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ON A HYPERBOLIC KELLER-SEGEL SYSTEM WITH DEGENERATE NONLINEAR FRACTIONAL DIFFUSION

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Abstract. We investigate a Keller-Segel model with quorum sensing and a fractional diffusion operator. This model describes the collective cell movement due to chemical sensing with flux limitation for high cell densities and with anomalous media represented by a nonlinear, degenerate fractional diffusion operator. The purpose of this paper is to introduce and prove the existence of a properly defined entropy solution.

1. **Introduction.** Chemotaxis is the active motion of organisms influenced by chemical gradients and it has been studied by many authors. The Keller-Segel model, introduced around 1970 [37, 38] (cf. also [47]), is used to describe aggregation of slime mold amoebae and this model has become one of the most widely studied models in mathematical biology. A special case of the Keller-Segel model is the following system of partial differential equations:

$$u_t = \operatorname{div}(\varepsilon\nabla u - u(1-u)\nabla S), \qquad -\Delta S + S = u. \tag{1}$$

It has been considered very recently by several authors [15, 16, 28, 48]. The cell-flux on the right hand side of (1) comprises two counteracting phenomena: random motion of cells described by the Fick's law and cell movement in the direction of the gradient of the chemical S.

A chemotaxis model of the form (1), but with a parabolic equation for S, was first introduced by Hillen and Painter [34], where it was shown that the additional volume-filling term $u(1-u)$ in the cell-flux leads to global existence of solutions. See [15, 16, 28, 34, 48] and some of the references cited therein for further background information about Keller-Segel type systems. Diffusion effects are generally included in these kind of models to account for the dispersal of organisms. Traditionally, a linear diffusion operator is employed but more recently there is a growing interest in

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models with degenerate and nonlinear diffusion terms accounting for over-crowding effects, see [3, 5, 11, 14, 50, 51].

Nonlocal diffusion operators, such as the fractional Laplacian, have received a lot of attention recently from the PDE community, see for example [1, 6, 8, 17, 18, 19, 20, 29, 36] and the references cited therein.

The spatiotemporal distribution of a population density of random walkers leads to diffusion [49]. On the other hand, biological organisms sometimes seem to adopt Lévy-flights and superdiffusion (the mean-square displacement of the position of the dispersing population depends superlinearly on time) to enhance their search strategies [2, 40, 43], in which case dispersal becomes better described by fractional diffusion operators [2, 7, 9, 10, 13, 31, 32, 40, 44, 45, 52]. Indeed, the so-called Lévy flight foraging hypothesis states that since Lévy flights optimize random searches, organisms have evolved to exploit Lévy flights. For example, there is evidence that fractional diffusion better explains the feeding strategies of microzooplankton [13, 32]. Likewise, there are circumstances in which a cellular population self-interacting chemotactically, is well described by a fractional diffusion system [32]. Moreover, fractional diffusion models have been proposed for cardiac electrical propagation in structurally heterogeneous excitable media [12].

Motivated by the works just mentioned, in the model below the density of cells move with a collective chemotactic attraction through a chemical potential in an anomalous medium, represented by a nonlinear nonlocal diffusion term.

The subject of this paper is the existence of properly defined generalized solutions of the following Keller-Segel type model, which contains a hyperbolic term as well as a nonlinear, degenerate nonlocal (fractional) diffusion term:

$$u_t + \operatorname{div}(\nabla S f(u)) = \Delta_\alpha \beta(u), \qquad -\Delta S + S = u, \tag{2}$$

where $\Delta_\alpha := -(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian with $\alpha \in (0,2)$ and

$$\begin{cases} \beta : \mathbb{R} \to \mathbb{R} \text{ Lipschitz, non-decreasing; } \beta(0) = 0; \\ f(u) = u(1-u). \end{cases} \tag{3}$$

In (2), $u = u(t,x)$, $S = S(t,x)$, $(t,x) \in Q_T := (0,T) \times \Omega$, $d > 1$, and $T > 0$ is a fixed final time. We augment (2) with initial data

$$\begin{cases} u(0,x) = u_0(x), \quad u_0 \in L^1(\Omega) \cap L^\infty(\Omega); \\ 0 \le u_0 \le 1 \text{ a.e. in } \Omega, \end{cases} \tag{4}$$

and the boundary condition

$$\nabla S \cdot n_\Omega = 0 \quad \text{on } \Omega. \tag{5}$$

The problem is posed on Ω, which is a bounded domain in $\mathbb{R}^d$, with smooth (say, C^2) boundary and $n_\Omega(x)$ is the outward normal to Ω at $x \in \partial\Omega$. We note that the normal flux in the equation for u vanishes at $\partial\Omega$ and thus the boundary is characteristic, for this reason one does not need boundary conditions for u. This prevents some specific difficulties, see [46, 48].

Since there is a nonlocal term in the equation for u we have to specify u in the whole space. We use a (complementary) Dirichlet boundary condition on $\mathbb{R}^d \setminus \Omega$:

$$u \equiv 0 \quad \text{on} \quad \mathbb{R}^d \setminus \Omega. \tag{6}$$

The nonlinearity β is assumed to be non-decreasing; in particular, $\beta \equiv \text{Const}$ is allowed and thus the hyperbolic Keller-Segel model studied in [48] is a special case of (2). The model in (2) represents the density $u(t,x)$ of cells moving with a collective chemotactic attraction through the chemical potential S in an "anomalous medium" represented by the nonlocal diffusion term $\Delta_\alpha \beta(u)$. The sensitivity of the cells is limited by the "quorum sensing" part $(1-u)$ of $f(u)$. The restrictions on f, u_0 will imply that the solutions satisfy $0 \le u(\cdot,\cdot) \le 1$.

The fractional Laplacian Δ_α on $\mathbb{R}^d$ can be defined by

$$\widehat{\Delta_\alpha u}(\omega) = |\omega|^\alpha\, \widehat{u}(\omega), \qquad \omega \in \mathbb{R}^d, \tag{7}$$

where $\widehat{u}$ represents the Fourier transform of u. Alternatively, it can be defined using the Cauchy principal value:

$$\Delta_\alpha u(x) = C_{\alpha,d}\,\text{P.V.} \int_{|z|>0} \frac{u(x+z)-u(x)}{|z|^{d+\alpha}}\, dz, \tag{8}$$

for some normalizing factor $C_{\alpha,d}$, $\alpha \in (0,2)$, and v is a sufficiently regular function (say, C_c^∞). A proof of (8) for the case $d=2$ can be found in, e.g., [25]. The formula still remains valid in the general case $d>2$, see [26]. For sufficiently regular v the above formulas are equivalent to the following representation:

$$\Delta_\alpha u(x) = C_{\alpha,d} \int_{\mathbb{R}^d \setminus \{0\}} \left[u(x+z) - u(x) - z \cdot \nabla u\, \mathbf{1}_{|z|<1} \right] \pi(dz),$$

where

$$\pi(dz) = \frac{1}{|z|^{d+\alpha}}\, \mathbf{1}_{|z|<1}\, dz, \qquad \alpha \in (0,2),$$

and $C_{\alpha,d}$ is a normalizing factor. For a proof, see for example [30]. In view of this, we will in the following interpret $\Delta_\alpha \beta(u)$ as

$$\begin{aligned} \Delta_\alpha \beta(u) &= \mathcal{L}\left[\beta(u(t,x))\right] \\ &:= \int_{\mathbb{R}^d\setminus\{0\}} \left[\beta(u(t,x+z)) - \beta(u(t,x)) - z \cdot \nabla\beta(u) 1_{|z|<1}\right] \pi(\,dz). \end{aligned} \tag{9}$$

Since the analysis does not depend on $C_{\alpha,d}$, we have set $C_{\alpha,d} = 1$.

Since we assume $\beta(0)=0$ the integral representation of the nonlinear nonlocal operator restricted to the bounded domain Ω makes sense. In fact, the nonlocal operator $\mathcal{L}$ has to be replaced by the Freidrich's extension of the restriction of $\mathcal{L}$ to the space $C_c^\infty(\Omega) \subset L^2(\Omega)$. In the following we identify $L^2(\Omega)$ with the space

$$\left\{ u \in L^2(\mathbb{R}^d) : u \equiv 0 \text{ on } \mathbb{R}^d \setminus \Omega \right\}.$$

This new operator has the form domain ("the largest domain on which a self-adjoint operator makes sense weakly")

$$H_0^{\alpha/2}(\Omega) = \left\{ u \in H^{\alpha/2}(\mathbb{R}^d) : u \equiv 0 \text{ on } \mathbb{R}^d \setminus \Omega \right\}.$$

In probabilistic terms, the operator coincides with the generator of the α-stable process in Ω killed upon leaving Ω. We note that for $u \in C_c^\infty(\Omega)$, we have the representation (neglecting the positive constant)

$$\begin{aligned} (-\Delta)^{\alpha/2} u(x) &= \text{P. V.} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}}\, dy \\ &= \text{P. V.} \int_{\Omega} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}}\, dy + u(x)\kappa_\Omega(x), \end{aligned} \tag{10}$$

for a.e. $x \in \Omega$, where

$$\kappa_\Omega(x) := \int_{\mathbb{R}^d\setminus\Omega} |x-y|^{-d-\alpha}\, dy, \qquad \text{for } x \in \Omega.$$

The first term in (10) describes the jump behavior of the α-stable process inside Ω. The function κ_Ω is called the killing measure. See [35] and some of the references cited therein for the motivation of this definition. Although the function u is 0 outside of Ω and $\beta(0) = 0$ we note that z can be any nonzero real number in (9) and this representation of the nonlocal operator is still valid. Note also that though $x \in \Omega$ and $z \in \mathbb{R}^d \setminus \{0\}$, $x + z$ may be in the set Ω so that we have the integral with respect to z over all $\mathbb{R}^d \setminus \{0\}$. Thus, we define our nonlocal operator as in (9). We note that we can split the integral in (9) over an integral in Ω and on Ω^c as in (10) but this is unnecessary.

