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SINGULAR PERTURBATION AND BIFURCATION OF DIFFUSE TRANSITION LAYERS IN INHOMOGENEOUS MEDIA, PART II

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Abstract. In this paper, we study the connection between the bifurcation of diffuse transition layers and that of the underlying limit interfacial problem in a *degenerate* spatially inhomogeneous medium. In dimension one, we prove the existence of bifurcation of diffuse interfaces in a pitchfork spatial inhomogeneity for a partial differential equation with bistable type nonlinearity. Bifurcation point is characterized quantitatively as well. The main conclusion is that the bifurcation diagram of the diffuse transition layers inherits mostly from that of the zeros of the spatial inhomogeneity. However, explicit examples are given for which the bifurcation of these two are different in terms of (im)perfection. This is a continuation of [8] which makes use of bilinear nonlinearity allowing the use of explicit solution formula. In the current work, we extend the results to a general smooth nonlinear function. We perform detail analysis of the principal eigenvalue and eigenfunction of some singularly perturbed eigenvalue problems and their interaction with the background inhomogeneity. This is the first result that takes into account simultaneously the interaction between *singular perturbation*, *spatial inhomogeneity* and *bifurcation*.

1. Introduction and motivation.

The current work concerns the existence and properties of stationary transition layer solutions to reaction diffusion equation with spatial inhomogeneity in the presence of *singular perturbation* and *bifurcation*. Our results are proved for the one-dimensional bounded interval $\Omega = (-1, 1)$. But in order to describe and motivate the problem, we find it convenient to first consider arbitrary dimensions. The following is a typical example of equation to be considered:

$$\epsilon^2 \triangle u + f(u, x, \beta) = 0, \quad x \in \Omega \tag{1.1}$$

where $f(u, x)$ has two stable states, say $u = +1$ and $u = -1$, i.e., f is of bistable type. In the above, ϵ is a small constant signifying singular perturbation and β is another parameter describing some variation of the dependence on the spatial variable x. The general description of solution u^ϵ to (1.1) is that the domain Ω is

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partitioned into two subdomains Ω_+ and Ω_- in which u^ϵ takes values close to $+1$ and -1 and make a rapid transition across $\partial\Omega_+ = \partial\Omega_-$. The transition layer u^ϵ typically has width ϵ. The location and property of the transition layer are some of the main questions of interests. This paper, a continuation of [8], analyzes the behavior of the transition layer when the underlying spatial inhomogeneity undergoes some bifurcation process. As a specific example, we consider the following nonlinearity function:

$$f(u, x) = (1 - u^2)(u - a(x; \beta)), \quad \text{with } |a| < 1.$$

Then under very general condition, for $\epsilon \ll 1$, the transition layer can be shown to be located near the zeros of a. The main goal of this paper is to investigate if the zeros of a bifurcate, to what extent the transition layer solution also bifurcates. As in [8], β is often referred as the *bifurcation* parameter. We are interested in those spatial term a, such that at the bifurcation point β_*, $a(x; \beta_*)$ contains *degenerate* zero (throughout this paper, we say a zero is degenerate if the derivative of a evaluated at that zero is 0). The idea of adding a bifurcation parameter β into the spatial term a enables us to find transition layers around the *degenerate* zeros of a. To the best of our knowledge, this is the first type of result that combines the effects of *singular perturbation*, *degenerate spatial inhomogeneity* and *bifurcation*. We refer to [8] for further motivation and references. Here we simply list the results which are most relevant to the current paper.

In Angenent-Mallet-Paret-Peletier [2], the existence of *stable* transition layer solutions near *non-degenerate* zeros of the spatial inhomogeneous term $a(x)$ is constructed via comparison principle. In Hale-Sakamoto [7], Lyapunov-Schmidt reduction technique and the reduced equation are used to construct both stable and unstable diffuse layers. In higher dimensions, such questions are considered in [5]. In view of all the previous works, various kinds of non-degeneracy conditions on the spatial term a are assumed. On the other hand, it is interesting and important to investigate the behavior and existence of layers when *degeneracy* appears. In many cases, *degeneracy* is related to the phenomena of bifurcation, see for example [6, 3, 4]. It is well-known that bifurcation plays a very important role in understanding dynamical systems arising from applications in science and engineering. The series of papers [13, 11] suggest that for reaction diffusion systems with spatial homogeneous nonlinearity, there exists pitchfork bifurcation diagram for stationary solutions against some bifurcation parameter in the system. Motivated by these results, we want to investigate the occurrence of bifurcations of diffuse layers in inhomogeneous media and compute the bifurcation point(s) from which the transition layers start to bifurcate. Using a piece-wise linear example of f, the work [8] analyzes quantitatively the similarities and differences between the bifurcation behaviors of u^ϵ and a. This paper is to extend the result to more general smooth nonlinear function f. It requires careful analysis of the principal eigenvalue and eigenfunction of some singularly perturbed eigenvalue problems and their interaction with the background inhomogeneity.

This paper concentrates on the following *pitchfork inhomogeneity*:

$$\textbf{Pitchfork} : a(x; \beta) = -\frac{1}{2}x^3 + \beta x. \tag{1.2}$$

The above function is *degenerate* in the sense that $a_x(0; \beta = 0) = 0$. Pitchfork spatial inhomogeneity is chosen as it has *odd* symmetry and is seen in many bifurcation studies. In general, *symmetry* property plays a crucial role in reducing the complexity of bifurcation problem. As a variation of the theme, we also consider perturbation of this symmetric spatial inhomogeneity of the following form,

$$\textbf{Perturbed Pitchfork}: a(x;\beta,\gamma) = -\frac{1}{2}x^3 + \beta x + \gamma x^n, n \geq 4 \tag{1.3}$$

where $|\gamma|$ is some small number. It turns out that adding higher-order variations can change the geometrical properties of the bifurcation diagrams in the perspective of *imperfection*. We refer to [8, Section 2] for an explanation of various concepts of bifurcation related to this problem. In that work, we also considered a complete list of spatial inhomogeneity coming from universal unfoldings with co-dimensions no more than three.

Now we introduce the precise problem considered in this work. It is described as

Problem [G]. (Bifurcation of transition layer to bistable reaction diffusion equation with spatial inhomogeneity). *Find u^ϵ which solves*

$$\epsilon^2 u_{xx} + (1-u^2)(u - a(x;\beta)) = 0, \quad -1 < x < 1, \quad u_x(\pm 1) = 0. \tag{1.4}$$

The solution u^ϵ is called a transition layer *if $u^\epsilon \sim 1\,(-1)$ as $x \longrightarrow 1\,(-1)$. The above equation is the Euler-Lagrange equation of the following functional:*

$$\mathcal{F}_\epsilon^G(u) := \int_{-1}^{1} \left[\frac{\epsilon}{2}u_x^2 + \frac{1}{4\epsilon}\left(1-u^2\right)^2 + \frac{a(x;\beta)}{\epsilon}\int_{-1}^{u}\left(1-s^2\right)ds \right] dx. \tag{1.5}$$

The goal is to understand how the bifurcation behavior of the solution resembles or differs from that of the zero(s) of the spatial inhomogeneity function $a(x;\beta)$ as the parameter β passes through some *critical value*, or *bifurcation point*.

In order to obtain precise analytical answer, in [8] we consider the following version of the above problem:

Problem [E]. (Bifurcation of transition layer with bilinear nonlinearity). *Find $u^\epsilon(x;\beta)$ and x_* which satisfy*

$$\begin{cases} \epsilon^2 u_{xx} = u - 1 - a(x;\beta), & \text{if } x_* < x \leq 1 \\ \epsilon^2 u_{xx} = u + 1 - a(x;\beta), & \text{if } -1 < x \leq x_* \\ u_x(\pm 1) = 0, & \text{Neumann boundary condition} \\ u(x_*^-) = u(x_*^+) = 0, & C^0 - \text{matching} \\ u_x(x_*^+) - u_x(x_*^-) = 0. & C^1 - \text{matching} \end{cases} \tag{1.6}$$

Similar to **[G]***, the above is the Euler-Lagrange equation of the following functional:*

$$\mathcal{F}_\epsilon^E(u) := \int_{-1}^{1} \left[\frac{\epsilon}{2}u_x^2 + \frac{1}{2\epsilon}(1-|u|)^2 - \frac{a(x;\beta)u}{\epsilon} \right] dx. \tag{1.7}$$

Problem **[E]** is proposed since the functional $\mathcal{F}_\epsilon^E$ is a good approximation of $\mathcal{F}_\epsilon^G$. Furthermore, the system (1.6) consists of ODE with constant coefficients which facilitates the use of explicit solution formula. The connection between the solution u and that of a is revealed concretely. In particular, we have demonstrated the similarities and differences between them in realm of normal forms and universal unfoldings.

To obtain statements for **Problem [G]** compatible to those for **Problem [E]** would require very precise computation related to some singular eigenvalue problems. Hence we will concentrate on the one example – pitchfork bifurcation – mentioned above. We believe the analysis for other types of bifurcation can be done similarly.

To tackle **Problem [G]**, we use the Lyapunov-Schmidt reduction (**LS**) technique to construct transition layers as a perturbation of some good initial approximation in the direction of some principal eigenfunction. Due to the presence of degeneracy, one has to choose approximate solutions with sufficient degree of accuracy. Moreover, unlike standard implementation of **LS** where usually only the linear term matters, we have to work harder to obtain information about nonlinear terms in order to obtain meaningful *reduced bifurcation equation*. This equation is pivotal to analyze the bifurcation of transition layers. Its derivation relies on careful study of some singularly perturbed eigenvalue problems.

The essence of **LS** is to reduce the solutions of partial differential equations to the zeros of some finite dimensional function. This function is usually denoted by $B(\alpha;\beta)$. We want to solve for α in $B(\alpha;\beta) = 0$ and investigate how the solution depends on β. To obtain quantitative results, we consider the Taylor expansion of B:

$$B(\alpha;\beta) = \sum_{i=1}^{N} B_i(\alpha;\beta)\alpha^i + O(\alpha^{N+1})$$

where the order N of the expansion is usually problem specific and is related to how degenerate the situation is. The computation of the Taylor coefficients is intimately related to some singularly perturbed linearized operator L_U^ϵ which depends not only on the singular parameter ϵ, but also on the spatial inhomogeneity $a(x;\beta)$ as well as some initial choice of approximate solution U. These dependences have not been carefully studied in [12, 18, 19, 15, 13, 10, 11, 16, 17]. There works are primarily concerned with the uniform boundedness and invertibility of L_U^ϵ. This is the main theme of the SLEP – singularly limit eigenvalue problem – method. These results are sufficient if the spatial inhomogeneity is non-degenerate leading to the non-vanishing property of B_1. On the other hand, when degeneracy arises, we will need high order terms B_n $(n \geq 2)$ which are out of the scope of SLEP. For the case of the pitchfork spatial inhomogeneity, we target at the B_3 term which requires much more detail analysis of the interaction between the eigenfunction and the spatial inhomogeneity function. We found that the form of the reduced bifurcation equation *mostly* inherits from that of the spatial inhomogeneity. The inheritance explains why Problem **[G]** shares many similarities with the underlying spatial inhomogeneity function $a(x;\beta)$.

It is also worthwhile to note that our results can handle the case when β depends on the singular parameter ϵ so that the bifurcation diagram is characterized with accuracy down to ϵ-neighborhood of the bifurcation point. To accurately calculate the bifurcation point for the singularly perturbed problem (1.1), it is often necessary to *stretch* the bifurcation parameter in order to derive a meaningful reduced bifurcation equation. A work, using similar technique is Matkowsky and Reiss [14], in which they analyze the *perturbation of bifurcation* resulting from imperfections or inhomogeneities. In particular, they considered nonlinear algebraic problem of the form,

$$F[y;\lambda,\delta] = 0$$

where λ is the bifurcation parameter and δ describes some perturbation of the underlying function. However, it does not consider singular perturbations. Nevertheless, we borrow from [14] the idea of *stretching* δ near the bifurcation point and treat different regimes of δ separately and then *combine* them together.

Our approach to construct transition layers to **Problem [G]** consists of three main steps: (1) construction of initial approximate solutions with sufficient high-degree of accuracy; (2) careful convergence study of the linearization around the approximate solutions and (3) implementation of Lyapunov-Schmidt reduction technique. Very often, geometric methods like invariant manifold theories are developed to investigate the solution structures to general PDEs. For example, Hale-Sakamoto [7, Remark 4.9] pointed out that transition layers can be constructed using *invariant manifold theory.* Essentially, invariant manifold theory is introduced so that the dynamics of transition layers can be described by a reduced ODEs along the manifold. Then the solution structures are revealed by studying the reduced ODEs. It has the same flavor as studying the reduced bifurcation equation used in the current paper. Specifically, the first equation in (4.17) of [7] (in the case of degenerate pitchfork spatial inhomogeneity) in general yields $\lambda_1^\epsilon = O(\epsilon^3)$ and $\widetilde{G}(\epsilon)$ can be made sufficiently small, provided very good approximate solutions are available. Bifurcation structure of transition layers will be revealed unless the nonlinear term (given by $Y(y, v, \epsilon)$) are carefully studied. However, these can be nontrivial to tackle, in particular in the presence of singular perturbation. **LS** reduction method is chosen in our work, as it can produce useful formulas for the coefficients $B_i(\alpha;\beta)$ which then enables us to study the connection between the degeneracy of the spatial inhomogeneity and the bifurcation phenomena.

With the above introduction, we now describe more quantitatively our main results. On the other hand, we still find it instructive to first recall the related results from [8] concerning the pitchfork inhomogeneity. In that work, we have explicitly written down the equation of the zeros x_* of the solution u for **Problem [E]** and compare it with those of a. We collect here the following three statements:

(A) Perfect Pitchfork Bifurcation. This happens for $a(x;\beta) = -\frac{1}{2}x^3 + \beta x + cx^{2k+1}$, $k \geq 2$. When $c = 0$, then x_* satisfies:

$$-\frac{1}{2}x_*^3 + (\beta - 3\epsilon^2)x_* = \text{e.s.t.}$$

where e.s.t. refers to exponentially small terms of the form $e^{-\frac{C}{\epsilon}}$. The above demonstrates that the bifurcation point for u is at $3\epsilon^2$, i.e. it is delayed by an order of $O(\epsilon^2)$. But due to the oddness of a, it still gives a perfect pitchfork bifurcation. For $c \neq 0$, the delay in the bifurcation point is given by $3\epsilon^2 + \tau_{o*}^\epsilon(c)$ for some higher order terms $\tau_{o*}^\epsilon(c)$ (see Fig. 1(a)).

(B) Imperfect Bifurcation without Hysteresis. This happens for $a(x;\beta) = -\frac{1}{2}x^3 + \beta x + cx^{2k}$, $k \geq 2$. We follow the notations in [8]. The reduced location equation of x_* is given by

$$G_{2k}^\epsilon(x,\beta;c) := A_{2k}^\epsilon(\beta;c)x^3 + B_{2k}^\epsilon(\beta;c)x + \alpha_{1,2k}^\epsilon(\beta,c) + \alpha_{2,2k}^\epsilon(\beta,c)x^2 + \text{ h.o.t} = 0 \tag{1.8}$$

where h.o.t refers to power functions (e.g., x^{2k}) of order greater than 3. In the case $k \geq 2$, then the coefficients satisfy

$$A_{2k}^\epsilon(\beta;c) = \frac{1}{2}, B_{2k}^\epsilon(\beta;c) = -(\beta - 3\epsilon^2), \alpha_{1,2k}^\epsilon(\beta,c) \propto -c\epsilon^{2k}, \alpha_{2,2k}^\epsilon(\beta,c) \propto -c\epsilon^{2k-2}.$$

where the notation $a(\epsilon) \propto b(\epsilon)$ meaning that $\liminf_{\epsilon\downarrow 0} \dfrac{a(\epsilon)}{b(\epsilon)} > 0$. The equation (1.8) is similar to the perfect pitchfork case **(A)**, except that the coefficients for both x^0 and x^2 terms are nonzero. They are sources of imperfection in the bifurcation diagram for equation (1.8). Specifically, (1.8) gives imperfect pitchfork bifurcation without hysteresis if

$$\frac{\alpha_{1,2k}^{\epsilon}(\beta,c)}{A_{2k}^{\epsilon}(\beta;c)} < \frac{1}{27}\left[\frac{\alpha_{2,2k}^{\epsilon}(\beta,c)}{A_{2k}^{\epsilon}(\beta;c)}\right]^3 \quad \text{for} \quad c>0. \tag{1.9}$$

Similar conclusion applies to $c<0$. See the detailed discussion in [8]. In either case, for $c\neq 0$, the delay in the bifurcation point is given by $3\epsilon^2+\tau_{e*}^{\epsilon}(c)$ for some higher order terms $\tau_{e*}^{\epsilon}(c)$ (see Fig. 1(b)).

(C) Imperfect Bifurcation with Hysteresis. This happens for $a(x;\beta) = -\frac{1}{2}x^3+\beta x+cx^2$. Using the same notations as in (1.8), the coefficients for $G_2^{\epsilon}(x,\beta;c) = 0$ satisfy

$$A_2^{\epsilon}(\beta;c)=\frac{1}{2},\ B_2^{\epsilon}(\beta;c)=-(\beta-3\epsilon^2),\ \alpha_{1,2}^{\epsilon}(\beta,c) = -2\epsilon^2 c,\ \alpha_{2,2}^{\epsilon}(\beta,c) = -c.$$

The analysis here is similar to the previous case **(B)**, except that the following relationship is used,

$$\frac{\alpha_{1,2}^{\epsilon}(\beta,c)}{A_{22}^{\epsilon}(\beta;c)} > \frac{1}{27}\left[\frac{\alpha_{2,2}^{\epsilon}(\beta,c)}{A_{2}^{\epsilon}(\beta;c)}\right]^3 \quad \text{for} \quad c>0. \tag{1.10}$$

Hence, the resulting bifurcation for x_* is imperfect pitchfork bifurcation with hysteresis in the primary branch. See the details in [8] and the imperfect pitchfork bifurcation digram in Fig. 1(c).

In principle, for **Problem [G]**, with careful asymptotic analysis, we can reproduce all of the above results to any degree of accuracy. But to keep the current paper within reasonable scope, we will demonstrate the key features of the bifurcation diagram. This already requires many intricate analysis. We will concentrate on cases **(A)** and **(B)** above. A more complete understanding of the interaction between singular perturbation and bifurcation in spatial inhomogeneity will be deferred for future work.

Our first main result is the validation of case **(A)** above for **Problem [G].** The overall results are divided into the following statements depending on how close β is to the actual bifurcation point (denoted as β_*^{ϵ}):

Propositions 3.1 and 3.2: *construction of good initial approximate solution. They lead to three translational shifts for some standard profile u_0.*

Theorem 3.7: *construction of exact solution around each of the shifts found previously. This works for β not too close to the actual bifurcation point $\epsilon^2\widetilde{\beta}_*$ (see (3.11)). Specifically, it requires:*

$$\beta = \epsilon^2\widetilde{\beta}_{\mathsf{q}} + \frac{1}{2}\mu^2$$

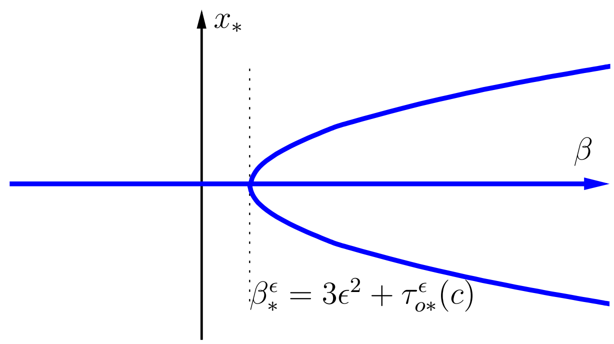
for $\mu \gg \epsilon^{\frac{5}{2}}$.

