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BOUNDARY VALUE PROBLEM FOR A PHASE TRANSITION MODEL

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Abstract. We consider the boundary value problem for the phase transition (PT) model, introduced in [4] and in [7]. By using the wave-front tracking technique, we prove existence of solutions when the initial and boundary conditions have finite total variation.

1. **Introduction.** The paper deals with the initial-boundary value problem for the phase transition model (PT for short), introduced in [4] and in [7] for modeling car traffic in an unidirectional road. Traffic models based on differential equations can be divided mainly in two classes: microscopic and macroscopic. The PT model belongs to the class of macroscopic traffic models, and it is composed by a system of two differential equations, which impose the conservation of the number of vehicles and of a momentum.

The fluid dynamic and macroscopic approach for car traffic was, in turn, initiated by the seminal work of Lighthill-Whitham and Richards [12, 13], known as the LWR model. It consists in a differential equation stating the conservation of the number of vehicles. It well describes the evolution of free traffic, but it is not accurate for congested traffic. Hence second order models, i.e. system with two equations, were introduced in the literature. Among these we mention the model proposed by Aw and Rascle in [3] and independently by Zhang in [14]. In 2002, Colombo proposed a second order model with phase transitions [6], in order to describe the different behaviors of traffic in free and congested regimes. In free flow the model is a classical LWR model, whereas in the congested flow it is a system of two equations: the first one is the conservation of the number of vehicles, while the second one describes the evolution of a linearized momentum. The system considered in this paper is a modification of the Colombo phase-transition model.

Here we consider the following initial-boundary value problem for the PT model

$$\left\{\begin{array}{ll} \partial_t \rho + \partial_x(\rho V) = 0, & \text{if } (\rho,q)\in\Omega_f, \\ \left\{\begin{array}{l}\partial_t \rho + \partial_x(\rho v(\rho,q)) = 0, \\ \partial_t q + \partial_x(q v(\rho,q)) = 0.\end{array}\right. & \text{if } (\rho,q)\in\Omega_c, \\ (\rho,q)\,(t,a) = (\rho_a(t), q_a(t)) & \\ (\rho,q)\,(t,b) = (\rho_b(t), q_b(t)) & \\ (\rho,q)\,(0,x) = (\rho_0(x), q_0(x)), & \end{array}\right. \qquad \text{if } x\in(a,b),\ t>0$$

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where $(\rho_0, q_0) : (a,b) \to \Omega_f \cup \Omega_c$ is the initial datum, $(\rho_a, q_a) : (0,+\infty) \to \Omega_f \cup \Omega_c$ and $(\rho_b, q_b) : (0,+\infty) \to \Omega_f \cup \Omega_c$ are the boundary data and Ω_f, Ω_c denote respectively the free and congested phase (see Section 2 for a rigorous definition). We prove existence of solutions by means of the wave-front tracking technique, provided the initial and boundary conditions are BV functions. The wave-front tracking technique consists in constructing a piecewise constant approximate solution and in proving that every limit point is indeed a solution of the problem. The key estimate to obtain compactness for the sequence of approximated solutions is a uniform bound of a functional measuring the strength of waves. Moreover bounds on the number of waves and interactions for wave-front tracking approximate solutions are given. We remark that imposing boundary conditions for systems of balance laws is a quite delicate issue, especially for characteristic boundary conditions; see [1, 2, 9] and the references therein. In particular, the PT system with boundary is characteristic, since phase-transition waves can travel with zero speed.

The paper is organized as follows. In Section 2 the phase transition model is presented, and, in Section 3, a theorem about the existence of solutions for the Cauchy problems with boundary data is stated and proved.

2. **Description of the phase transition model.**

We describe the phase transition model, introduced in [4], with the Newell-Daganzo velocity function. The PT model describes the evolution of traffic trough the macroscopic variables ρ and q, representing respectively the *density* and the *linearized momentum*; see also [6]. In this model, it is assumed that cars behave differently, depending on the fact that the traffic is low or heavy. Therefore there are two phases: the free phase, denoted by Ω_f, and the congested one, denoted by Ω_c. These two phases are described by the sets

$$\begin{cases} \Omega_f = \left\{(\rho,q) \in [0,R] \times [0,+\infty[\colon\ q = \frac{R(\rho-\sigma)}{\sigma(R-\rho)},\ 0 \le \rho \le \sigma_+\right\} \\ \Omega_c = \left\{(\rho,q) \in [0,R] \times [0,+\infty[\colon\ v(\rho,q) \le V,\ \frac{q^-}{R} \le \frac{q}{\rho} \le \frac{q^+}{R}\right\} \end{cases} \tag{1}$$

where $V > 0$ is the velocity in the free phase, $R > 0$ is the maximal density, $-1 < q^- < 0 < q^+ < 1$, $\frac{R}{5} < \sigma < R$ and $v : \Omega_f \cup \Omega_c \to [0,+\infty[$, defined by

$$\begin{cases} v(\rho,q) = V, & \text{if } (\rho,q) \in \Omega_f, \\ v(\rho,q) = v^{eq}(\rho)(1+q) = \frac{V\sigma}{R-\sigma}\left(\frac{R}{\rho} - 1\right)(1+q), & \text{if } (\rho,q) \in \Omega_c, \end{cases} \tag{2}$$

is the velocity; see Figure 1. The PT model assumes that the velocity of cars in the free phase Ω_f is constantly equal to V, while in the congested phase Ω_c is a perturbation of the *equilibrium velocity* $v^{eq}(\rho)$. The equilibrium velocity $v^{eq}(\rho)$ represents the desired speed of cars when the density is ρ. Moreover, the constant σ is called *critical density* and corresponds to the density for which the flux is maximal in scalar models. The constants q^- and q^+ are respectively the minimum and the maximum value of the momentum. When $q = 0$, the velocity of cars v coincides with the equilibrium velocity. Finally, the constants $\sigma_- > 0$ and $\sigma_+ > 0$ are defined respectively by the ρ-component of the solutions in Ω_c to the systems

$$\begin{cases} v(\rho,q) = V, \\ \frac{q}{\rho} = \frac{q^-}{R}, \end{cases} \qquad \text{and} \qquad \begin{cases} v(\rho,q) = V, \\ \frac{q}{\rho} = \frac{q^+}{R}. \end{cases}$$

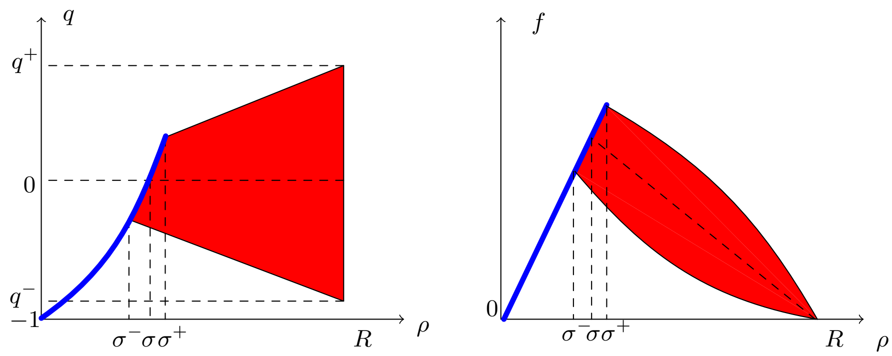

FIGURE 1. The fundamental diagram for the phase transition model in (ρ, q) and $(\rho, \rho v)$ coordinates. The free phase Ω_f is the one-dimensional region in blue. The congested phase Ω_c is the two-dimensional region in red. Note that $\Omega_f \cap \Omega_c \neq \emptyset$.