As mentioned above, without the fractional diffusion operator ($\beta \equiv \text{const}$), (2) is a hyperbolic-elliptic system first analyzed by Perthame and Dalibard [48]. One can view the the first equation in (2) ($\beta \equiv \text{Const}$) as a quasilinear hyperbolic equation with a nonlinear flux function $V(t,x)f(u)$ depending on a velocity field $V = \nabla S$ with minimal regularity. This system, as well as our system (2), does not possess BV estimates and lacks contraction properties. In [48], the authors circumvent the lack of strong compactness by passing to the weak limit in a kinetic formulation of the viscosity approximation of the system, thereby arriving at a kinetic formulation for the weak limit. Exploiting the specific structure of this formulation, the authors deduce a posteriori the strong subsequential convergence. For other applications of this "propagation of compactness" argument, see [22, 24] for some models describing two-phase flow in porous media, and [21] for a mean-field vortex model arising in superconductivity theory.

The purpose of this paper is to prove the existence of an entropy solution, i.e., a pair (u, S) of functions satisfying (2) in the weak sense as well as a family of entropy inequalities adapted from [36]. As in [48], we will prove existence of solution by passing to the limit as $\varepsilon \to 0$ in a sequence u^ε of solutions to a parabolic regularized system. However, we will employ a completely different argument to arrive at the required strong convergence. Inspired by a "continuous dependence on the nonlinearities" argument in [23] yielding fractional BV estimates (see also [27]), in combination with a recent result of Ben Belgacem and Jabin [4] addressing "low regularity" velocity fields $V = \nabla S$, we will exhibit an ε-uniform spatial modulus of continuity: for $0 < t < T$ and any $\delta > 0$,

$$\sup_{|y|\le\delta} \int |u^\varepsilon(t, x+y) - u^\varepsilon(t,x)| \, dx = o(\delta).$$

Relying on a well-known technique of Kružkov [41], we use this spatial estimate to establish an ε-uniform temporal L^1-continuity estimate. These estimates are enough to ensure the strong convergence of $(u^\varepsilon, S^\varepsilon)$ along a subsequence to a limit point (u, S). The limit is easily shown to be an entropy solution.

Finally, we mention that a number of authors have studied questions regarding existence, uniqueness, regularity, and temporal asymptotics for (scalar) quasilinear hyperbolic equations perturbed by Lévy operators, see [1, 6, 8, 20, 29, 36].

The remaining part of this paper is organized as follows: In Section 2 we present the notion of solution and state the main result (Theorem 2.2). The proof of the main theorem is given in Section 3, while Section A is devoted to a sketch of an existence result for the parabolic regularized system.

2. **Notion of solution and the main result.**

Given any convex C^2 entropy function $\eta : \mathbb{R} \to \mathbb{R}$, we define the corresponding entropy flux $q : \mathbb{R} \to \mathbb{R}$ by

$q'(u) = \eta'(u)f'(u)$. We refer to (η, q) as an entropy pair. Given an entropy η and the function β we define $q_\beta : \mathbb{R} \to \mathbb{R}$ by $q_\beta' = \eta'\beta'$.

Fix a small number $\kappa > 0$, and let us split the non-local diffusion operator $\mathcal{L}$, which is defined in (9), into two parts

$$\begin{aligned}\mathcal{L}[\phi] = &\int_{|z|\le\kappa} \left[\phi(t,x+z) - \phi(t,x) - z\cdot\nabla\phi\mathbf{1}_{|z|<1}\right]\,\pi(dz)\\ &+ \int_{|z|>\kappa} \left[\phi(t,x+z) - \phi(t,x) - z\cdot\nabla\phi\mathbf{1}_{|z|<1}\right]\,\pi(dz)\\ =:&\, \mathcal{L}_\kappa[\phi] + \mathcal{L}^\kappa[\phi], \qquad \forall\phi\in C^2.\end{aligned}$$

Before we continue we need to introduce some notation. Let $\varphi = \varphi(t,x,s,y)$ be a test function in the doubled variables (t,x,s,y). To simplify the presentation, we introduce the following notation (with ∇_{x+y} being short-hand for $\nabla_x + \nabla_y$)

$$\begin{aligned}\mathcal{L}_x[\varphi] &:= \int_{\mathbb{R}^d\setminus\{0\}} \left[\varphi(t,x+z,s,y) - \varphi - z\cdot\nabla_x\varphi\mathbf{1}_{|z|<1}\right]\,\pi(dz),\\ \mathcal{L}_y[\varphi] &= \int_{\mathbb{R}^d\setminus\{0\}} \left[\varphi(t,x,s,y+z) - \varphi - z\cdot\nabla_y\varphi\mathbf{1}_{|z|<1}\right]\,\pi(dz),\\ \mathcal{L}_{x+y}[\varphi] &= \int_{\mathbb{R}^d\setminus\{0\}} \Big[\varphi(t,x+z,s,y+z) - \varphi - z\cdot\nabla_{x+y}\varphi\mathbf{1}_{|z|<1}\Big]\,\pi(dz).\end{aligned}$$

Similar to the splitting for $\mathcal{L}$, we also let

$$\mathcal{L}_x = \mathcal{L}_{x,\kappa} + \mathcal{L}_x^\kappa, \quad \mathcal{L}_y = \mathcal{L}_{y,\kappa} + \mathcal{L}_y^\kappa, \quad \mathcal{L}_{x+y} = \mathcal{L}_{x+y,\kappa} + \mathcal{L}_{x+y}^\kappa.$$

We also need to introduce the operator $\widetilde{\mathcal{L}}^\kappa$ defined by writing

$$\mathcal{L}^\kappa[\varphi] = \widetilde{\mathcal{L}}^\kappa[\varphi] - \left(\int_{|z|>\kappa} z\,\mathbf{1}_{|z|<1}\,\pi(dz)\right)\cdot\nabla_x\varphi,$$

with similar definitions for $\widetilde{\mathcal{L}}_x^\kappa$, $\widetilde{\mathcal{L}}_y^\kappa$, and $\widetilde{\mathcal{L}}_{x+y}^\kappa$. Let us motivate the notion of entropy solution to be presented in Definition 2.1 below. Define $Q_\beta : \mathbb{R} \to \mathbb{R}$ by

$$Q_\beta' = \eta'\circ\beta^{-1},$$

where β^{-1} denotes, say, the left-continuous inverse of the non-decreasing function β. One can check that

$$Q_\beta(\beta(u)) = q_\beta(u).$$

By the convexity of Q_β,

$$\begin{aligned}&\eta'(u(t,x))\Big(\beta(u(t,x+z)) - \beta(u(t,x)) - z\cdot\nabla\beta(u(t,x))\mathbf{1}_{|z|<1}\Big)\\ &\quad= Q_\beta'(\beta(u(t,x)))\Big(\beta(u(t,x+z)) - \beta(u(t,x)) - z\cdot\beta(u(t,x))\mathbf{1}_{|z|<1}\Big)\\ &\quad\le Q_\beta(\beta(u(t,x+z))) - Q_\beta(\beta(u(t,x))) - z\cdot\nabla Q_\beta(\beta(u(t,x)))\mathbf{1}_{|z|<1}\\ &\quad= q_\beta(u(t,x+z)) - q_\beta(u(t,x)) - z\cdot\nabla q_\beta(u(t,x))\mathbf{1}_{|z|<1}.\end{aligned}$$

Therefore,

$$\eta'(u(t,x))\mathcal{L}_\kappa[\beta(u)] \le \mathcal{L}_\kappa[q_\beta(u)],$$

and so for any non-negative $\phi\in C_c^\infty(\Omega)$,

$$\int_\Omega \eta'(u(t,x))\mathcal{L}_\kappa[\beta(u)]\phi(x)\,dx$$

$$\leq \int_\Omega \mathcal{L}_\kappa \left[q_\beta(u(x))\right] \phi(x)\, dx = \int_\Omega q_\beta(u(x)) \mathcal{L}_\kappa[\phi]\, dx.$$

Summarizing, for any $\kappa \in (0,1)$,

$$\begin{aligned}
&\int_\Omega \eta'(u(x)) \mathcal{L}[\beta(u)] \phi(x)\, dx \\
&\quad \leq \int_\Omega q_\beta(u(x)) \mathcal{L}_\kappa[\phi]\, dx + \int_\Omega \eta'(u(x)) \mathcal{L}^\kappa[\beta(u)]\phi(x)\, dx \\
&\quad = \int_\Omega q_\beta(u(x)) \mathcal{L}_\kappa[\phi] dx + \int_\Omega \eta'(u(x)) \widetilde{\mathcal{L}}^\kappa[\beta(u)]\phi(x)\, dx \\
&\qquad\qquad + \int_\Omega q_{\beta,\kappa}(u(x)) \cdot \nabla \phi\, dx,
\end{aligned}$$

where $q_{\beta,\kappa} : \mathbb{R} \to \mathbb{R}^d$ is defined by

$$q_{\beta,\kappa}(u) = q_\beta(u) \left(\int_{|z|>\kappa} z\, \mathbf{1}_{|z|<1}\, \pi(dz) \right),$$

and we have used integration by parts.

The formal calculations above motivate the following definition.