Theorem 3.11: *construction of three exact solutions around the primary (center) solution branch corresponding to the zero shift. This works for β close to the actual bifurcation point:*

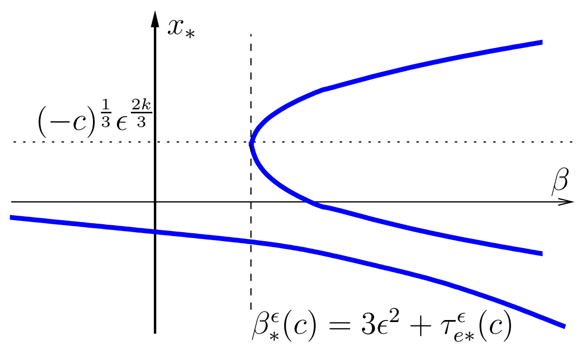
$$\beta = \epsilon^2\left(\widetilde{\beta}_{\mathsf{q}} + \delta\right)$$

for $\delta \ll \epsilon^{2+p}$ for any $p>0$.

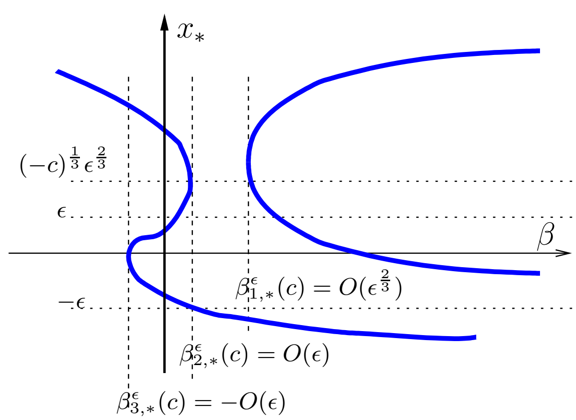
Note that there is an overlapping region which is covered by both Theorems above. This is stated precisely in Proposition 3.12. The bifurcation diagram is depicted in Figure 2.



(a) Perfect pitchfork bifurcation, $c = 0$ or n is odd



(b) Imperfect pitchfork bifurcation without hysteresis, $c < 0$ and $n = 2k, k \geq 2$



(c) Imperfect pitchfork bifurcation with hysteresis, $c < 0$ and $n = 2$

FIGURE 1. Pitchfork Bifurcations of Problem **[E]**. (**a**): perfect pitchfork bifurcation where the spatial inhomogeneity has odd symmetry. (**b**): imperfect pitchfork bifurcation without hysteresis which results from adding higher-order even perturbation and $c < 0$. (**c**): imperfect pitchfork bifurcation with hysteresis which results from adding cx^2 and $c < 0$. The terms $\tau_{o*}^\epsilon, \tau_{e*}^\epsilon$ are lower order corrections to the bifurcation points.

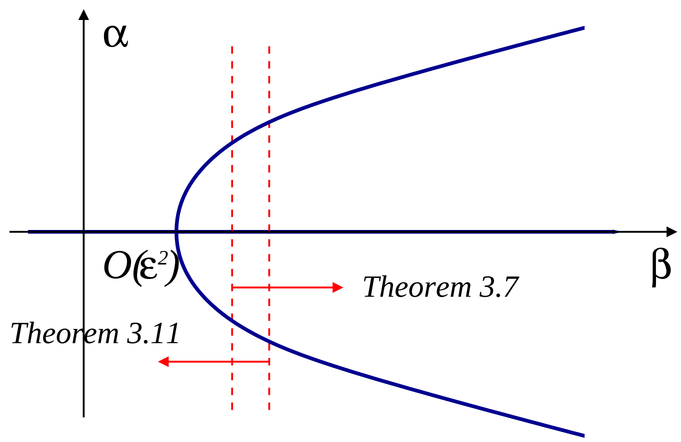


FIGURE 2. Perfect Pitchfork Over Different Regions

Our second result **Theorem 5.3** concerns case **(B)** above. Many of the analysis is similar to the previous case so the exposition there will be brief.

The next section explains our main strategy and the difficulties we encounter.

2. The method.

In this part, we first formulate the Lyapunov-Schmidt reduction technique and give the definition of reduced (bifurcation) equation in (2.15). We explain the main difficulties in obtaining a useful form of such an equation when degeneracy arises. The reduction of the full problem (1.4) to a scalar equation is possible due to some crucial technical results. Our overall scheme resembles that of [7], but we need much more elaborate asymptotic expansions for the principal eigenvalue and eigenfunction in order to obtain precise information on the coefficients of the reduced (bifurcation) equation.

2.1. Lyapunov-Schmidt (LS) formulation.

Using the following *shift stretched variable*,

$$s = t - z = \frac{x}{\epsilon} - z^{\epsilon}(\beta), \tag{2.1}$$

the original problem (1.4) becomes

$$u_{ss} + (1-u^2)u + (1-u^2)\left(-a(\epsilon(s+z);\beta)\right) = 0, \quad u_s\Big(\pm\frac{1}{\epsilon} - z\Big) = 0. \tag{2.2}$$

The following spatially homogeneous version of the above in infinite real line provides useful information to initiate the investigation:

$$\begin{cases} u_{ss} + \left(1-u^2\right)u & = & 0, \quad -\infty < s < \infty \\ u(0) & = & 0, \\ u(\pm\infty) & = & \pm 1. \end{cases} \tag{2.3}$$

It has the solution

$$u_0(s) = \tanh\left(\frac{s}{\sqrt{2}}\right) \tag{2.4}$$

which is often called the *standard transition layer profile*. The linearization of (2.3) at u_0 is given as follows.

Definition 1 (Linearized operator at standard transition profile u_0)**.**

$$L_0\left[V\right] := V_{tt} + \left(1 - 3u_0^2\right)V. \tag{2.5}$$

Note that its derivative $\frac{d}{ds}u_0(s)$, often denoted as $\dot{u}_0$, lies in the kernel of L_0. As $\dot{u}_0$ is a positive function, it is also the principal eigenfunction.

We next note two useful identities which are obtained by integrating and differentiating (2.3):

$$\sqrt{2}\dot{u}_0 = (1 - u_0^2) \tag{2.6}$$

$(\ddot{u}_0\dot{u}_0 + (1-u_0^2)u_0\dot{u}_0 = 0,\ \ \frac{1}{2}\frac{d}{ds}(\dot{u}_0{}^2) = \frac{1}{4}\frac{d}{ds}\left(1-u_0^2\right)^2,\ \ 2\dot{u}_0{}^2 = \left(1-u_0^2\right)^2$ and noting $\dot{u}_0 > 0.)$

$$L_0(\ddot{u}_0) = 6u_0\dot{u}_0^2. \tag{2.7}$$

$((\dot{u}_0)_{ss} + (1-3u_0^2)\dot{u}_0 = 0,\ \ (\ddot{u}_0)_{ss} + (1-3u_0^2)\dot{u}_0 - 6u_0\dot{u}_0^2 = 0.)$

Using an appropriate shift z^ϵ, we will construct a good initial approximate solution $U(s;\epsilon)$ of (2.2). If we write the exact solution u as $u = U + w(s;\beta,\epsilon)$ where w is some correction term, then problem (2.2) is recast into finding w which solves

$$L_U^\epsilon\left(w\right) + G^\epsilon(U) + F(w,\beta;\epsilon) = 0 \tag{2.8}$$

where

$$\begin{aligned}
L_U^\epsilon\left(w\right) \ &:=\ w_{ss} + \left(1 - 3U^2 + 2a(\epsilon(s+z);\beta)U\right)w, &(2.9)\\
G^\epsilon(U) \ &:=\ \left[U_{ss} + (1-U^2)U\right] + (1-U^2)\left(-a(\epsilon(s+z);\beta)\right), &(2.10)\\
f(U;x,\beta) \ &:=\ (1-U^2)(U - a(x;\beta)),\\
F(w,\beta;\epsilon) \ &:=\ f(U+w,\epsilon(s+z),\beta) - f\left(U,\epsilon(s+z),\beta\right) - f_u\left(U,\epsilon(s+z),\beta\right)w.
\end{aligned}$$

For the pitchfork spatial inhomogeneity (1.2), we explicitly have

$$\begin{aligned}
a(\epsilon(s+z);\beta) \ &=\ -\frac{1}{2}\epsilon^3(s+z)^3 + \epsilon\beta(s+z), &(2.11)\\
F(w,\beta;\epsilon) \ &=\ \left[-3U + a(\epsilon(s+z);\beta)\right]w^2 - w^3. &(2.12)
\end{aligned}$$

For equation (2.2), with good initial approximate solution U, the linearized operator L_U^ϵ enjoys the nice property that only its principal eigenvalue (denoted as λ_1^ϵ) is critical, i.e., it goes to zero as ϵ approaches zero. In this case, Lyapunov-Schmidt reduction technique is very convenient to transform equation (2.2) into a system of equations one of which is the reduced (bifurcation) equation. This is performed as follows. Let λ_1^ϵ and ϕ^ϵ be the principal eigenvalue and its associated normalized eigenfunction i.e., $L_U^\epsilon\left(\phi^\epsilon\right) = \lambda_1^\epsilon\phi^\epsilon$. We further decompose the correction function w as

$$w = \alpha\phi^\epsilon + v$$

where ϕ^ϵ and v are orthogonal in the L^2-sense, i.e., $\langle v,\phi^\epsilon\rangle = 0$. Now solving problem (2.2) is equivalent to finding α and $v(\alpha;\beta;\epsilon)$ such that

$$L_U^\epsilon\left(\alpha\phi^\epsilon + v\right) + G^\epsilon(U) + F(\alpha\phi^\epsilon + v;\beta,\epsilon) \ = \ 0. \tag{2.13}$$

They are found by two steps.

For any given α, first project (2.13) onto $\left(\phi^\epsilon\right)^\perp$. We seek a v which solves:

Definition 2 (Equation for orthogonal component)**.**

$$L_U^\epsilon(v) + (I - E_{\phi^\epsilon})\left\{G^\epsilon(U)\ +\ F(\alpha\phi^\epsilon + v;\beta,\epsilon)\right\}\ =\ 0 \tag{2.14}$$

where E_{ϕ^ϵ} is the L^2-projection operator onto $\mathrm{Span}(\phi^\epsilon)$.

The spectrum analysis of the operator L_U^ϵ implies that v is always solvable in the space orthogonal to ϕ^ϵ. With the v just found (which depends on α), then the correct value of α is found by projecting (2.13) onto ϕ^ϵ. This leads to the following equation.

Definition 3 (Reduced (bifurcation) equation for Problem [**G**])**.**

$$B^\epsilon(\alpha;\beta) = 0 \quad \text{where} \quad B^\epsilon(\alpha;\beta) := \alpha\,\lambda_1^\epsilon(\beta) + \frac{\langle G^\epsilon(U) \,+\, F(\alpha\phi + v), \phi^\epsilon\rangle}{\langle \phi^\epsilon, \phi^\epsilon\rangle}. \tag{2.15}$$

where $\langle\cdot,\,\cdot\rangle$ refers to the standard L^2-inner product on $(-1, 1)$.

In the above, α is the unknown scalar variable and β is regarded as a parameter. As to be seen, the reduced equation may take on different forms for different regimes of β. In particular, for β close to the bifurcation point, the equation involves much more accurate quantification of higher-order coefficients. Their analysis require very precise information on the spectrum of L_U^ϵ. In Section 2.2, we discuss the main difficulties in obtaining a useful reduced (bifurcation) equation in our degenerate setting.

Remark 2.1. Equation (2.15) is called the *reduced* or *reduced bifurcation equation* depending on its usage and outcome. Our overall construction starts by first finding some *macroscopic shifts* z^ϵ's – there might be multiple of them. Then finer adjustment in the form of $\alpha\phi^\epsilon$ is required. The correct value of α is found by solving equation (2.15). When the macroscopic shift is sufficiently large, we call the equation *reduced equation* as only one value of α will be found around each z^ϵ. However, when the macroscopic shift is too small – corresponding to β very close to the bifurcation value, then multiple solutions of α might exist for a single value of z^ϵ. In this case, (2.15) will be called the *reduced bifurcation equation*. Which one to use depends on the regimes of β. This will be explained in more detail in Section 2.2.

The above implementation starts by constructing a good initial approximate solution to (2.2). This is achieved by considering perturbation of the standard transition layer profile u_0 defined in (2.4). However, we need to proceed with great care due to degeneracy. The perturbation consists of *macroscopic* part ψ^ϵ and *microscopic* part $\alpha\phi + v$. They are modified appropriately in the following form so as to fulfill the Neumann boundary condition:

$$u_{\text{exact}}(x) = \Big[\xi_0(\epsilon s)\,(u_0(s) + \psi^\epsilon(s;\beta)) + \xi_1(\epsilon s)\Big] + \Big[\alpha\phi^\epsilon\,(s) + v\Big], \tag{2.16}$$

where $x = \epsilon s + \epsilon z^\epsilon(\beta)$ (from (2.1)). The cut-off functions ξ_0 and ξ_1 are defined as follows (illustrated also in Figure 3).

$$\xi_0(r) := \begin{cases} 1, & \text{if } |r| \le \frac14 \\ 0, & \text{if } |r| > \frac12 \\ \text{smooth extension} & \text{if } |r| \in [\frac14, \frac12] \end{cases}, \quad \xi_1(r) := \begin{cases} 0, & \text{if } |r| \le \frac14 \\ 1 - \xi_0(r), & \text{if } r > \frac14 \\ -1 + \xi_0(r). & \text{if } r < -\frac14 \end{cases}.$$

The function $\psi^\epsilon(s;\beta)$ appeared in the macroscopic perturbation is used to annihilate the leading order term of the error $G^\epsilon(u_0)$. It is determined by the spatial inhomogeneity $a(x;\beta)$. Its solvability is paired with a suitable choice of the shift $z^\epsilon(\beta)$ using the Fredholm alternative leading to an algebraic equation for $z^\epsilon(\beta)$. For convenience, we formally define the following

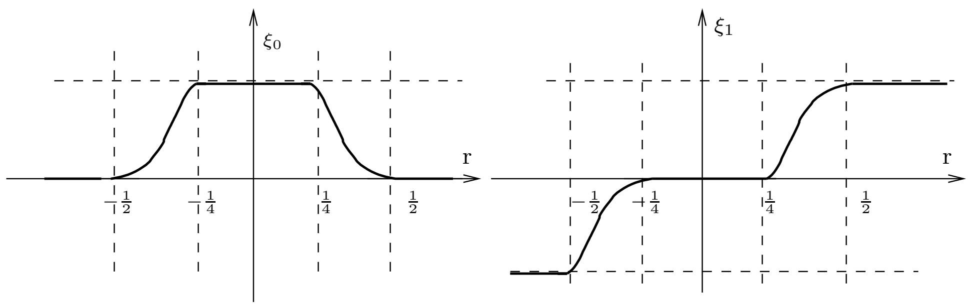


FIGURE 3. Graph of Cutoffs $\xi_0(r), \xi_1(r)$

Definition 4 (Initial good approximate solution)**.**

$$U^{\epsilon;\beta} := \xi_0(\epsilon s)\,(u_0(s) + \psi^\epsilon(s;\beta)) + \xi_1(\epsilon s) \tag{2.17}$$

and

$$L^\epsilon_{\text{app}} := L^\epsilon_{U^{\epsilon;\beta}} = \partial_{ss} + \Big(1 - 3\left(U^{\epsilon;\beta}\right)^2 + 2a(\epsilon(s+z);\beta)U^{\epsilon;\beta}\Big). \tag{2.18}$$

In the above, the function ψ^ϵ will be constructed in Propositions 3.1 and 3.2.

Note that both of the above implicitly depends on the shift z^ϵ. In addition, we have

$$U^{\epsilon;\beta}(s) \quad = \quad u_0(s) + \psi^\epsilon(s;\beta), \quad \text{for} \quad |s| \le \frac{1}{4\epsilon}; \tag{2.19}$$

$$\big||U^{\epsilon;\beta}(s)| - 1\big| \quad \le \quad Ce^{-Cs}. \quad \text{for } |s| \ge \frac{1}{4\epsilon} \text{ for some constant } C. \tag{2.20}$$

The microscopic perturbation $\alpha\phi^\epsilon + v$ is to incorporate further adjustment to make $U^{\epsilon;\beta}$ into an exact solution to (2.2). It also consists of a microscopic shift $\alpha\phi^\epsilon$ in the direction of the principal eigenfunction and v perpendicular to ϕ^ϵ. Overall, we expect that $\|\alpha\phi^\epsilon\| \ll |z^\epsilon|$ and $\|v\| \ll \|\psi^\epsilon + \alpha\phi^\epsilon\|$. Even though the above decompositions are very intuitive, the actual construction is quite elaborate and it depends on the scaling of β, in particular how far β is away from the actual bifurcation point.

2.2. **Strategy to handle the degeneracy.** The main technical difficulties of our analysis come from the fact that the spatial inhomogeneity function $a(\cdot;\beta)$ is *degenerate* in the sense that $a_x(x_*;\beta)$ can vanish at the zeros of a, in particular, at bifurcation point. All the existing results do not apply to this case. The key to obtain globally the bifurcation of the singularly perturbed transition layer problem, is to treat different regimes of β separately.

Given a normal form $a(x;\beta)$ (as in [6]) as the spatial inhomogeneity, our goal is to prove that bifurcation of transition layers is similar to that of the zeros of the underlying spatial inhomogeneity $a(x;\beta)$. Moreover, we also like to study the dependence of bifurcation point β^ϵ_* on ϵ. In the current degenerate setting, the choice of approximate solution and the form of the reduced equation are often closely related to the value of β. Singularly perturbed problem (1.4) calls for scale of $\beta = \widetilde{\beta}\epsilon^m$ with $m > 0$. The followings are the major steps in our strategy.

Step I. First we will find an accurate value of the shift z^ϵ appeared in (2.1). It solves a *location equation*. For each shift, we find the corresponding macroscopic perturbation ψ^ϵ leading to an initial good approximate good solution $U^{\epsilon;\beta}$ as written in (2.17). This is the basis of the *shift variable* [**SV**] approach. From here we can also identity a *critical scale* m_* for $\beta = \widetilde{\beta}\epsilon^m$. What follows next depend only on $m < m_*$ or $m > m_*$.

Step II. In the *sub-critical* case, $m < m_*$, i.e. relatively large β, *near each shift* z^ϵ found above, we make use of the *reduced equation* [**RE**] (2.15) to find the microscopic perturbation $\alpha\phi^\epsilon + v$ which would then give an exact solution to (2.2). The outcome is that there is one solution near each zero of $a(x;\beta)$.

Step III. At the *critical scale* $m = m_*$, we need to consider the bifurcation point $\widetilde{\beta}_*$ of the location equation. We introduce another parameter δ:

$$\beta = \widetilde{\beta}\epsilon^{m_*} = [\widetilde{\beta}_* + \delta]\epsilon^{m_*}.$$

When δ is not too small, we can still use the [**RE**] approach to find the the corresponding α and v near each shift. In fact, in this case, the analysis of **Step II** can be combined with the current step via transformation $\mu = \epsilon\sqrt{2\delta}$ in (3.28). However, when δ is much smaller, then we need much more careful analysis of the reduced equation, in particular we will make use of the precise asymptotics of the principal eigenvalue. The overall procedure resembles the bifurcation analysis of the reduced equation. Hence we call it the *reduced bifurcation equation* approach [**RB**].

In the above scheme, the dissection of bifurcation parameter β into sub-critical, critical regions is really necessary. Both the [**SV**] and [**RB**] approaches can be regarded as instantiations of the Lyapunov-Schmidt reduction technique. However, they are different in several aspects. Precisely, the two approaches start with different approximate solutions. Consequently, the associated reduced equations take on completely different forms. The one obtained by [**RB**] method is much more degenerate and involved. It requires more elaborate information on high-order terms. This overall treatment is similar to [14] where inner, outer and matching regions are considered to obtain global representation of solutions.