The solution of each system exists and is unique. The first system describes the point $(\rho, q) \in \Omega_c$ such that $v(\rho, q) = V$, i.e. (ρ, q) belongs also to Ω_f, and $\frac{q}{\rho} = \frac{q^-}{R}$, i.e. (ρ, q) belongs to the lower side of Ω_c; see Figure 1. The second system, instead, describes the point $(\rho, q) \in \Omega_c$ such that $v(\rho, q) = V$, i.e. (ρ, q) belongs also to Ω_f, and $\frac{q}{\rho} = \frac{q^+}{R}$, i.e. (ρ, q) belongs to the upper side of Ω_c; see Figure 1.

The model in the free phase Ω_f reads

$$\partial_t \rho + \partial_x(\rho V) = \partial_t \rho + V \partial_x(\rho) = 0, \tag{3}$$

while in the congested phase Ω_c reads

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v(\rho, q)) = 0. \end{cases} \tag{4}$$

The eigenvalues for (4) are

$$\lambda_1(\rho, q) = (1 + 2q)v^{eq}(\rho) + \rho(1 + q)(v^{eq})'(\rho), \quad \lambda_2(\rho, q) = (1 + q)v^{eq}(\rho), \tag{5}$$

while the respectively eigenvectors are

$$r_1 = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad r_2 = \begin{pmatrix} v^{eq}(\rho) \\ -(1 + q)(v^{eq})'(\rho) \end{pmatrix}. \tag{6}$$

Moreover the first characteristic speed is genuinely nonlinear if $q \neq 0$ and it is linearly degenerate if $q = 0$. Instead the second characteristic speed is linearly degenerate. A deeper analysis of (4) is contained in [4]. We denote with $L_1(\rho; \rho_0, q_0)$ and $L_2(\rho; \rho_0, q_0)$ respectively the Lax curves of the first and second family for system (4); see Figure 2. We recall that $(\rho, L_1(\rho; \rho_0, q_0))$ are lines in the (ρ, q) plane passing through the origin and $(\rho, L_2(\rho; \rho_0, q_0))$ are lines in the $(\rho, \rho v)$ plane passing through the origin. Let us introduce the functions

$$\psi_1 : \Omega_c \to \Omega_c, \qquad \psi_2^- : \Omega_c \to \Omega_c, \qquad \psi_2^+ : \Omega_c \to \Omega_c, \tag{7}$$

such that, for every $(\rho_0, q_0) \in \Omega_c$, $\psi_1(\rho_0, q_0)$, $\psi_2^-(\rho_0, q_0)$ and $\psi_2^+(\rho_0, q_0)$ are defined respectively by the solutions in Ω_c to the systems

$$\begin{cases} \frac{q}{\rho} = \frac{q_0}{\rho_0}, \\ v(\rho, q) = V, \end{cases} \qquad \begin{cases} \frac{q}{\rho} = \frac{q^-}{R}, \\ v(\rho, q) = v(\rho_0, q_0), \end{cases} \qquad \begin{cases} \frac{q}{\rho} = \frac{q^+}{R}, \\ v(\rho, q) = v(\rho_0, q_0). \end{cases} \tag{8}$$

Remark 1. As in the case of the definitions of σ^- and σ^+, each system in (8) admits a unique solution, due to the fact that the Lax curve of the first and second families are transverse.

The first system in (8) selects the point $(\hat{\rho}, \hat{q}) \in \Omega_c$ on the Lax curve of the first family through (ρ_0, q_0) such that $v(\hat{\rho}, \hat{q}) = V$, i.e. $(\hat{\rho}, \hat{q})$ belongs also to the free phase Ω_f; see Figure 2.

Similarly the second and third system in (8) select, the point $(\hat{\rho}, \hat{q}) \in \Omega_c$ on the Lax curve of the second family through (ρ_0, q_0), i.e. $v(\hat{\rho}, \hat{q}) = v(\rho_0, q_0)$, such that it belongs respectively on the lower and upper side of Ω_c; see Figure 2.

The following lemma implies that waves of the first family have negative speed, while waves of second family have positive speed.

Lemma 2.1. *The first eigenvalue λ_1 is strictly negative in Ω_c. The second eigenvalue λ_2 is positive in Ω_c.*

For a proof, see [10, Lemma 3.6]. The next proposition states that the Lax curves are Lipschitz continuous.

Proposition 1. *Since $-1 < q^- < 0 < q^+ < 1$, the Lax curves of the first family in the $(\rho, \rho v)$ plane, parametrized by ρ, are uniformly bi-Lipschitz in Ω_c. For every $0 < \bar{v} < V$, the Lax curves of the second family in the $(\rho, \rho v)$ plane, parametrized by ρ, are uniformly bi-Lipschitz in the set $\{(\rho, q) \in \Omega_c : v(\rho, q) \geq \bar{v}\}$.*

For a proof, see [10, Proposition 3.7]. For the PT model, there is a third kind of waves, namely phase-transition waves. They are waves connecting a state $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ (on the left) with a state $(\rho^r, q^r) \in \Omega_c$ (on the right). More precisely, if one considers the following Riemann problem on the whole line $\mathbb{R}$

$$\begin{cases} \partial_t \rho + \partial_x(\rho V) = 0, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v(\rho, q)) = 0. \end{cases} & \text{if } (\rho, q) \in \Omega_c, \qquad \text{if } x \in \mathbb{R},\ t > 0 \\ (\rho, q)(0, x) = \begin{cases} (\rho^l, q^l) & \text{if } x < 0, \\ (\rho^r, q^r) & \text{if } x > 0, \end{cases} \end{cases} \tag{9}$$

with $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) \in \Omega_c \setminus \Omega_f$, then its solution is given by the following phase-transition wave

$$(\rho, q)(t, x) = \begin{cases} (\rho^l, q^l) & \text{if } x < \lambda_1 t,\ t > 0 \\ (\rho^m, q^m) & \text{if } \lambda_1 t < x < \lambda_2 t,\ t > 0 \\ (\rho^r, q^r) & \text{if } x > \lambda_2 t,\ t > 0, \end{cases} \tag{10}$$

where $(\rho^m, q^m) \in \Omega_c$ is the unique solution to

$$\begin{cases} \frac{q}{\rho} = \frac{q^-}{R} \\ v(\rho, q) = v(\rho^r, q^r) \end{cases}$$

i.e. (ρ^m, q^m) is the point belonging to the lower part of the congested phase Ω_c and to the Lax curve of second family through (ρ^r, q^r), and

$$\lambda_1 = \frac{\rho^m v(\rho^m, q^m) - \rho^l v(\rho^l, q^l)}{\rho^m - \rho^l} \qquad \lambda_2 = v(\rho^r, q^r).$$

Note that λ_1 is given by the Rankine-Hugoniot condition; for complete details see [4].

Introduce the following definition.

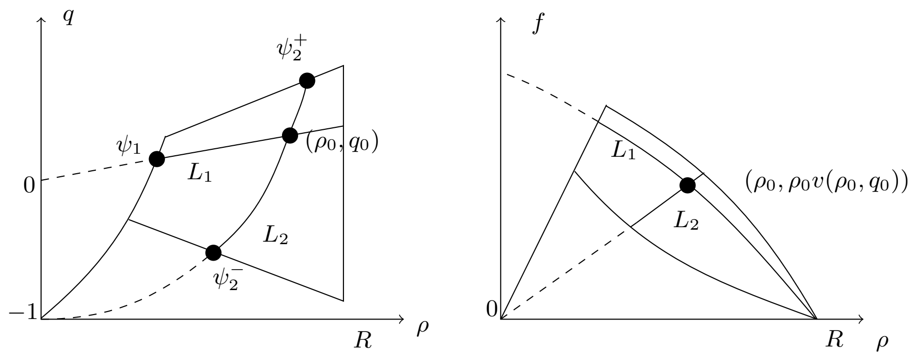

FIGURE 2. Shape of Lax curves in (ρ, q) and $(\rho, \rho v(\rho, q))$ planes.

Definition 2.2. Let I be a real interval. We say that a function $(\rho, q) : I \to \Omega_f \cup \Omega_c$ satisfies the assumption **(H-1)** in I if there exists $0 < \bar{v} < V$ such that, for a.e. $x \in I$, $v((\rho, q)(x)) \geq \bar{v}$.