Definition 2.1. An entropy solution of the initial-value problem (2)-(4) is a pair (u, S) of measurable functions $u, S : Q_T \to \mathbb{R}$ satisfying the following conditions:

(**D**.1) $u \in L^\infty(Q_T)$, $u \in L^\infty(0,T; L^1(\Omega))$, $S \in L^\infty(0,T; W^{2,p}(\Omega))$ for any p in $[1, \infty)$, and $0 \leq u(t,x)$, $S(t,x) \leq 1$ for a.e. $(t,x) \in Q_T$;

(**D**.2) For any entropy pair (η, q),

$$\begin{aligned}
\iint_{Q_T} \Bigg(&\eta(u)\varphi_t + \Big(q(u)\nabla S + q_{\beta,\kappa}(u) \Big) \cdot \nabla \varphi \\
&\qquad + (u - S)\big[q(u) - f(u)\eta'(u)\big]\varphi \Bigg)\, dx\, dt \\
+ \iint_{Q_T} &q_\beta(u) \mathcal{L}_\kappa[\varphi]\, dx\, dt + \iint_{Q_T} \eta'(u) \widetilde{\mathcal{L}}^\kappa[\beta(u)]\varphi\, dx\, dt \\
&\qquad + \int_\Omega \eta(u_0)\varphi(0,x)\, dx \geq 0,
\end{aligned} \tag{11}$$

for all non-negative $\varphi \in C_c^\infty([0,T) \times \Omega)$, for any $\kappa \in (0,1)$.

Remark 1. We comment that entropy inequalities like the ones above imply uniqueness and L^1 contraction properties for a class of scalar equations, see for example [1, 20, 36].

Without the fractional diffusion operator, the above notion of solution coincides with the one utilized in [48] for a hyperbolic-elliptic Keller-Segel system.

Here is the main result of our paper.

Theorem 2.2. *Suppose that the nonlinearities f, β satisfy (3), and consider the system (2) with initial condition (4), Neumann boundary condition (5), and complementary Dirichlet condition (6). Then there exists an entropy solution according to Definition 2.1.*

3. **Proof of Theorem 2.2.** For the proof of Theorem 2.2 we will regularize the system (2) by adding an artificial viscosity term $\varepsilon\Delta$, $\varepsilon > 0$. Indeed, we will consider classical solutions $(u^\varepsilon, S^\varepsilon)$ of the parabolic regularized system

$$\begin{cases} u_t^\varepsilon + \operatorname{div}\left(\nabla S^\varepsilon(t,x) f(u^\varepsilon)\right) = \mathcal{L}\left[\beta(u^\varepsilon)\right] + \varepsilon \Delta u^\varepsilon, \quad t > 0, x \in \Omega, \\ -\Delta S^\varepsilon + S^\varepsilon = u^\varepsilon, \quad \text{in } \Omega, \\ u^\varepsilon\big|_{t=0} = u_0^\varepsilon, \\ u^\varepsilon \equiv 0 \quad \text{in } \mathbb{R}^d \setminus \Omega, \\ \nabla S^\varepsilon \cdot n_\Omega = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{12}$$

with smooth initial data u_0^ε ($\to u_0$ in L^1) satisfying the conditions in (4). The non-local diffusion operator $\mathcal{L}$ is defined in (9). It is standard to construct a unique solution $(u^\varepsilon, S^\varepsilon) \in L^2_{\text{loc}}(\mathbb{R}_+; H^1(\Omega)) \times L^\infty(\mathbb{R}_+; H^1(\Omega))$ to this system for each fixed $\varepsilon > 0$. Indeed, it can be done using the Galerkin method and the compactness argument, see Chapter 5 in [33] and [39]. See [48] for a very similar argument, which uses the semigroup approach. For the convenience of the reader we sketch a proof in the Appendix A.

Our goal is to pass to the limit $\varepsilon \to 0$ in the viscous problem (12) and show that the limit function is an entropy solution of the original problem (2) with initial condition (4). To achieve that we will derive a series of a priori estimates implying strong convergence of $(u^\varepsilon, S^\varepsilon)$, at least along a subsequence as $\varepsilon \to 0$. The main step of our analysis is to establish spatial and temporal translation estimates for u^ε implying strong L^1 compactness.

However, before we do that let us list the more immediate a priori estimates satisfied by the solution $(u^\varepsilon, S^\varepsilon)$ of (17) (cf. the Appendix): For all $1 \le p < \infty$ and for any finite $T > 0$, the following ε-uniform estimates hold:

$$\begin{aligned} & 0 \le u^\varepsilon(t,x) \le 1 \text{ a.e. in } \mathbb{R}_+ \times \Omega; \quad \|u^\varepsilon\|_{L^\infty([0,T];L^1(\Omega))} \le C_1(T); \\ & 0 \le S^\varepsilon(t,x) \le 1 \text{ a.e. in } \mathbb{R}_+ \times \Omega; \quad \|S^\varepsilon\|_{L^\infty(\mathbb{R}_+;W^{2,p}(\Omega))} \le C_2(d,p); \\ & \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} \sqrt{\varepsilon}\, \|\nabla u^\varepsilon\|_{L^2((0,T)\times\Omega)} + \|\partial_t S^\varepsilon\|_{L^2((0,T);H^1(\Omega))} \le C_3(d,T). \end{aligned} \tag{13}$$

3.1. **Spatial translation estimate.** The goal is to prove an L^1 spatial translation estimate for u^ε that is independent of the regularizing parameter ε. As a first main step in that direction, we will prove

Lemma 3.1. *Let $\tilde{J}_\delta$ be a nonnegative $C^\infty(\mathbb{R}^d)$ kernel with support in $\{|x| \le 2\}$, where we assume without loss of generality that the set $\{|x| \le 2\}$ is included in Ω (otherwise we just need to modify the arguments slightly),*

$$\tilde{J}_\delta(x) = \frac{1}{(|x|^2 + \delta^2)^{d/2}}, \qquad \text{for } |x| \le 1, \tag{14}$$

and $\tilde{J}_\delta$ is independent of δ on $\mathbb{R}^d \setminus \{|x| < 1\}$. Define $J_\delta : \mathbb{R}^d \to \mathbb{R}$ by

$$J_\delta(x) = \frac{C_\delta}{|\log \delta|} \tilde{J}_\delta(x), \quad \text{with } C_\delta \text{ chosen such that } \int_{\mathbb{R}^d} J_\delta(\tfrac{x}{2})\, dx = 1. \tag{15}$$

Let $\psi \in C_c^\infty(\Omega)$ be an arbitrary nonnegative cut-off function. For any $t \in [0,T]$ with $T > 0$ finite,

$$\begin{aligned} \iint_{\Omega\times\Omega} &|u^\varepsilon(t,x) - u^\varepsilon(t,y)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy \\ &\le C_T\Big(\left\|D^2\psi\right\|_{L^\infty} + \frac{C}{|\log(\delta)|} + \varepsilon\left\|\Delta\psi\right\|_{L^\infty} + \tau(\delta)\Big), \end{aligned} \tag{16}$$

where $C_T > 0$ is a constant that is independent of ε and $\tau(\delta)$ is a non-decreasing continuous function with $\tau(0) = 0$.

Remark 2. The function $J_\delta(\cdot/2)$ is a smooth convolution kernel, and the constant C_δ defined in (15) is bounded from below and from above independently of δ.

Proof. To prove Lemma 3.1 we will employ a doubling-of-variables argument *à la* Kružkov [42], along with some recent ideas [4] for handling velocity fields with low regularity. To this end, fix $\varepsilon > 0$, and let $u^\varepsilon(t) = u^\varepsilon(t,x)$ be the solution to the regularized problem

$$\partial_t u^\varepsilon + \mathrm{div}_x\left(\nabla S^\varepsilon(t,x) f(u^\varepsilon)\right) = \mathcal{L}\left[\beta(u^\varepsilon)\right] + \varepsilon\Delta_x u^\varepsilon. \tag{17}$$

with initial data $u^\varepsilon(0,x) = u_0(x)$ and $-\Delta_x S^\varepsilon(t,x) + S^\varepsilon(t,x) = u^\varepsilon(t,x)$.

Furthermore, let $v^\varepsilon(t) = v^\varepsilon(t,y)$ be the solution to the regularized problem

$$\partial_t v^\varepsilon + \mathrm{div}_y\left(\nabla S^\varepsilon(t,y) f(v^\varepsilon)\right) = \mathcal{L}\left[\beta(v^\varepsilon)\right] + \varepsilon\Delta_y v^\varepsilon, \tag{18}$$

with initial data $v^\varepsilon(0,y) = u_0(y)$ and $-\Delta_y S^\varepsilon(t,y) + S^\varepsilon(t,y) = u^\varepsilon(t,y)$. Clearly, by uniqueness of regular solutions to (12), $v^\varepsilon(t,y) = u^\varepsilon(t,y)$.

In what follows, C denotes a generic constant, which may depend on the time interval considered, uniform bounds on the initial data or on ∇S^ε, which never depends on the regularization parameters δ or ε.

