dominates the nonlinearity $\mathrm{div}(f(u))$ of lower order in many respects. Typically, existence of solutions may be shown by a fixed-point argument based on the Duhamel formula with $\mathrm{div}(f(u))$ as r.h.s. Moreover, these solutions are unique and identical with weak solutions understood in a distributional sense including the initial datum. Exactly like in purely local parabolic PDEs, the equation turns out to have an immediate regularizing effect: for any $u_0 \in L^\infty(\Omega)$ the solution u is smooth (C^∞) w.r.t. $t > 0$ and $x \in \Omega$. This topic has been investigated on $\mathbb{R}^N$ in [12] for instance. The purpose of Section 3 below is to establish the same regularizing effect for domains. Our strategy of proof differs significantly from [12] in the iterative scheme leading to higher and higher regularity, since it seems impossible to perform space differentiation (and/or truncation) in the nonlocal equation because of some non-commutativity between Δ and ∇. Of course, this obstacle is specific to domains with boundary conditions, and does not appear in $\mathbb{R}^N$. However, there is a way to get more and more regularity (on u), through some special quantities (of u) which are indeed well-behaved.

- When $s < 1/2$ the boundary condition $u = 0$ is not expected to be pointwise (hyperbolic PDEs are generally overdetermined by boundary conditions coming from elliptic PDEs). Moreover, it is known that classical/regular solutions need not exist, that weak solutions need not be unique [2], that jump singularities may develop from smooth data [5] ... All this has been shown rigorously for (1) viewed in $\mathbb{R}^N$. Since there is no hope for better behavior in domains, the standpoint is rather to seek existence and uniqueness in an intermediate class of entropy solutions in the spirit of [16]. This direction has been illustrated on $\mathbb{R}^N$ by the papers [1] [10] [13] for instance. Section 4 below follows [1] very closely and explains how to develop a similar theory of entropy solutions for which existence and uniqueness hold.

- The critical case $s = 1/2$ remains unclear at least for us. In recent years, there has been some evidence that this case should fall within the parabolic side, since many works on similar problems on $\mathbb{R}^N$ conclude to the C^∞-character of solutions. The closest example is probably the critical Burgers's equation studied in [8] (and also in [11] [17] previously). Similar studies [15] [9] concerning the quasi-geostrophic equation also lead to the same conclusion.

Nevertheless, the preceding discussion should not oppose too much the regimes $s < 1/2$ and $s > 1/2$. Indeed, there are many common features of great interest, which are quite insensitive to the value of $s \in (0, 1)$. For instance, the entropy solution concept yields existence and uniqueness for any $s \in (0, 1)$ and provides maximum principles as well as L^1-contraction principles for entropy solutions. The main reason for presenting entropy solutions in the hyperbolic case $s < 1/2$ as an alternative to weak solutions is the uniqueness result within the entropic class. But this equally applies for any $0 < s < 1$. Distinctions really appear on the issue of regularity of solutions and accordingly on whether Dirichlet boundary conditions persist or not when s jumps over $1/2$. To illustrate that some properties are insentitive to the value of s, we have recalled in Section 5 a very simple proof of the strong maximum principle for classical solutions.

A final notice is in order about possible generalizations of (1) viewed as a simplified problem. Because of its abstract nature, the spectral approach developed below calls for extensions, inasmuch as the local operator Δ comes into play through its resolvent $(\lambda - \Delta)^{-1}$ *only*. For instance, a possible technical extension allowed by our method consists in replacing the Dirichlet laplacian by a general second-order elliptic operator with *nonconstant* coefficients. This can be done at the price of very minor changes at the start and does not affect the final results. A less academic question is to enlarge the class of powers of second-order elliptic operators by identifying clearly the few qualitative operator-properties of the resolvent $(\lambda - \Delta)^{-1}$ really needed for our techniques. Even the (integro)differential character of Δ does not seem essential, if $(\lambda - \Delta)^{-1}$ has some 'good' properties. Highlighting these properties is still an open question out of the scope of this note.

2. **Construction of the nonlocal operator.** The general construction of powers of operators in abstract functional analysis is often developed along the following line also known as *subordination*. An elementary text on the subject is [19, Ch.IX, Sec.11, p.264-272] (or also [18, Ch.II, Sec.2.6, p.69]). Let A be the infinitesimal generator of a C^0-semigroup of operators $(S(t))_{t\geq 0}$ acting on a Banach space E with a uniform bound

$$\sup_{t\in\mathbb{R}_+^*} \|S(t)\|_{\mathcal{L}(E)} \leq M < \infty.$$

The (normalized) resolvents $K_\lambda := \lambda(\lambda - A)^{-1}$ for $0 < \lambda < \infty$ are also uniformly bounded by M as (normalized) Laplace transforms on S. Now, for a given $s \in (0,1)$, let