There is another important issue deserving special attention. In **Step III**, we establish bifurcation of the diffuse transition layers for the bifurcation parameter close to the bifurcation point *locally*. For bifurcation parameter in the sub-critical region, [**SV**] has already resulted the existence of multiple transition layers. One of the main difficulty in concluding the existence of bifurcation of diffuse layer *globally* is to make sure that the two approaches [**SV**] and [**RB**] have *overlapping* region. For that, one must push both methods to their limits. For instance, in the critical scale region, in terms of the offset variable δ, there is always a lower limit $\delta_l^\epsilon > 0$ bounding the applicability of [**SV**]. On the other hand, there is always an upper limit $\delta_u^\epsilon > 0$ restricting the applicability of [**RB**]. We need to ensure that $\delta_l^\epsilon < \delta_u^\epsilon$. Once this holds, it implies that there is an overlapping (or matching) region where both methods work.

To make the above discussion more quantitative, we give an outline of the key results.

1. The shift variable approach is captured by the location equation (Proposition 3.2 (3.12)):

$$\frac{1}{2}z^3 - (\widetilde{\beta} - \widetilde{\beta}_*)z = 0, \quad \text{where} \quad \beta = \epsilon^2\widetilde{\beta}$$

which gives three solutions for $\widetilde{\beta} > \widetilde{\beta}_*$:

$$z_1 = 0, \quad z_2 = \sqrt{2(\widetilde{\beta} - \widetilde{\beta}_*)}, \quad z_3 = -\sqrt{2(\widetilde{\beta} - \widetilde{\beta}_*)}.$$

2. Linearized at an approximate solution centered around each location found above, the principal eigenvalue is given by (Proposition 3.8):

$$\begin{aligned} \lambda_1^\epsilon &= \tfrac{1}{2}K_1\epsilon\mu^2 + O(\epsilon^6+\epsilon^2\mu^4) \quad (\text{at } z_1),\\ \lambda_1^\epsilon &= -K_1\epsilon\mu^2 + O(\epsilon^6+\epsilon^2\mu^4) \quad (\text{at } z_2, z_3) \end{aligned} \qquad \text{where } \mu^2 = 2(\widetilde{\beta}-\widetilde{\beta}_*) \text{ and } K_1>0.$$

3. If μ is "not too small" in the sense that $\mu \gg \epsilon^{\frac{5}{2}}$, then the form of the reduced equation (2.15) is dominated by the asymptotics of λ_1^ϵ (Proposition 3.6):

$$B^\epsilon(\alpha;\mu) = \alpha\lambda_1^\epsilon(\mu) + \frac{\langle G^\epsilon(U)+F(\alpha\phi+v),\,\phi\rangle}{\langle\phi,\,\phi\rangle} \approx \alpha\lambda_1^\epsilon(\mu) \approx 0.$$

As $\lambda_1^\epsilon \neq 0$, the above leads to a solution α (close to zero) for each location, giving an exact solution of (2.2) or (2.13) (Theorem 3.7).

4. On the other hand, if μ is "too small" in the sense that $\mu \ll \epsilon^{\frac{5}{2}}$, then both the eigenvalue and the nonlinearity in the form of the reduced equation (2.15) play equally important role. Specifically, we linearized at the approximate solution with zero shift z_1. Then we have (Proposition 4.1):

$$\lambda_1^\epsilon = K_1\epsilon^3\delta + O(\epsilon^6) \quad \text{where } \delta = \widetilde{\beta}-\widetilde{\beta}_* \text{ or } \mu = \epsilon\sqrt{2\delta}$$

so that

$$\partial_\alpha B^\epsilon(\alpha,\beta)\Big|_{\alpha=0} \approx \lambda_1^\epsilon.$$

Furthermore, the quantity $\partial_\alpha^3 B^\epsilon(\alpha,\beta)\Big|_{\alpha=0}$ is crucial. It is given by (Proposition 3.9):

$$\partial_\alpha^3 B^\epsilon(\alpha,\beta)\Big|_{\alpha=0} = (3!)\left[-K_2\epsilon^3 + O(\epsilon^4)\right].$$

The above leads to the following form of the reduced bifurcation equation and its three solutions demonstrating a pitchfork bifurcation (Theorem 3.11):

$$B^\epsilon(\alpha,\beta) \approx K_1\epsilon^3\delta\alpha - K_2\epsilon^3\alpha^3 \approx 0,$$

$$\alpha_0 = 0, \quad \alpha_2 \approx \sqrt{\frac{K_1}{K_2}\delta}, \quad \alpha_3 \approx -\sqrt{\frac{K_1}{K_2}\delta}.$$

5. In Proposition 3.12, we show that there is an overlapping region for the parameter μ and δ so that both Theorems 3.7 and 3.11 are applicable so that there is a solution branch "connecting" large value of μ to the actual bifurcation point.

Note that our results are much more refined than those in [7] which only deals with non-degenerate spatial inhomogeneity. The somewhat "crude" asymptotics of the principal eigenvalue $\lambda_1^\epsilon = K_1\epsilon + o(\epsilon)$ is sufficient for their purpose.

In Section 5, we investigate the imperfect pitchfork bifurcation when higher order term is added to the spatial inhomogeneity: $a(x,\beta,\gamma) = -\frac{1}{2}x^3 + \beta x + \gamma x^4$.

2.3. **Notations and conventions.** In order to facilitate effective presentation, we list here some frequently used notations and conventions.

(i) We use $a \ll 1$ to indicate that for a sufficiently small, i.e., $0 < a < a_0$ for some $a_0 > 0$. Unless specified explicitly, the smallness requirements between multiple parameters are independent of each other. For example, for $a, b \ll 1$, then the choice of a_0 and b_0 are independent.

Almost all the results require the singular parameter ϵ be small. In this case, we will write $\epsilon \ll 1$.

(ii) The symbol $O(1)$ refers to some fixed (unspecified or not relevant) constant. Then $a < O(1)$ means that a is less than some constant. The expression $b = O(a)$ means that $b \leq Ca$ for some constant C whose value is not important.

(iii) Unless otherwise specified, the domain of integration in $\displaystyle\int$ is always $(-\infty, \infty)$. We understand that for the original spatial variable x, it lies in $(-1, 1)$, while for the shift stretched variable s, its range is $(-\frac{1}{\epsilon} - z^\epsilon, \frac{1}{\epsilon} - z^\epsilon)$. However, as to be seen, all the functions in the integration will decay exponentially in the integration variable and the shift z^ϵ is at most of order $O(\epsilon^{m-1})$ for some $m > 0$. By means of simple extension of functions from $(-\frac{1}{\epsilon} - z^\epsilon, \frac{1}{\epsilon} - z^\epsilon)$ to $(-\infty, \infty)$, only exponentially small quantities of the form $e^{-\frac{C}{\epsilon}}$ is involved and they will be neglected throughout the paper.

(iv) Exponential decay means that $|f(x)| \leq A_\epsilon e^{-\frac{C|x|}{\epsilon}}$ (in the x variable), or $|f(s)| \leq A_\epsilon e^{-C|s|}$ (in the s variable) for some constant C. Note that the multiplicative constant A_ϵ might depend on ϵ. Its dependence can be important and will be characterized when needed.

(v) As we will be dealing with different regimes for the bifurcation parameter β, different symbols will be used. For example, in the sub-critical regime, we will use $\widetilde{\beta}$. However, in the critical regime, as mentioned above, it is more convenient to use the offset variable δ introduced in (3.15). For simplicity, for a given function $f(\beta, \cdots)$ that depends on β, we will use both the notations $f(\widetilde{\beta}, \cdots)$ and $f(\delta, \cdots)$ to denote the *same* function without introducing the precise definition of the change of variable.

For example, depending on which parameters or regimes we want to emphasize, both the notations $\widetilde{\beta}$, $z(\widetilde{\beta})$, $\phi^{\epsilon;\widetilde{\beta}}$, $U^{\epsilon;\widetilde{\beta}}$, $B^\epsilon(\widetilde{\beta})$ and $\widetilde{\beta}_* + \delta$, $z(\delta)$, $\phi^{\epsilon;\delta}$, $U^{\epsilon;\delta}$, $B^\epsilon(\delta)$ will be used. The same comment applies to other functions. In fact, in the critical regime, we will also consider other parameters μ and τ to indicate how far β is away from the bifurcation point.

2.4. **Preliminary technical results.** In this part, we list some important results on the spectrum analysis of the linearized operator L_U^ϵ and the solvability of the function v as function of α in (2.14). We apply Theorem 3.1 in [7] to show that L_U^ϵ has unique critical eigenvalue which approaches to zero as the singular parameter ϵ goes to zero. Assumptions **(A-1, A-2**) in [7] are needed for the proof. In regards of our nonlinearity $f(u, x; \beta) = (1 - u^2)(u - a(x; \beta))$, we assume $|2a(x; \beta)| < 1$ so that the corresponding two assumptions hold.
(A-1-m) $f : R \times [-1, 1] \times R \longrightarrow R$ is C^∞-function of $(u, x; \beta)$ with $f(\pm 1, x; \beta) = 0$.
(A-2-m) There exists a positive number γ such that

$$f_u(\pm 1, x; \beta) \leq -3\gamma^2 \text{ for } x \in [-1, 1].$$

For spatial inhomogeneity with bifurcation parameter $\beta = \epsilon^m \widetilde{\beta}, m > 0$, then it is *degenerate* in the sense that the assumption **(A-3)** in [7] is violated as ϵ goes to zero. More precisely, let $J(x) = \displaystyle\int_{-1}^{1} f(u, x) du$, at a zero x_i of $a(x; \beta) = -\dfrac{1}{2}x^3 + \epsilon^m \widetilde{\beta} x$, then $J(x_i) = 0$. However $\displaystyle\lim_{\epsilon \downarrow 0} \frac{d}{dx} J(x)\bigg|_{x_i} = 0$. Hence the techniques in[7] are not applicable. As mentioned before, such degeneracy calls for better initial approximate solution $U^{\epsilon;\beta}$ and more accurate derivation of the reduced (bifurcation) equation. On the other hand, up to some order, the equation is still in some sense non-degenerate and its form inherits mostly from the form of the spatial inhomogeneity function.

For self-containedness, we state some of the key technical results from [7] concerning the linearized operator L_U^ϵ in (2.9). As we will only linearize equation (2.2) near functions resembling the profile u_0, in particular functions U that converges to the end states ± 1 exponentially fast near $x = \pm 1$, the following result is applicable. Its proof relies on condition **(A-2-m)**. A similar condition is used in the proof of Theorem 3.1 in [7] (see (3.7) of page 376 in [7]). With spatial inhomogeneity $a(x;\beta)$, then $f_u = (1 - 3u^2 + 2a(x;\beta)u)$. To have **(A-2-m)**, we suppose $|2a(x;\beta)| < 1$ for $x \in [-1, 1]$. This is indeed the case for our pitchfork spatial inhomogeneity (1.2).

Lemma 2.2 ([7, Theorem 3.1])**.** *Assume $|2a(x;\beta)| < 1$. For $\epsilon \ll 1$, the following statements hold:*

1. *The principal eigenvalue λ_1^ϵ of L_U^ϵ is simple and approaches zero as $\epsilon \to 0$.*
2. *Let $\phi^\epsilon(x)$ be any eigenfunction associated with λ_1^ϵ and let s be the stretched shift variable (2.1), then $\phi^\epsilon(\epsilon z + \epsilon s)$ decays exponentially fast as $|s|$ goes to infinity, i.e., there exists positive constants k, such that*
$$|\phi^\epsilon(\epsilon z + \epsilon s)| \le k|\phi^\epsilon(\epsilon z)|e^{-\gamma|s|} \quad \textit{for} \quad s \in \left(-\frac{1}{\epsilon} - z^\epsilon, \frac{1}{\epsilon} - z^\epsilon\right).$$
3. *There is a $\mu_0 > 0$ such that the second eigenvalue $\lambda_2(\epsilon)$ of L_U^ϵ satisfies*
$$\lambda_2(\epsilon) \le -\mu_0.$$

To see the solvability of equation (2.14), define
$$X = \{u \in C^2[-1, 1] : u_x(-1) = 0 = u_x(1)\} \quad \text{and} \quad Y = C^0[-1, 1]. \tag{2.21}$$
Associated with the projection operator E_{ϕ^ϵ}, we consider the following decomposition of X and Y:
$$X = \mathrm{Span}\,\{\phi^\epsilon\} \oplus X_1 \quad \text{and} \quad Y = \mathrm{Span}\,\{\phi^\epsilon\} \oplus Y_1. \tag{2.22}$$
Then $L_U^\epsilon : X_1 \longrightarrow Y_1$ is a one to one map. Now define
$$\|v\|_0 = \sup\{|v(x)|, x \in [-1, 1]\} \quad \text{and} \quad \|v\|_{2;\epsilon} := \|v\|_0 + \epsilon\|v_x\|_0 + \epsilon^2\|v_{xx}\|_0. \tag{2.23}$$
Then we have the following statements.

Lemma 2.3 ([7, Lemma 4.1])**.** *For $\epsilon \ll 1$ and $p \in Y_1$, the equation $L_U^\epsilon v = p$ has a unique solution $v = v(p) \in X_1$. Moreover, there exists a constant $C > 0$ such that*
$$\|v(p)\|_{2;\epsilon} \le C\|p\|_0.$$

Lemma 2.4 ([7, Lemma 4.2])**.** *Equation (2.14) for $v = v(\alpha, \beta; \epsilon)(s)$ is uniquely solvable as a function of $v(\alpha, \beta; \epsilon)(s)$ for $(\alpha; \epsilon)$ in a neighborhood of $(0; 0)$. Moreover, it is smooth in α and for for $|\epsilon|$, $|\alpha| \ll 1$ it holds that*
$$\|v(\alpha, \beta; \epsilon)\|_{2;\epsilon} \le O\left(\|(I - E_{\phi^\epsilon})\,[G^\epsilon(U)]\|_0 + \alpha^2\right) \tag{2.24}$$

See the cited reference for details of the proofs. Here we just want to remark that in (2.24), the α^2 term comes from the fact that $F(w)$ depends on $w = \alpha\phi + v$ at least with quadratic order – see (2.12). Hence the equation for v roughly equals:
$$L_{U^\epsilon}^\epsilon(v) + (I - E_{\phi^\epsilon})\left[G^\epsilon(U^\epsilon) + O(\alpha^2 + v^2)\right] = 0.$$
The result then follows from an application of implicit function theorem.

Supplementing Lemma 2.2, in the appendix, we will provide further technical results such as the exponential decay of solutions, and expansion theorems for the principal eigenvalue and eigenfunction for a related linear operator.

3. **Perfect pitchfork bifurcation.** We begin in this section our investigation of the bifurcation phenomena of the diffuse transition layer with the pitchfork spatial inhomogeneity (1.2). The overall behavior is depicted in Figure 2. As explained previously, it is necessary to consider different regimes for the bifurcation parameter β.

We first explain how to determine the *critical scaling* for β: $\beta = \epsilon^2\widetilde{\beta}$. Consider the problem in the stretched variable $t = x/\epsilon$ and $\beta = \epsilon^m\widetilde{\beta}$. Then (2.2) becomes

$$u_{tt} + (1-u^2)\left(u + \frac{1}{2}\epsilon^3 t^3 - \epsilon^{1+m}\,\widetilde{\beta} t\right) = 0, \quad \text{for } -\frac{1}{\epsilon} < t < \frac{1}{\epsilon}, \quad u_t(\pm\frac{1}{\epsilon}) = 0. \tag{3.1}$$

As a first approximation, using the shift variable (2.1) and the standard transition layer (2.4), the main error term induced by $u_0(s)$ in the above is given by

$$G(u_0(s)) = (1 - u_0^2(s))\left(\frac{1}{2}\epsilon^3(s+z)^3 - \epsilon^{1+m}\widetilde{\beta}(s+z)\right).$$

For the critical case, we want z to be an $O(1)$-quantity so that the terms $\frac{1}{2}\epsilon^3 z^3$ and $\epsilon^{1+m}\widetilde{\beta}z$ balance with each other. This uniquely gives that $m_* = 2$. With this, we call the scaling $\beta = \epsilon^m\widetilde{\beta}$ with $m > 0$,

sub-critical ($m < 2$), *critical* ($m = 2$), and *super-critical* ($m > 2$).

Now we present the construction of the initial good approximate solution.

3.1. Good approximate solutions $U^{\epsilon;\beta}$.

In this section, we demonstrate how to determine the macroscopic shift z^ϵ and perturbation $\psi^{\epsilon;m}$ so as to eliminate the main error term in $G(u_0(s))$. Recall the form of (2.17) for the initial approximate solution $U^{\epsilon;\beta}$. In both sub-critical and critical regimes, three such solutions with good accuracy are constructed in Propositions 3.1 and 3.2. However, in the super-critical scale, where the bifurcation parameter is very small, there exists only one candidate for good approximate solution. We find it convenient to separate the results into sub-critical and critical cases even though they are very similar (and in fact only the latter result will be used).

Proposition 3.1 (Good approximate solutions, sub-critical scale)**.** *Consider $\beta = \epsilon^m\widetilde{\beta}$ with $0 < m < 2$. For $\epsilon \ll 1$, the following hold.*

1. *For $\widetilde{\beta} > 0$, there exist three pairs of $(z^\epsilon, \psi^{\epsilon;m})$ each of which satisfies*

$$\epsilon^3\frac{A}{2}z^3 - \epsilon^{1+m}\widetilde{\beta}Az + 3B\epsilon^3 z = 0, \tag{3.2}$$

and

$$L_0\left(\psi^{\epsilon;m}\right) + [1-u_0^2]\left[\epsilon^3\frac{1}{2}\left(s + z^\epsilon(\widetilde{\beta})\right)^3 - \epsilon^{1+m}\widetilde{\beta}\left(s + z^\epsilon(\widetilde{\beta})\right)\right] = 0 \tag{3.3}$$

with $\int \psi^{\epsilon;m}\dot{u}_0\,ds = 0$. In the above, A and B are positive numbers defined as:

$$A := \int \dot{u}_0^2\,ds > 0, \quad \text{and} \quad B := \frac{1}{2}\int s^2\dot{u}_0^2\,ds > 0. \tag{3.4}$$

2. *If we further rescale the shift variable $z^\epsilon(\widetilde{\beta}) = \epsilon^{\frac{m}{2}-1}z_0(\widetilde{\beta})$, then z_0 satisfies:*

$$\frac{A}{2}z_0^3 - (A\widetilde{\beta} - 3B\epsilon^{2-m})z_0 = 0. \tag{3.5}$$

The solutions of the above are given by

$$z_0(\widetilde{\beta}) = \begin{cases} \pm\sqrt{2\widetilde{\beta}} + O(\epsilon^{2-m}) \ \text{ or } 0, & \text{for } \widetilde{\beta} > \frac{3B}{A}\epsilon^{2-m}, \\ 0 & \text{for } \widetilde{\beta} \le \frac{3B}{A}\epsilon^{2-m}. \end{cases} \tag{3.6}$$

3. *In all of the above cases, $\psi^{\epsilon;m}$ and the associated approximate solution $U^{\epsilon;\widetilde{\beta}}$ satisfy the following estimates:*

$$\|\psi^{\epsilon;m}(s;\widetilde{\beta})\|_0 \le O\left(\epsilon^{1+m}\right) \quad \text{and} \quad \|G^{\epsilon;\widetilde{\beta}}(U^{\epsilon;\beta})\|_0 \le O(\epsilon^{2+2m}). \tag{3.7}$$

Furthermore, $\psi^{\epsilon;m}$ and $G^{\epsilon;\beta}(U^{\epsilon;\beta})$ decay exponentially fast in s: there exist C_1, $C_2 > 0$ such that

$$|\psi^{\epsilon;m}(s)| \le C_1\epsilon^{1+m}e^{-C_2|s|}, \quad \text{and} \quad \left|G^{\epsilon;\tilde{\beta}}(U^{\epsilon;\beta}(s))\right| \le C_1\epsilon^{2+2m}e^{-C_2|s|}. \tag{3.8}$$

Proof. To reduce the main error term in $G(u_0)$, we choose $\psi^{\epsilon;m}$ solving the equation

$$L_0\left(\psi^{\epsilon;m}\right) = [1-u_0^2]\, a(\epsilon(s+z^\epsilon(\widetilde{\beta}), \epsilon^m\widetilde{\beta}), \qquad \int \psi^{\epsilon;m}\dot{u}_0\, ds = 0. \tag{3.9}$$

This is exactly the equation (3.3). Recall the L_0 in Definition 1 which has kernel spanned by $\dot{u}_0(s)$. Then by Fredholm alternative, the macroscopic correction function $\psi^{\epsilon;m}(s;\widetilde{\beta})$ exists if and only if

$$\int [1-u_0^2]\left[\frac{1}{2}\epsilon^3(s+z^\epsilon(\widetilde{\beta}))^3 - \epsilon^{1+m}\widetilde{\beta}(s+z^\epsilon(\widetilde{\beta}))\right]\dot{u}_0(s)\, ds = 0.$$

Expanding the above and making use of (2.6), it gives precisely the location equation (3.2), with A, B defined in (3.4).