Given an entropy function $\eta(\cdot)$ with $\eta(0) = 0$ and $\eta'(\cdot)$ odd, we introduce the associated entropy flux $q(u,v) = \int_v^u \eta'(\xi - v) f'(\xi)\,d\xi$. Subtracting (18) from (17) and subsequently applying the chain rule, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\eta(u^\varepsilon - v^\varepsilon) \\ &\quad = -\eta'(u^\varepsilon - v^\varepsilon)\Big(\mathrm{div}_x\left(\nabla S^\varepsilon(x) f(u^\varepsilon)\right) - \mathrm{div}_y\left(\nabla S^\varepsilon(y) f(v^\varepsilon)\right)\Big) \\ &\qquad + \eta'(u^\varepsilon - v^\varepsilon)\Big(\varepsilon\Delta_x u^\varepsilon - \varepsilon\Delta_y v^\varepsilon\Big) \\ &\qquad\quad + \eta'(u^\varepsilon - v^\varepsilon)\Big(\mathcal{L}\left[\beta(u^\varepsilon)\right] - \mathcal{L}\left[\beta(v^\varepsilon)\right]\Big). \end{aligned} \tag{19}$$

We first observe that

$$\begin{aligned} &\eta'(u^\varepsilon - v^\varepsilon)\mathrm{div}_x\left(\nabla S^\varepsilon(x) f(u^\varepsilon)\right) \\ &\quad = \mathrm{div}_x\left(\nabla S^\varepsilon(x) q(u^\varepsilon, v^\varepsilon)\right) + \Delta S^\varepsilon(x)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(u^\varepsilon) - q(u^\varepsilon, v^\varepsilon)\Big). \end{aligned}$$

Moreover,

$$\begin{aligned} &\eta'(u^\varepsilon - v^\varepsilon)\mathrm{div}_y\left(\nabla S^\varepsilon(y) f(v^\varepsilon)\right) \\ &\quad = -\mathrm{div}_y\left(\nabla S^\varepsilon(y) q(v^\varepsilon, u^\varepsilon)\right) + \Delta S^\varepsilon(y)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(v^\varepsilon) + q(v^\varepsilon, u^\varepsilon)\Big) \\ &\quad = -\mathrm{div}_y\left(\nabla S^\varepsilon(x) q(v^\varepsilon, u^\varepsilon)\right) + \mathrm{div}_y\Big(\left(\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y)\right) q(v^\varepsilon, u^\varepsilon)\Big) \\ &\qquad + \Delta S^\varepsilon(y)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(v^\varepsilon) + q(v^\varepsilon, u^\varepsilon)\Big) \end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{div}_y\left(\nabla S^\varepsilon(x) q(u^\varepsilon, v^\varepsilon)\right) + \mathrm{div}_y\Big(\nabla S^\varepsilon(x)\left(q(u^\varepsilon, v^\varepsilon) - q(v^\varepsilon, u^\varepsilon)\right)\Big) \\
&\qquad + \mathrm{div}_y\Big(\left(\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y)\right) q(v^\varepsilon, u^\varepsilon)\Big) \\
&\qquad + \Delta S^\varepsilon(y)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(v^\varepsilon) + q(v^\varepsilon, u^\varepsilon)\Big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&-\eta'(u^\varepsilon - v^\varepsilon)\Big(\mathrm{div}_x\left(\nabla S^\varepsilon(x) f(u^\varepsilon)\right) - \mathrm{div}_y\left(\nabla S^\varepsilon(y) f(v^\varepsilon)\right)\Big) \\
&\quad = -(\nabla_x + \nabla_y)\cdot\left(\nabla S^\varepsilon(x) q(u^\varepsilon, v^\varepsilon)\right) \\
&\qquad + \mathrm{div}_y\Big(\nabla S^\varepsilon(x)\left(q(u^\varepsilon, v^\varepsilon) - q(v^\varepsilon, u^\varepsilon)\right)\Big) \\
&\qquad + \mathrm{div}_y\Big(\left(\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y)\right) q(v^\varepsilon, u^\varepsilon)\Big) \\
&\qquad - \Delta S^\varepsilon(x)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(u^\varepsilon) - q(u^\varepsilon, v^\varepsilon)\Big) \\
&\qquad + \Delta S^\varepsilon(y)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(v^\varepsilon) + q(v^\varepsilon, u^\varepsilon)\Big).
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
&\eta'(u^\varepsilon - v^\varepsilon)\Big(\varepsilon\Delta_x u^\varepsilon - \varepsilon\Delta_y v^\varepsilon\Big) \\
&\quad = \Big(\varepsilon\Delta_x + 2\varepsilon\nabla_x\cdot\nabla_y + \varepsilon\Delta_y\Big)\eta(u^\varepsilon - v^\varepsilon) \\
&\qquad - \eta''(u^\varepsilon - v^\varepsilon)\left|\sqrt{\varepsilon}\nabla_x u^\varepsilon - \sqrt{\varepsilon}\nabla_y v^\varepsilon\right|^2.
\end{aligned}
$$

Inserting our findings into (19), we arrive at

$$
\begin{aligned}
&\frac{\partial}{\partial t}\eta(u^\varepsilon - v^\varepsilon) \\
&\quad = -(\nabla_x + \nabla_y)\cdot\left(\nabla S^\varepsilon(x) q(u^\varepsilon, v^\varepsilon)\right) \\
&\qquad + \mathrm{div}_y\Big(\nabla S^\varepsilon(x)\left(q(u^\varepsilon, v^\varepsilon) - q(v^\varepsilon, u^\varepsilon)\right)\Big) \\
&\qquad + \mathrm{div}_y\Big(\left(\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y)\right) q(v^\varepsilon, u^\varepsilon)\Big) \\
&\qquad - \Delta S^\varepsilon(x)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(u^\varepsilon) - q(u^\varepsilon, v^\varepsilon)\Big) \\
&\qquad + \Delta S^\varepsilon(y)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(v^\varepsilon) + q(v^\varepsilon, u^\varepsilon)\Big) \\
&\qquad\quad + \Big(\varepsilon\Delta_x + 2\varepsilon\nabla_x\cdot\nabla_y + \varepsilon\Delta_y\Big)\eta(u^\varepsilon - v^\varepsilon) \\
&\qquad\quad - \eta''(u^\varepsilon - v^\varepsilon)\left|\sqrt{\varepsilon}\nabla_x u^\varepsilon - \sqrt{\varepsilon}\nabla_y v^\varepsilon\right|^2 \\
&\qquad\quad + \eta'(u^\varepsilon - v^\varepsilon)\Big(\mathcal{L}\left[\beta(u^\varepsilon)\right] - \mathcal{L}\left[\beta(v^\varepsilon)\right]\Big).
\end{aligned} \tag{20}
$$

We take $0 \le \phi_\delta = \phi_\delta(x, y) \in C_c^\infty(\Omega \times \Omega)$ to be of the form

$$
\phi_\delta(x, y) =: J_\delta\left(\tfrac{x-y}{2}\right)\psi\left(\tfrac{x+y}{2}\right), \tag{21}
$$

where $J_\delta(\cdot)$ is the kernel defined in (15) and $0 \le \psi \in C_c^\infty(\Omega)$ is a cut-off function. By an approximation argument, we can allow for cut-off functions $\psi \in W^{2,\infty}$ that are compactly supported in Ω and satisfy

$$
|\nabla\psi(\mathbf{x})| \le C_0\psi(\mathbf{x}).
$$

We integrate (20) against the test function ϕ_δ defined in (21), yielding

$$\int_0^t \iint_{\Omega\times\Omega} \eta(u^\varepsilon(t,x) - v^\varepsilon(t,y))\phi_\delta(x,y)\,dx\,dy - \int_0^t \iint_{\Omega\times\Omega} \eta(u_0^\varepsilon(x) - u_0^\varepsilon(y))\phi_\delta(x,y)\,dx\,dy \le I_c^1 + I_c^2 + I_c^3 + I_c^4 + I_d + I_l,$$

where

$$\begin{aligned}
I_c^1 &:= -\int_0^t \iint_{\Omega\times\Omega} (\nabla_x + \nabla_y)\cdot(\nabla S^\varepsilon(x) q(u^\varepsilon, v^\varepsilon))\,\phi_\delta(x,y)\,dx\,dy\,ds,\\
I_c^2 &:= \int_0^t \iint_{\Omega\times\Omega} \operatorname{div}_y\Big(\nabla S^\varepsilon(x)\,(q(u^\varepsilon,v^\varepsilon) - q(v^\varepsilon,u^\varepsilon))\Big)\phi_\delta(x,y)\,dx\,dy\,ds,\\
I_c^3 &:= \int_0^t \iint_{\Omega\times\Omega} \operatorname{div}_y\Big((\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y))\,q(v^\varepsilon,u^\varepsilon)\Big)\phi_\delta(x,y)\,dx\,dy\,ds,\\
I_c^4 &:= \int_0^t \iint_{\Omega\times\Omega} \Bigg(-\Delta S^\varepsilon(x)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(u^\varepsilon) - q(u^\varepsilon,v^\varepsilon)\Big)\\
&\qquad + \Delta S^\varepsilon(y)\Big(\eta'(u^\varepsilon - v^\varepsilon) f(v^\varepsilon) + q(v^\varepsilon,u^\varepsilon)\Big)\Bigg)\phi_\delta(x,y)\,dx\,dy\,ds,\\
I_d &:= \int_0^t \iint_{\Omega\times\Omega} \Big(\varepsilon\Delta_x + 2\varepsilon\nabla_x\cdot\nabla_y + \varepsilon\Delta_y\Big)\eta(u^\varepsilon - v^\varepsilon)\phi_\delta(x,y)\,dx\,dy\,ds,\\
I_l &:= \int_0^t \iint_{\Omega\times\Omega} \eta'(u^\varepsilon - v^\varepsilon)\Big(\mathcal{L}[\beta(u^\varepsilon)] - \mathcal{L}[\beta(v^\varepsilon)]\Big)\phi_\delta(x,y)\,dx\,dy\,ds.
\end{aligned}$$

One can easily check that

$$\begin{aligned}
(\nabla_x + \nabla_y)\phi_\delta(x,y) &= J_\delta(\tfrac{x-y}{2})\nabla\psi(\tfrac{x+y}{2}),\\
\Big(\Delta_x + 2\nabla_x\cdot\nabla_y + \Delta_y\Big)\phi_\delta(x,y) &= J_\delta(\tfrac{x-y}{2})\Delta\psi(\tfrac{x+y}{2}).
\end{aligned}$$