$$\Upsilon_t(\tau) = \frac{1}{t^{1/s}}\Upsilon(\frac{\tau}{t^{1/s}})$$

be the standard s-function obtained by homogeneity from the inverse Laplace transform $\Upsilon := \mathcal{L}^{-1}(e^{-z^s})$ on $\mathbb{R}_+^*$. To be more specific, a (semi)explicit formula for $\Upsilon_t(\tau)$ is known in term of some contour integral

$$\Upsilon_t(\tau) = \frac{1}{\pi}\int_0^\infty e^{+\tau r\cos(\vartheta)} e^{-tr^s\cos(s\vartheta)} \sin\Big(\vartheta + \tau r\sin(\vartheta) - tr^s\sin(s\vartheta)\Big) dr,$$

which turns out to be independent of the choice of $0 < \vartheta < \pi$ such that $\cos(\vartheta) < 0 < \cos(s\vartheta)$. The main properties of $\Upsilon_t(\tau)$ are well-explained in [19, Ch.IX, Sec.11, p.259-263] for instance. We shall only retain here the averaging effect due to $\Upsilon \in L^1_+(\mathbb{R}_+^*) : \int_0^\infty \Upsilon = 1$. In this sense, s-subordination (i.e. integration against the probability measure $\Upsilon_t(\tau)d\tau$ on $\mathbb{R}_+^*$) just appears as a special convex combination w.r.t. time (τ).

It is kwown that s-subordination applied to the original semigroup

$$S_s(t) := \int_0^\infty \Upsilon_t(\tau) S(\tau) d\tau \qquad (0 < t < \infty) \tag{2}$$

defines an analytic C^0-semigroup of operators on E with the same uniform bound

$$\sup_{t\in\mathbb{R}_+^*} \|S_s(t)\|_{\mathcal{L}(E)} \leq M < \infty.$$

Remark 1. The analyticity of S_s holds even if S is not analytic itself.

At the level of resolvents, there is also a nice formula based on some convex combination

$$\lambda(\lambda - A_s)^{-1} = \frac{\sin(s\pi)}{\pi}\int_0^\infty \frac{\lambda r^s}{\lambda^2 + 2\lambda r^s\cos(s\pi) + r^{2s}} K_r \frac{dr}{r} \qquad (0 < \lambda < \infty). \tag{3}$$

These two identities (2)-(3) in $\mathcal{L}(E)$ provide an easy way to pass from to the pure case $s = 1$ to the fractional case $s \in (0,1)$ as far as semigroups and resolvents are concerned. On the contrary, the infinitesimal generator A_s of the semigroup

$(S_s(t))_{t\geq 0}$ is less easy to handle, in the sense that the corresponding integral representations

$$-A_s u = \frac{s\Gamma(s)\sin(s\pi)}{\pi}\int_0^\infty \Big(u - S(t)u\Big)\frac{dt}{t^{1+s}} = \frac{\sin(s\pi)}{\pi}\int_0^\infty \Big(u - K_\lambda u\Big)\frac{d\lambda}{\lambda^{1-s}} \quad (4)$$

are more singular, and only make sense for special elements u of E. Typically, the belonging $u \in D(A)$ to the domain of A itself is a sufficient condition for both integrals to converge in E, whatever $s \in (0,1)$ be.

We now turn to deeper considerations on the characterization of the domain of A_s, and recall its link with interpolation theory. Let the domain $D(A)$ be endowed with the graph norm $\|u\|_{D(A)} := \|u\|_E + \|Au\|_E$. In real interpolation theory as developped and reviewed in Chapter III of [6] for instance, the K-method consists in defining a large variety of "intermediate spaces" between E and $D(A)$ by means of the so-called K_t functional

$$K_t : u \in E \mapsto \inf_{(v,w)\in E\times D(A):u=v+w} \|v\|_E + t\|w\|_{D(A)} \in \mathbb{R}_+ \qquad (0 < t < \infty).$$

Among those numerous spaces are two extreme examples

$$[E; D(A)]_{s,1} := \{u \in E : \frac{K_t(u)}{t^s} \in L^1(\mathbb{R}_+^*, \frac{dt}{t})\},$$

$$[E; D(A)]_s := \{u \in E : \lim_{t\to 0^+}\frac{K_t(u)}{t^s} = \lim_{t\to +\infty}\frac{K_t(u)}{t^s} = 0\},$$

respectively normed by the natural quantities

$$\|u\|_{[E;D(A)]_{s,1}} := \|u\|_E + \int_0^\infty \frac{K_t(u)}{t^s}\frac{dt}{t},$$

$$\|u\|_{[E;D(A)]_s} := \|u\|_E + \sup_{t\in\mathbb{R}_+^*}\frac{K_t(u)}{t^s}.$$

A far-reaching result on the characterization of domains for general powers asserts that $D(A_s)$ for the graph norm $\|u\|_{D(A_s)} := \|u\|_E + \|A_s u\|_E$ is nested between those two extreme spaces, in the sense of the double inclusion

$$[E; D(A)]_{s,1} \subset D(A_s) \subset [E; D(A)]_s \qquad \text{with continuous imbeddings.}$$