Part (2) is a direct result of (3.2). For part (3), the size of $\psi^{\epsilon;m}$ depends on the spatial inhomogeneity and the shift variable $z^\epsilon(\widetilde{\beta})$. Using $z^\epsilon = \epsilon^{\frac{m}{2}-1}z_0$, then

$$\begin{aligned} a\left(\epsilon(s+\epsilon^{\frac{m}{2}-1}z_0);\epsilon^m\widetilde{\beta}\right) &= -\frac{1}{2}\epsilon^3\left(s+\epsilon^{\frac{m}{2}-1}z_0\right)^3 + \widetilde{\beta}\epsilon^{1+m}\left(s+\epsilon^{\frac{m}{2}-1}z_0\right) \\ = \quad \left[-\frac{1}{2}\epsilon^{\frac{3m}{2}}z_0^3 + \widetilde{\beta}\epsilon^{\frac{3m}{2}}z_0\right] &+ \left[-\frac{3}{2}\epsilon^{1+m}sz_0^2 + \widetilde{\beta}\epsilon^{1+m}s\right] - \frac{3}{2}\epsilon^{2+\frac{m}{2}}s^2z_0 - \frac{1}{2}\epsilon^3s^3. \end{aligned}$$

By the asymptotic form of z_0 in (3.6), the first bracket in its absolute value is bounded by a $\epsilon^{2+\frac{m}{2}}$-order term. Moreover, the fact $m<2$ implies $\epsilon^{1+m} \gg \epsilon^{2+\frac{m}{2}}, \epsilon^3$. Hence the term in the middle bracket dominates. This, together with Lemmas 2.3 and A.1, and the exponential decay of $1-u_0^2(s)$, give that $\|\psi^{\epsilon;m}(s;\widetilde{\beta})\|_0 \le O\left(\epsilon^{1+m}\right)$ and its exponential decay in s.

It remains to verify that $\|G^{\epsilon;\widetilde{\beta}}(U^{\epsilon;\widetilde{\beta}})\|_0 \le O(\epsilon^{2+2m})$. For $|\epsilon s| > \dfrac{1}{4}$, by the exponential decaying properties of $\psi^{\epsilon;m}(s)$, $(\operatorname{sgn}(s) - u_0(s))$ and $\dot{u}_0(s)$, the quantity $\|G^{\epsilon;\widetilde{\beta}}(U^{\epsilon;\widetilde{\beta}})\|_{|\epsilon s|>\frac{1}{4}}$ is exponentially small. On the other hand, for $|\epsilon s| \le \dfrac{1}{4}$, by (2.10) and (3.9), it holds that

$$\begin{aligned} G^{\epsilon;\tilde{\beta}}(U^{\epsilon;\widetilde{\beta}}) &= G^{\epsilon;\tilde{\beta}}\left(u_0+\psi^{\epsilon;m}\right) \\ &= 2u_0\psi^{\epsilon;m}a(\epsilon(s+z_0);\widetilde{\beta}) + \left[-3u_0 + a(\epsilon(s+z_0);\widetilde{\beta})\right](\psi^{\epsilon;m})^2 - (\psi^{\epsilon;m})^3. \end{aligned} \tag{3.10}$$

Since $\|\psi^{\epsilon;m}a(\epsilon(s+z_0);\widetilde{\beta})\|_0 \le O(\epsilon^{2+2m})$, we have $\|G^{\epsilon;\tilde{\beta}}(U^{\epsilon;\widetilde{\beta}})\|_{|\epsilon s|\le\frac{1}{4}} \le O(\epsilon^{2+2m})$. Its exponential decay property follows from Lemma A.1. ◻

We next list explicitly the choice of good approximate solutions in the *critical* scaling region. The proof is essentially the same as above, except the shift variable $z^{m=2}(\widetilde{\beta}) = z_0(\widetilde{\beta})$ has a different scaling size than in the sub-critical region.

Proposition 3.2 (Good approximate solutions, critical scale)**.** *Consider $\beta = \epsilon^2\widetilde{\beta} > 0$. For $\epsilon \ll 1$, the following hold.*

1. *For*
$$\widetilde{\beta} > \widetilde{\beta}_* := \frac{3B}{A} = \frac{3}{2}\frac{\int s^2 \dot{u}_0^2\, ds}{\int \dot{u}_0^2\, ds}, \tag{3.11}$$
there exist three pairs of $(z_0, \psi^{\epsilon;2})$ each of which solves
$$\frac{A}{2}z_0^3 - \left(\widetilde{\beta}A - 3B\right) z_0 = 0 \quad \textit{(where } A, B \textit{ are defined as in (3.4))} \tag{3.12}$$
and
$$L_0(\psi^{\epsilon;2}) + [1-u_0^2]\left[\frac{1}{2}\epsilon^3(s+z_0)^3 - \epsilon^3\widetilde{\beta}(s+z_0)\right] = 0, \quad \int \psi^{\epsilon;2}(s)\dot{u}_0\, ds = 0. \tag{3.13}$$
2. *For $\widetilde{\beta} \le \widetilde{\beta}_*$, (3.12) has unique solution which is the trivial zero, $z_0 = 0$.*
3. *In all of the above cases, the exponential decaying function $\psi^{\epsilon;2}$ and the associated approximate solution $U^{\epsilon;\widetilde{\beta}}$ satisfy the following estimates:*
$$\|\psi^{\epsilon;2}(s;\widetilde{\beta})\|_0 \le O\left(\epsilon^3\right) \quad \textit{and} \quad \|G^{\epsilon;\widetilde{\beta}}(U^{\epsilon;\widetilde{\beta}})\|_0 \le O(\epsilon^6). \tag{3.14}$$

Remark 3.3.

1. In both of the above results, the location equations demonstrate pitchfork bifurcations: at $\widetilde{\beta} = O(\epsilon^{2-m})$ for (3.5) and at $\widetilde{\beta} = \widetilde{\beta}_*$ for (3.12).
2. Note that at the critical scale $m = 2$, the shift z is an $O(1)$-quantity that does not depend on ϵ — this is the original motivation for the introduction of this regime. As we expect that the corresponding transition layer problem will bifurcate in a similar way as the location equation, we introduce a convenient offset variable δ
$$\delta := \widetilde{\beta} - \widetilde{\beta}_*. \tag{3.15}$$
Then equation (3.12) becomes $z_0^3 - 2\delta z_0 = 0$.
3. The super-critical scale $\beta = \epsilon^m\widetilde{\beta}$ with $m > 2$ can be viewed as a form of the critical scale $\beta = \epsilon^2\left(\widetilde{\beta}\epsilon^{m-2}\right)$. For $\epsilon \ll 1$, $\widetilde{\beta}\epsilon^{m-2} \ll \widetilde{\beta}_*$, hence there is only one good candidate approximate solution. Due to this, we do not consider the super-critical case separately.
4. It turns out that for the analysis of the reduced equation (2.15), *the sub-critical case* and *the critical case with δ not too small* can be combined together. Hence only Proposition 3.2 above will be used.

3.2. **Key formulas and expressions.** To implement the Lyapunov-Schmidt ([**LS**]) method described in Section 2, we need very precise information about the reduced (bifurcation) equation (2.15). In terms of $\widetilde{\beta}$, we write $B^\epsilon(\alpha;\beta)$ as
$$B^\epsilon\left(\alpha;\widetilde{\beta}\right) = \sum_{n=0}^{N} B_n^\epsilon(\alpha=0;\widetilde{\beta})\alpha^n + B_{N+1}^\epsilon(\overline{\alpha};\widetilde{\beta})\alpha^{N+1}, \quad \text{where } |\bar{\alpha}| < |\alpha|. \tag{3.16}$$

For the convenience of giving formulas for the coefficients $B_n^\epsilon(\alpha;\widetilde{\beta})$, we write
$$\mathfrak{p} := \epsilon(s + z(\widetilde{\beta})), \quad \mathfrak{q} := \alpha\phi^{\epsilon;\widetilde{\beta}} + v, \quad \mathfrak{r} := \partial_\alpha(\mathfrak{q}) = \phi^{\epsilon;\widetilde{\beta}} + v_\alpha. \tag{3.17}$$

Using the above, we have (from (2.12)):

$$F(\mathfrak{q},\widetilde{\beta};\epsilon) := \left[-3U^{\epsilon;\widetilde{\beta}} + a(\mathfrak{p};\widetilde{\beta})\right](\mathfrak{q})^2 - (\mathfrak{q})^3. \tag{3.18}$$

Note that F can be bounded by $\|\mathfrak{q}\|^2$ for $\|\mathfrak{q}\| \ll 1$. In addition, with the good initial approximation $U^{\epsilon,\widetilde{\beta}}$, the form (3.10) for the error term $G(U)$ is repeated here:

$$G^{\epsilon;\widetilde{\beta}}(U^{\epsilon;\widetilde{\beta}}) = 2u_0\,\psi^{\epsilon;m}a(\mathfrak{p};\widetilde{\beta}) + \left[-3u_0 + a(\mathfrak{p};\widetilde{\beta})\right](\psi^{\epsilon;m})^2 - (\psi^{\epsilon;m})^3. \tag{3.10}$$

Next we write down explicitly the coefficients for B^ϵ in the critical regime (in terms of δ). We will need terms up to $N = 5$. We first record the following expressions,

$$\begin{aligned}
(F(\mathfrak{q}))_\alpha \;&:=\; 2\left[-3U^{\epsilon;\delta} + a(\mathfrak{p};\delta)\right](\mathfrak{q})(\mathfrak{r}) - 3(\mathfrak{q})^2(\mathfrak{r}). &(3.19)\\
(F(\mathfrak{q}))_{\alpha\alpha} \;&:=\; \left\{-3(\mathfrak{q})^2 v_{\alpha\alpha} - 6(\mathfrak{q})(\mathfrak{r})^2\right\} &(3.20)\\
&\quad + \left\{\left[-3U^{\epsilon;\delta} + a(\mathfrak{p};\delta)\right]2(\mathfrak{r})^2 + \left[-3U^{\epsilon;\delta} + a(\mathfrak{p};\delta)\right]2(\mathfrak{q})v_{\alpha\alpha}\right\}.
\end{aligned}$$

$$\begin{aligned}
(F(\mathfrak{q}))_{\alpha^{(3)}} \;&:=\; -6(\mathfrak{r})^3 - 18(\mathfrak{q})(\mathfrak{r})v_{\alpha\alpha} - 3(\mathfrak{q})^2 v_{\alpha^{(3)}} &(3.21)\\
&\quad + \left[-3U^{\epsilon;\delta} + a(\mathfrak{p};\delta)\right]6(\mathfrak{r})v_{\alpha\alpha} + \left[-3U^{\epsilon;\delta} + a(\mathfrak{p};\delta)\right]2(\mathfrak{q})v_{\alpha^{(3)}}.\\
(F(\mathfrak{q}))_{\alpha^{(4)}} \;&:=\; \left\{\left[-3U^{\epsilon;\delta} + a(\mathfrak{p};\delta)\right]\left(6v_{\alpha\alpha}^2 + 8(\mathfrak{r})v_{\alpha^{(3)}} + 2(\mathfrak{q})v_{\alpha^{(4)}}\right)\right\} &(3.22)\\
&\quad - \left\{36v_{\alpha\alpha}(\mathfrak{r})^2 + 24(\mathfrak{q})v_{\alpha^{(3)}}(\mathfrak{r}) + 18(\mathfrak{q})v_{\alpha\alpha}^2 + 3(\mathfrak{q})^2 v_{\alpha^{(4)}}\right\}.\\
(F(\mathfrak{q}))_{\alpha^{(5)}} \;&:=\; \left\{\left[-3U^{\epsilon;\delta} + a(\mathfrak{p};\delta)\right]\left(20v_{\alpha\alpha}v_{\alpha^{(3)}} + 10(\mathfrak{r})v_{\alpha^{(4)}} + 2(\mathfrak{q})v_{\alpha^{(5)}}\right)\right\} &(3.23)\\
&\quad - \left\{90v_{\alpha\alpha}^2(\mathfrak{r}) + 60v_{\alpha^{(3)}}(\mathfrak{r})^2\right\}\\
&\quad - \left\{60(\mathfrak{q})v_{\alpha^{(3)}}v_{\alpha\alpha} + 30(\mathfrak{r})v_{\alpha^{(4)}}(\mathfrak{q}) + 3(\mathfrak{q})^2 v_{\alpha^{(5)}}\right\}.
\end{aligned}$$

Making use of the above and recall $B_n^\epsilon(\alpha=0;\delta) = \dfrac{1}{n!}\partial_{\alpha^{(n)}}B^\epsilon(\alpha;\delta)\Big|_{\alpha=0}$, we have

$$\begin{aligned}
B_0^\epsilon(\alpha=0;\delta) \;&:=\; \frac{\left\langle G^{\epsilon;\delta}(U^{\epsilon;\delta}), \phi^{\epsilon;\delta}\right\rangle}{\langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle} + \frac{\left\langle F(v;\delta,\epsilon), \phi^{\epsilon;\delta}\right\rangle}{\langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle}, &(3.24)\\
B_1^\epsilon(\alpha=0;\delta) \;&:=\; \lambda_1^\epsilon(\delta) + \frac{\left\langle (F(\mathfrak{q}))_\alpha, \phi^{\epsilon;\delta}\right\rangle}{\langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle}, &(3.25)\\
\text{and}\quad B_n^\epsilon(\alpha=0;\delta) \;&:=\; \frac{\left\langle (F(\mathfrak{q}))_{\alpha^{(n)}}, \phi^{\epsilon;\delta}\right\rangle}{n!\,\langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle} \quad \text{for } n \ge 2. &(3.26)
\end{aligned}$$

Note that v and its derivatives $\partial_{\alpha^{(n)}} = v_{\alpha^{(n)}}$ with respect to α appear in these formulas. By (2.14), they satisfy the following equation:

$$L^{\epsilon;\delta}[v_{\alpha^{(n)}}] + \left(I - E_{\phi^{\epsilon;\delta}}\right)\left[F\left(\alpha\phi^{\epsilon;\delta} + v\right)_{\alpha^{(n)}}\right] = 0 \quad \text{with} \quad \int v_{\alpha^{(n)}}\phi^{\epsilon;\delta} = 0. \tag{3.27}$$

We point out here that for the analysis of the reduced equation, the function v and its derivatives do not play a decisive role as they are much smaller compared to the approximate solution. On the other hand, when working on the *reduced bifurcation equation*, we do need the functions $v_{\alpha^{(n)}}$ with $n \ge 2$. Their estimates will be given the Proposition 4.2.

3.3. **Main result at subcritical and critical scales — reduced equation.** Consider the new bifurcation parameter δ introduced in (3.15). Then for $\delta > 0$, the solution of the location equation (3.12) in Proposition 3.2 are given by $z_0 = \pm\sqrt{2\delta}$ and $z_0 = 0$. Here we present the existence of transition layer solution near each of these shifts when δ is *relatively large* (and *positive*). The case when δ is small is dealt with in the next section.

We find it convenient to introduce the following variable

$$\mu = \epsilon\sqrt{2\delta} \tag{3.28}$$

so that the original bifurcation parameter β is written as:

$$\beta = \epsilon^2(\widetilde{\beta}_* + \delta) = \epsilon^2\widetilde{\beta}_* + \frac{1}{2}\mu^2. \tag{3.29}$$

In terms of μ, the three shifts become $z_0 = \pm\dfrac{\mu}{\epsilon}$ and $z_0 = 0$. We will consider μ bounded as ϵ goes to zero. This allows δ to take the form of $\delta = \epsilon^\theta$ with $\theta > -2$. Note that the subcritical scaling where $\beta = \epsilon^m\widetilde{\beta}$ with $0 < m < 2$ corresponds to $\theta = \dfrac{m}{2} - 1 > -2$. Thus the result of this section can also cover the sub-critical case corresponding to Proposition 3.1. In this way, the analysis of the sub-critical and critical regime with relatively *large* δ can be combined. In Section 3.5, we will derive the *reduced bifurcation equation* for the most interesting case where the offset variable δ is *small*.

Now for $a(x;\beta) = -\dfrac{1}{2}x^3 + \beta x$, upon choosing $z_0 = \dfrac{\mu}{\epsilon}$, we have

$$a\left(\epsilon(s \pm z_0);\delta\right) = a\left(\epsilon s + \mu; \epsilon^2\widetilde{\beta}_* + \frac{1}{2}\mu^2\right) = \epsilon^3\widetilde{a}_1^{\text{odd}} + \epsilon\mu^2\widetilde{a}_2^{\text{odd}} + \epsilon^2\mu\widetilde{a}^{\text{even}} \tag{3.30}$$

where

$$\widetilde{a}_1^{\text{odd}} = -\frac{1}{2}s^3 + \widetilde{\beta}_* s, \quad \widetilde{a}_2^{\text{odd}} = -s, \quad \widetilde{a}^{\text{even}} = \pm(-\frac{3}{2}s^2 + \widetilde{\beta}_*). \tag{3.31}$$

We then decompose the macroscopic perturbation $\psi^{\epsilon;\mu}(s)$ as

$$\psi^{\epsilon;\mu}(s) = \epsilon^3\widetilde{\psi}_1^{\text{odd}} + \epsilon\mu^2\widetilde{\psi}_2^{\text{odd}} + \epsilon^2\mu\widetilde{\psi}^{\text{even}} \tag{3.32}$$

so that

$$L_0\left(\widetilde{\psi}^{\text{even}}\right) = [1-u_0^2]\widetilde{a}^{\text{even}}, \;\; L_0\left(\widetilde{\psi}_i^{\text{odd}}\right) = [1-u_0^2]\widetilde{a}_i^{\text{odd}}, \; i = 1,2. \tag{3.33}$$

Note that $\widetilde{\psi}^{\text{even}}$ is solvable as the right hand side is orthogonal to $\dot{u}_0$:

$$\int(1-u_0^2)\widetilde{a}^{\text{even}}\dot{u}_0\,ds = \sqrt{2}\int\left(-\frac{3}{2}s^2 + \widetilde{\beta}_*\right)\dot{u}_0^2\,ds$$

which equals zero by the definition of $\widetilde{\beta}_* := \dfrac{3B}{A}$ in Proposition 3.2(1).

By Lemma 2.3, we have that

$$\begin{aligned}
\|\psi^{\epsilon;\mu}\|_0 &\leq \epsilon^3\left\|\widetilde{\psi}_1^{\text{odd}}\right\|_0 + \epsilon\mu^2\left\|\widetilde{\psi}_2^{\text{odd}}\right\|_0 + \epsilon^2\mu\left\|\widetilde{\psi}^{\text{even}}\right\|_0 \\
&\leq \epsilon^3\left\|(1-u_0^2)\widetilde{a}_1^{\text{odd}}\right\|_0 + \epsilon\mu^2\left\|(1-u_0^2)\widetilde{a}_2^{\text{odd}}\right\|_0 + \epsilon^2\mu\left\|(1-u_0^2)\widetilde{a}^{\text{even}}\right\|_0 \\
&\leq O(\epsilon^3 + \epsilon\mu^2 + \epsilon^2\mu) \leq O(\epsilon\mu^2 + \epsilon^3)
\end{aligned} \tag{3.34}$$

where the inequality follows by $\epsilon^2\mu \leq \epsilon^3 + \epsilon\mu^2$.