In view of these two relations, integrating by parts gives

$$\begin{aligned}
I_c^1 &= \int_0^t \iint_{\Omega\times\Omega} J_\delta(\tfrac{x-y}{2})\nabla\psi(\tfrac{x+y}{2})\cdot\nabla S^\varepsilon(x)\,q(u^\varepsilon,v^\varepsilon)\,dx\,dy\,ds,\\
I_c^2 &= -\int_0^t \iint_{\Omega\times\Omega} (q(u^\varepsilon,v^\varepsilon) - q(v^\varepsilon,u^\varepsilon))\,\nabla S^\varepsilon(x)\cdot\nabla_y\phi_\delta(x,y)\,dx\,dy\,ds,\\
I_c^3 &= -\int_0^t \iint_{\Omega\times\Omega} q(v^\varepsilon,u^\varepsilon)\,(\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y))\cdot\nabla_y\phi_\delta(x,y)\,dx\,dy\,ds,\\
I_d &= -\varepsilon\int_0^t \iint_{\Omega\times\Omega} \eta(u^\varepsilon(s,x) - v^\varepsilon(s,y))J_\delta(\tfrac{x-y}{2})\Delta\psi(\tfrac{x+y}{2})\,dx\,dy\,ds,
\end{aligned}$$

In what follows, we will utilize the Kružkov entropy and corresponding entropy fluxes, i.e.,

$$\eta(u-v) = |u-v|, \qquad q(u,v) = \operatorname{sgn}(u-v)(f(u) - f(v)).$$

We continue as follows:

$$\left|I_c^1\right| \le \int_0^t \iint_{\Omega\times\Omega} J_\delta(\tfrac{x-y}{2})\left|\nabla\psi(\tfrac{x+y}{2})\right|\left|\nabla S^\varepsilon(x)\right|\left|u^\varepsilon(s,x) - v^\varepsilon(s,y)\right|\,dx\,dy\,ds$$

$$
\begin{aligned}
&\leq C_0 \left\|\nabla S^\varepsilon\right\|_{L^\infty} \int_0^t \iint_{\Omega\times\Omega} \left|u^\varepsilon(s,x) - v^\varepsilon(s,y)\right| J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\, dx\, dy\, ds,\\
I_c^2 &= 0,\\
\left|I_c^3\right| &\leq \int_0^t \iint_{\Omega\times\Omega} \left|u^\varepsilon(s,x) - v^\varepsilon(s,y)\right| \left|(\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y))\cdot \nabla_y J_\delta(\tfrac{x-y}{2})\right|\\
&\qquad\qquad \times \psi(\tfrac{x+y}{2})\, dx\, dy\, ds\\
&\quad + C \left\|\nabla S^\varepsilon\right\|_{L^\infty} \int_0^t \iint_{\Omega\times\Omega} \left|u^\varepsilon(s,x) - v^\varepsilon(s,y)\right| \psi(\tfrac{x+y}{2}) J_\delta(\tfrac{x-y}{2})\, dx\, dy\, ds,\\
\left|I_d\right| &\leq \tilde{C}\,\varepsilon \int_0^t \iint_{\Omega\times\Omega} \left|u^\varepsilon(s,x) - v^\varepsilon(s,y)\right| J_\delta(\tfrac{x-y}{2}) \Delta\psi(\tfrac{x+y}{2})\, dx\, dy\, ds \leq C\,\varepsilon \left\|\Delta\psi\right\|_{L^\infty}.
\end{aligned}
$$

Observe next that

$$
\begin{aligned}
I_c^4 &= \int_0^t \iint_{\Omega\times\Omega} \Big(-\Delta S^\varepsilon(x)\operatorname{sgn}(u^\varepsilon - v^\varepsilon) f(v^\varepsilon) + \Delta S^\varepsilon(y)\operatorname{sgn}(u^\varepsilon - v^\varepsilon) f(u^\varepsilon)\Big)\\
&\qquad\qquad \times \phi_\delta(x,y)\, dx\, dy\, ds\\
&= \int_0^t \iint_{\Omega\times\Omega} \Big(\Delta S^\varepsilon(x) q(u^\varepsilon, v^\varepsilon) + (\Delta S^\varepsilon(y) - \Delta S^\varepsilon(x))\operatorname{sgn}(u^\varepsilon - v^\varepsilon) f(u^\varepsilon)\Big)\\
&\qquad\qquad \times \phi_\delta(x,y)\, dx\, dy\, ds.
\end{aligned}
$$

Regarding the “Lévy term” I_l, let $\widetilde{\mathcal{L}}^\kappa, \mathcal{L}_{x,\kappa}$, and $\mathcal{L}_{y,\kappa}$ be the quantities defined at the beginning of Section 2. Let us also introduce the functions

$$
\begin{aligned}
&q_\beta(u,v) = \left|\beta(u) - \beta(v)\right|,\\
&q_{\beta,\kappa}(u,v) = \left|\beta(u) - \beta(v)\right| \left(\int_{|z|>\kappa} z\,\mathbf{1}_{|z|<1}\,\pi(dz)\right), \qquad \kappa \in (0,1).
\end{aligned}
$$

Now following the arguments leading up to Definition 2.1, we can write

$$
I_l := \int_0^t \iint_{\Omega\times\Omega} q_{\beta,\kappa}(u^\varepsilon, v^\varepsilon)\cdot \nabla_{x+y}\phi_\delta(x,y)\, dx\, dy\, ds + I_{l,\kappa} + I_l^\kappa,
$$

with

$$
I_{l,\kappa} := \int_0^t \iint_{\Omega\times\Omega} q_\beta(u^\varepsilon, v^\varepsilon)\Big(\mathcal{L}_{x,\kappa}\left[\phi_\delta\right] + \mathcal{L}_{y,\kappa}\left[\phi_\delta\right]\Big)\, dx\, dy\, ds,
$$

and

$$
\begin{aligned}
I_l^\kappa = \int_0^t \iint_{\Omega\times\Omega} &\operatorname{sgn}(u^\varepsilon - v^\varepsilon)\left(\widetilde{\mathcal{L}}^\kappa\left[\beta(u^\varepsilon)\right](t,x) - \widetilde{\mathcal{L}}^\kappa\left[\beta(v^\varepsilon)\right](s,y)\right)\\
&\times \phi_\delta(x,y)\, dx\, dy\, ds.
\end{aligned}
$$

One can easily verify that

$$
|I_{l,\kappa}| = o(\kappa), \qquad \text{independently of } \delta.
$$

Moreover, arguing as in [36], it follows that

$$
I_l^\kappa \leq \int_0^t \iint_{\Omega\times\Omega} q_\beta(u^\varepsilon, v^\varepsilon)\widetilde{\mathcal{L}}^\kappa_{x+y}[\phi_\delta]\, dx\, dy\, ds.
$$

Let us introduce a function $Q_{\varepsilon,\delta}(t)$, defined for $t \geq 0$ by

$$Q_{\varepsilon,\delta}(t) := \iint_{\Omega\times\Omega} |u^\varepsilon(t,x) - v^\varepsilon(t,y)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy,$$

$$Q_{\varepsilon,\delta}(0) := \iint_{\Omega\times\Omega} |u_0^\varepsilon(x) - u_0^\varepsilon(y)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy.$$

Summarizing our findings so far, we have arrived at the inequality

$$\begin{aligned}
Q_{\varepsilon,\delta}(t) \leq Q_{\varepsilon,\delta}(0) &+ C\,\varepsilon\, \|\Delta\psi\|_{L^\infty} + \frac{C}{|\log(\delta)|} + C\int_0^t Q_{\varepsilon,\delta}(s)\,ds \\
&+ \int_0^t \iint_{\Omega\times\Omega} |u^\varepsilon(s,x) - v^\varepsilon(s,y)| \\
&\qquad \times \left|(\nabla S^\varepsilon(x) - \nabla S^\varepsilon(y))\cdot \nabla_y J_\delta(\tfrac{x-y}{2})\right| \psi(\tfrac{x+y}{2})\,dx\,dy\,ds \\
&+ C\int_0^t \iint_{\Omega\times\Omega} |\Delta S^\varepsilon(y) - \Delta S^\varepsilon(x)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy\,ds \\
&+ \int_0^t \iint_{\Omega\times\Omega} |\beta(u^\varepsilon(s,x)) - \beta(v^\varepsilon(s,y))|\, \mathcal{L}_{x+y}[\phi_\delta]\,dx\,dy\,ds,
\end{aligned} \tag{22}$$

where C depends on $\Delta\psi$, $\|\nabla S^\varepsilon\|_{L^\infty}$, and $\|\Delta S^\varepsilon\|_{L^\infty}$ as well as $\|u^\varepsilon\|_{L^\infty([0,T];L^1_x\cap L^\infty_x)}$ and $\|v^\varepsilon\|_{L^\infty([0,T];L^1_x\cap L^\infty_x)}$, and we have moreover sent $\kappa \to 0$ to obtain (22). Since S^ε solves the elliptic equation $\Delta S^\varepsilon = S^\varepsilon - u^\varepsilon$, it follows that

$$\|\Delta S^\varepsilon\|_{L^\infty} \leq \|u^\varepsilon\|_{L^\infty} + \|S^\varepsilon\|_{L^\infty}.$$

By Sobolev embedding, $\|\nabla S^\varepsilon\|_{L^\infty}$ is bounded independently of ε.

We rewrite (22) as

$$\begin{aligned}
Q_{\varepsilon,\delta}(t) \leq Q_{\varepsilon,\delta}(0) &+ C\,\varepsilon\,\|\Delta\psi\|_{L^\infty} + \frac{C}{|\log(\delta)|} \\
&+ C\int_0^t Q_{\varepsilon,\delta}(s)\,ds + A^1_{\varepsilon,\delta}(t) + A^2_{\varepsilon,\delta}(t) + A^3_{\varepsilon,\delta}(t).
\end{aligned} \tag{23}$$

We will estimate the terms on the right-hand side of (23).