This result is essentially contained in the last page of Ch.II in [6], which describes what the exact domain of the power is in terms of abstract convergence of integrals. Furthermore, a general comparison theorem (see e.g. Thm 3.2.12 and Cor 3.2.13 in [6]) between the spaces obtained by real interpolation gives an easy continuous imbedding

$$[E; D(A)]_s \subset [E; D(A)]_{\theta,1} \qquad \text{for any } 0 < \theta < s.$$

To sum up:

$$[E; D(A)]_{s,1} \subset D(A_s) \subset \cap_{0<\theta<s}[E; D(A)]_{\theta,1}, \quad (5)$$

a relation that will allow us to have a rather good idea of what $D(A_s)$ is, even if an exact characterization is missing. The tendency to prefer spaces of the type $[E; D(A)]_{\theta,1}$ $(0 < \theta < 1)$ will be explained later on by a simple coincidence of this scale of spaces with a well-known Sobolev class.

Given a bounded domain $\Omega \subset \mathbb{R}^N$ with regular boundary, we shall apply this general construction to the Dirichlet laplacian Δ properly defined on $L^p(\Omega)$ to create a generator $(1 \leq p \leq \infty)$:

- $E = L^p(\Omega)$ with $1 < p < \infty$: $D(\Delta) = W^{2,p} \cap W_0^{1,p}(\Omega)$

- $E = L^1(\Omega)$: $D(\Delta) = \{u \in W_0^{1,1}(\Omega) : \Delta u \in L^1(\Omega)\}$
- $E = C_0^0(\Omega)$: $D(\Delta) = \{u \in C_0^0(\Omega) : \Delta u \in C_0^0(\Omega)$ and $u \in W^{2,N+\varepsilon}(\Omega)\}$ (independently of $\varepsilon > 0$).

Here $C_0^0(\Omega) \subset L^\infty(\Omega)$ denotes the subspace of continuous functions in Ω with null limit at every point of $\partial\Omega$. For references on this classical subject see e.g. Ch VII in [18]. All these realizations give rise to an analytic C^0-semigroup $S(t) = e^{t\Delta}$ of contractions ($M = 1$) on E. Moreover, their action is coherent within these functional spaces, in the sense that the construction on $L^1(\Omega)$ extends the others. Application of the general s-power to this setting yields an analytic C^0-semigroup $S_s(t) = e^{t\Delta_s}$ of contractions on $L^1(\Omega)$, which are also contractions on $L^p(\Omega)$ for any $1 \le p \le \infty$ and particularly contractions on $L^\infty(\Omega)$. The opposite of the corresponding generator $-\Delta_s =: (-\Delta)^s$ is known as the *Dirichlet fractional laplacian* on Ω. To make it clearer, we choose to define one single operator Δ_s to be the realization of the s-power in the functional space $E = L^1(\Omega)$: it is indeed the 'largest' version of the s-power one can hope for.

Remark 2. Another equivalent way to get Δ_s on $L^1(\Omega)$ is to build it as the adjoint of the power defined in the $C_0^0(\Omega)$-setting. This seems less natural even if it could lead to a more general action on measures.

The main interest for switching from the original space $L^\infty(\Omega)$ to the wider space $L^1(\Omega)$ is that Δ_s may now operate on functions $u \in L^\infty(\Omega)$ which are not necessarily zero on the boundary. More specifically, we have

$$W^{2s,1}(\Omega) \subset D(\Delta_s) \subset W^{s,1}(\Omega) \qquad \text{for any } 0 < s < 1/2,$$
$$W_0^{2s,1}(\Omega) \subset D(\Delta_s) \subset W^{s,1}(\Omega) \qquad \text{for any } 1/2 < s < 1,$$

and in particular $D(\Delta_s)$ contains any constant function whenever $s < 1/2$ (this phenomenon may appear as rather surprising ...). The proof is essentially a combination of (5) with the trivial imbeddings $W_0^{2,1}(\Omega) \subset D(\Delta) \subset W_0^{1,1}(\Omega)$. Indeed, this gives

$$[L^1(\Omega); W_0^{2,1}(\Omega)]_{s,1} \subset D(\Delta_s) \subset \cap_{0<\theta<s}[L^1(\Omega); W_0^{1,1}(\Omega)]_{\theta,1},$$

where the spaces on the left and on the right are known to be standard Sobolev spaces, namely

$$[L^1(\Omega); W_0^{2,1}(\Omega)]_{s,1} = W_0^{2s,1}(\Omega) \text{ and } [L^1(\Omega); W_0^{1,1}(\Omega)]_{s,1} = W_0^{s,1}(\Omega).$$

Remark 3. More generally $[L^p(\Omega); W_0^{m,p}(\Omega)]_{\theta,p} = W_0^{m\theta,p}(\Omega)$ for any $1 \le p < \infty$ and any $0 < \theta < 1$, provided that $m\theta$ is not an integer. The statement for $1 < p < \infty$ may be found in books by H. Triebel or P. Grisvard, the case $p = 1$ being excluded for the sake of simplicity. In fact, $p = 1$ has nothing special in this result, even if it is really hard to find a reference in the literature.