If we choose $z_0 = 0$, then

$$a\,(\epsilon s;\delta) = a\left(\epsilon s; \epsilon^2\widetilde{\beta}_* + \frac{1}{2}\mu^2\right) = \epsilon^3\widetilde{a}_1^{\text{odd}} - \frac{1}{2}\epsilon\mu^2\widetilde{a}_2^{\text{odd}}, \tag{3.35}$$

$$\text{and} \quad \psi^{\epsilon;\mu} = \epsilon^3\widetilde{\psi}_1^{\text{odd}} - \frac{1}{2}\epsilon\mu^2\widetilde{\psi}_2^{\text{odd}}. \tag{3.36}$$

Many of the estimates for $z_0 = 0$ are similar to $z_0 = \frac{\mu}{\epsilon}$. Hence for the following we will mainly concentrate on the latter. We now reformulate Proposition 3.2 in terms of μ for the approximate solution $U^{\epsilon;\mu} = \xi_0(s)\,(u_0(s) + \psi^{\epsilon;\mu}) + \xi_1(s)$.

Proposition 3.4 (Good approximate solutions in μ)**.** *Consider $\beta = \epsilon^2(\widetilde{\beta}_* + \delta) = \epsilon^2\widetilde{\beta}_* + \dfrac{1}{2}\mu^2$. For ϵ, $\mu \ll 1$, with the choice of $\psi^{\epsilon;\mu}$ in (3.32), the following estimates hold:*

$$\|\psi^{\epsilon;\mu}(s)\|_0 = O(\epsilon\mu^2 + \epsilon^2\mu + \epsilon^3) \le O(\epsilon\mu^2 + \epsilon^3) \quad \textit{and} \quad \|G^{\epsilon;\mu}(U^{\epsilon;\mu})\|_0 \le O(\epsilon^6 + \epsilon^2\mu^4). \tag{3.37}$$

The above leads to following estimates for the next-order correction function v.

Proposition 3.5. *Let v be the solution of (2.14). Then for $\epsilon, |\alpha|, |\mu| \ll 1$, it holds that*

$$\|v\|_0 \le O(\alpha^2 + \epsilon^6 + \epsilon^2\mu^4), \quad \|v_\alpha\|_0 \le O(|\alpha| + \epsilon^6 + \epsilon^2\mu^4), \quad \textit{and} \quad \|v_{\alpha\alpha}\|_0 \le O(1).$$

Proof. The upper bound estimate for $\|v\|_0$ follows from Lemma 2.4 and the above fact $\|G^{\epsilon;\mu}(U^{\epsilon;\mu})\|_0 \le O(\epsilon^6 + \epsilon^2\mu^4)$.

For v_α, one takes derivative of (2.9) with respect to α, then

$$L^{\epsilon;\mu}\,[v_\alpha] + (I - E_{\phi^{\epsilon;\mu}})\,[F\,(\alpha\phi^{\epsilon;\mu} + v)]_\alpha \,=\, 0.$$

By Lemma 2.3, we have $\|v_\alpha\|_0 \le \|\,(I - E_{\phi^{\epsilon;\mu}})\,F\,(\alpha\phi^{\epsilon;\mu} + v)_\alpha\,\|_0$ where $F\,(\alpha\phi^{\epsilon;\mu} + v)_\alpha$ (in (3.19)) is explicitly written as

$$F_\alpha = (\alpha\phi^{\epsilon;\mu} + v)_\alpha \left\{2\big[-3U^{\epsilon;\mu} + a(\epsilon s + \mu; \epsilon^2\widetilde{\beta}_* + \frac{1}{2}\mu^2)\big](\alpha\phi^{\epsilon;\mu} + v) - 3(\alpha\phi^{\epsilon;\mu} + v)^2\right\}\phi^{\epsilon;\mu}$$

$$+ \left\{2\big[-3U^{\epsilon;\mu} + a(\epsilon s + \mu; \epsilon^2\widetilde{\beta}_* + \frac{1}{2}\mu^2)\big](\alpha\phi^{\epsilon;\mu} + v) - 3(\alpha\phi^{\epsilon;\mu} + v)^2\right\} v_\alpha.$$

Note that for α and ϵ small,

$$\left\|2\left[-3U^{\epsilon;\mu} + a(\epsilon s + \mu; \epsilon^2\widetilde{\beta}_* + \frac{1}{2}\mu^2)\right](\alpha\phi^{\epsilon;\mu} + v)\phi^{\epsilon;\mu} - 3(\alpha\phi^{\epsilon;\mu} + v)^2\phi^{\epsilon;\mu}\right\|_0$$

$$\le C\left(\|\alpha\phi^{\epsilon;\mu} + v\|_0 + \|\alpha\phi^{\epsilon;\mu} + v\|_0^2\right) \le C\,\|\alpha\phi^{\epsilon;\mu} + v\|_0 \le C(|\alpha| + \|v\|_0).$$

Hence

$$\|v_\alpha\|_0 \le \|\,(I - E)_{\phi^{\epsilon;\mu}}\,F_\alpha\,(\alpha\phi^{\epsilon;\mu} + v)\,\|_0 \le C(|\alpha| + \|v\|_0) + C(|\alpha| + \|v\|_0)\,\|v_\alpha\|_0$$

so that

$$\|v_\alpha\|_0 \le \frac{C}{1 - C(|\alpha| + \|v\|_0)}(|\alpha| + \|v\|_0)$$

The desired estimates then follows from that of $\|v\|_0$ (with α and ϵ small).

The estimate for $v_{\alpha\alpha}$ can be proved similarly by considering

$$L^{\epsilon;\mu}\,[v_{\alpha\alpha}] + (I - E_{\phi^{\epsilon;\mu}})\,[F\,(\alpha\phi^{\epsilon;\mu} + v)]_{\alpha\alpha} \,=\, 0$$

and making use of (3.20):

$$(F(\mathfrak{q}))_{\alpha\alpha} := \left\{-3(\mathfrak{q})^2 v_{\alpha\alpha} - 6(\mathfrak{q})\,(\mathfrak{r})^2\right\} + \left\{[-3U^{\epsilon;\mu} + a(\mathfrak{p};\delta)]\,2\,(\mathfrak{r})^2 + [-3U^{\epsilon;\mu} + a(\mathfrak{p};\delta)]\,2(\mathfrak{q})v_{\alpha\alpha}\right\}.$$

The $O(1)$-term for $\|v_{\alpha\alpha}\|_0$ comes from the term $[-3U^{\epsilon;\mu} + a(\mathfrak{p};\delta)]\,2\,(\mathfrak{r})^2$. □

Using (3.16), we write the reduced equation in the critical scale case using μ as:

$$B^\epsilon(\alpha;\mu) = 0 \quad \text{with} \quad B^\epsilon(\alpha;\mu) := B_0^\epsilon(\alpha = 0;\mu) + B_1^\epsilon(\alpha = 0;\mu)\,\alpha + B_2^\epsilon(\overline{\alpha},\mu)\alpha^2 \tag{3.38}$$

where $\overline{\alpha}$ is some number satisfying $0 \le |\overline{\alpha}| \le |\alpha|$. Note that all the coefficients implicitly depend on z_0. We next give the expansion of (3.38).

Proposition 3.6. *Consider $\epsilon \ll 1$ and $\epsilon^p \le \mu \ll 1$ with $0 < p < \frac{5}{2}$. Then,*

1. *at $z_0 = \pm\dfrac{\mu}{\epsilon}$,*

$$B_0^\epsilon(\alpha = 0;\mu) = M_{0,1}\epsilon^5\mu + M_{0,2}\epsilon^3\mu^3 + O(\epsilon^9 + \epsilon^3\mu^6); \tag{3.39}$$

$$B_1^\epsilon(\alpha = 0;\mu) = -K_1\epsilon\mu^2 + O(\epsilon^6 + \epsilon^2\mu^4); \tag{3.40}$$

$$|B_2^\epsilon(\alpha = \overline{\alpha};\mu)| \le O(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2 + |\overline{\alpha}|) \le O(\epsilon^3 + \epsilon\mu^2 + |\overline{\alpha}|). \tag{3.41}$$

2. *at $z_0 = 0$,*

$$B_0^\epsilon(\alpha = 0;\mu) = 0; \tag{3.42}$$

$$B_1^\epsilon(\alpha = 0;\mu) = \frac{1}{2}K_1\epsilon\mu^2 + O(\epsilon^6 + \epsilon^2\mu^4); \tag{3.43}$$

$$|B_2^\epsilon(\alpha = \overline{\alpha};\mu)| \le O(\epsilon^3 + \epsilon\mu^2 + |\overline{\alpha}|). \tag{3.44}$$

In the above, K_1 is the same positive number appearing in Proposition 3.8 (in (3.49)) for the asymptotics of the principal eigenvalue. $M_{0,1}$ and $M_{0,2}$ are finite numbers.

Note that in order for the above result to be useful, we need $\epsilon\mu^2 \gg \epsilon^6$ which gives the range of μ in the statement. This range corresponds to $\delta := \widetilde{\beta} - \widetilde{\beta}_* = O(\epsilon^\theta)$ with $\theta \in (-2, 3)$. The proof of the above will be given in Section 3.4. The main result of this section is a simple consequence of the above statement.

Theorem 3.7. *There exists a $\alpha_0 > 0$ such that for $\epsilon \ll 1$ and $\epsilon^p \le \mu \ll 1$ with $0 < p < \frac{5}{2}$, then at $z_0 = \pm\frac{\mu}{\epsilon}$ and 0, the reduced equation (3.38) has a unique solution $\alpha^\epsilon(\mu)$ satisfying $|\alpha^\epsilon(\mu)| \le \alpha_0$.*

Proof. We re-write here the reduced equation:

$$B_0^\epsilon(\alpha = 0;\mu) + B_1^\epsilon(\alpha = 0;\mu)\alpha + B_2^\epsilon(\bar{\alpha};\mu)\alpha^2 = 0.$$

Recall that the range of μ guarantees that $K_1\epsilon\mu^2$ is the dominating term in the expansion of $B_1^\epsilon(\alpha = 0;\mu)$, i.e., $K_1\epsilon\mu^2 \gg O(\epsilon^6 + \epsilon^2\mu^4)$. Using Implicit Function Theorem, one then shows the equation (3.38) has the following solution:

1. at $z_0 = \pm\frac{\mu}{\epsilon}$:

$$\begin{aligned}
\alpha^\epsilon(\mu) &= -\frac{B_0^\epsilon(\alpha = 0;\mu) + B_2^\epsilon(\bar{\alpha};\mu)\alpha^\epsilon(\mu)^2}{B_1^\epsilon(\alpha = 0;\mu)} \\
&= -\frac{B_0^\epsilon(\alpha = 0;\mu)}{B_1^\epsilon(\alpha = 0;\mu)} + \frac{B_2^\epsilon(\bar{\alpha};\mu)}{B_1^\epsilon(\alpha = 0;\mu)}\alpha^\epsilon(\mu)^2 \\
&\approx \frac{M_{0,1}\epsilon^5\mu + M_{0,2}\epsilon^3\mu^3 + O(\epsilon^9 + \epsilon^3\mu^6)}{K_1\epsilon\mu^2 + O(\epsilon^6 + \epsilon^2\mu^4)}.
\end{aligned}$$

Note that the above solution of α satisfies $|\alpha^\epsilon(\mu)| \le O(\epsilon^4\mu^{-1}+\epsilon^2\mu) \ll 1$ as $\epsilon^4 \ll \epsilon^{\frac{5}{2}} \ll \mu \ll 1$.

2. at $z_0 = 0$: since $B_0^\epsilon(\alpha=0;\mu)=0$, $\alpha^\epsilon(\mu)=0$ is always a solution of

$$B_1^\epsilon(\alpha=0;\mu)\alpha + B_2^\epsilon(\overline{\alpha};\mu)\alpha^2 = 0.$$

Furthermore, the solutions found above are unique in a small neighborhood of $\alpha = 0$. $\square$

We give here some heuristic but intuitive remarks about the above result. First it indicates the validity of splitting the solution into macroscopic shift (z) and perturbation (ψ) versus microscopic shift (α) and perturbation (v):

$$u(x) = \left[U_0\left(\frac{x}{\epsilon} - z\right) + \psi\right] + [\alpha\phi + v]$$

and naturally we have

$$\begin{aligned} &\alpha^\epsilon(\mu) \le O(\epsilon^4\mu^{-1}+\epsilon^2\mu) \ll \tfrac{\mu}{\epsilon} = z_1^\epsilon \\ \text{and}\quad &\|v\|_{2;\epsilon} \le O(\alpha^2+\epsilon^6+\epsilon^2\mu^4) \ll O(\epsilon\mu^2+\epsilon^2\mu+\epsilon^3) = \|\psi^{\epsilon;\mu}\|_0 \end{aligned}$$

where $\epsilon^{\frac{5}{2}} \ll \mu \ll 1$ is used. The fact that $|\alpha| \ll |z|$ also means that the three exact solutions constructed are actually distinct.

Second, we can also infer the stability of the solutions. For this, we will make use of Proposition 3.8. We say that u^{exact} is *stable* if all the eigenvalues of the linearized operator at u^{exact} are *negative*. Otherwise, it is *unstable*. We thus need to study the eigenvalue problem (3.45) below where $U^{\epsilon;\mu}$ is replaced by u^{exact}. Note that by Lemma 2.2, other than the principal eigenvalue, all other eigenvalues of the linearized operator are negative. Thus the sign of the principal eigenvalue determines the stability of the transition layers. Noting that $u^{\text{exact}} = U^{\epsilon;\mu}+\alpha\phi^\epsilon+v$. Since $\|v\|_0 \le O(\alpha^2+\epsilon^6+\epsilon^2\mu^4)$ and ϕ^ϵ converges to $\dot{u}_0$ in C^2_{loc}, we can roughly use the following representation:

$$u^{\text{exact}} \approx U^{\epsilon;\mu} + \alpha\dot{u}_0(s) \ \text{ with } \ |\alpha| \ll 1.$$

One can redo the proof of the Proposition 3.8, except that in the definition of p_1 (3.51), u_0 is replaced $u_0+\alpha\dot{u}_0$. Now (A.3) reads as

$$L_0\Phi_1^\epsilon + \left(p_1 - 6\alpha\dot{u}_0\psi^{\epsilon;\mu} + 2a\alpha\dot{u}_0\right)\dot{u}_0 = \lambda_1^\epsilon(\mu)\dot{u}_0 + o(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2).$$

As $|\alpha| \ll 1$ from the construction, it follows that the leading term of order $\epsilon\mu^2$ in $\lambda_1^\epsilon(\mu)$ remains unchanged as

$$\|-6\alpha\dot{u}_0\psi^{\epsilon;\mu} + 2a\alpha\dot{u}_0\|_0 \ll O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2).$$

Therefore, (3.47) and (3.48) hold, except that the error $O(\epsilon^6+\epsilon^2\mu^4)$ there is replaced by $o(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2)$. It then follows the stability, that is, the two with nonzero shifts $z_0 = \pm\frac{\mu}{\epsilon}$ are *stable*, and the one with $z_0 = 0$ is *unstable*.

3.4. **Asymptotics of the reduced equation — proof of proposition 3.6.** We will need the asymptotic expansions of the principal eigenvalue and eigenfunction $\lambda_1^\epsilon(\mu)$ and $\phi^\epsilon(s;\mu)$ for the linearized operator L_U^ϵ (2.18). For concreteness, we consider the following form of normalization:

$$L_U^{\epsilon;\mu}(\phi^\epsilon(s;\mu)) = \lambda_1^\epsilon(\mu)\,\phi^\epsilon(s;\mu), \qquad \int \phi^\epsilon(s;\mu)\dot{u}_0\,ds = \int \dot{u}_0^2(s)\,ds. \tag{3.45}$$

It suffices to analyze the above for $|\epsilon s| \leq \dfrac{1}{4}$ of the following operator

$$L_U^{\epsilon;\mu} := \partial_{ss} + \left[1 - 3\left(u_0 + \psi^{\epsilon;\mu}(s)\right)^2 + 2a\left(u_0 + \psi^{\epsilon;\mu}(s)\right)\right] \tag{3.46}$$

as only exponential small error is neglected (recall that $\xi_0(s)\left(u_0 + \psi^{\epsilon;\mu}\right) + \xi_1(s)$ converges to ± 1 for $|\epsilon s| > \dfrac{1}{4}$ exponentially fast in s).

Proposition 3.8 (Principal Eigenvalue)**.** *For the linearized eigenvalue problem (3.45), and for $\epsilon \ll 1$, then its principal eigenvalue satisfies:*

$$\lambda_1^\epsilon(\mu) = -K_1\epsilon\mu^2 + O(\epsilon^6 + \epsilon^2\mu^4) \quad \text{at } z_0 = \pm\frac{\mu}{\epsilon}, \tag{3.47}$$

$$\text{and} \quad \lambda_1^\epsilon(\mu) = \frac{1}{2}K_1\epsilon\mu^2 + O(\epsilon^6 + \epsilon^2\mu^4) \quad \text{at } z_0 = 0, \tag{3.48}$$

where

$$K_1 = \frac{\int \dot{u}_0(1 - u_0^2)\, ds}{\int \dot{u}_0^2\, ds} = \frac{\int \sqrt{2}\dot{u}_0^2\, ds}{\int \dot{u}_0^2\, ds} = \sqrt{2}. \tag{3.49}$$

Proof. As a preparation, we expand $L_{\text{app}}^{\epsilon;\mu}$ in (3.46) as

$$L_U^{\epsilon;\mu} = L_0 + p_1 + p_2 \tag{3.50}$$

where $L_0 = \partial_s^2 + (1 - 3u_0^2)$ (from Definition 1),

$$\begin{aligned} p_1 &= -6u_0\psi^{\epsilon;\mu} + 2au_0 \\ &= -6u_0\left(\epsilon^3\widetilde{\psi}_1^{\text{odd}} + \epsilon\mu^2\widetilde{\psi}_2^{\text{odd}} + \epsilon^2\mu\widetilde{\psi}^{\text{even}}\right) \\ &\quad +2u_0\left(\epsilon^3\widetilde{a}_1^{\text{odd}} + \epsilon\mu^2\widetilde{a}_2^{\text{odd}} + \epsilon^2\mu\widetilde{a}^{\text{even}}\right) \\ \text{and} \quad p_2 &= -3(\psi^{\epsilon;\mu})^2 + 2a\psi^{\epsilon;\mu} \end{aligned}$$

Note that

$$\|p_1\dot{u}_0\|_0 \leq O(\epsilon^3 + \epsilon\mu^2 + \epsilon^2\mu) \quad \text{and} \quad \|p_2\|_0 \leq O(\epsilon^6 + \epsilon^2\mu^4). \tag{3.51}$$

From Theorem A.2, we have that

$$\lambda_1^\epsilon(\mu) = \Lambda_1^\epsilon(\mu) + \Lambda_2^\epsilon(\mu) \quad \text{and} \quad \phi^\epsilon(s;\mu) = \dot{u}_0(s) + \Phi_1^\epsilon(s;\mu) + \Phi_2^\epsilon(s;\mu) \tag{3.52}$$

with $|\Lambda_1^\epsilon(\mu)|,\ \|\Phi_1^\epsilon(\cdot;\mu)\|_0 \leq O(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2)$ and $|\Lambda_2^\epsilon(\mu)|,\ \|\Phi_2^\epsilon(\cdot,\mu)\|_0 \leq O(\epsilon^6 + \epsilon^2\mu^4)$. The quantity Λ_1^ϵ and function Φ_1^ϵ are defined in (A.3) which is recalled here for convenience:

$$\Lambda_1^\epsilon(\mu) = \left(\int \dot{u}_0^2\right)^{-1}\int p_1\dot{u}_0^2, \quad \text{and} \quad L_0\Phi_1^\epsilon + p_1\dot{u}_0 = \Lambda_1^\epsilon(\mu)\dot{u}_0, \quad \int \Phi_1^\epsilon\dot{u}_0\, ds = 0.$$

Here we compute Λ_1^ϵ explicitly. To proceed, we have,

$$\begin{aligned} &\int p_1\dot{u}_0^2\, ds \\ =& \int -6u_0\left(\epsilon^3\widetilde{\psi}_1^{\text{odd}} + \epsilon\mu^2\widetilde{\psi}_2^{\text{odd}} + \epsilon^2\mu\widetilde{\psi}^{\text{even}}\right)\dot{u}_0^2 \\ &+ \int 2u_0\left(\epsilon^3\widetilde{a}_1^{\text{odd}} + \epsilon\mu^2\widetilde{a}_2^{\text{odd}} + \epsilon^2\mu\widetilde{a}^{\text{even}}\right)\dot{u}_0^2 \\ =& \int -6u_0\left(\epsilon^3\widetilde{\psi}_1^{\text{odd}} + \epsilon\mu^2\widetilde{\psi}_2^{\text{odd}}\right)\dot{u}_0^2\, ds + \int 2u_0\left(\epsilon^3\widetilde{a}_1^{\text{odd}} + \epsilon\mu^2\widetilde{a}_2^{\text{odd}}\right)\dot{u}_0^2\, ds \\ =& \int -6u_0\dot{u}_0^2\psi^{\text{odd}}\, ds + \int 2u_0\dot{u}_0^2 a^{\text{odd}}\, ds \end{aligned}$$

where we have made use of the oddness and evenness of the integrands and notations $\psi^{\text{odd}} = \epsilon^3\widetilde{\psi}_1^{\text{odd}} + \epsilon\mu^2\widetilde{\psi}_2^{\text{odd}}$ and $a^{\text{odd}} = \epsilon^3\widetilde{a}_1^{\text{odd}} + \epsilon\mu^2\widetilde{a}_2^{\text{odd}}$.