The strong compactness of $\{u_0^\varepsilon\}_{\varepsilon>0}$ and $\{S^\varepsilon\}_{\varepsilon>0}$ implies, via Lemma A.2 in the appendix, cf. also the proof of Corollary 1 below, the existence of a non-decreasing, continuous function $\tau(\delta)$ with $\tau(0)=0$, independent of ε, such hat

$$\begin{aligned}
&\iint_{\Omega\times\Omega} |u_0^\varepsilon(x) - u_0^\varepsilon(y)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy \\
&\quad + \int_0^t \iint_{\Omega\times\Omega} |S^\varepsilon(s,x) - S^\varepsilon(s,y)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy\,ds \leq \tau(\delta).
\end{aligned} \tag{24}$$

Repeating the main arguments of Ben Belgacem and Jabin [4], we can establish the crucial estimate

$$A^1_{\varepsilon,\delta}(t) \leq C\tau(\delta) + C\int_0^t Q_{\varepsilon,\delta}(s)\,ds.$$

Since S^ε satisfies the elliptic equation $\Delta S^\varepsilon = S^\varepsilon - u^\varepsilon$, it follows that

$$A^2_{\varepsilon,\delta}(t) \leq \int_0^t \iint_{\Omega\times\Omega} |S^\varepsilon(s,x) - S^\varepsilon(s,y)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy\,ds$$

$$
\begin{aligned}
&+ C\int_0^t \iint_{\Omega\times\Omega} |u^\varepsilon(s,x) - v^\varepsilon(s,y)|\, J_\delta(\tfrac{x-y}{2})\psi(\tfrac{x+y}{2})\,dx\,dy\,ds \\
\le\; & C\tau(\delta) + C\int_0^t Q_{\varepsilon,\delta}(s)\,ds,
\end{aligned}
$$

where the last inequality follows from (24).

Finally, by the Lipschitz continuity of $\beta(\cdot)$, the definition of the nonlocal operator $\mathcal{L}[\cdot]$, and since $\psi(\cdot)$ is smooth and compactly supported, we arrive at

$$
\begin{aligned}
A^3_{\varepsilon,\delta}(t) &\le \tilde{C}\int_0^t \iint_{\Omega\times\Omega} |u^\varepsilon(s,x) - v^\varepsilon(s,y)|\, J_\delta(\tfrac{x-y}{2})\mathcal{L}_{x+y}[\psi(\tfrac{x+y}{2})]\,dx\,dy\,ds \\
&\le C\left\|D^2\psi\right\|_{L^\infty}.
\end{aligned}
$$

In view of the above estimates,

$$
Q_{\varepsilon,\delta}(t) \le C\left\|D^2\psi\right\|_{L^\infty} + C\,\varepsilon\left\|\Delta\psi\right\|_{L^\infty} + \frac{C}{|\log(\delta)|} + C\tau(\delta) + C\int_0^t Q_{\varepsilon,\delta}(s)\,ds,
$$

and hence, by Gronwall's lemma, (16) follows. ☐

Corollary 1 (Spatial compactness)**.** *For any $t\in[0,T]$ with $T>0$ finite,*

$$
\sup_{|y|\le\delta} \|u^\varepsilon(t,\cdot+y) - u^\varepsilon(t,\cdot)\|_{L^1(\Omega)} \le \nu_T(\delta)\to 0, \quad \text{as } \delta\to 0, \tag{25}
$$

where $\nu_T:\mathbb{R}_+\to[0,\infty)$ is a modulus of continuity, i.e., $\nu_T(\cdot)$ is a non-decreasing, continuous function with $\nu_T(0+)=0$ and not depending on ε.

Proof. By a change of variables ($\tilde{x} = \frac{x+y}{2}$, $z=\frac{x-y}{2}$ so $x=\tilde{x}+z$, $y=\tilde{x}-z$) and letting $\psi\to 1$ in (16), it follows that (dropping the tildes)

$$
\int_\Omega\int_\Omega J_\delta(z)\,|u^\varepsilon(t,x+z) - u^\varepsilon(t,x-z)|\,dx\,dz \le \tilde{\nu}_T(\delta), \tag{26}
$$

where $\tilde{\nu}_T(\delta) := C_T\tau(\delta)\to 0$ as $\delta\to 0$. Thanks to Lemma A.2 in the appendix, (26) implies the existence a modulus of continuity $\nu_T(\cdot)$ such that (25) holds. ☐

3.2. **Temporal translation estimate.** Corollary 1 above establishes a uniform spatial translation estimate in L^1 implying spatial compactness of u^ε. Next, we will establish a uniform temporal L^1-continuity estimate, adapting an argument of Kružkov [41, 42].

Lemma 3.2 (Temporal compactness)**.** *For any $t\in[0,T]$ with $T>0$ finite,*

$$
\sup_{t+s\in[0,T],|s|\le\tau} \|u^\varepsilon(t+s,\cdot) - u^\varepsilon(t,\cdot)\|_{L^1(\Omega)} \le \omega_T(\tau)\to 0,
$$

as $\tau\to 0$, where $\omega_T(\cdot)$ is a temporal modulus of continuity (independent of ε).

Proof. To prove this lemma, fix $\Delta t>0$ and set

$$
d^\varepsilon(t,x) := u^\varepsilon(t+\Delta t,x) - u^\varepsilon(t,x), \qquad t\in[0,T-\Delta t].
$$

For any $\varphi\in L^\infty([0,T];C_0^\infty(\Omega))$ we compute

$$
\int_\Omega d^\varepsilon(t,x)\varphi(t,x)\,dx = \int_\Omega\left(\int_t^{t+\Delta t}\partial_s u^\varepsilon(s,x)\,ds\right)\varphi(t,x)\,dx
$$

$$
\begin{aligned}
&= \int_t^{t+\Delta t} \int_\Omega f(u^\varepsilon(s,x)) \nabla S^\varepsilon \cdot \nabla \varphi(t,x)\, dx\, ds \\
&\quad + \int_t^{t+\Delta t} \int_\Omega \beta(u^\varepsilon(s,x)) \mathcal{L}[\varphi(t,x)]\, dx\, ds \\
&\quad - \int_t^{t+\Delta t} \int_\Omega \varepsilon \nabla u^\varepsilon(s,x) \cdot \nabla \varphi(t,x)\, dx\, ds.
\end{aligned} \tag{27}
$$

Let $J \in C_c^\infty(\mathbb{R}^d)$ be a standard mollifier, for example take

$$
J(x) = \begin{cases} C \exp(\frac{1}{|x|^2-1}), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}
$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}^d} J(x)\, dx = 1$. As above, we assume without loss of generality that the support of J is included in the set Ω, otherwise minor modifications have to be done. For any fixed $\delta > 0$, we specify in (27) the test function

$$
\varphi := \varphi_\delta(t,x) = \delta^{-d} \int_\Omega J(\tfrac{x-y}{\delta}) \operatorname{sgn}(d^\varepsilon(t,y))\, dy.
$$

Clearly, $\|\varphi_\delta(t,\cdot)\|_{L^\infty} + \delta \|\nabla \varphi_\delta(t,\cdot)\|_{L^\infty} + \delta^2 \left\|D^2 \varphi_\delta(t,\cdot)\right\|_{L^\infty} \leq C$, uniformly in t, for some finite constant $C > 0$ independent of δ.

Now it follows that

$$
\begin{aligned}
\int_\Omega |d^\varepsilon(t,x)|\, dx &= \int_\Omega \Big(|d^\varepsilon(t,x)| - \varphi_\delta(t,x) d^\varepsilon(t,x) \Big)\, dx \\
&\quad + \int_t^{t+\Delta t} \int_\Omega f(u^\varepsilon(s,x)) \nabla S^\varepsilon \cdot \nabla \varphi_\delta(t,x)\, dx\, ds \\
&\quad + \int_t^{t+\Delta t} \int_\Omega \beta(u^\varepsilon(s,x)) \mathcal{L}[\varphi_\delta(t,x)]\, dx\, ds \\
&\quad - \varepsilon \int_t^{t+\Delta t} \int_\Omega \nabla u^\varepsilon(s,x) \cdot \nabla \varphi_\delta(t,x)\, dx\, ds \\
&=: \sum_{j=1}^4 I_j^\delta.
\end{aligned} \tag{28}
$$

We examine each term in (28) separately.

For the term I_4^δ we have

$$
\left|I_4^\delta\right| = \left| \int_t^{t+\Delta t} \int_\Omega \varepsilon u^\varepsilon(s,x) \Delta \varphi_\delta\, dx\, ds \right| \leq C \frac{\Delta t}{\delta^2}, \tag{29}
$$

where we used that u^ε is uniformly bounded in $L_t^\infty(L_x^1)$.

By Hölder's inequality,

$$
\left|I_2^\delta\right| \leq \tilde{C} \frac{\Delta t}{\delta} \|f(u^\varepsilon)\|_{L_t^\infty(L_x^2)} \|\nabla S^\varepsilon\|_{L_t^\infty(L_x^2)} \leq C \frac{\Delta t}{\delta}, \tag{30}
$$

where we have used the bound $|f(u)| \leq |u| + |u|^2$, and that u^ε and ∇S^ε are uniformly bounded respectively in $L_t^\infty(L_x^1 \cap L_x^\infty)$ and in $L_t^\infty(L_x^2)$, cf. Theorem A.1.