To conclude, it is enough to recall that the lower subscript (for Dirichlet condition) may be dropped in $W_0^{\sigma,1}$ when σ is small, i.e.

$$W_0^{\sigma,1}(\Omega) = W^{\sigma,1}(\Omega) \qquad \text{for any } \sigma \in (0,1).$$

Working in $L^1(\Omega)$ is especially useful for this kind of simplification.

Remark 4. More generally $W_0^{\sigma,p}(\Omega) = W^{\sigma,p}(\Omega)$ for any $1 \le p < \infty$ and any $0 < \sigma < 1/p$. This coincidence is classical in the theory of (nonintegral) Sobolev spaces and reflects that the trace operator from $W^{\sigma,p}(\Omega)$ to $L^p(\partial\Omega)$ only makes sense for $\sigma > 1/p$.

To sum up: we have $W_0^{2s,1}(\Omega) \subset D(\Delta_s) \subset W^{s,1}(\Omega)$ for any $s \in (0,1) : s \neq 1/2$. (the right inclusion into $W^{s,1}(\Omega)$ is not sharp but easy to obtain).

Remark 5. More generally, the domain of the s-power built on $L^p(\Omega)$ is nested between $W_0^{2s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ for any $1 \leq p < \infty$, by the same argument (again whenever $s \neq 1/2$).

Coming back to the L^∞-setting, the preceding discussion sheds some light on the action of Δ_s on (regular) functions with arbitrary behavior on the boundary:

Lemma 2.1. *If $s < 1/2$ then $W^{1,\infty}(\Omega) \subset D(\Delta_s)$ and $\|\Delta_s u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{1,\infty}(\Omega)}$ for some constant C independent of $u \in W^{1,\infty}(\Omega)$.*

Note again that this result has been allowed by a fundamental change of functional spaces. In other words, using only L^∞-norms would make it impossible to drop the Dirichlet boundary condition on u and still let $\Delta_s u$ well-defined. Roughly speaking, for the term $\Delta_s u$ to make sense, it was "natural" to think that u should meet some requirement on the boundary $\partial\Omega$ in addition to some local regularity in Ω, but this belief is only true for $s > 1/2$ and not so much for $s < 1/2$. In any case, even if the Dirichlet boundary condition disappears from the domain $D(\Delta_s)$ when $s < 1/2$, it is still encoded in the operator itself (which really depends on the Dirichlet realization of the laplacian).

We now return to the solutions $u \in L^\infty(I \times \Omega)$ to the formal Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{div}_x\big(f(u)\big) - \Delta_s u = 0 \qquad \text{in } I \times \Omega, \\ u(t=0) = u_0 \in L^\infty(\Omega), \\ u = 0 \qquad \text{on } I \times \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where Δ_s has been built through (4) i.e.

$$(-\Delta)^s u := \frac{\sin(s\pi)}{\pi} \int_0^\infty \left(u - (1 - \frac{\Delta}{\lambda})^{-1} u \right) \frac{d\lambda}{\lambda^{1-s}}. \tag{6}$$

Remark 6. When $\Omega = \mathbb{R}^N$, the group invariance due to translations implies that $K_\lambda = (1 - \frac{\Delta}{\lambda})^{-1}$ is a convolution operator (by some Bessel-type function scaled by λ), so that (6) becomes in this case

$$\begin{aligned} -\Delta_s u = (-\Delta)^s u &= c\, \mathcal{F}^{-1}\Big(|\xi|^{2s} \mathcal{F}u(\xi)\Big) \\ &= c \lim_{H \to 0^+} \int_{\mathbb{R}^N} \mathbb{I}_{\{|x-y|>H\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy. \end{aligned} \tag{7}$$

for some (irrelevant) constant c depending on N and s. This representation of Δ_s as a 'principal value integral' in space is just one realization of Δ_s among many others.

3. The parabolic case ($2s > 1$).

Suppose $2s > 1$ in this section.

Definition 3.1. We say that $u \in L^\infty(I \times \Omega)$ is a semigroup-solution to problem ($\mathcal{P}$) when

$$u(t) = S_s(t) u_0 + \int_0^t \sigma_s(t - \tau) f(u(\tau)) d\tau \qquad (t \in I),$$

the equality being inderstood in the $L^\infty(I \times \Omega)$-sense (i.e. almost everywhere in $I \times \Omega$).