By means of $L_0\left[\ddot{u}_0\right] = 6u_0\dot{u}_0^2$, the first term of the above becomes

$$\begin{aligned}\int -6u_0\dot{u}_0^2\psi^{\text{odd}} &= \int -L_0\left[\ddot{u}_0\right]\psi^{\text{odd}} = \int -\ddot{u}_0 L_0\left[\psi^{\text{odd}}\right] = \int -\ddot{u}_0\left[1-u_0^2\right]a^{\text{odd}}\\ &= \int \dot{u}_0[1-u_0^2]a_s^{\text{odd}}\,ds + \int \dot{u}_0(-2u_0\dot{u}_0)a^{\text{odd}}\,ds.\end{aligned}$$

Combining with the second term, we get

$$\begin{aligned}\int p_1\dot{u}_0^2\,ds &= \int -6u_0\dot{u}_0^2\psi^{\text{odd}} + \int 2u_0\dot{u}_0^2a^{\text{odd}}\,ds = \int \dot{u}_0[1-u_0^2]a_s^{\text{odd}}\,ds\\ &= \int \dot{u}_0[1-u_0^2]\left[\epsilon^3\left(-\frac{1}{2}s^3 + \widetilde{\beta}_* s\right) - \epsilon\mu^2 s\right]_s\,ds\\ &= \int \dot{u}_0\left[1-u_0^2\right]\left[\epsilon^3\left(-\frac{3}{2}s^2 + \widetilde{\beta}_*\right) - \epsilon\mu^2\right]ds\end{aligned}$$

so that,

$$\begin{aligned}\int p_1\dot{u}_0^2\,ds &= -\int \dot{u}_0\left[1-u_0^2\right]\left[\epsilon^3\left(\frac{3}{2}s^2 - \widetilde{\beta}_*\right) + \epsilon\mu^2\right]\\ &= -\epsilon^3\int\left(\frac{3}{2}s^2 - \widetilde{\beta}_*\right)\dot{u}_0\left[1-u_0^2\right] - \epsilon\mu^2\int \dot{u}_0\left[1-u_0^2\right].\end{aligned}$$

Using (3.11), we finally conclude that

$$\begin{aligned}\lambda_1^\epsilon(\mu) &:= -\epsilon\mu^2\left(\int \dot{u}_0^2\,ds\right)^{-1}\int \dot{u}_0(1-u_0^2) + O(\epsilon^6 + \epsilon^2\mu^4)\\ &= -\sqrt{2}\epsilon\mu^2 + O(\epsilon^6 + \epsilon^2\mu^4) \quad (\text{note } (1-u_0^2) = \sqrt{2}\dot{u}_0).\end{aligned}$$

Similar computation gives (3.48) for the case $z_0 = 0$. In fact $a\left(\epsilon s; \epsilon^2\widetilde{\beta}_* + \frac{1}{2}\mu^2\right) = \epsilon^3\widetilde{a}_1^{\text{odd}} - \frac{1}{2}\epsilon\mu^2\widetilde{a}_2^{\text{odd}}$ (as defined in (3.35)) which is odd in s. Hence, we still have

$$\begin{aligned}\int p_1\dot{u}_0^2\,ds &= \int \dot{u}_0[1-u_0^2]\left[\epsilon^3\widetilde{a}_1^{\text{odd}} - \frac{1}{2}\widetilde{a}_2^{\text{odd}}\right]_s\,ds\\ &= \int \dot{u}_0\left[1-u_0^2\right]\left[\epsilon^3\left(-\frac{3}{2}s^2 + \widetilde{\beta}_*\right) + \frac{1}{2}\epsilon\mu^2\right]ds\\ &= \epsilon^3\int \dot{u}_0[1-u_0^2]\left(-\frac{3}{2}s^2 + \widetilde{\beta}_*\right)ds + \frac{1}{2}\epsilon\mu^2\int \dot{u}_0\left[1-u_0^2\right]ds\\ &= \frac{\sqrt{2}}{2}\epsilon\mu^2\end{aligned}$$

giving the desired result. $\square$

Now we are ready to proceed to the

Proof of Proposition 3.6. First we recall the set of formula (3.19)–(3.22) and (3.24)–(3.26) for $F(\cdot)_{\alpha^{(n)}}$ and $B_n(\cdot,\cdot)$. It turns out that all the asymptotics is dominated by that of the principal eigenvalue. The details follow.

Asymptotics of $B_0^\epsilon(\alpha = 0; \mu)$ (3.39). By the formula for B_0 in (3.24), we need to estimate

$$\left|\frac{\langle F\left(v;\mu,\epsilon\right),\phi^{\epsilon;\mu}\rangle}{\langle\phi^{\epsilon;\mu},\phi^{\epsilon;\mu}\rangle}\right| \quad \text{and} \quad \left|\frac{\langle G^{\epsilon;\mu}\left(u_0+\psi^{\epsilon;\mu}\right),\phi^{\epsilon;\mu}\rangle}{\langle\phi^{\epsilon;\mu},\phi^{\epsilon;\mu}\rangle}\right|.$$

For the former, by Proposition 3.5 at $\alpha = 0$, we have $\|v\|_0 \le O(\epsilon^6 + \epsilon^2\mu^4)$. Recall the formula of F in (3.18), which is at least quadratic in v. Hence

$$\|F\left(v;\mu,\epsilon\right)\|_0 \le O(\|v\|_0^2) \le O(\epsilon^{12} + \epsilon^4\mu^8).$$

For the latter, recall the form (3.10), i.e.,

$$\begin{aligned} G^{\epsilon;\mu}(U^{\epsilon;\mu}) &= 2u_0\,\psi^{\epsilon;\mu}a(\epsilon(s+z_0);\mu) + \left[-3u_0 + a(\epsilon(s+z_0);\mu)\right](\psi^{\epsilon;\mu})^2 - (\psi^{\epsilon;\mu})^3 \\ &= 2u_0\,\psi^{\epsilon;\mu}a(\epsilon(s+z_0);\mu) - 3u_0(\psi^{\epsilon;\mu})^2 + a(\epsilon(s+z_0);\mu)(\psi^{\epsilon;\mu})^2 - (\psi^{\epsilon;\mu})^3 \end{aligned}$$

Using the facts that $\|\psi^{\epsilon;\mu}\| \le O\left(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2\right) \le O\left(\epsilon^3 + \epsilon\mu^2\right)$ and the exponential decay of $\psi^{\epsilon;\mu}$, thus

$$\|a(\epsilon(s+z_0);\mu)(\psi^{\epsilon;\mu})^2 - (\psi^{\epsilon;\mu})^3\| \le O(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2)^3 \le O\Big(\epsilon^9 + \epsilon^3\mu^6\Big)$$

which can be absorbed into the smaller term on the right hand side of (3.39), we just need to consider the following expression:

$$\Big\langle 2u_0\psi^{\epsilon;\mu}a - 3u_0\left(\psi^{\epsilon;\mu}\right)^2, \phi^{\epsilon;\mu}\Big\rangle.$$

Recall the form (A.2) for $\phi^{\epsilon;\mu}$. Then the above can be further simplified to

$$\Big\langle 2u_0\psi^{\epsilon;\mu}a - 3u_0\left(\psi^{\epsilon;\mu}\right)^2, \dot{u}_0\Big\rangle$$

where $\left\|\Big\langle 2u_0\psi^{\epsilon;\mu}a - 3u_0\left(\psi^{\epsilon;\mu}\right)^2, \Phi_1^\epsilon(s;\mu) + \Phi_2^\epsilon(s;\mu)\Big\rangle\right\| \le O(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2)^3$ is already absorbed into the error term on the right hand side of (3.39). Next making use of the decompositions (3.30) and (3.32) and the oddness and evenness of the functions, we compute

$$\begin{aligned} &\int \Big(2\psi^{\epsilon;\mu}a - 3\left(\psi^{\epsilon;\mu}\right)^2\Big)\,u_0\dot{u}_0\,ds \\ =\;& \int \left[2\left(\psi^{\text{odd}} + \psi^{\text{even}}\right)\left(a^{\text{odd}} + \psi^{\text{even}}\right) - 3\left(\psi^{\text{odd}} + \psi^{\text{even}}\right)^2\right]u_0\dot{u}_0\,ds \\ =\;& \int \left[2\left(\psi^{\text{odd}}a^{\text{even}} + \psi^{\text{even}}a^{\text{odd}}\right) - 6\psi^{\text{odd}}\psi^{\text{even}}\right]u_0\dot{u}_0\,ds \\ =\;& O(\epsilon^3 + \epsilon\mu^2)(\epsilon^2\mu) = O(\epsilon^5\mu + \epsilon^3\mu^3) \\ =\;& M_{0,1}\epsilon^5\mu + M_{0,2}\epsilon^3\mu^3 \end{aligned}$$

where $M_{0,1}, M_{0,2}$ are finite in the former computation.

Asymptotics of $B_1^\epsilon(\alpha = 0; \mu)$ (3.40). The result states that $B_1^\epsilon(\alpha = 0;\mu)$ is dominated by $\lambda_1^\epsilon(\mu)$. Indeed, using the formula for B_1 and $(F(\mathfrak{q}))_\alpha$ from (3.25) and (3.19), we have

$$\left|\frac{\langle 2\left(-3U^{\epsilon;\mu} + a(\mathfrak{p};\mu)\right)v(\mathfrak{r}),\phi^{\epsilon;\mu}\rangle}{\langle\phi^{\epsilon;\mu},\phi^{\epsilon;\mu}\rangle} - \frac{\big\langle 3v^2(\mathfrak{r}),\phi^{\epsilon;\mu}\big\rangle}{\langle\phi^{\epsilon;\mu},\phi^{\epsilon;\mu}\rangle}\right| \le O(\|v\|_0) \le O(\epsilon^6 + \epsilon^2\mu^4),$$

where we have used the estimate for v from Proposition 3.5. The claim is proved with the asymptotic of $\lambda_1^\epsilon(\mu)$ in (3.47). Note that the condition $\epsilon^{\frac{5}{2}} \ll \mu \ll 1$ is used such that non-vanishing $O(\epsilon\mu^2)$ term is dominating in (3.40).

Asymptotics of $B_2^\epsilon(\alpha;\mu)$ (3.41). As all the functions $v, v_\alpha, v_{\alpha\alpha}$, and $\alpha\phi^{\epsilon;\mu}$ depend continuously on the α, we have $B_2^\epsilon(\alpha=\overline{\alpha};\mu) = B_2^\epsilon(\alpha=0;\mu) + O(|\overline{\alpha}|)$. Hence it suffices to prove $|B_2^\epsilon(\alpha=0;\mu)| \leq O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2)$. Indeed, using the formula for B_2 and $(F(\mathfrak{q}))_{\alpha\alpha}$ from (3.26) with $n=2$ and (3.20), we have

$$|B_2^\epsilon(\alpha=0;\mu)| = \left|\frac{\left\langle (F(\mathfrak{q}))_{\alpha\alpha}|_{\alpha=0}\,, \phi^{\epsilon;\mu}\right\rangle}{\langle \phi^{\epsilon;\mu}, \phi^{\epsilon;\mu}\rangle}\right|$$

where $(F(\mathfrak{q}))_{\alpha\alpha}|_{\alpha=0} = -3v^2 v_{\alpha\alpha} - 6v\,(\mathfrak{r})^2 + (-6U^{\epsilon;\mu}+2a)\,(\mathfrak{r})^2 + (-6U^{\epsilon;\mu}+2avv_{\alpha\alpha})$. To continue, one uses the estimates for $\|v(\alpha=0;\mu,\epsilon)\|_0$, $\|v_\alpha(\alpha=0;\mu,\epsilon)\|_0$ and $\|v_{\alpha\alpha}(\alpha=0;\mu,\epsilon)\|_0$ from Proposition 3.5, together with $U^{\epsilon;\mu} = u_0 + \psi^{\epsilon;\mu}$ for $|\epsilon s| < \frac{1}{4}$. Then, $|B_2^\epsilon(\alpha=0;\mu)|$ can be simplified as

$$|B_2^\epsilon(\alpha=0;\mu)| = O(1) + O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2).$$

In the above, the $O(1)$-term is given explicitly as

$$O(1) = \frac{\left\langle -6u_0(\phi^{\epsilon;\mu})^2, \phi^{\epsilon;\mu}\right\rangle}{\langle \phi^{\epsilon;\mu}, \phi^{\epsilon;\mu}\rangle}.$$

To compute it, recall the expansion of $\phi^\epsilon(\mu)$ in (A.2), it is then reduced to

$$O(1) = -\frac{\int 6u_0\dot{u}_0^3 ds}{\int u_0^2 ds} + O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2).$$

As $\displaystyle\int -6u_0\dot{u}_0^3 = 0$, it concludes that $|B_2^\epsilon(\alpha=0;\mu)| = O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2) \leq O(\epsilon^3+\epsilon\mu^2)$.

At zero shift $z_0=0$. Noting that with $z_0=0$, from (3.35) and (3.36), both a and $\psi^{\epsilon;\mu}$ are *odd* in s. Consequently the approximate solution $U^{\epsilon;\mu}$ and $G^{\epsilon;\mu}(U^{\epsilon;\mu})$ in (3.10) are *odd* in s. Furthermore, at $\alpha=0$, the solution v to (2.14) is *odd*, with estimate $\|v\|_0 \leq O(\epsilon^6+\epsilon^2\mu^4)$. These symmetries lead to that the associated principle eigenfunction $\phi^{\epsilon;\mu}$ is then *even* in s, and the nonlinear term $F(v;\mu,\epsilon)$ in (3.18) is *odd*. As a consequence, B_0 in (3.24) equals zero at the zero shift. Here we take the advantages of the symmetry obtaining B_0 identically equal to zero. After this, the proof of (3.43)–(3.44) at $z_0=0$ are similar, except that for the proof of (3.43) we need (3.48) instead. ☐

3.5. **Main result at critical scale — reduced bifurcation equation.** Note that in Theorem 3.7, the existence of solutions α in the ball of radius of $O(\epsilon\mu^2)$ requires that $|K\epsilon\mu^2| \gg O(\epsilon^6+\epsilon^2\mu^4)$ in the expansion of $B_1^\epsilon(\alpha=0;\mu)$. Hence Proposition 3.6 is not useful if $\mu^2 \leq O(\epsilon^5)$. Such a situation of small μ calls for a much more careful treatment of the high-order terms in the reduced equation $B^\epsilon(\alpha;\delta)=0$. In order to have a desirable reduced equation, we choose the approximate solution $U^{\epsilon;\delta}$ in Proposition 3.2 with $z_0(\delta)\equiv 0$. Note that $\delta=0$ is exactly the bifurcation point of the location equation (3.12) which is presumably very close to the actual bifurcation point of the transition layer problem. We now state the analysis for (2.15) in terms of δ. Recall Remark 2.1 that (2.15) is now called the *reduced bifurcation equation* as it will lead to multiple (three) solutions for α around $z_0=0$.

Proposition 3.9. *At $z_0 = 0$, for $\epsilon, \delta \ll 1$, the coefficients of the reduced bifurcation equation (2.15) and (3.16) read*

$$\begin{aligned} B_n^\epsilon(\alpha = 0; \delta) &\equiv 0, \quad \text{for } n = 0, 2, 4; \\ B_1^\epsilon(\alpha = 0; \delta) &= \left[N_{1,0}\epsilon^6 + O(\epsilon^7)\right] + \left[K_1\epsilon^3\right]\delta + O(\epsilon^6\delta^2); \\ B_3^\epsilon(\alpha = 0; \delta) &= \left[-K_2\epsilon^3 + O(\epsilon^6)\right] + O(\epsilon^3\delta); \\ |B_5^\epsilon(\alpha; \delta)| &\leq O(\epsilon + |\alpha|). \end{aligned}$$

where K_1 has appeared in Proposition 4.1, $K_2 = \dfrac{\int t u_0(1-u_0^2)^2\,dt}{\int \dot{u}_0^2\,dt} > 0$. ($N_{1,0}$ is some constant.)

The proof of the above is given in Section 4. We first state the following result concerning the bifurcation point for the reduced equation (2.15) with above form of coefficients.