Remark 2. Note that, since $0 < \bar{v} < V$, a point $(\hat{\rho}, \hat{q})$ satisfying $v(\hat{\rho}, \hat{q}) \geq \bar{v}$ either belongs to the free phase Ω_f or belongs to a subset of the congested phase Ω_c. Indeed if $(\hat{\rho}, \hat{q}) \in \Omega_f$, then $v(\hat{\rho}, \hat{q}) = V$ and so $v(\hat{\rho}, \hat{q}) \geq \bar{v}$. Instead if $(\hat{\rho}, \hat{q}) \in \Omega_c$, then $(\hat{\rho}, \hat{q})$ is at the left of the Lax curve of the second family $v(\rho, q) = \bar{v}$.

Remark 3. For every $\bar{v} > 0$, the set

$$\{(\rho, q) \in \Omega_f \cup \Omega_c : v(\rho, q) \geq \bar{v}\}$$

in invariant for the solution to the Riemann problem. Moreover, assumption **(H-1)** of Definition 2.2 guarantees that the total variation of a solution can be controlled by the total variation of the Riemann invariants of the solution itself.

We introduce also a distance in the set $\Omega_f \cup \Omega_c$ in the following way. First define the function $\omega : \Omega_f \cup \Omega_c \to \mathbb{R}$

$$\omega(\rho, q) = \begin{cases} \frac{q}{\rho} & \text{if } \frac{q}{\rho} \geq \frac{q^-}{R}, \\ V\rho - \sigma_- V + \frac{q^-}{R} & \text{otherwise.} \end{cases} \tag{11}$$

Secondly, given two states (ρ^l, q^l), (ρ^r, q^r) in the set $\Omega_f \cup \Omega_c$:

1. if $(\rho^l, q^l) \in \Omega_f$ and $(\rho^r, q^r) \in \Omega_f$, then
$$d\left(\left(\rho^l, q^l\right), \left(\rho^r, q^r\right)\right) = \left|\omega\left(\rho^l, q^l\right) - \omega\left(\rho^r, q^r\right)\right|; \tag{12}$$
2. if $(\rho^l, q^l) \in \Omega_c$ and $(\rho^r, q^r) \in \Omega_c$, then, denoting by $(\rho^m, q^m) \in \Omega_c$ the point satisfying $q^m = L_1\left(\rho^m; \rho^l, q^l\right)$ and $q^r = L_2\left(\rho^r; \rho^m, q^m\right)$,
$$d\left(\left(\rho^l, q^l\right), \left(\rho^r, q^r\right)\right) = \left|v\left(\rho^l, q^l\right) - v\left(\rho^m, q^m\right)\right| + \left|\omega\left(\rho^m, q^m\right) - \omega\left(\rho^r, q^r\right)\right|; \tag{13}$$
3. in the remaining case, i.e. $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) \in \Omega_c \setminus \Omega_f$ or viceversa, we put
$$\begin{aligned} d\left(\left(\rho^l, q^l\right), \left(\rho^r, q^r\right)\right) = {} & d\left(\left(\rho^l, q^l\right), \left(\sigma_-, \frac{q^-\sigma_-}{R}\right)\right) \\ & + d\left(\left(\sigma_-, \frac{q^-\sigma_-}{R}\right), \left(\rho^r, q^r\right)\right). \end{aligned} \tag{14}$$

Remark 4. Note that equations (12) and (13) can be both rewritten as

$$d\left(\left(\rho^l,q^l\right),\left(\rho^r,q^r\right)\right)=\left|v\left(\rho^l,q^l\right)-v\left(\rho^r,q^r\right)\right|+\left|\omega\left(\rho^l,q^l\right)-\omega\left(\rho^r,q^r\right)\right|, \tag{15}$$

since in case 1. $v\left(\rho^l,q^l\right)=v\left(\rho^r,q^r\right)$, and in case 2. $\omega\left(\rho^l,q^l\right)=\omega\left(\rho^m,q^m\right)$ and $v\left(\rho^m,q^m\right)=v\left(\rho^r,q^r\right)$.

Remark 5. The function ω, defined in (11), is indeed a continuous function. It is clear that $\frac{q}{\rho}$ and $V\rho-\sigma_-V+\frac{q^-}{R}$ are both continuous functions. There exists exactly one point $(\rho,q)\in\Omega_f\cap\Omega_c$ satisfying $\frac{q}{\rho}=\frac{q^-}{R}$. This point is $\left(\sigma_-,\frac{q^-\sigma_-}{R}\right)$. We have:

$$\left(\frac{q}{\rho}\right)_{|\left(\sigma_-,\frac{q^-\sigma_-}{R}\right)}=\frac{q^-}{R}=\left(V\rho-\sigma_-V+\frac{q^-}{R}\right)_{|\left(\sigma_-,\frac{q^-\sigma_-}{R}\right)}.$$

This proves that ω is continuous.

Lemma 2.3. *The function* $d:(\Omega_f\cup\Omega_c)\times(\Omega_f\cup\Omega_c)\to\mathbb{R}^+$, *defined in (12), (13) and (14), is a metric on* $\Omega_f\cup\Omega_c$.

Proof. First we prove that $d\left(\left(\rho^l,q^l\right),\left(\rho^r,q^r\right)\right)=0$ if and only if $\left(\rho^l,q^l\right)=\left(\rho^r,q^r\right)$. It is clear that if $\left(\rho^l,q^l\right)=\left(\rho^r,q^r\right)$, then $d\left(\left(\rho^l,q^l\right),\left(\rho^r,q^r\right)\right)=0$. Assume therefore that $\left(\rho^l,q^l\right)\neq\left(\rho^r,q^r\right)$. We have different possibilities.

1. $\left(\rho^l,q^l\right)\in\Omega_f$ and $\left(\rho^r,q^r\right)\in\Omega_f$. We have that the function ω restricted to the set Ω_f can be viewed as a strictly increasing function of the variable ρ. Since $\left(\rho^l,q^l\right)\neq\left(\rho^r,q^r\right)$, we may suppose, without loss of generalities, that $\rho^l<\rho^r$ and so
$$d\left(\left(\rho^l,q^l\right),\left(\rho^r,q^r\right)\right)=\left|\omega\left(\rho^l,q^l\right)-\omega\left(\rho^r,q^r\right)\right|=\omega\left(\rho^r,q^r\right)-\omega\left(\rho^l,q^l\right)>0.$$
2. $\left(\rho^l,q^l\right)\in\Omega_c$ and $\left(\rho^r,q^r\right)\in\Omega_c$. Call $\left(\rho^m,q^m\right)\in\Omega_c$ the point defined in the point 2. of the construction of the function d. Since $\left(\rho^l,q^l\right)\neq\left(\rho^r,q^r\right)$, either $\left(\rho^l,q^l\right)\neq\left(\rho^m,q^m\right)$ and so
$$d\left(\left(\rho^l,q^l\right),\left(\rho^r,q^r\right)\right)>\left|v\left(\rho^l,q^l\right)-v\left(\rho^m,q^m\right)\right|>0$$
or $\left(\rho^r,q^r\right)\neq\left(\rho^m,q^m\right)$ and so
$$d\left(\left(\rho^l,q^l\right),\left(\rho^r,q^r\right)\right)>\left|\omega\left(\rho^l,q^l\right)-\omega\left(\rho^m,q^m\right)\right|>0.$$
3. $\left(\rho^l,q^l\right)\in\Omega_f$ and $\left(\rho^r,q^r\right)\in\Omega_c$. Since $\left(\rho^l,q^l\right)\neq\left(\rho^r,q^r\right)$, we have $\left(\rho^l,q^l\right)\neq\left(\sigma_-,\frac{q^-\sigma_-}{R}\right)$ or $\left(\rho^r,q^r\right)\neq\left(\sigma_-,\frac{q^-\sigma_-}{R}\right)$. Therefore we conclude that
$$\begin{aligned}d\left(\left(\rho^l,q^l\right),\left(\rho^r,q^r\right)\right)&=d\left(\left(\rho^l,q^l\right),\left(\sigma_-,\frac{q^-\sigma_-}{R}\right)\right)\\&\quad+d\left(\left(\sigma_-,\frac{q^-\sigma_-}{R}\right),\left(\rho^r,q^r\right)\right)>0.\end{aligned}$$
4. $\left(\rho^l,q^l\right)\in\Omega_c$ and $\left(\rho^r,q^r\right)\in\Omega_f$. This case is completely analogous to the previous one.