Here $\sigma_s(t)$ stands for the formal operator $S_s(t)\circ \mathrm{div}$ extended as a bounded operator acting on the whole of $L^p(\Omega)$ for any $1\le p\le\infty$ thanks to the following result:

Lemma 3.2. $\|\sigma_s(t)\|_{\mathcal{L}[L^p(\Omega)]}\le \dfrac{C}{t^{1/2s}} \qquad (0<t<\infty).$

The proof of Lemma 3.2 consists in applying the s-subordination to the local case ($s=1$), for which one can prove directly by different methods from local PDEs that the operator $\sigma(t):=S(t)\circ\mathrm{div}$ extends from $C_c^\infty(\Omega)$ to $L^p(\Omega)$ into a bounded operator satisfying

$$\|\sigma(t)\|_{\mathcal{L}[L^p(\Omega)]}\le C/\sqrt{t} \qquad (0<t<\infty)$$

for some constant C independent of t (and p). In fact, the same property holds for any first-order derivative applied on the left or on the right of S, namely

$$\|S(t)\circ\nabla\|_{\mathcal{L}[L^p(\Omega)]}+\|\nabla\circ S(t)\|_{\mathcal{L}[L^p(\Omega)]}\le C/\sqrt{t} \qquad (0<t<\infty).$$

Here the effect of s-subordination is just to exhibit the power $t^{1/2s}$ out of $\sqrt{t}$.

The existence and uniqueness of semigroup-solutions results from the fact that the mapping

$$\mathcal{F}: u\in L^\infty(I\times\Omega)\mapsto S_s(t)u_0+\int_0^t \sigma_s(t-\tau)f(u(\tau))d\tau\in L^\infty(I\times\Omega)$$

has a unique fixed point. The proof of this consists in showing that $\mathcal{F}$ has some strictly contractive iterate, by using some combinatorial properties of iterated convolutions (on $\mathbb{R}_+^*$) with the locally integrable power $t^{-1/2s}$ given by Lemma 3.2. In semigroup theory, performing fixed points in Duhamel's formula is a very common idea: the existence-uniqueness of semigroup-solutions claimed here is not new and may be read as Thm 3.3.3 of [14] for instance.

In the same spirit, $\mathcal{F}$ can be studied w.r.t. the finer norms $u\mapsto\|u\|_{L^\infty(I\times\Omega)}+\|t^{1/2s}\nabla_x u\|_{L^\infty(I\times\Omega)}$ and $u\mapsto\|u\|_{L^\infty(I\times\Omega)}+\|t^\gamma\partial_t u\|_{L^\infty(I\times\Omega)}$ for any fixed $0<\gamma<1$. As a result, the fixed point u of $\mathcal{F}$ is shown to have the additional regularity $t^{1/2s}\nabla_x u\in L^\infty(I\times\Omega)$ for any $u_0\in L^\infty(\Omega)$ and $t^\gamma\partial_t u\in L^\infty(I\times\Omega)$ for any $u_0\in L^\infty(\Omega)$ s.t. $\Delta_s^{1-\gamma}u_0\in L^\infty(\Omega)$. Given $t>0$, Lemma 2.1 then applies to $u(t,\cdot)\in W^{1,\infty}(\Omega)$ for any arbitrarily small value of the power $s(1-\gamma)$ (fix γ close to 1), to the effect that $\partial_t u\in L^\infty([\varepsilon,T]\times\Omega)$ (for any $\varepsilon>0$). The consequence of this first step is that u has first order-derivatives in $t>0$ and $x\in\Omega$. This shows also that $\Delta_s u=\partial_t u+\mathrm{div}(f(u))$ makes sense for any $t>0$, and accordingly that u evolves in the domain of the s-power viewed on any $L^p(\Omega)$, since

$$u(t)\in L^\infty(\Omega)\subset\cap_{1<p<\infty}L^p(\Omega):\Delta_s u(t)\in L^\infty(\Omega)\subset\cap_{1<p<\infty}L^p(\Omega) \qquad (t>0).$$

In particular, $u(t)\in\cap_{1<p<\infty}W_0^{s,p}(\Omega)$ by Remark 5, and the Dirichlet boundary condition persists for any positive time:

Theorem 3.3. *Suppose $f\in C^1(\mathbb{R})^N$. For any $u_0\in L^\infty(\Omega)$, the semigroup-solution u is continuous in $t>0$ and $x\in\overline{\Omega}$, and satisfies the Dirichlet boundary condition pointwise on $\partial\Omega$, i.e. $u\in C^0(I^*\times\Omega):\forall t>0\quad u(t)\in C_0^0(\Omega)$.*

A second step in the analysis consists in iterations of the additional regularity argument explained before, in order to get higher and higher regularity for u w.r.t. time and space. With this goal, it is tempting to differentiate the equation of u w.r.t. space, and try to check that the first-order derivatives $\nabla_x u$ satisfy a similar equation as u. This 'standard' way does not seem to work at all, since commuting symbols in

$$\nabla_x \sigma_s(t-\tau) f(u) = \nabla_x S_s(t-\tau)\Big(f'(u).\nabla_x u\Big) \neq S_s(t-\tau)\nabla_x\Big(f'(u).\nabla_x u\Big)$$