We handle the I_3^δ similarly:

$$
\left|I_3^\delta\right| = \left| \int_t^{t+\Delta t} \int_\Omega \beta(u^\varepsilon(s,x)) \mathcal{L}[\varphi_\delta](t,x)\, dx\, ds \right|
$$

$$\leq \tilde{C} \int_t^{t+\Delta t} \int_\Omega |u^\varepsilon(s,x)| \, |\mathcal{L}[\varphi_\delta](t,x)| \, dx \, ds \leq C \frac{\Delta t}{\delta^2}, \tag{31}$$

where we have used that β is globally Lipschitz with $\beta(0) = 0$, u^ε is uniformly bounded in $L_t^\infty(L_x^1)$, and

$$|\mathcal{L}[\varphi_\delta](t,x)| \leq \tilde{C}_L \left(\int_{\mathbb{R}^d \setminus \{0\}} |z|^2 \, \pi(dz) \right) \left\| D^2 \varphi_\delta \right\|_{L^\infty} \leq C_L / \delta^2.$$

Finally, we estimate I_1^δ as follows:

$$\begin{aligned} \left| I_1^\delta \right| &\leq \int_\Omega \left| |d^\varepsilon(t,x)| - d^\varepsilon(t,x) \int_\Omega \delta^{-d} J(\tfrac{x-y}{\delta}) \operatorname{sgn}(d^\varepsilon(t,y)) \, dy \right| \, dx \\ &\leq \int_\Omega \int_\Omega \delta^{-d} J(\tfrac{x-y}{2}) \left| |d^\varepsilon(t,x)| - d^\varepsilon(t,x) \operatorname{sgn}(d^\varepsilon(t,y)) \right| \, dy \, dx \\ &\leq 2 \int_\Omega \int_\Omega \delta^{-d} J(\tfrac{x-y}{\delta}) \, |d^\varepsilon(t,x) - d^\varepsilon(t,y)| \, dy \, dx \\ &= 2 \int_\Omega \int_\Omega \delta^{-d} J(z) \, |d^\varepsilon(t,x) - d^\varepsilon(t,x-\delta z)| \, dx \, dz \leq C \, \nu_T(\delta), \end{aligned} \tag{32}$$

where we have used the inequality $||a| - a \operatorname{sgn}(b)| \leq 2 \, |a-b|$, valid for any $a, b \in \mathbb{R}$, and Corollary 1.

In view of (29), (30), (31), and (32), it follows from (28) that

$$\int_\Omega |d^\varepsilon(t,x)| \, dx \leq \omega_T(\Delta t), \quad \omega_T(\Delta t) := C \inf_{\delta > 0} \left\{ \frac{\Delta t}{\delta^2} + \Delta t + \frac{\Delta t}{\delta} + \nu_T(\delta) \right\}.$$

This concludes the proof of Lemma 3.2. □

3.3. **Concluding the proof of Theorem 2.2.** In view of the a priori estimates listed in (13), as well as Corollary 1 and Lemma 3.2 in combination with the Riesz-Fréchet-Kolmogorov theorem, there exists a subsequence $\varepsilon \to 0$ (not relabeled) and limit functions $u \in L^\infty(\mathbb{R}_+ \times \Omega) \cap L^\infty([0,T]; L^1(\Omega))$ and $S \in L^\infty(\mathbb{R}_+ \times \Omega) \cap L^\infty([0,T]; H^1(\Omega)) \cap L^\infty(\mathbb{R}_+; W^{2,p}(\Omega))$, for all $1 \leq p < \infty$ and for any finite $T > 0$, such that

$$\begin{aligned} &u^\varepsilon \to u \quad \text{in } L^p_{\text{loc}}(\mathbb{R}_+ \times \Omega); \\ &S^\varepsilon \to S \quad \text{in } L^p([0,T]; W^{1,p}(\Omega)) \text{ for any finite } T > 0. \end{aligned} \tag{33}$$

Moreover, for any entropy pair (η, q),

$$\begin{aligned} &\iint_{Q_T} \Bigg(\eta(u^\varepsilon)\varphi_t + \Big(q(u^\varepsilon) \nabla S^\varepsilon + q_{\beta,\kappa}(u^\varepsilon) \Big) \cdot \nabla \varphi \\ &\qquad\qquad + (u^\varepsilon - S^\varepsilon) \big[q(u^\varepsilon) - f(u^\varepsilon) \eta'(u^\varepsilon) \big] \varphi \Bigg) \, dx \, dt \\ &\quad + \iint_{Q_T} q_\beta(u^\varepsilon) \mathcal{L}_\kappa[\varphi] \, dx \, dt + \iint_{Q_T} \eta'(u^\varepsilon) \widetilde{\mathcal{L}}^\kappa[\beta(u^\varepsilon)] \varphi \, dx \, dt \\ &\qquad\qquad + \int_\Omega \eta(u_0^\varepsilon) \varphi(0,x) \, dx \geq 0, \end{aligned} \tag{34}$$

for all non-negative $\varphi \in C_c^\infty([0,T) \times \Omega)$, for any $\kappa \in (0,1)$; cf. Section 2. Thanks to the strong convergence (33), we can send $\varepsilon \to 0$ in (34), reaching easily the conclusion that (u, S) is an entropy solution according to Definition 2.1. This ends the proof of Theorem 2.2. □

Appendix A. **On the viscous regularized system.** In this appendix we outline a proof for the existence of a solution to the following viscous regularized problem:

$$\begin{cases} u_t + \operatorname{div}(\nabla S f(u)) = \mathcal{L}[\beta(u)] + \varepsilon \Delta u, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ -\Delta S + S = u, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ \nabla S \cdot n_\Omega = 0 \quad \text{on } \partial\Omega, \\ u = 0 \quad \text{in } \mathbb{R}^d \setminus \Omega \text{ for a.e. } t > 0, \end{cases} \tag{35}$$

where $f(u) = u(1-u)$, the degenerate, nonlinear integral operator $\mathcal{L}[\cdot]$ is defined in (9), and $\varepsilon > 0$ is a fixed number. For the notational simplicity we suppress the dependence of ε on the solution of (35) and use the notation (u, S). We have the following theorem:

Theorem A.1. *Suppose $u_0 \in (L^1 \cap L^\infty \cap H^1)(\Omega)$. Then there exists a unique weak solution $(u,S) \in L^2_{loc}(\mathbb{R}_+; H^1(\Omega)) \times L^\infty(\mathbb{R}_+; H^1(\Omega))$ of (35). For all $1 \le p < \infty$ and any finite $T > 0$, the following ε-uniform estimates hold:*

$$0 \le u(t,x) \le 1 \text{ a.e. in } \mathbb{R}_+ \times \Omega; \quad \|u\|_{L^\infty([0,T];L^1(\Omega))} \le C_1(T); \tag{36}$$

$$0 \le S(t,x) \le 1 \text{ a.e. in } \mathbb{R}_+ \times \Omega; \quad \|S\|_{L^\infty(\mathbb{R}_+;W^{2,p}(\Omega))} \le C_2(d,p); \tag{37}$$

$$\begin{aligned} &\int_0^T \iint_{\Omega \times (\mathbb{R}^d \setminus \{0\})} \left(\beta(u(t,x+z)) - \beta(u(t,x))\right)^2 \pi(dz)\, dx\, dt \\ &\qquad + \sqrt{\varepsilon}\, \|\nabla u\|_{L^2((0,T)\times\Omega)} + \|\partial_t S\|_{L^2((0,T);H^1(\Omega))} \le C_3(d,T). \end{aligned} \tag{38}$$

Proof. To prove this theorem, note that the system (35) can be written as one single equation for u by means of the convolution representation formula

$$S(t,x) = (\mathcal{B} \underset{(x)}{\star} u)(t,x) = \int_\Omega \mathcal{B}(x-y) u(t,y)\, dy,$$

where $\mathcal{B}$ is the Bessel potential:

$$\mathcal{B}(x) := \frac{1}{(4\pi)^{d/2}} \int_0^{+\infty} \frac{e^{-r-|x|^2/4r}}{r^{d/2}}\, dr, \qquad x \in \mathbb{R}^d.$$

We shall also need the heat kernel associated with the operator $\partial_t - \varepsilon\Delta$:

$$\mathcal{G}(t,x) := \frac{1}{(4\pi\varepsilon t)^{d/2}} e^{-|x|^2/4\varepsilon t}, \qquad x \in \mathbb{R}^d, t > 0.$$

The function u is extended to all of $\mathbb{R}^d$ by letting it be 0 outside of Ω. So the heat kernel estimates over $\mathbb{R}^d$ can be employed. To this end, we also note that the following estimates follow by straightforward computations, see page 1291 in [15]:

$$\begin{aligned} &\|\mathcal{B}\|_{L^1(\mathbb{R}^d)} = 1; \qquad \|\nabla \mathcal{B}\|_{L^1(\mathbb{R}^d)} < \infty; \\ &\|\mathcal{G}(t,\cdot)\|_{L^p(\mathbb{R}^d)} \le C t^{-\frac{d(p-1)}{2p}} \quad (p \ge 1); \\ &\|\nabla \mathcal{G}(t,\cdot)\|_{L^p(\mathbb{R}^d)} \le C t^{-\frac{d(p-1)}{2p} - \frac{1}{2}} \quad (p \ge 1); \\ &\left\|D^2 \mathcal{G}(t,\cdot)\right\|_{L^p(\mathbb{R}^d)} \le C t^{-\frac{d(p-1)}{2p} - 1} \quad (p \ge 1). \end{aligned}$$

The above estimates enable us to prove local existence to $L^1 \cap L^\infty$ weak solutions of (17) by means of the implicit representation formula

$$
\begin{aligned}
\mathbf{u}(t,x) = {} & (\mathcal{G} \star u_0)(t,x) \\
& + \int_0^t \int_\Omega \nabla \mathcal{G}(t-s, x-y) f(\mathbf{u}(s,y)) \left(\nabla (\mathcal{B} \underset{(x)}{\star} \mathbf{u}) \right)(s,y)\, dy\, ds \\
& + \int_0^t \int_\Omega \beta(\mathbf{u}(s,y)) \mathcal{L}[\mathcal{G}](t-s, x-y)\, dy\, ds,
\end{aligned}
$$

utilizing a standard fixed point argument. We note here that this argument follows very much the approach in [15] and the arguments so far do not require having a bounded domain Ω. On the other hand, for further regularity estimates, needed in the technical proofs above, we need to consider the problem on a bounded domain with the specific restrictions mentioned above.