We prove now that d is symmetric. Fix two states $\left(\rho^l,q^l\right)$ and $\left(\rho^r,q^r\right)$ in $\Omega_f\cup\Omega_c$. If both states belong to Ω_f, then the conclusion easily follows by (12).

If both states belong to Ω_c, then define the states $(\rho^m, q^m) \in \Omega_c$ and $(\rho^s, q^s) \in \Omega_c$ by

$$q^m = L_1\left(\rho^m; \rho^l, q^l\right) \qquad q^r = L_2\left(\rho^r; \rho^m, q^m\right)$$
$$q^s = L_1\left(\rho^s; \rho^r, q^r\right) \qquad q^l = L_2\left(\rho^l; \rho^s, q^s\right).$$

One can easily deduce that

$$\omega\left(\rho^l, q^l\right) = \omega\left(\rho^m, q^m\right) \qquad \omega\left(\rho^r, q^r\right) = \omega\left(\rho^s, q^s\right)$$
$$v\left(\rho^r, q^r\right) = v\left(\rho^m, q^m\right) \qquad v\left(\rho^l, q^l\right) = v\left(\rho^s, q^s\right)$$

and so

$$\begin{aligned} d\left(\left(\rho^l, q^l\right), \left(\rho^r, q^r\right)\right) &= \left|v\left(\rho^l, q^l\right) - v\left(\rho^m, q^m\right)\right| + \left|\omega\left(\rho^m, q^m\right) - \omega\left(\rho^r, q^r\right)\right| \\ &= \left|v\left(\rho^s, q^s\right) - v\left(\rho^r, q^r\right)\right| + \left|\omega\left(\rho^l, q^l\right) - \omega\left(\rho^s, q^s\right)\right| \\ &= d\left(\left(\rho^r, q^r\right), \left(\rho^l, q^l\right)\right). \end{aligned}$$

In the remaining case, the conclusion easily follows from the fact that the point $\left(\sigma_-, \frac{q^-\sigma_-}{R}\right)$ belongs to both Ω_f and Ω_c.

We prove now that the triangular inequality holds for d. Fix three states (ρ_1, q_1), (ρ_2, q_2), and (ρ_3, q_3) in $\Omega_f \cup \Omega_c$. If they both belong either to Ω_f or to Ω_c, then, by equation (15), the triangular inequality easily follows. The triangular inequality for the general case follows from the previous case. □

3. The initial-boundary value problem.

Fix $a, b \in \mathbb{R}$ with $a < b$. Consider the following initial boundary value problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho V) = 0, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v(\rho, q)) = 0. \end{cases} & \text{if } (\rho, q) \in \Omega_c, \\ (\rho, q)(t, a) = (\rho_a(t), q_a(t)) \\ (\rho, q)(t, b) = (\rho_b(t), q_b(t)) \\ (\rho, q)(0, x) = (\rho_0(x), q_0(x)), \end{cases} \quad \text{if } x \in (a, b),\ t > 0 \qquad (16)$$

where $(\rho_0, q_0) : (a, b) \to \Omega_f \cup \Omega_c$, $(\rho_a, q_a) : (0, +\infty) \to \Omega_f \cup \Omega_c$ and $(\rho_b, q_b) : (0, +\infty) \to \Omega_f \cup \Omega_c$. First introduce the definition of solution to (16).

Definition 3.1. The function

$$(\hat{\rho}, \hat{q}) \in C([0, +\infty[; L^1((a, b); \Omega_f \cup \Omega_c))$$

is a solution to (16) if

1. the function $(\hat{\rho}, \hat{q})$ is a weak solution to
$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v(\rho, q)) = 0. \end{cases} & \text{if } (\rho, q) \in \Omega_c, \end{cases}$$
for $(t, x) \in (0, +\infty) \times (a, b)$;
2. for a.e. $t > 0$, the function $x \mapsto (\hat{\rho}(t, x), \hat{q}(t, x))$ has a version with bounded total variation;

3. for a.e. $t > 0$, the Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v(\rho, q)) = 0. \end{cases} & \text{if } (\rho, q) \in \Omega_c, \\ (\rho, q)(0, x) = \begin{cases} (\rho_a, q_a)(t) & \text{if } x < a \\ (\hat{\rho}, \hat{q})(t, a+) & \text{if } x > a \end{cases} \end{cases} \qquad \tau > 0, x \in \mathbb{R}$$

admits a self similar solution $(\tilde{\rho}, \tilde{q})$ such that, for a.e. $\tau > 0$,

$$(\tilde{\rho}, \tilde{q})(\tau, a+) = (\hat{\rho}, \hat{q})(t, a+).$$

4. for a.e. $t > 0$, the Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v(\rho, q)) = 0. \end{cases} & \text{if } (\rho, q) \in \Omega_c, \\ (\rho, q)(0, x) = \begin{cases} (\hat{\rho}, \hat{q})(t, b-) & \text{if } x < b \\ (\rho_b, q_b)(t) & \text{if } x > b \end{cases} \end{cases} \qquad \tau > 0, x \in \mathbb{R}$$

admits a self similar solution $(\tilde{\rho}, \tilde{q})$ such that, for a.e. $\tau > 0$,

$$(\tilde{\rho}, \tilde{q})(\tau, b-) = (\hat{\rho}, \hat{q})(t, b-).$$

5. for a.e. $x \in (a, b)$, $(\hat{\rho}(0, x), \hat{q}(0, x)) = (\rho_0(x), q_0(x))$.

Remark 6. As usual in conservation laws, boundary conditions are a delicate issue. In this paper the boundaries are characteristic, since there are phase-transition waves with zero speed. Conditions 3 and 4 of Definition 3.1 are exactly the same as the boundary condition in the characteristic case in [1].

We state the main result of the paper whose proof is contained in the next subsections.

Theorem 3.2. *Fix the initial condition $(\rho_0, q_0) \in BV((a, b); \Omega_f \cup \Omega_c)$, and the boundary data $(\rho_a, q_a), (\rho_b, q_b) \in \left(BV \cap L^1\right)((0, +\infty); \Omega_f \cup \Omega_c)$. Assume that both the initial condition (ρ_0, q_0) and the boundary data (ρ_a, q_a) and (ρ_b, q_b) satisfy the assumption* ***(H-1)****, in the sense of Definition 2.2. Then there exists $(\hat{\rho}, \hat{q})$, a solution to (16) in the sense of Definition 3.1.*

3.0.1. *Wave-front tracking technique.* We construct piecewise constant approximations via the wave-front tracking technique; see [5, 8, 11] for the general theory.

Definition 3.3. Given $\varepsilon > 0$, we say that the map $\bar{u}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{q}_\varepsilon)$ is an ε-approximate wave-front tracking solution to (16) if there exist $\bar{u}_{a,\varepsilon} = (\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})$ and $\bar{u}_{b,\varepsilon} = (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})$ such that the following conditions hold.