is a really serious obstacle: quantities like $f'(u).\nabla_x u$ (and later on $f_i'(u)\partial_i\partial_j u + f_i''(u)\partial_i u \partial_j u$ ... for higher-order derivatives) do not satisfy Dirichlet boundary conditions any more. For a precise statement of the 'standard' procedure, the reader may refer to Exercises 1 and 2 p.77 of [14], which fail to apply here, not because of a loss in local regularity but because of a loss in boundary conditions (in the notation of [14], the scale of spaces $(X^t)_{t\in\mathbb{R}_+}$ is destroyed by the nonlinearity $u \mapsto f'(u).\nabla u$, even if its 'regularity' looks good enough). From the viewpoint of abstract functional analysis, this phenomenon is essentially due to the fact that the operator (div) governing the scalar conservation law is not obtainable by functional calculus on the generator Δ. Besides, truncations do not appear easy to handle due to the nonlocality. Instead, our idea is rather to focus on time derivatives and show that

$$\Delta_s u, \quad \Delta_s\Big(\Delta_s u + \mathrm{div} f(u)\Big), \quad \Delta_s\Big(\Delta_s\big(\Delta_s u + \mathrm{div} f(u)\big) + \mathrm{div} f'(u)\partial_t u\Big)...$$

all make sense. This cascade of regularity relations is enough to make u infinitely smooth:

Theorem 3.4. *Suppose $f \in C^\infty(\mathbb{R})^N$. For any $u_0 \in L^\infty(\Omega)$, the semigroup-solution u is infinitely regular in $t > 0$ and $x \in \Omega$, i.e. $u \in C^\infty(I^* \times \Omega)$.*

We close this part with a simple remark on semigroup and weak solutions.

Definition 3.5. We say that $u \in L^\infty(I \times \Omega)$ is a weak solution to problem $(\mathcal{P})$ when

$$\int_{I\times\Omega} u\partial_t\varphi + f(u).\nabla_x\varphi + u\Delta_s\varphi = -\int_\Omega u_0\varphi(t=0)$$

for any test φ in the space $\mathcal{D}(I\times\Omega)$ made of all $C^\infty(I\times\Omega)$ functions with compact support in $[0,T[\times\Omega$.

Proposition 1. *Semigroup-solutions and weak solutions coincide.*

Combining Theorem 3.4 and Proposition 1, we get: (i) the existence of C^∞-smooth solutions satisfying classically the evolution equation as well as the Dirichlet boundary condition (only the initial value $u_0 \in L^\infty$ has to be interpreted weakly) and (ii) the uniqueness of weak solutions satisfying the problem in the sense of distributions (with boundary terms). This most favorable picture explains why the concept of entropy solutions, which yet makes sense for any $s \in (0,1)$, is more relevant to the hyperbolic case.

4. The hyperbolic case ($2s < 1$).

Suppose $2s < 1$ to fix ideas (in fact this section applies to any $0 < s < 1$ as explained in the introduction).

4.1. Convexity property of the nonlocal term.

Denoting by $k_\lambda = k_\lambda(x,y)$ the kernel of the operator $K_\lambda = (1 - \dfrac{\Delta}{\lambda})^{-1} : \phi \mapsto \int_\Omega k_\lambda(\cdot,y)\phi(y)dy$, it will prove fruitful to rewrite $-\Delta_s$ with operators of the type

$$L_r^R : \phi \mapsto \int_r^R \int_\Omega k_\lambda(x,y)\Big(\phi(x) - \phi(y)\Big)dy\frac{d\lambda}{\lambda^{1-s}} \qquad (0 < r < R < \infty),$$

for which holds the following convexity principle:

Lemma 4.1. $\forall \eta \in C^2(\mathbb{R})$ *convex* $\qquad L_r^R[\eta(\phi)] \leq \eta'(\phi) L_r^R[\phi]$.

Remark 7. This convexity principle is obviously the analog of Kato's inequality for the laplacian, namely

$$-\Delta[\eta \circ \phi] = -(\eta' \circ \phi)\Delta\phi - (\eta'' \circ \phi)\sigma\nabla\phi.\nabla\phi \leq -(\eta' \circ \phi)\Delta\phi.$$

With this end in mind, we introduce

$$(B_r\phi)(x) := \int_0^r \int_\Omega k_\lambda(x,y)\Big(\phi(x) - \phi(y)\Big)dy\frac{d\lambda}{\lambda^{1-s}},$$
$$(C_r\phi)(x) := \int_r^\infty \int_\Omega k_\lambda(x,y)\Big(\phi(x) - \phi(y)\Big)dy\frac{d\lambda}{\lambda^{1-s}},$$
$$b_r(x) := \int_0^r \Big(1 - \int_\Omega k_\lambda(x,y)dy\Big)\frac{d\lambda}{\lambda^{1-s}},$$
$$c_r(x) := \int_r^\infty \Big(1 - \int_\Omega k_\lambda(x,y)dy\Big)\frac{d\lambda}{\lambda^{1-s}},$$

as well as $a := b_r + c_r$, and we refer to the formal identity $-\Delta_s = a + B_r + C_r$ as the r-decomposition of Δ_s. To sum up: a is a multiplicative operator and B_r is a mild integral operator, while C_r acts as a kind of derivative of order $2s$.