Proposition 3.10. *The reduced bifurcation equation $B^\epsilon(\alpha; \delta) = 0$ in Proposition 3.9 has a bifurcation point at $(\alpha_*^\epsilon = 0, \delta_*^\epsilon)$ satisfying $|\delta_*^\epsilon| \leq O(\epsilon^3)$.*

Proof. The reduced bifurcation equation can be written as:

$$B_1^\epsilon(\alpha = 0; \delta)\alpha + B_3^\epsilon(\alpha = 0; \delta)\alpha^3 + B_5^\epsilon(\bar{\alpha}; \delta)\alpha^5 = 0.$$

Note that the necessary conditions for $(\alpha_*^\epsilon, \delta_*^\epsilon)$ being a bifurcation point are

$$B^\epsilon(\alpha_*^\epsilon; \delta_*^\epsilon) = 0, \quad B_\alpha^\epsilon(\alpha_*^\epsilon; \delta_*^\epsilon) = 0.$$

As $B^\epsilon(\alpha; \delta)$ is an odd function in α, $\alpha = 0$ is always a solution. We take $\alpha_*^\epsilon = 0$. The other equation $B_\alpha^\epsilon(\alpha_*^\epsilon = 0; \delta_*^\epsilon) = 0$ becomes $B_1^\epsilon(\alpha_{\mathsf{q}}; \delta) = 0$, i.e.,

$$[N_{1,0}\epsilon^6 + O(\epsilon^7)] + K_1\epsilon^3\delta_* + O(\epsilon^6\delta_*^2) = 0.$$

As $K_1\epsilon^3 > 0$, it is equivalent to solving

$$K_1\delta_* = -[N_{1,0}\epsilon^3 + O(\epsilon^6)] + O(\epsilon^3\delta_*^2).$$

It has a solution δ_*^ϵ, corresponding to the bifurcation point, satisfying the property $|\delta_*^\epsilon| \leq O(\epsilon^3)$. ☐

Note that the bifurcation point $\delta_{\mathsf{q}} = O(\epsilon^3)$ is so small that, the reduced bifurcation equation is *structurally robust* in the sense that if we introduce a new bifurcation parameter

$$\tau := \delta - \delta_*^\epsilon, \tag{3.53}$$

then in terms of τ, coefficients of the reduced bifurcation equation reads asymptotically the same as that in Proposition 3.9. In fact,

$$\begin{aligned} B_1^\epsilon(\alpha = 0; \tau + \delta_*^\epsilon) &= [N_{1,0}\epsilon^6 + O(\epsilon^7)] + \left[K_1\epsilon^3\right](\tau + \delta_*^\epsilon) + O(\epsilon^6(\tau + \delta_*^\epsilon)^2) \\ &= \Big([N_{1,0}\epsilon^6 + O(\epsilon^7)] + \left[K_1\epsilon^3\right]\delta_*^\epsilon + O(\epsilon^6(\delta_*^\epsilon)^2)\Big) \\ &\quad + \Big(\left[K_1\epsilon^3\right] + O(\epsilon^6\delta_*^\epsilon)\Big)\tau + O(\epsilon^6\tau^2). \end{aligned}$$

Recall the equation in determining of δ_*^ϵ, so the first bracket becomes zero. And the second coefficient does not change at $O(\epsilon^3)$ (in fact, the change is at most of $O(\epsilon^9)$).

Hence in terms of new parameter τ, the reduced bifurcation equation becomes (with abuse of notation of B_i^ϵ),

$$\begin{aligned} B_n^\epsilon(\alpha=0;\tau) &\equiv 0, \text{ for } n=0,2,4; \\ B_1^\epsilon(\alpha=0;\tau) &= \left[K_1\epsilon^3+O(\epsilon^9)\right]\tau+O(\epsilon^6\tau^2); \\ B_3^\epsilon(\alpha=0;\tau) &= \left[-K_2\epsilon^3+O(\epsilon^6)\right]+O(\epsilon^3\tau); \\ |B_5^\epsilon(\alpha;\tau)| &\le O(\epsilon+|\alpha|). \end{aligned}$$

With the above, the reduced bifurcation equation becomes

$$B_1^\epsilon(\alpha=0;\tau)\,\alpha+B_3^\epsilon(\alpha=0;\tau)\,\alpha^3+B_5^\epsilon(\overline{\alpha},\tau)\alpha^5=0. \tag{3.54}$$

Using the form of the coefficients, the above equation roughly equals

$$K_1\epsilon^3\tau\alpha-K_2\epsilon^3\alpha^3=0$$

from which we find three solutions demonstrating a pitchfork bifurcation:

$$\alpha_1=0, \quad \alpha_2=-\alpha_3\approx\sqrt{\frac{K_1}{K_2}\tau}.$$

The above conclusion holds for τ is small. The following statement makes this more rigorous and quantitative.

Theorem 3.11. *For any positive exponent p, there are two positive constants C_1, C_2, such that the following two statements hold for the solution of the reduced bifurcation equation.*

1. *if $0<\tau\le C_1\,\epsilon^{2+p}$, then equation (3.54) gives the following three solutions,*

$$\alpha_1=0, \quad \alpha_2=-\alpha_3=\sqrt{\frac{K_1}{K_2}\tau}+o(\sqrt{\tau}).$$

2. *If $-C_2\,\epsilon^{2+p}\le\tau\le0$, then (3.54) has only trivial solution $\alpha=0$.*

Proof. Clearly $\alpha=0$ is a zero to (3.54). Next we seek only non-zero solutions to (3.54). Without loss of generality, consider

$$\frac{B^\epsilon(\alpha;\tau)}{\alpha}=0, \quad \text{i.e.,} \quad B_1^\epsilon(\alpha=0;\tau)+B_3^\epsilon(\alpha=0;\tau)\,\alpha^2+B_5^\epsilon(\overline{\alpha},\tau)\alpha^4\;=\;0.$$

For convenience, let $y=\alpha^2\ge0$. Noting the bound for B_5^ϵ, then the above becomes

$$B_1^\epsilon(\alpha=0;\tau)+B_3^\epsilon(\alpha=0;\tau)\,y+O(\epsilon+|\sqrt{y}|)y^2\;=\;0.$$

Anticipating that there are non-zero solutions given by $\alpha=\pm O(\sqrt{\tau})$, we further introduce $y=\tau\widetilde{y}$. Upon factoring out $\epsilon^3\tau$, we have

$$\Big(K_1+O(\epsilon^6+\epsilon^3\tau)\Big)+\Big(-K_2+O(\epsilon^3+\tau)\Big)\widetilde{y}+O\left(\frac{\tau}{\epsilon^2}+\frac{\tau^{\frac{3}{2}}}{\epsilon^3}\sqrt{\widetilde{y}}\right)\widetilde{y}^2=0.$$

Now if $\tau\ll\epsilon^2$ (for example if we impose $\tau=O(\epsilon^{2+p})$ for any $p>0$ as stated in the hypothesis), then the coefficient in front of $\widetilde{y}^2$ is small, i.e., $O\left(\frac{\tau}{\epsilon^2}+\frac{\tau^{\frac{3}{2}}}{\epsilon^3}\right)\ll O(1)$. In this case, the quadratic term can be neglected comparing to the linear terms. Thus for $\epsilon,\tau\ll1$, an application of the implicit function theorem implies that there is unique solution $\widetilde{y}^*(\tau;\epsilon)$ in a ball $|\widetilde{y}|<R_0$ for some sufficiently large constant R_0. More specifically, one has that

$$\widetilde{y}^*(\tau;\epsilon)=\frac{K_1}{K_2}+o(1)>0$$

where $o(1)$ tends to zero as both ϵ, τ approach zero. Rescale back to variable $\alpha = \pm\sqrt{\tau}\widetilde{y}$, it gives two non-trivial solutions $\alpha_2 = -\alpha_3 = \sqrt{\frac{K_1}{K_2}\tau} + o(\sqrt{\tau})$.

The same analysis can be done for the case $\tau < 0$. $\square$

As mentioned in page 908, another difficulty in establishing the global pitchfork bifurcation of regular transition layer in Figure 2 is to obtain an overlap region where both the *reduced equation* method in Section 3.3 and the *reduced bifurcation equation* method in Section 3.5 can be applied. If there is such an overlap, then the local pitchfork bifurcation diagram established in Theorem 3.11 can be extended into the regime of Theorem 3.7.

Proposition 3.12 (Existence of overlapping region)**.** *For any τ in the following range*

$$\left\{\tau \middle|\, \epsilon^{3-\gamma} \leq \tau \leq \epsilon^{2+\gamma}\right\} \text{ with } 0 < \gamma \ll 1, \tag{3.55}$$

there exists a $\mu_\tau = O(\epsilon^p)$ with $0 < p < \frac{5}{2}$ such that

$$\beta = \epsilon^2(\widetilde{\beta}_* + \delta^\epsilon_{\mathsf{q}} + \tau) = \epsilon^2(\widetilde{\beta}_* + \frac{1}{2}\mu_\tau^2). \tag{3.56}$$

(Note that the above ranges of τ and μ_τ fall in the regimes such that Theorems 3.11 and 3.7 are simultaneously applicable.)

Proof. The range of β for the validity of Theorem 3.7 is given by

$$\beta = \epsilon^2\tilde{\beta}_{\mathsf{q}} + \frac{1}{2}\mu^2 = \epsilon^2\left(\tilde{\beta}_{\mathsf{q}} + \frac{1}{2}\left(\frac{\mu}{\epsilon}\right)^2\right) \quad \text{for } \mu = O(\epsilon^p) \text{ with } 0 < p < \frac{5}{2}$$

while that for Theorem 3.11 is given by:

$$\beta = \epsilon^2\left(\tilde{\beta}_{\mathsf{q}} + \delta^\epsilon_{\mathsf{q}} + \tau\right) \quad \text{for } -O(\epsilon^{2+p}) \leq \tau \leq O(\epsilon^{2+p}) \text{ with } 0 < p.$$

Using the facts that $\left(\frac{\mu}{\epsilon}\right)^2 = \epsilon^{2p-2}$ with $-2 < 2p-2 < 3$, $|\delta^\epsilon_{\mathsf{q}}| = O(\epsilon^3)$ and $(2,3) \subseteq (-2,3)$, we deduce that any τ in the range (3.55) corresponds to a μ such that Theorem 3.7 can be applied. $\square$

The above statement implicates that in the overlapping region, the two *non-trivial* solutions can be constructed either by making use of the *macroscopic shift z* or *microscopic shift α*. The former is given by $\pm O\left(\frac{\mu}{\epsilon}\right)$ while the latter is given by $\pm O(\sqrt{\delta})$:

$$u_{\text{exact}} \approx u_0\left(\frac{x}{\epsilon} \pm \frac{\mu}{\epsilon}\right) \quad \text{or} \quad u_{\text{exact}} \approx u_0\left(\frac{x}{\epsilon}\right) \pm O(\sqrt{\delta})\phi^\epsilon.$$

The above is also consistent with the relation that $\mu \propto \epsilon\sqrt{\delta}$. With more analysis, we expect to have some "uniqueness" statement in the sense that the two approaches give the same solution. In the interest of the length of the article, we do not prove this.

4. **The analysis of the reduced bifurcation equation at critical scale.** This section is devoted to prove Proposition 3.9. The asymptotic form of the reduced bifurcation is derived with respect to the singular parameter ϵ, bifurcation variable δ and scalar variable α. The main effort will be spent in the derivation of $B_3^\epsilon\,(\alpha = 0; \delta)$.

As to be seen, the result is intimately connected to the underlying pitchfork spatial inhomogeneity. From its formula

$$B_3^\epsilon(\alpha=0;\delta) := \frac{\left\langle (F(\mathfrak{q}))_{\alpha^{(3)}}, \phi^{\epsilon;\delta}\right\rangle}{3!\,\langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle} \tag{3.26}$$

and (3.21) for $F(\mathfrak{q})_{\alpha^{(3)}}$, in order to have an accurate description, we need to analyze the interaction between the principal eigenfunction $\phi^{\epsilon;\delta}$ and $v_{\alpha^{(n)}}$.

First note that the statements of Proposition 3.2 imply that with the choice of $z_0(\delta)=0$, there is always an odd middle approximate solution $U^{\epsilon;\delta}(x)$ regardless of the sign of δ. We will construct the exact solution as a perturbation of this $U^{\epsilon;\delta}(x)$. For convenience, we will recall some key definitions.

The pitchfork spatial inhomogeneity is written as

$$a(\epsilon t;\delta) = \epsilon^3\widetilde{a}(t;\delta) \quad \text{where} \quad \widetilde{a}(t;\delta) := -\frac{1}{2}t^3 + \left(\widetilde{\beta}_* + \delta\right)t \quad \text{for} \quad t\in[-\frac{1}{\epsilon},\frac{1}{\epsilon}]. \tag{4.1}$$

Hence by (3.13), we can write the macroscopic perturbation as $\psi^{\epsilon;2} = \epsilon^3\widetilde{\psi}(\cdot,\delta)$ where $\widetilde{\psi}$ solves

$$L_0(\widetilde{\psi}) = (1-u_0^2)\widetilde{a}(t;\delta) \quad \text{with} \quad \int \widetilde{\psi}(t;\delta)\dot{u}_0\,dt = 0. \tag{4.2}$$

We have the following decomposition,

$$\widetilde{\psi}(t;\delta) := u_1(t) + \delta\,u_2(t) \tag{4.3}$$

where $u_1(t)$ and $u_2(t)$ are determined respectively by

$$L_0(u_1) = (1-u_0^2)\left[-\frac{t^3}{2} + \widetilde{\beta}_* t\right], \quad \int u_1(t)\dot{u}_0\,dt = 0 \tag{4.4}$$

$$L_0(u_2) = (1-u_0^2)\,t, \quad \int u_2(t)\dot{u}_0\,dt = 0. \tag{4.5}$$

Note that u_1, u_2 and hence $\widetilde{\psi}$ decay exponentially in t.

With the above notations, the initial good approximation solution (2.17) and the linearized operator (2.18) are given by (for simplicity, we omit the subscript app):

$$\begin{aligned} U^{\epsilon;\delta}(x) &:= \xi_0(x)\left[u_0\left(\frac{x}{\epsilon}\right) + \epsilon^3\,\widetilde{\psi}\left(\frac{x}{\epsilon};\delta\right)\right] + \xi_1(x), \quad x\in(-1,1); &(4.6)\\ L^{\epsilon;\delta} &:= \partial_{tt} + \left[1 - 3\left(U^{\epsilon;\delta}\right)^2 + 2a(\epsilon t;\widetilde{\beta}_* + \delta)U^{\epsilon;\delta}\right]. &(4.7)\end{aligned}$$

For the analysis, we also need the expansion of $L^{\epsilon;\delta}$. Over the interval $|t|\le \frac{1}{4\epsilon}$, we have

$$L^{\epsilon;\delta} = \partial_{tt} + \left[1 - 3(u_0+\epsilon^3\widetilde{\psi})^2 + 2\epsilon^3\widetilde{a}(u_0+\epsilon^3\widetilde{\psi})\right] := L_0 + \epsilon^3\widetilde{q}_1 + \epsilon^6\widetilde{q}_2 \tag{4.8}$$

where

$$\begin{aligned} \widetilde{q}_1 &= -6u_0\widetilde{\psi} + 2u_0\widetilde{a} = -6u_0(u_1+\delta u_2) + u_0\left(-t^3 + 2(\widetilde{\beta}_*+\delta)t\right) &(4.9)\\ \widetilde{q}_2 &= -3\widetilde{\psi}^2 + 2\widetilde{a}\widetilde{\psi} = -3(u_1+\delta u_2)^2 + \left(-t^3 + 2(\widetilde{\beta}_*+\delta)t\right)(u_1+\delta u_2) &(4.10)\end{aligned}$$

Furthermore, we have

$$\widetilde{q}_1(\delta=0) = -6u_0u_1 + u_0\left(-t^3 + 2\widetilde{\beta}_* t\right) \tag{4.11}$$

$$\frac{d\widetilde{q}_1}{d\delta} = -6u_0u_2 + 2u_0 t \quad \text{and} \quad \frac{d^2\widetilde{q}_1}{d\delta^2} = 0 \tag{4.12}$$

and

$$\widetilde{q}_2(\delta=0) = -3u_1^2 + \left(-t^3 + 2\widetilde{\beta}_* t\right) u_1 \tag{4.13}$$

$$\frac{d\widetilde{q}_2}{d\delta} = -6(u_1+\delta u_2)u_2 + 2t(u_1+\delta u_2) + \left(-t^3 + 2(\widetilde{\beta}_* + \delta)t\right) u_2 \tag{4.14}$$

$$\frac{d\widetilde{q}_2}{d\delta}\Big|_{\delta=0} = -6u_1u_2 + 2tu_1 + \left(-t^3 + 2\widetilde{\beta}_* t\right) u_2 \tag{4.15}$$

$$\frac{d^2\widetilde{q}_2}{d\delta^2} = -6u_2^2 + 4tu_2. \tag{4.16}$$

4.1. **Preliminary analysis.** Many of the results here can be obtained directly by substituting $\mu = \epsilon\sqrt{2\delta}$. But to better control the error terms, we need more information on high-order terms. The first technical result of this kind concerns the asymptotic expansion of the principal eigenvalue $\lambda_1^\epsilon(\delta)$ and eigenfunction $\phi^\epsilon(t;\delta)$ of $L^{\epsilon;\delta}$ which satisfy:

$$L^{\epsilon;\delta}\phi^\epsilon(t;\delta) \;=\; \lambda_1^\epsilon(\delta)\,\phi^\epsilon(t;\delta), \quad \text{and} \quad \int \phi^\epsilon(t;\delta)\dot{u}_0(t)\,dt = \int \dot{u}_0^2(t)dt. \tag{4.17}$$

We remark that many of the results here hold for $|\delta| \ll 1$, even if δ is negative. To keep our presentation in a reasonable scope, we derive many of them for $0 \le \delta \ll 1$ using transformation $\mu = \epsilon\sqrt{2\delta}$.

Proposition 4.1. *For $\epsilon, |\delta| \ll 1$, the principal eigenvalue $\lambda_1^\epsilon(\delta)$ satisfies*

$$\lambda_1^\epsilon(\delta) \;:=\; O(\epsilon^6) + \left[K_1\epsilon^3\right]\delta + O(\epsilon^6\delta^2) \tag{4.18}$$

where $K_1 > 0$ is the same constant appeared in Proposition 3.8 (3.49).

The principal eigenfunction $\phi^\epsilon(t;\delta)$ is an even function of t and has the following form

$$\phi^\epsilon(t;\delta) \quad = \quad \dot{u}_0 + \epsilon^3\widetilde{\phi}_1(t;\delta) + \epsilon^6\widetilde{\phi}_r^\epsilon(t;\delta) \tag{4.19}$$

where the function $\widetilde{\phi}_1(t;\delta)$ solves

$$L_0\widetilde{\phi}_1 = -\widetilde{q}_1(t;\delta)\dot{u}_0 + \widetilde{\lambda}_1(\delta)\dot{u}_0, \quad \int \widetilde{\phi}_1\dot{u}_0 = 0. \tag{4.20}$$

The function $\widetilde{q}_1(t;\delta)$ is from (4.9) and $\widetilde{\lambda}_1(\delta) = K_1\delta$. In addition (denote $\phi^{\epsilon;\delta}(t) = \phi^\epsilon(t;\delta)$)

$$\|\phi_\delta^{\epsilon;\delta}\|_0 \le O(\epsilon^3), \quad \textit{and} \quad \|\phi_{\delta\delta}^{\epsilon;\delta}\|_0 \le O(\epsilon^6). \tag{4.21}$$

Proof. The statement for $\lambda_1^\epsilon(\delta)$ comes from Proposition 3.8 by substituting $\mu = \epsilon\sqrt{2\delta}$ to the case $z_0 = 0$. To determine the expansion of eigenfunction, it is convenient to rewrite (4.18) as

$$\lambda_1^\epsilon(\delta) := \epsilon^3\widetilde{\lambda}_1(\delta) + \widetilde{\lambda}_r^\epsilon(\delta)$$

where $\widetilde{\lambda}_1(\delta) = K_1\delta$ is linear in δ, and $|\widetilde{\lambda}_r^\epsilon(\delta)| \le O(\epsilon^6)$.

Substituting $\mu = \epsilon\sqrt{2\delta}$ into equation (A.3) gives $\Phi_1^\epsilon(\delta) = \epsilon^3\widetilde{\phi}_1$ with $\widetilde{\phi}_1$ being the unique solution to equation (4.20). Note that $\widetilde{q}_1(t;\delta)$ in (4.9) and $\widetilde{\lambda}_1(\delta)$ in (4.20) are both *linear* in δ, so does $\widetilde{\phi}_1$ where δ is considered as a parameter. In addition, $\|\widetilde{\phi}_1(t;\delta)\|_0,\ \|\widetilde{\phi}_r^\epsilon(t;\delta)\|_0 = O(1)$ and both of them decay exponentially fast in t.