1. $\bar{u}_\varepsilon \in C((0, +\infty); L^1((a, b); \Omega_f \cup \Omega_c))$ and $\bar{u}_{a,\varepsilon}, \bar{u}_{b,\varepsilon} \in L^1((0, +\infty); \Omega_f \cup \Omega_c)$.
2. $(\bar{\rho}_\varepsilon, \bar{q}_\varepsilon)$ is piecewise constant, with discontinuities occurring along finitely many straight lines in $(0, +\infty) \times (a, b)$. Moreover the jumps can be of the first family, of the second family, or phase-transition waves. They are indexed by $\mathcal{J}(t) = \mathit{1}(t) \cup \mathit{2}(t) \cup \mathcal{PT}(t)$.
3. $\bar{u}_{a,\varepsilon}$ and $\bar{u}_{b,\varepsilon}$ are piecewise constant with a finite number of discontinuities.

4. It holds that

$$\begin{cases} \|(\bar\rho_\varepsilon(0,\cdot),\bar q_\varepsilon(0,\cdot))-(\rho_0(\cdot),q_0(\cdot))\|_{L^1(a,b)}<\varepsilon \\ \|(\bar\rho_{a,\varepsilon},\bar q_{a,\varepsilon})-(\rho_a,q_a)\|_{L^1(0,+\infty)}<\varepsilon \\ \|(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})-(\rho_b,q_b)\|_{L^1(0,+\infty)}<\varepsilon \\ \text{Tot.Var. } (\bar\rho_\varepsilon(0,\cdot),\bar q_\varepsilon(0,\cdot)) \le \text{Tot.Var. } (\rho_0(\cdot),q_0(\cdot)) \\ \text{Tot.Var. } (\bar\rho_{a,\varepsilon},\bar q_{a,\varepsilon}) \le \text{Tot.Var. } (\rho_a,q_a) \\ \text{Tot.Var. } (\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon}) \le \text{Tot.Var. } (\rho_b,q_b)\,. \end{cases}$$

5. It holds that, for a.e. $t>0$, the Riemann problem with initial condition

$$\begin{cases} (\rho_{a,\varepsilon},q_{a,\varepsilon})\,(t) & \text{if } x<a \\ (\bar\rho_\varepsilon,\bar q_\varepsilon)\,(t,a+) & \text{if } x>a \end{cases}$$

is solved with waves with non positive speed.
6. It holds that, for a.e. $t>0$, the Riemann problem with initial condition

$$\begin{cases} (\bar\rho_\varepsilon,\bar q_\varepsilon)\,(t,b-) & \text{if } x<b \\ (\rho_{b,\varepsilon},q_{b,\varepsilon})\,(t) & \text{if } x>b \end{cases}$$

is solved with waves with non negative speed.

Consider three sequences $(\rho_{0,\nu},q_{0,\nu})$, $(\rho_{a,\nu},q_{a,\nu})$, and $(\rho_{b,\nu},q_{b,\nu})$ of piecewise constant functions with a finite number of discontinuities such that

1. $(\rho_{0,\nu},q_{0,\nu}):(a,b)\to\Omega_f\cup\Omega_c$ and $(\rho_{a,\nu},q_{a,\nu}),(\rho_{b,\nu},q_{b,\nu}):(0,+\infty)\to\Omega_f\cup\Omega_c$;
2. the following limits hold

$$\lim_{\nu\to+\infty}(\rho_{0,\nu},q_{0,\nu})=(\rho_0,q_0) \qquad \text{in } L^1((a,b);\Omega_f\cup\Omega_c)$$
$$\lim_{\nu\to+\infty}(\rho_{a,\nu},q_{a,\nu})=(\rho_a,q_a) \qquad \text{in } L^1((0,+\infty);\Omega_f\cup\Omega_c)$$
$$\lim_{\nu\to+\infty}(\rho_{b,\nu},q_{b,\nu})=(\rho_b,q_b) \qquad \text{in } L^1((0,+\infty);\Omega_f\cup\Omega_c);$$

3. the following inequalities hold

$$\text{Tot.Var.}(\rho_{0,\nu},q_{0,\nu})\le\text{Tot.Var.}(\rho_0,q_0)$$
$$\text{Tot.Var.}(\rho_{a,\nu},q_{a,\nu})\le\text{Tot.Var.}(\rho_a,q_a)$$
$$\text{Tot.Var.}(\rho_{b,\nu},q_{b,\nu})\le\text{Tot.Var.}(\rho_b,q_b).$$

For every $\nu\in\mathbb{N}\setminus\{0\}$, we apply the following procedure. At time $t=0$, we solve the boundary Riemann problems at $x=a$ and $x=b$ and all Riemann problems for $x\in(a,b)$. We approximate every rarefaction wave with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ traveling with the Rankine-Hugoniot speed. At every discontinuity time for $(\rho_{a,\nu},q_{a,\nu})$ or for $(\rho_{b,\nu},q_{b,\nu})$, we solve the corresponding Riemann problem at $x=a$ or $x=b$. At every interaction between two waves, we solve the corresponding Riemann problem. Finally, when a wave interacts with the boundary $x=a$ or $x=b$, we solve the corresponding boundary Riemann problem.

Remark 7. As usual, by slightly modifying the speed of waves or the position of the discontinuities for the boundary values, we may assume that, at every positive time t, at most one of the following possibilities happens:

1. two waves interact together at a point $x\in(a,b)$;
2. a wave interacts with the boundary $x=a$ or with the boundary $x=b$;

3. t is a point of discontinuity either for $(\rho_{a,\nu}, q_{a,\nu})$ or for $(\rho_{b,\nu}, q_{b,\nu})$.

Remark 8. For interactions at a point $x \in (a, b)$, we split rarefaction waves into rarefaction fans just at time $t = 0$. At the boundary $x = a$ or $x = b$, instead, we allow the formation of rarefaction fans at every positive time, but only for changes of the boundary data.

Remark 9. By assumption, the initial condition and the boundary data satisfy hypothesis **(H-1)**. Therefore there exists $\bar{v} > 0$ such that the velocity v of every states in Ω_c, generated by the previous construction, is greater than or equal to $\bar{v}$.

Given an ε-approximate wave-front tracking solution $\bar{u}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{q}_\varepsilon)$ with boundary data $\bar{u}_{a,\varepsilon} = (\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})$ and $\bar{u}_{b,\varepsilon} = (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})$, define, for a.e. $t > 0$, the following functionals (a detailed explanation is contained in Remark 10)

$$W_1(t) = \sum_{x \in \mathit{1}_r(t)} d\left((\bar{\rho}_\varepsilon(t, x+), \bar{q}_\varepsilon(t, x+)), (\bar{\rho}_\varepsilon(t, x-), \bar{q}_\varepsilon(t, x-))\right) \tag{17}$$

$$W_2(t) = \sum_{x \in \mathit{2}_r(t)} d\left((\bar{\rho}_\varepsilon(t, x+), \bar{q}_\varepsilon(t, x+)), (\bar{\rho}_\varepsilon(t, x-), \bar{q}_\varepsilon(t, x-))\right) \tag{18}$$

$$W_{PT}(t) = \sum_{x \in \mathcal{PT}(t)} d\left((\bar{\rho}_\varepsilon(t, x+), \bar{q}_\varepsilon(t, x+)), (\bar{\rho}_\varepsilon(t, x-), \bar{q}_\varepsilon(t, x-))\right) \tag{19}$$

$$W_a(t) = d\left((\bar{\rho}_\varepsilon(t, a+), \bar{q}_\varepsilon(t, a+)), (\bar{\rho}_{a,\varepsilon}(t), \bar{q}_{a,\varepsilon}(t))\right) \tag{20}$$

$$W_b(t) = d\left((\bar{\rho}_\varepsilon(t, b-), \bar{q}_\varepsilon(t, b-)), (\bar{\rho}_{b,\varepsilon}(t), \bar{q}_{b,\varepsilon}(t))\right) \tag{21}$$

$$W(t) = W_1(t) + W_2(t) + W_{PT}(t) + W_a(t) + W_b(t) \tag{22}$$

Note that the previous functionals may vary only at times at which the boundary datum changes or at times $\bar{t}$ when two waves interact or a wave reaches the boundary.

Remark 10. The functional W is composed by 5 terms. The first and second term measure the strength of waves of first and second family. The third term measures the phase transition waves. Finally, the last two terms measure the distance of the boundary term from the trace at the boundary of the approximate solution.