Proposition 2. $\forall \phi \in D(\Delta) \quad -\Delta_s\phi = a\phi + B_r\phi + C_r\phi$, *Where each term is meaningful in $L^1(\Omega)$ individually. In general, $a \in L^1_{loc}(\Omega)$ is a nonnegative function whose global integrability is not garanteed, however the product $a\phi$ always falls in $L^1(\Omega)$.*

Remark 8. In $\mathbb{R}^N$, it is standard to analyze the integral representation (7) of Δ_s according to the proximity to the singularity, i.e.

$$-\Delta_s u = \int_{\mathbb{R}^N} \mathbb{I}_{\{|x-y|>H\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}}dy$$
$$+ \int_{\mathbb{R}^N} \mathbb{I}_{\{|x-y|<H\}} \frac{u(x) - u(y) - (x-y).\nabla u(x)}{|x-y|^{N+2s}}dy \qquad (0 < H < \infty).$$

Here, in some sense, a similar decomposition is made in Ω, but at a spectral level. Note also that the multiplicative part due to a vanishes in $\mathbb{R}^N$ because of the corresponding value of the marginal $\int_{\mathbb{R}^N} k_\lambda(x,y)dy = 1$ in this case.

4.2. **Entropy solutions.** We are now in a position to introduce the concept of entropy solution for problem $(\mathcal{P})$.

Definition 4.2. We say that $u \in L^\infty(I \times \Omega)$ is an entropic solution to $(\mathcal{P})$ if and only if

(*i*) $\forall \varphi \in \mathcal{D}_+(I \times \Omega) \quad \forall r \in \mathbb{R}_+^* \quad \forall \eta \in C^\infty(\mathbb{R})$ convex

$$\int_{I\times\Omega} \eta(u)\partial_t\varphi + \Phi(u).\nabla_x\varphi - \int_{I\times\Omega} \varphi a u \eta'(u) + \varphi\eta'(u)B_r u + \eta(u)C_r\varphi$$
$$+ \int_\Omega \eta(u_0)\varphi(t=0) \geq 0 \tag{8}$$

where Φ is the η-flux ($\Phi' = \eta' f'$) and where (a, B_r, C_r) is the r-decomposition of Section 4.1.

(*ii*) $u \in L^\infty(I; BV(\Omega))$ satisfies

$$\forall k \in \mathbb{R}_+ \qquad n_\Omega.\Big(f(\gamma(u)) - f(k)\Big)\, \mathbb{I}_{\{\gamma(u)>k\}} \geq 0 \qquad \text{a.e. on } I \times \partial\Omega,$$

$$\forall k \in \mathbb{R}_- \qquad n_\Omega.\Big(f(\gamma(u)) - f(k)\Big)\ \mathbb{I}_{\{\gamma(u)<k\}} \leq 0 \qquad \text{a.e. on } I \times \partial\Omega,$$

where $\gamma : BV(\Omega) \to L^1(\partial\Omega)$ stands for the usual trace operator (interpreted in the BV-sense).

Recall that $\mathcal{D}_+(I \times \Omega)$ is the positive cone of the space of Definition 3.5. Note that each term in (i) makes sense because of the boundedness of all quantities in u and because of the integrability of all quantities in φ (including the product $a\varphi$ of Proposition 2). Note also that the restrictions on the geometry of $\partial\Omega$ and on the BV-regularity of the solution only appear through the BLN-conditions of (ii). For BLN-conditions in purely local conservation laws, see the original paper [7].

The notion originates from the corresponding study [1] in $\mathbb{R}^N$. It has been designed mainly to lead to the following well-posedness and stability result:

Theorem 4.3. *Problem $(\mathcal{P})$ admits a unique entropic solution.*

Moreover, if u and v are two entropy solutions with respective initial data $u_0 \in L^\infty(\Omega)$ and $v_0 \in L^\infty(\Omega)$ then

$$\begin{cases} \textit{(maximum principle)} & |u| \leq \|u_0\|_{L^\infty(\Omega)} \qquad \textit{a.e. in } I \times \Omega \\ \textit{(contraction principle)} & \|u(t) - v(t)\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)} \\ \textit{(finite-infinite speed)} & \int_{\Omega\cap B(x_0,\rho)} |u - v|(t) \leq \int_{\Omega \cap B(x_0, \rho+Lt)} T_s(t)\ |u_0 - v_0| \end{cases}$$

for any $t \in I$ and $\rho > 0$, where

$$L := \sup_{(-l,+l)} |f'| \qquad \textit{with } l := \max\{\|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}\}.$$

Note that the 'finite-infinite speed' estimate is a refined L^1-contraction principle involving some localisations in balls. Therein, the symbol $T_s(t)$ refers to the semi-group generated by the s-power defined *on the whole* of $\mathbb{R}^N$ and not only in Ω as before (i.e. $T_s(t) \in \mathcal{L}[L^\infty(\mathbb{R}^N)]$ and $S_s(t) \in \mathcal{L}[L^\infty(\Omega)]$ should not be confused). For further discussion, see [1] and also [13].