For the additional regularity properties listed in the theorem, for which we follow the approach of [48], we note that the first part of (36) follows from the maximum principle because 0 and 1 are solutions of the system. The second part follows by "testing" the u-equation in (35) against $\operatorname{sgn}(u)$. The first part of (37) also follows from the maximum principle, while the second part is a consequence of the regularizing effect for elliptic equations with smooth coefficients. The first two bounds in (38) are obtained by "testing" the u-equation in (35) against u. The third bound follows by differentiating the equation for S with respect to t, yielding

$$
-\Delta(\partial_t S) + (\partial_t S) = -\nabla \cdot (\nabla S f(u)) + \mathcal{L}[\beta(u)] + \varepsilon \Delta u,
$$

and noting that the right-hand side is uniformly bounded in $L^2([0,T]; H^{-1}(\Omega))$. $\square$

Fix a bounded time interval $[0,T]$, $T > 0$. Let $\{u^\varepsilon(t,x)\}_{\varepsilon>0}$ be a sequence of functions that is bounded in $L^\infty([0,T]; L^p(\Omega))$ uniformly in ε, for $p \in [1,\infty)$. We recall that a function $\nu_T : \mathbb{R}_+ \to [0,\infty)$, not depending on ε but possibly on T, which is non-decreasing, continuous, and satisfies $\nu_T(0+) = 0$, is called a modulus of continuity. The next lemma lists two equivalent characterizations of L^p compactness. The proof is essentially taken from [4].

Lemma A.2. *The spatial translation estimate*

$$
\sup_{|y|\le\delta} \|u^\varepsilon(t, \cdot + y) - u^\varepsilon(t,\cdot)\|_{L^p} \le \nu_{p,T}(\delta) \to 0, \quad \text{as } \delta \to 0, \quad t \in [0,T], \tag{39}
$$

where $\nu_{p,T} : [0,\infty) \to [0,\infty)$ is a modulus of continuity, holds if and only if there exists another modulus of continuity $\tilde{\nu}_{p,T}(\cdot)$ such that

$$
\int J_\delta(z) \left| u^\varepsilon(t, x+z) - u^\varepsilon(t, x-z) \right|^p \, dx\, dz \le \tilde{\nu}_{p,T}(\delta), \quad t \in [0,T], \tag{40}
$$

where $J_\delta(\cdot)$ is defined via (14) and (15).

Proof. First we will prove that (40) implies the existence a modulus of continuity $\nu_T(\cdot)$, independent of ε, such that (39) holds, that is, $\{u^\varepsilon\}_{\varepsilon>0}$ is compact in $L^p(\Omega)$. To this end, we write

$$
u^\varepsilon = u^{\varepsilon,\kappa} + \left(u^{\varepsilon,\kappa} - u^\varepsilon\right), \quad u^{\varepsilon,\kappa}(t,x) := \int J_\kappa\left(\tfrac{x-y}{2}\right) u^\varepsilon(t,y)\, dy, \quad \kappa > 0. \tag{41}
$$

Observe that

$$
\begin{aligned}
&\left\| u^{\varepsilon,\kappa}(t,\cdot) - u^{\varepsilon}(t,\cdot) \right\|_{L^p} \\
&\qquad \le \int \left(\int J_\kappa\left(\tfrac{x-y}{2}\right) \left| u^{\varepsilon}(t,y) - u^{\varepsilon}(t,x) \right| \, dy \right)^p \, dx \\
&\qquad = \int \left(\int J_\kappa(z) \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right| \, dz \right)^p \, dx \\
&\qquad \overset{\text{(Hölder ineq.)}}{\le} \left\| J_\kappa \right\|_{L^1(\mathbb{R}^d)}^{p-1} \int \int J_\kappa(z) \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz \\
&\qquad\qquad \overset{(40)}{\le} C \, \tilde{\nu}_{p,T}(\kappa) \to 0 \quad \text{as } \kappa \to 0, \text{ uniformly in } \varepsilon.
\end{aligned}
$$

Since $J_\kappa(\cdot/2)$ is a smooth convolution kernel, the ε-uniform $L^\infty([0,T];L^p(\Omega))$ bound implies that $u^{\varepsilon,\kappa}(t,\cdot)$ is compact in $L^p(\Omega)$ as $\varepsilon \to 0$, for each fixed $\kappa > 0$, that is, there exists a modulus of continuity $\nu_{p,T}^{\kappa}$, which is independent of ε but dependent on κ, such that

$$
\sup_{|y| \le \delta} \left\| u^{\varepsilon,\kappa}(t,\cdot+y) - u^{\varepsilon,\kappa}(t,\cdot) \right\|_{L^p} \le \nu_{p,T}^{\kappa}(\delta) \to 0 \quad \text{as } \delta \to 0.
$$

Consequently, in view of (41),

$$
\sup_{|y| \le \delta} \left\| u^{\varepsilon}(t,\cdot+y) - u^{\varepsilon}(t,\cdot) \right\|_{L^1} \le \inf_{\kappa > 0} \left\{ \nu_{p,T}^{\kappa}(\delta) + 2\tilde{\nu}_{p,T}(\kappa) \right\} =: \nu_T(\delta),
$$

which is (25).

Next, let us assume that $\{u^{\varepsilon}\}_{\varepsilon > 0}$ is compact in $L^p(\Omega)$, i.e., (39) holds for some ε-independent modulus of continuity $\nu_{p,T}(\cdot)$. Clearly,

$$
\begin{aligned}
&\iint J_\delta(z) \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz \\
&\qquad \le \frac{C_T}{|\log \delta|} + \iint\limits_{|z| \le 1} J_\delta(z) \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz.
\end{aligned}
$$

We split the integral on the right-hand side as follows:

$$
\begin{aligned}
&\iint\limits_{|z| \le 1} J_\delta(z) \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz \\
&\quad = \iint\limits_{|z| \le 1} \frac{C_\delta}{|\log \delta| \left(|z|^2 + \delta^2 \right)^{d/2}} \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz \\
&\quad = \iint\limits_{|z| \le \delta} \frac{C_\delta}{|\log \delta| \left(|z|^2 + \delta^2 \right)^{d/2}} \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz \\
&\quad + \iint\limits_{\delta < |z| \le 1} \frac{C_\delta}{|\log \delta| \left(|z|^2 + \delta^2 \right)^{d/2}} \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz =: A + B.
\end{aligned}
$$

To estimate A, note that $|z| \le \delta$ implies $\frac{1}{|z|^2+\delta^2} \le \frac{1}{2|z|^2}$, and hence

$$
|A| \le \frac{\tilde{C}}{|\log \delta|} \iint\limits_{|z| \le \delta} \frac{1}{|z|^d} \left| u^{\varepsilon}(t,x+z) - u^{\varepsilon}(t,x-z) \right|^p \, dx \, dz
$$

$$= \frac{\tilde{C}}{|\log h|} \int\limits_{|z|\le\delta} \frac{1}{|z|^d} \left(\int |u^\varepsilon(t,x+z) - u^\varepsilon(t,x-z)|^p \, dx \right) dz$$

$$\le \frac{\tilde{C}}{|\log \delta|} \left(\int\limits_{|z|\le\delta} \frac{1}{|z|^d} \, dz \right) \nu_{p,T}(\delta) \le C\, \nu^A_{p,T}(\delta) \to 0 \quad \text{(independently of } \varepsilon),$$

where we have used (39). To estimate B, note that $|z| \ge \delta \iff |z| \ge K\, 2^{-|\log\delta|}$ for some constant $K > 1$, and hence

$$|B| \le \iint\limits_{K\,2^{-|\log\delta|}<|z|\le 1} \frac{C_\delta}{|\log\delta|\,|z|^d} |u^\varepsilon(t,x+z) - u^\varepsilon(t,x-z)|^p \, dx\, dz$$

$$\le \frac{C_\delta}{|\log\delta|} \int\limits_{2^{-|\log\delta|}\le|z|\le 1} \frac{1}{|z|^d} \left(\int |u^\varepsilon(t,x+z) - u^\varepsilon(t,x-z)|^p \, dx \right) dz$$

$$\le \frac{C_\delta}{|\log\delta|} \sum_{n=0}^{|\log\delta|-1} \int\limits_{2^{-(n+1)}\le|z|\le 2^{-n}} \frac{1}{|z|^d} \left(\int |u^\varepsilon(t,x+z) - u^\varepsilon(t,x-z)|^p \, dx \right) dz$$

$$\le \frac{2^d C_\delta}{|\log\delta|} \sum_{n=0}^{|\log\delta|-1} \int\limits_{2^{-(n+1)}\le|z|\le 2^{-n}} 2^{dn} \left(\int |u^\varepsilon(t,x+z) - u^\varepsilon(t,x-z)|^p \, dx \right) dz$$

$$\le \frac{2^d C_\delta}{|\log\delta|} \sum_{n=0}^{|\log\delta|} \int\limits_{|z|\le 2^{-n}} 2^{dn} \left(\int |u^\varepsilon(t,x+z) - u^\varepsilon(t,x-z)|^p \, dx \right) dz$$

$$\overset{(39)}{\le} \frac{2^d C_\delta}{|\log\delta|} \sum_{n=0}^{|\log\delta|} \left(\int\limits_{|z|\le 2^{-n}} 2^{dn} \, dz \right) \nu_{p,T}(2^{-n}),$$

$$\le \frac{\tilde{C}}{|\log\delta|} \sum_{n=0}^{|\log\delta|} \nu_{p,T}(2^{-n}) =: \nu^B_{p,T}(\delta) \to 0 \quad \text{as } \delta \to 0, \text{ uniformly in } \varepsilon.$$

Summarizing, we can produce a non-decreasing and continuous function $\tilde{\nu}_{p,T}(\cdot)$,

$$\tilde{\nu}_{p,T}(\delta) := \frac{C_T}{|\log\delta|} + \nu^A_{p,T}(\delta) + \nu^B_{p,T}(\delta), \qquad \tilde{\nu}_T(0+) = 0,$$

which is independent of ε, such that (40) holds. □

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