3.0.2. *Interaction estimates.* We consider here estimates for wave interactions. In the following, as in [10], we describe wave interactions by the nature of the involved waves. For example, if a wave of the second family hits a wave of the first family producing a phase-transition wave, we write **2-1/PT**. Here the symbol "/" divides the waves before and after the interaction.

The functional W does not increment for wave interaction inside the phase transition system; hence it can only increases for interaction of waves with the interface $x = 0$.

Lemma 3.4. *Assume that the waves* $((\rho^l, q^l), (\rho^m, q^m))$ *and* $((\rho^m, q^m), (\rho^r, q^r))$ *interact at the point* $(\bar{t}, \bar{x})$ *with* $\bar{t} > 0$ *and* $\bar{x} \in (a, b)$. *Then* $W(\bar{t}+) \le W(\bar{t}-)$.

The proof is completely identical to that of Lemma 4.29 of [10]; hence we omit it.

Lemma 3.5. *Assume that the wave* $((\rho^l, q^l), (\rho^r, q^r))$ *interacts with the boundary at the point* $(\bar{t}, a)$. *Then* $W(\bar{t}+) \le W(\bar{t}-)$. *Moreover at time* $\bar{t}$ *either no wave or one wave is generated. In the latter case, the possible interactions are* ***1/PT*** *(if* $(\rho^r, q^r) \notin \Omega_f$*) or* ***1/2*** *(if* $(\rho^r, q^r) \in \Omega_f$*).*

Proof. First note that $\Delta W_b(\bar t) = 0$, since the interaction happens at $x = a$. Moreover, since the boundary Riemann problem does not generate waves, then the states $(\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(t)$ and (ρ^l, q^l) are connected through waves with non positive speed. We have therefore several possibilities.

1. $(\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(t) = (\rho^l, q^l)$. Since the wave $((\rho^l, q^l), (\rho^r, q^r))$ has negative speed, then, at time $\bar t$, it is absorbed and no other wave is generated. In the case $((\rho^l, q^l), (\rho^r, q^r))$ is a wave of the first family, then $\Delta W_1(\bar t) = -\Delta W_a(\bar t) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$, $\Delta W_2(\bar t) = \Delta W_{PT}(\bar t) = 0$ and so
$$\Delta W(\bar t) = 0.$$
In the case $((\rho^l, q^l), (\rho^r, q^r))$ is a phase-transition wave with non positive speed, then $\Delta W_{PT}(\bar t) = -\Delta W_a(\bar t) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$, $\Delta W_1(\bar t) = \Delta W_2(\bar t) = 0$ and so
$$\Delta W(\bar t) = 0.$$
2. The states $(\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)$ and (ρ^l, q^l) are connected by a wave of the first family. In this case both the states $(\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)$ and (ρ^l, q^l) belong to Ω_c and so the interacting wave also is of the first family; at time $\bar t$, it is absorbed and no other wave is generated.
   Thus $\Delta W_1(\bar t) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$, $\Delta W_2(\bar t) = \Delta W_{PT}(\bar t) = 0$, $\Delta W_a(\bar t) = d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right) - d\left((\rho^l, q^l), (\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)\right)$. Consequently
$$\begin{aligned}\Delta W(\bar t) &= d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right) - d\left((\rho^l, q^l), (\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)\right) \\ &\quad - d\left((\rho^l, q^l), (\rho^r, q^r)\right) \le 0.\end{aligned}$$
3. The states $(\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)$ and (ρ^l, q^l) are connected by a phase-transition wave with non positive speed.
   In the case no wave is produced at time $\bar t$, we have $\Delta W_2(\bar t) = \Delta W_{PT}(\bar t) = 0$, $\Delta W_1(\bar t) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$, and $\Delta W_a(\bar t) = d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right) - d\left((\rho^l, q^l), (\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)\right)$. Consequently
$$\begin{aligned}\Delta W(\bar t) &= d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right) - d\left((\rho^l, q^l), (\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)\right) \\ &\quad - d\left((\rho^l, q^l), (\rho^r, q^r)\right) \le 0.\end{aligned}$$
   In the case a phase transition wave, connecting the states $(\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)$ to (ρ^r, q^r), is produced at time $\bar t$, then $(\rho^r, q^r) \notin \Omega_f$. Moreover $\Delta W_1(\bar t) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$, $\Delta W_2(\bar t) = 0$, $\Delta W_{PT}(\bar t) = d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right)$, and $\Delta W_a(\bar t) = -d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^l, q^l)\right)$. Then
$$\begin{aligned}\Delta W(\bar t) &= d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right) - d\left((\rho^l, q^l), (\rho^r, q^r)\right) \\ &\quad - d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^l, q^l)\right) \le 0.\end{aligned}$$
   In the case a wave of the second family, connecting $(\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t)$ to (ρ^r, q^r), is produced at time $\bar t$, then $(\rho^r, q^r) \in \Omega_f$, $\Delta W_1(\bar t) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$, $\Delta W_{PT}(\bar t) = 0$, $\Delta W_2(\bar t) = d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right)$, and $\Delta W_a(\bar t)$ is equal to $-d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^l, q^l)\right)$. Since $\omega(\rho^l, q^l) = \omega(\rho^r, q^r)$ and $v((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(t)) = v(\rho^r, q^r)$, then
$$\begin{aligned}\Delta W(\bar t) &= d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^r, q^r)\right) - d\left((\bar\rho_{a,\varepsilon}, \bar q_{a,\varepsilon})(\bar t), (\rho^l, q^l)\right) \\ &\quad - d\left((\rho^l, q^l), (\rho^r, q^r)\right) \le 0.\end{aligned}$$

4. The states $(\bar\rho_{a,\varepsilon},\bar q_{a,\varepsilon})(\bar t)$ and (ρ^l,q^l) are connected by a phase-transition wave with non positive speed (connecting $(\bar\rho_{a,\varepsilon},\bar q_{a,\varepsilon})(\bar t)$ with the state $(\tilde\rho,\tilde q)$) followed by a wave of the first family (connecting $(\tilde\rho,\tilde q)$ with the state (ρ^l,q^l)). The situation here is completely analogous to that of the previous point.

The proof is so finished. □

Lemma 3.6. *Assume that the wave* $((\rho^l,q^l),(\rho^r,q^r))$ *interacts with the boundary at the point* $(\bar t,b)$*. Then* $W(\bar t+)\le W(\bar t-)$*. The possible interactions, producing at least one wave, are:* ***2/PT**, **2/1**, and **2/PT-1**.*

Proof. First note that $\Delta W_a(\bar t)=0$, since the interaction happens at $x=b$. Moreover, since the boundary Riemann problem does not generate waves, then the states (ρ^r,q^r) and $(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)$ are connected through waves with non negative speed. We have therefore several possibilities.

1. $(\rho^r,q^r)=(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)$. Since the wave $((\rho^l,q^l),(\rho^r,q^r))$ has positive speed, then, at time $\bar t$, it is absorbed and no other wave is generated. In the case $((\rho^l,q^l),(\rho^r,q^r))$ is a wave of the first family, then $\Delta W_2(\bar t)=-\Delta W_b(\bar t)=-d\left((\rho^l,q^l),(\rho^r,q^r)\right)$, $\Delta W_1(\bar t)=\Delta W_{PT}(\bar t)=0$ and so
$$\Delta W(\bar t)=0.$$
In the case $((\rho^l,q^l),(\rho^r,q^r))$ is a phase-transition wave with positive speed, then $\Delta W_{PT}(\bar t)=-\Delta W_b(\bar t)=-d\left((\rho^l,q^l),(\rho^r,q^r)\right)$, $\Delta W_1(\bar t)=\Delta W_2(\bar t)=0$ and so
$$\Delta W(\bar t)=0.$$
2. The states (ρ^r,q^r) and $(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)$ are connected by a wave of the second family.