The existence may be proved by a method of splitting like in [1] and [12]. The uniqueness results from the L^1-contraction principle which can be established by Kruzhkov's method of doubling variables. For this, the general idea is to follow the argument of [1] at a spectral level (applying Lemma 4.1 to C_r and optimizing in $r \to \infty$), in order to infer from (8) the so-called Kato inequality: $\forall \varphi \in \mathcal{D}_+(I \times \Omega)$

$$\int_{I\times\Omega} |u - v|\ \partial_t\varphi + F(u,v).\nabla_x\varphi + |u - v|\ \Delta_s\varphi + \int_\Omega |u_0 - v_0|\ \varphi(t=0) \geq 0. \quad (9)$$

where $F(u,v) := \operatorname{sign}(u-v)(f(u) - f(v))$. Afterwards, it turns out that local and nonlocal terms do not really interfere: the BLN-condition (ii) is only used to deal with the boundary terms that come out from testing (9) against a special approximation of the constant in space ($\mathbb{I} = \mathbb{I}_\Omega$) of the type

$$\phi_\mu := K_\mu \mathbb{I} \in D(\Delta) \uparrow \mathbb{I} \qquad \text{as } \mu \uparrow \infty.$$

Indeed, given $\psi \in \mathcal{D}_+(I)$, any nonnegative element $\phi = \phi(x)$ in $D(\Delta) = \{\phi \in W_0^{1,1}(\Omega) : \Delta\phi \in L^1(\Omega)\}$ yields a test function $\varphi(t,x) := \psi(t)\phi(x)$ which is admissible in the entropic formulation (8). By approximation and continuity of each term in (8), this just results from the density of $\mathcal{D}(\Omega)$ in $D(\Delta)$ w.r.t. the graph norm.

We close this part by a simple remark connecting Definition 4.2 to weak solutions:

Lemma 4.4. *Every entropic solution u of ($\mathcal{P}$) is a weak solution in the sense of the following distributional formulation:* $\forall\varphi \in \mathcal{D}(I \times \Omega)$

$$\int_{I\times\Omega} u\partial_t\varphi + f(u).\nabla_x\varphi + u\Delta_s\varphi = -\int_\Omega u_0\varphi(t=0).$$

5. **Maximum principle.** For classical solutions to ($\mathcal{P}$) the maximum principle is a typical instance of a property common to all powers $s \in (0,1)$, and which can be checked by easy means:

Proposition 3. *Assume Ω bounded. If u is a solution to ($\mathcal{P}$) which is continuous in $I \times \overline{\Omega}$ then*

$$\inf_\Omega -u_0^- \leq u \leq \sup_\Omega u_0^+ \qquad \textit{pointwise in } I \times \Omega,$$

and in particular $\|u\|_{L^\infty(I\times\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$.

Proof. For fixed $\varepsilon > 0$, let $(t,x) \in I \times \overline{\Omega}$ be a point where the maximum

$$M_\varepsilon := \sup_{(t,x)\in I\times\overline{\Omega}} u(t,x) - \frac{\varepsilon}{T-t}$$

is attained. If $t = 0$ then $M_\varepsilon \leq u(t,x) \leq \|u_0^{(+)}\|_{L^\infty(\Omega)}$. If $0 < t < T$ and $x \in \partial\Omega$ then $M_\varepsilon \leq 0 \leq \|u_0^{(+)}\|_{L^\infty(\Omega)}$. Otherwise, $(t,x) \in]0,T[\times\Omega$ is an extremal point at which all first-order derivatives are necessarily zero

$$\partial_t\left(u(t,x) - \frac{\varepsilon}{T-t}\right) = 0 = \nabla_x\left(u(t,x) - \frac{\varepsilon}{T-t}\right),$$

hence

$$(\Delta_s u)(t,x) = \Big(\partial_t u + \mathrm{div}_x(f(u))\Big)(t,x) = +\frac{\varepsilon}{(T-t)^2} > 0.$$

Now, by the maximum principle for the Dirichlet laplacian in Ω, we can say that the function $u(t,\cdot)$, which is bounded above in Ω by the constant $u(t,x)$, is transformed into a function $K_\lambda u(t,\cdot)$, which is also bounded above in Ω by the same constant $u(t,x)$. In particular, we recover the nonnegativity $(u - K_\lambda u)(t,x) \geq 0$ at (t,x), and consequently

$$(-\Delta_s u)(t,x) = c\int_0^\infty (u - K_\lambda u)(t,x)\frac{d\lambda}{\lambda^{1-s}} \geq 0.$$

This contradiction on the sign of $\Delta_s u$ shows that the maximum in M_ε cannot occur in the interior of the cylinder $I \times \Omega$. We have thus proved $M_\varepsilon \leq \|u_0^{(+)}\|_{L^\infty(\Omega)}$ in any case, so that $\varepsilon \to 0^+$ gives the expected pointwise inequality $u \leq \|u_0^{(+)}\|_{L^\infty(\Omega)}$ on $I \times \Omega$. The estimate from below follows the same pattern. □

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