   In the case the interacting wave is a wave of the second family, then no wave is generated at time $\bar t$. Therefore we have $\Delta W_1(\bar t)=\Delta W_{PT}(\bar t)=0$, $\Delta W_2(\bar t)=-d\left((\rho^l,q^l),(\rho^r,q^r)\right)$ and $\Delta W_b(\bar t)=d\left((\rho^l,q^l),(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)\right)-d\left((\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t),(\rho^r,q^r)\right)$; hence
$$\Delta W(\bar t)\le 0.$$
In the case the interacting wave is a phase-transition wave with positive speed, then no wave is generated at time $\bar t$ and $\Delta W_1(\bar t)=\Delta W_2(\bar t)=0$, $\Delta W_{PT}(\bar t)=-d\left((\rho^l,q^l),(\rho^r,q^r)\right)$ and $\Delta W_b(\bar t)=d\left((\rho^l,q^l),(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)\right)-d\left((\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t),(\rho^r,q^r)\right)$. Hence
$$\Delta W(\bar t)\le 0.$$
3. The states (ρ^r,q^r) and $(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)$ are connected by a phase-transition wave, possibly followed by a wave of the second family. In this case $(\rho^r,q^r)\in\Omega_f$ and necessarily also $(\rho^l,q^l)\in\Omega_f$.

   If $(\rho^l,q^l)\in\Omega_f\cup\Omega_c$, then at time $\bar t$ a wave of the first family, connecting (ρ^l,q^l) to a state (ρ^m,q^m), is generated. The states (ρ^m,q^m) and $(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)$ either coincide or belong to the same Lax curve of the second family. In this case $\Delta W_1(\bar t)=d\left((\rho^l,q^l),(\rho^m,q^m)\right)$, $\Delta W_2(\bar t)=-d\left((\rho^l,q^l),(\rho^r,q^r)\right)$, $\Delta W_{PT}(\bar t)=0$ and $\Delta W_b(\bar t)=d\left((\rho^m,q^m),(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)\right)-d\left((\rho^r,q^r),(\bar\rho_{b,\varepsilon},\bar q_{b,\varepsilon})(t)\right)$. Hence
$$\Delta W(\bar t)\le 0.$$

If $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and a phase-transition wave is generated at time $\bar{t}$, then $\Delta W_1(\bar{t}) = 0$, $\Delta W_2(\bar{t}) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$. Moreover $\Delta W_{PT}(\bar{t}) = d\left((\rho^l, q^l), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t)\right)$ and $\Delta W_b(\bar{t}) = -d\left((\rho^r, q^r), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t)\right)$. Thus

$$\Delta W(\bar{t}) \leq 0.$$

If $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and a phase-transition wave (connecting (ρ^l, q^l) to $(\hat{\rho}, \hat{q})$), followed by a wave of the first family (connecting $(\hat{\rho}, \hat{q})$ to $(\tilde{\rho}, \tilde{q})$), is generated at time $\bar{t}$, then $\Delta W_1(\bar{t}) = d\left((\hat{\rho}, \hat{q}), (\tilde{\rho}, \tilde{q})\right)$, $\Delta W_2(\bar{t}) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$. Moreover $\Delta W_{PT}(\bar{t}) = d\left((\rho^l, q^l), (\hat{\rho}, \hat{q})\right)$ and $\Delta W_b(\bar{t}) = d\left((\tilde{\rho}, \tilde{q}), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t)\right) - d\left((\rho^r, q^r), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t)\right)$. Thus

$$\Delta W(\bar{t}) \leq 0.$$

If $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and no wave is generated at time $\bar{t}$, then $\Delta W_1(\bar{t}) = 0$, $\Delta W_2(\bar{t}) = -d\left((\rho^l, q^l), (\rho^r, q^r)\right)$, $\Delta W_{PT}(\bar{t}) = 0$ and finally $\Delta W_b(\bar{t}) = d\left((\rho^l, q^l), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t)\right) - d\left((\rho^r, q^r), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t)\right)$. Thus

$$\Delta W(\bar{t}) \leq 0.$$

The proof is finished. □

Lemma 3.7. *Assume that $\bar{t}$ is a discontinuity point for the boundary datum at $x = a$. Then $W(\bar{t}+) \leq W(\bar{t}-) + d\left((\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})(t+), (\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})(t-)\right)$.*

Proof. In general, at time $\bar{t}$, a wave with positive speed emerges from the boundary $x = a$. In all the cases, denoting with $(\bar{\rho}, \bar{q})$ the trace of the approximate solution before time $\bar{t}$ at $x = a+$, we have

$$\begin{aligned}\Delta W(\bar{t}) &= d\left((\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})(t+), (\bar{\rho}, \bar{q})\right) - d\left((\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})(t-), (\bar{\rho}, \bar{q})\right) \\ &\leq d\left((\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})(t+), (\bar{\rho}_{a,\varepsilon}, \bar{q}_{a,\varepsilon})(t-)\right)\end{aligned}$$

by the triangular inequality. The proof is so concluded. □

Lemma 3.8. *Assume that $\bar{t}$ is a discontinuity point for the boundary datum at $x = b$. Then $W(\bar{t}+) \leq W(\bar{t}-) + d\left((\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t+), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t-)\right)$.*

Proof. In general, at time $\bar{t}$, a wave with negative speed emerges from the boundary $x = b$. In all the cases, denoting with $(\bar{\rho}, \bar{q})$ the trace of the approximate solution before time $\bar{t}$ at $x = b-$, we have

$$\begin{aligned}\Delta W(\bar{t}) &= d\left((\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t+), (\bar{\rho}, \bar{q})\right) - d\left((\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t-), (\bar{\rho}, \bar{q})\right) \\ &\leq d\left((\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t+), (\bar{\rho}_{b,\varepsilon}, \bar{q}_{b,\varepsilon})(t-)\right)\end{aligned}$$

by the triangular inequality. The proof is so concluded. □

Proposition 2. *For a.e. $t > 0$, we have*

$$W(t) \leq M, \tag{23}$$

where $M > 0$ is a constant.

Proof. It is a simple consequence of Lemmas 3.4, 3.5, 3.6, 3.7, and 3.8. □

3.0.3. *Existence of a wave-front tracking solution.* We now want to bound the number of waves and of interactions.

Definition 3.9. A wave of $\bar{u}_\varepsilon$, generated at time $t = 0$, is said an original wave.

Definition 3.10. A wave of the second family $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ is said special if $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) \in \Omega_f \cap \Omega_c \setminus \{(\sigma_-, \sigma_- q^-/R)\}$.

Definition 3.11. A wave of the second family $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ is said semi-special either if

- $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) = (\sigma_-, \sigma_- q^-/R)$

or

- $(\rho^r, q^r) \in \Omega_f \setminus \Omega_c$ and $(\rho^l, q^l) = (\sigma_-, \sigma_- q^-/R)$.

Lemma 3.12. *Special waves can not emerge by interactions of waves inside the phase transition system. They can be generated only at time $t = 0$ or at the boundary $x = a$.*

For a proof see [10].

Lemma 3.13. *A wave of the second family $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ with $(\rho^l, q^l), (\rho^r, q^r) \in \Omega_f \setminus \Omega_c$ either is an original wave or can be generated by a variation of the boundary data at $x = a$. This holds in particular for semi-special waves.*

Proof. Note first that the wave has positive speed; hence can not be generated at the boundary $x = b$.

Assume first that it is generated at a point $(\bar{t}, \bar{x})$ with $\bar{t} > 0$ and $\bar{x} \in (a, b)$. In this case, it should emerge also a wave of the first family or a phase-transition wave connecting a state $(\bar{\rho}, \bar{q})$ to (ρ^l, q^l). This is clearly not possible.

Assume now that a wave with negative speed $((\bar{\rho}, \bar{q}), (\rho^r, q^r))$ interacts with the boundary $x = a$ at a time $\bar{t} > 0$. Since $(\rho^r, q^r) \in \Omega_f$ this interactive wave is not a phase-transition wave. The remaining possibility is a wave of the first family, which is not possible, since $(\rho^r, q^r) \notin \Omega_c$. This completes the proof. □

The following proposition holds.

Proposition 3. *For every $\nu \in \mathbb{N} \setminus \{0\}$, the construction in Subsection 3.0.1 can be done for every positive time, producing a $\frac{1}{\nu}$-approximate wave-front tracking solution to (16).*

Proof. For $\nu \in \mathbb{N} \setminus \{0\}$, call $u_\nu = (\rho_\nu, q_\nu)$ the function built with the procedure of Subsection 3.0.1. It is sufficient to prove that the number of waves and interactions, generated by the construction, is finite. Define the functional $N_\nu(t)$, which counts the number of discontinuities of (ρ_ν, q_ν). $N_\nu(t)$ is locally constant in time and can vary at interaction times in the following way.

1. If at time $\bar{t} > 0$ two waves interact at $\bar{x} \in (a, b)$, then $\Delta N_\nu(\bar{t}) \le 1$. More precisely, $\Delta N_\nu(\bar{t}) = 1$ if and only if the interaction is of type **2-1/PT-1-2**; see [10].
2. If at time $\bar{t} > 0$ a wave interacts with the boundary at $x = a$, then $\Delta N_\nu(\bar{t}) \le 0$. In the case $\Delta N_\nu(\bar{t}) = 0$ the interaction is either **1/PT** or **1/2**.
3. If at time $\bar{t} > 0$ a wave interacts with the boundary at $x = b$, then $\Delta N_\nu(\bar{t}) \le 1$. Indeed in the case $\Delta N_\nu(\bar{t}) = 1$, the interaction is of type **2/PT-1**, while in the case $\Delta N_\nu(\bar{t}) = 0$, the interaction is either of type **2/PT** or of type **2/1**.

Interaction's position	Interaction's types
Left boundary	**1/PT**, **1/2**
Right boundary	**2/PT**, **2/1**
Inside the domain	**2-1/1-2**, **2-PT/PT-1**, **2-PT/1-2**, **PT-1/PT-1**, **2-1/PT-1**, **2-1/PT-2**

TABLE 1. List of interaction types, which can happen, in principle, an infinite number of times.

4. If the time $\bar{t} > 0$ is a point of discontinuity for the boundary value $(\rho_{a,\nu}, q_{a,\nu})$, then $\Delta N_\nu(\bar{t}) \leq 2$.
5. If the time $\bar{t} > 0$ is a point of discontinuity for the boundary value $(\rho_{b,\nu}, q_{b,\nu})$, then $\Delta N_\nu(\bar{t}) \leq 2$.

The claims 1., 4., and 5. are immediate. Claim 2. follows directly by Lemma 3.5, while claim 3. follows directly by Lemma 3.6.

The number of waves can increase in the cases 1., 3., 4., and 5. By construction, the case 4., and 5. happen at most a finite number of times. In case 1., the only interaction, which produces an increment of the number of waves, is **2-1/PT-1-2**. In this situation, a special wave interacts with a wave of the first family, producing three waves; see [10]. The wave of the second family, generated by the interaction is not special. By Lemma 3.12, the number of special waves is bounded by $N_\nu(0+)$ (number of original waves) plus the number of discontinuities of the boundary datum $(\rho_{a,\nu}, q_{a,\nu})$.

In case 3., the interaction, producing an increment of the number of waves, is **2/PT-1**. In this case the interacting wave of the second family $\left(\left(\rho^l, q^l\right), \left(\rho^r, q^r\right)\right)$ has the property that both $\left(\rho^l, q^l\right)$ and $\left(\rho^r, q^r\right)$ belong to $\Omega_f \setminus \Omega_c$. By Lemma 3.13, such a wave is either an original wave or a wave produced at the boundary $x = a$ and so, its number is bounded by $N_\nu(0+)$ (number of original waves) plus the number of discontinuities of the boundary datum $(\rho_{a,\nu}, q_{a,\nu})$.

Hence

$$N_\nu(t) \leq 3N_\nu(0+) + 4N_{a,\nu} + 2N_{b,\nu}$$

for a.e. $t > 0$, where $N_{a,\nu}$ and $N_{b,\nu}$ are respectively the number of discontinuities of the boundary data $(\rho_{a,\nu}, q_{a,\nu})$ and $(\rho_{b,\nu}, q_{b,\nu})$.

Now we want to prove that the number of interactions is finite.

The previous analysis shows that the interactions, producing an increment of the number of waves N_ν, can happen at most a finite number of times. Therefore the interactions (inside the domain (a, b)) **1-1/1**, **2-PT/PT**, **PT-1/2**, **PT-1/PT**, **2-1/PT** can happen at most a finite number of times, since we have a uniform bound on the number of waves. So it remains to consider and to bound the number of interactions of the types included in Table 1. Consider first the interactions of Table 1 generating a wave of the second family and happening inside the domain (a, b). In all such interactions, a wave of the second family is present also before the interaction itself. Moreover this interacting wave of the second family is not a semi-special wave; see Definition 3.11. Instead the interaction **1/2** at the boundary $x = a$ produces a semi-special wave. Therefore the interactions **2-1/1-2**, **2-PT/1-2**, and **2-1/PT-2** can happen a finite number of times, since the interacting waves of the second family can be generated either at time 0 or at the boundary $x = a$ when the boundary datum changes or by an interaction happening at most a finite number of times. For the same reason (the interacting wave of the second family is not semi-special), also the interaction **2-1/PT-1** inside the domain and **2/PT** at the right boundary can happen at most a finite number of times. We can therefore limit the study to the following interactions:

Interaction's position	Interaction's types
Left boundary	**1/PT**, **1/2**
Right boundary	**2/1**
Inside the domain	**PT-1/PT-1**

Let Λ be the set containing the speed of all possible waves in the phase-transition model and define $K = \sup\{|\lambda| : \lambda \in \Lambda\}$. It is sufficient to prove that the number of interactions is finite in each time interval $[t_1, t_2]$ such that $t_2 - t_1 < \frac{b-a}{2K}$. Indeed this choice implies that the boundary at $x = a$ does not influence the boundary at $x = b$ and viceversa. Under this assumption, the interactions **1/2** at $x = a$ and **2/1** at $x = b$ can happen at most a finite number of times.

Therefore only **PT-1/PT-1** (inside (a, b)) and **1/PT** (at $x = a$) can happen an infinite number of times.

If **1/PT** happens a finite number of times, then also **PT-1/PT-1** does; see [10, Proof of Proposition 4.37]. Thus the only possibility for having an infinite number of interactions is that the wave of the first family produced by **PT-1/PT-1** goes back to the boundary $x = a$ in order to produce **1/PT**. If this happens the number of waves strictly decreases, since the generated phase-transition wave of **PT-1/PT-1** is at left of the wave of the first family. Hence the combination of interactions can not happen an infinite number of times. The proof is so concluded. □

3.0.4. *Existence of a solution.* In this part we conclude the proof of Theorem 3.2.

Proof of Theorem 3.2. Fix an ε-approximate wave-front tracking solution $\bar{u}_\varepsilon$ to (16), in the sense of Definition 3.3. By Proposition 2, we deduce that there exists a constant $M > 0$, depending on the total variation of the flux of the initial datum, such that

$$W_1(t) + W_2(t) + W_{PT}(t) \leq M, \tag{24}$$

for a.e. $t > 0$. Since **(H-1)** holds, then inequality (24) implies that the functional $\mathrm{Tot.Var.}\,((\bar{\rho}_\varepsilon, \bar{q}_\varepsilon)(t, \cdot))$ is uniformly bounded for a.e. $t > 0$. Hence, at least by a subsequence, there is a function $(\bar{\rho}, \bar{q})$, which is a solution to (16) in the sense of Definition 3.1. This permits to conclude. □

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