In (4.17), $\phi^\epsilon(t;\delta)$ continuously depends on bifurcation parameter δ. Following the expansion form of (4.19), the first estimate in (4.21) holds. For the second estimate, noting that from (4.19),

$$\phi^{\epsilon;\delta}_{\delta\delta} = \epsilon^3(\widetilde{\phi}_1)_{\delta\delta} + \epsilon^6(\widetilde{\phi}_r)_{\delta\delta}.$$

Taking derivatives of equation (4.20) with respect to δ twice, and recalling $(\widetilde{q}_1)_{\delta\delta} = 0$, $\widetilde{\lambda}_1(\delta) = K_1\delta$, then

$$L_0(\widetilde{\phi}_1)_{\delta\delta} = 0, \quad \int (\widetilde{\phi}_1)_{\delta\delta}\dot{u}_0 = 0.$$

That is, $(\widetilde{\phi}_1)_{\delta\delta} = 0$. It follows $\phi^{\epsilon;\delta}_{\delta\delta} = \epsilon^6(\widetilde{\phi}_r)_{\delta\delta}$ and the second estimate in (4.21). ☐

The second technical result gives more accurate description of $v_{\alpha^{(n)}}$. The most important case is $n = 2$.

Proposition 4.2. *Let v be the solution of (2.14) in the critical scaling case. Then for $\epsilon, |\alpha|, |\delta| \ll 1$, it holds*

$$\|v\|_0 \le O(\alpha^2 + \epsilon^6), \quad \|v_\alpha\|_0 \le O(|\alpha| + \epsilon^6), \quad \text{and} \quad \|v_{\alpha\alpha}\|_0 \le O(1). \tag{4.22}$$

At $\alpha = 0$, we have the following more precise statements. The function $v_{\alpha\alpha}$ admits the form,

$$v_{\alpha\alpha} = \ddot{u}_0 + \epsilon^3 g_2 + O(\epsilon^6) \tag{4.23}$$

where g_2 satisfies the following equation:

$$L_0\left(g_2\right) = 12u_0\dot{u}_0\widetilde{\phi}_1 - \widetilde{q}_1\ddot{u}_0 + \left[6\widetilde{\psi} - 2\widetilde{a}(t;\widetilde{\beta}_*)\right]\dot{u}_0^2, \quad \text{and} \quad \int g_2\dot{u}_0(t)\,dt = 0. \tag{4.24}$$

Furthermore,

$$v_{\alpha^{(3)}}(\alpha = 0;\delta)(t) \quad := \quad g_3 + O(\epsilon^3) = \left(u_0^{(3)} + \frac{2}{5}\dot{u}_0\right) + O(\epsilon^3); \tag{4.25}$$

$$v_{\alpha^{(4)}}(\alpha = 0;\delta)(t) \quad := \quad g_4 + O(\epsilon^3) = \left(u_0^{(4)} + \frac{8}{5}\ddot{u}_0\right) + O(\epsilon^3). \tag{4.26}$$

All the error terms are measured in the norm $\|\cdot\|_0$.

Proof. Statement (4.22) is a direct application of the Proposition 3.5 with $\mu = \epsilon\sqrt{2\delta}$.

Asymptotics of $v_{\alpha\alpha}$ (4.23). Using (3.27) (with $n = 2$), at $\alpha = 0$, we have

$$(v_{\alpha\alpha})_{tt} + \left[1 - 3\left(u_0 + \epsilon^3\widetilde{\psi}\right)^2 + 2\epsilon^3\widetilde{a}(t;\widetilde{\beta}_* + \delta)\left(u_0 + \epsilon^3\widetilde{\psi}\right)\right]v_{\alpha\alpha} + (I - E)_{\phi^{\epsilon;\delta}}\{\cdots\} = 0$$

where $\cdots$ is the following expression,

$$\left[-3\left(u_0 + \epsilon^3\widetilde{\psi}\right) + \epsilon^3\widetilde{a}(t;\widetilde{\beta}_* + \delta)\right]\left[2\left(\phi^{\epsilon;\delta} + v_\alpha\right)^2 + 2vv_{\alpha\alpha}\right] - 3v^2v_{\alpha\alpha} - 6v(\phi^{\epsilon;\delta} + v_\alpha)^2.$$

At $\alpha = 0$, by (4.22), $v_{\alpha\alpha}$ satisfies an equation of the following form:

$$\begin{aligned}(v_{\alpha\alpha})_{tt} &+ \left[1 - 3\left(u_0 + \epsilon^3\widetilde{\psi}\right)^2 + 2\epsilon^3\widetilde{a}(t;\widetilde{\beta}_* + \delta)\left(u_0 + \epsilon^3\widetilde{\psi}\right)\right]v_{\alpha\alpha} \\ &+ (I - E)_{\phi^{\epsilon;\delta}}\left\{\left[-3\left(u_0 + \epsilon^3\widetilde{\psi}\right) + \epsilon^3\widetilde{a}(t;\widetilde{\beta}_* + \delta)\right]2\left(\phi^{\epsilon;\delta}\right)^2\right\} = O(\epsilon^6)\end{aligned}$$

which can be expanded as:

$$\begin{aligned}L_0(v_{\alpha\alpha}) &= (I - E)_{\phi^{\epsilon;\delta}}\left[6u_0\dot{u}_0^2\right] \\ &+ \epsilon^3(I - E)_{\phi^{\epsilon;\delta}}\left[12u_0\dot{u}_0\widetilde{\phi}_1 + (6\widetilde{\psi} - 2\widetilde{a})\dot{u}_0^2\right] - \epsilon^3\widetilde{q}_1(\delta)v_{\alpha\alpha} + O(\epsilon^6)\end{aligned} \tag{4.27}$$

where $\widetilde{q}_1(\delta) = -\left(6\widetilde{\psi} - 2\widetilde{a}\right) u_0$ in (4.9).

The $O(1)$-component of the above equation is:

$$L_0\left(v_{\alpha\alpha}\right) = (I-E)_{\dot{u}_0}\left[6u_0\dot{u}_0^2\right] = 6u_0\dot{u}_0^2 \qquad (\text{note } \langle u_0\dot{u}_0^2,\, \dot{u}_0\rangle = \int u_0\dot{u}_0^3 = 0).$$

Noting that $L_0(\ddot{u}_0) = 6u_0\dot{u}_0^2$ and $\ddot{u}_0 \perp \dot{u}_0$, the solution of the above is exactly $\ddot{u}_0$. Hence,

$$v_{\alpha\alpha} = \ddot{u}_0 + \epsilon^3 g_2 + O(\epsilon^6)$$

where the term $\epsilon^3 g_2$ solves an equation of the form

$$\begin{aligned} L_0(\epsilon^3 g_2) \;&=\; O(\epsilon^3)\text{-terms in} \\ &\left\{(I-E)_{\phi^{\epsilon;\delta}}\left[6u_0\dot{u}_0^2\right] + \epsilon^3 (I-E)_{\phi^{\epsilon;\delta}}\left[12u_0\dot{u}_0\widetilde{\phi}_1 + (6\widetilde{\psi} - 2\widetilde{a})\dot{u}_0^2\right] - \epsilon^3\widetilde{q}_1\ddot{u}_0\right\}. \end{aligned}$$

Noting that $6u_0\dot{u}_0^2$ is odd in t, and $\phi^{\epsilon;\delta}$ is even, thus

$$(I-E)_{\phi^{\epsilon;\delta}}\left[6u_0\dot{u}_0^2\right] = 6u_0\dot{u}_0^2 - \frac{\int 6u_0\dot{u}_0^2\phi^{\epsilon;\delta}}{\int(\phi^{\epsilon;\delta})^2}\phi^{\epsilon;\delta} = 6u_0\dot{u}_0^2.$$

On the other hand,

$$\begin{aligned} &\epsilon^3 (I-E)_{\phi^{\epsilon;\delta}}\left[12u_0\dot{u}_0\widetilde{\phi}_1 + (6\widetilde{\psi} - 2\widetilde{a})\dot{u}_0^2\right] \\ =\;& \epsilon^3 (I-E)_{\dot{u}_0}\left[12u_0\dot{u}_0\widetilde{\phi}_1 + (6\widetilde{\psi} - 2\widetilde{a})\dot{u}_0^2\right] + O(\epsilon^6) \\ =\;& \epsilon^3\left[12u_0\dot{u}_0\widetilde{\phi}_1 + (6\widetilde{\psi} - 2\widetilde{a})\dot{u}_0^2\right] + O(\epsilon^6), \end{aligned}$$

where the second equality is due to the oddness of $\left[12u_0\dot{u}_0\widetilde{\phi}_1 + (6\widetilde{\psi} - 2\widetilde{a})\dot{u}_0^2\right]$. Thus, g_2 uniquely solves

$$L_0\left(g_2\right) = 12u_0\dot{u}_0\widetilde{\phi}_1 - \widetilde{q}_1\ddot{u}_0 + \left[6\widetilde{\psi} - 2\widetilde{a}(t;\widetilde{\beta}_*)\right]\dot{u}_0^2, \;\text{ and} \int g_2\dot{u}_0(t)\,dt = 0.$$

which is exactly (4.24).

Asymptotics of $v_{\alpha^{(3)}}$ (4.25). In the following, we use the notation $u_0^{(n)}(s) := \frac{d^n u_0}{ds^n}$. Using equation (3.27) (with $n = 3$) at $\alpha = 0$ and the formula for $F_{\alpha^{(3)}}$ in (3.21), we have,

$$L_0 v_{\alpha^{(3)}} + (I-E)_{\phi^{\epsilon;\delta}}\left\{-6(\phi^{\epsilon;\delta})^3 - 18u_0\phi^{\epsilon;\delta}v_{\alpha\alpha}\right\} = O(\epsilon^3). \tag{4.28}$$

Noting that, as we only concern the $O(1)$-term, and the error neglected above is at most of $O(\epsilon^3)$. We have also used the estimates (at $\alpha = 0$) of (4.22). We continue the expansion using (4.19) and (4.23), then (4.28) becomes

$$L_0 v_{\alpha^{(3)}} + (I-E)_{\dot{u}_0}\left\{-6\dot{u}_0^3 - 18u_0\dot{u}_0\ddot{u}_0\right\} = O(\epsilon^3).$$

Noting that $\int\{-6\dot{u}_0^3 - 18u_0\dot{u}_0\ddot{u}_0\}\dot{u}_0 = 0$, therefore,

$$L_0 g_3 = \{6\dot{u}_0^3 + 18u_0\dot{u}_0\ddot{u}_0\}, \qquad \int g_3\dot{u}_0 dt = 0. \tag{4.29}$$

On the other hand, from $L_0(\ddot{u}_0) = 6u_0\dot{u}_0^2$, taking derivative on both sides, then

$$L_0(u_0^{(3)}) = 6\dot{u}_0^3 + 18u_0\dot{u}_0\ddot{u}_0.$$

To meet the orthogonality condition, we must choose

$$g_3 = u_0^{(3)} + c\dot{u}_0, \text{ where } c := -\frac{\int u_0^{(3)}\dot{u}_0}{\int \dot{u}_0^2}.$$

We can compute c in the following way. Noting that $L_0\dot{u}_0 = 0$, multiplying it by $\dot{u}_0$ and integrating, it then leads to

$$\int u_0^{(3)}\dot{u}_0 + (1-3u_0^2)\dot{u}_0^2 = 0 \text{ i.e., } \int u_0^{(3)}\dot{u}_0 - \int 3u_0^2\dot{u}_0^2 = -\int \dot{u}_0^2.$$

Noting also

$$\int u_0^{(3)}\dot{u}_0 = -\int \ddot{u}_0^2 = -2\int u_0^2\dot{u}_0^2 \text{ using } \ddot{u}_0 = -u_0(1-u_0^2) = -\sqrt{2}u_0\dot{u}_0.$$

Hence

$$-\int \dot{u}_0^2 = \int u_0^{(3)}\dot{u}_0 - \int 3u_0^2\dot{u}_0^2 = \int u_0^{(3)}\dot{u}_0 + \frac{3}{2}\int u_0^{(3)}\dot{u}_0 = \frac{5}{2}\int u_0^{(3)}\dot{u}_0,$$

which leads to $c = \frac{2}{5}$, as in (4.25).

Asymptotics of $v_{\alpha^{(4)}}$ (4.26). Using equation (3.27) (with $n = 4$), at $\alpha = 0$ and formula $F_{\alpha^{(4)}}$ in (3.22) we have, after denoting $v_{\alpha^{(4)}}(\alpha = 0; \delta)(t) = g_4 + O(\epsilon^3)$ and neglecting error no more than $O(\epsilon^3)$, that

$$L_0 g_4 = (I-E)_{\dot{u}_0}\left\{18u_0\ddot{u}_0^2 + 24u_0\dot{u}_0\,(g_3) + 36\dot{u}_0^2\ddot{u}_0\right\}, \quad \text{and} \quad \int g_4\dot{u}_0 = 0. \tag{4.30}$$

Recall that $g_3 = u_0^{(3)} + \frac{2}{5}\dot{u}_0$, and noting $\left\{18u_0\ddot{u}_0^2 + 24u_0\dot{u}_0\,(g_3) + 36\dot{u}_0^2\ddot{u}_0\right\} \perp \dot{u}_0$, therefore, (4.30) is reduced to

$$L_0 g_4 = \left\{18u_0\ddot{u}_0^2 + 24u_0\dot{u}_0\left(u_0^{(3)} + \frac{2}{5}\dot{u}_0\right) + 36\dot{u}_0^2\ddot{u}_0\right\}, \quad \text{and} \quad \int g_4\dot{u}_0 = 0. \tag{4.31}$$

From the identity $L_0(u_0^{(3)}) = 6\dot{u}_0^3 + 18u_0\dot{u}_0\ddot{u}_0$, taking derivative on both sides leads to

$$L_0(u_0^{(4)}) = \left\{18u_0\ddot{u}_0^2 + 24u_0\dot{u}_0u_0^{(3)} + 36\dot{u}_0^2\ddot{u}_0\right\}.$$

Recalling also, $L_0\left(\frac{8}{5}\ddot{u}_0\right) = \frac{8}{5}(6u_0\dot{u}_0^2) = \frac{48}{5}u_0\dot{u}_0^2$, it follows that $g_4 = u_0^{(4)} + \dfrac{8}{5}\ddot{u}_0$ uniquely solves equation (4.31), which is the expansion (4.26). $\square$

The following is an analog of Proposition 3.5 for v_δ and $v_{\delta\delta}$:

Proposition 4.3. *Let v be the solution of (2.14). Then at $\alpha = 0$, for $\epsilon, |\delta| \ll 1$, it holds that*

$$\|v,\ v_\delta,\ v_{\delta\delta}\|_0 \le O(\epsilon^6), \text{ and } \|v_{\alpha\delta}\|_0 \le O(\epsilon^9).$$

Proof. In the critical scaling regime, equation (2.14) for v reads as

$$L^{\epsilon;\delta}(v) + (I-E)_{\phi^{\epsilon;\delta}}\left[G^\epsilon(U^{\epsilon;\delta}) + F(\alpha\phi^{\epsilon;\delta} + v; \epsilon^2(\widetilde{\beta}_* + \delta))\right] = 0,$$

where $U^{\epsilon;\delta}, L^{\epsilon;\delta}$ are in (4.6, 4.7). The estimate for $\|v\|$ follows from Lemma 2.4 and the fact that $\|G^\epsilon(U^{\epsilon;\delta})\|_0 \le O(\epsilon^6)$ at $\alpha = 0$.

Next we take derivative of above equation with respect to δ, (noting that in general $\partial_\delta U$, $\partial_\delta\phi$, and $\partial_\delta((I-E)_{\phi^{\epsilon;\delta}}) \ne 0$.) We compute $\partial_\delta((I-E)_{\phi^{\epsilon;\delta}}(N))$ as follows. Given a function $N(\delta)$ (depending on δ), satisfying $\|N\|_0 \le O(\epsilon^6)$,

$$\begin{aligned}\partial_\delta((I-E)_\phi(N)) \;&=\; (I-E)_\phi(N_\delta) + \langle N, \phi_\delta\rangle\langle\phi,\phi\rangle^{-1}\phi \,-\, 2\langle N,\phi\rangle\langle\phi,\phi\rangle^{-2}\langle\phi,\phi_\delta\rangle\phi \\ &\quad+\; \langle N,\phi\rangle\langle\phi,\phi\rangle^{-1}\phi_\delta.\end{aligned}$$

Recall that $\|\phi_\delta^{\epsilon;\delta}\| \leq O(\epsilon^3)$ in Proposition 4.1, we then simplify $\partial_\delta((I-E)_\phi(N))$ as

$$\partial_\delta((I-E)_\phi(N)) = (I-E)_\phi(N_\delta) + O(\epsilon^9). \tag{4.32}$$

On the other hand,

$$\partial_\delta\left(L^{\epsilon;\delta}(v)\right) = L^{\epsilon;\delta}(v_\delta) + \left(\epsilon^3\widetilde{q}_{1,\delta} + \epsilon^6\widetilde{q}_{2,\delta}\right) v = L^{\epsilon;\delta}(v_\delta) + O(\epsilon^9), \tag{4.33}$$

as $\|\left(\epsilon^3\widetilde{q}_{1,\delta} + \epsilon^6\widetilde{q}_{2,\delta}\right) v\| \leq O(\epsilon^9)$. Now, taking derivative of the above equation with respect to δ (set $\alpha = 0$), we arrive at

$$L^{\epsilon;\delta}(v_\delta) + (I-E)_{\phi^{\epsilon;\delta}}\left[G^\epsilon(U^{\epsilon;\delta}) + F(v;\epsilon^2(\widetilde{\beta}_* + \delta))\right]_\delta = O(\epsilon^9), \tag{4.34}$$

where we have recalled $\|G^\epsilon(U^{\epsilon;\delta})\| \leq O(\epsilon^6)$ and $\|F(v;\epsilon^2(\widetilde{\beta}_*+\delta))\| \leq O(\epsilon^{12})$. Recalling G and F in (3.10), (3.18), respectively, then

$$G^\epsilon(U^{\epsilon;\delta})_\delta = \left(2u_0\epsilon^6\widetilde{\psi}\widetilde{a} + \left[-3u_0 + \epsilon^3\widetilde{a}\right](\epsilon^3\widetilde{\psi})^2 - (\epsilon^3\widetilde{\psi})^3\right)_\delta.$$

$$F_\delta(v,\epsilon^2(\widetilde{\beta}_*+\delta)) := \left\{\left[-3U^{\epsilon;\delta} + \epsilon^3\widetilde{a}\right]v^2 - v^3\right\}_\delta,$$

$\widetilde{a}, \widetilde{\psi}$ in (4.1, 4.2). A similar proof of Proposition 3.5 gives $\|v_\delta\| \leq O(\epsilon^6)$ as $\|G^\epsilon(U^{\epsilon;\delta})_\delta\| \leq O(\epsilon^6)$.

Lastly, taking derivative of equation (4.34) with respect to δ again yields,

$$L^{\epsilon;\delta}(v_{\delta\delta}) + (I-E)_{\phi^{\epsilon;\delta}}\left[G^\epsilon(U^{\epsilon;\delta}) + F(v;\epsilon^2(\widetilde{\beta}_*+\delta))\right]_{\delta\delta} = O(\epsilon^9)$$

here the $O(\epsilon^9)-$term absorbs all the small terms in simplifying processes (4.32) and (4.33). It then follows $\|v_{\delta\delta}\| \leq O(\epsilon^6)$ as $\|G^\epsilon(U^{\epsilon;\delta})_{\delta\delta}\| \leq O(\epsilon^6)$.

For $v_{\alpha\delta}$, we start with the equation for v_α, i.e.,

$$L^{\epsilon;\delta}(v_\alpha) + (I-E)_{\phi^{\epsilon;\delta}}\left[F_\alpha(\alpha\phi^{\epsilon;\delta} + v)\right] = 0, \text{ where}$$

$$F_\alpha(\alpha\phi + v) = \left\{[-6U^{\epsilon;\delta} + 2\epsilon^3\widetilde{a}](\alpha\phi+v) - 3(\alpha\phi+v)^2\right\}(\phi + v_\alpha).$$

The estimates for $v_{\alpha\delta}$ can be proved similarly by considering

$$\partial_\delta\left(L^{\epsilon;\delta}(v_\alpha)\right) + \partial_\delta(I-E)_{\phi^{\epsilon;\delta}}\left[F_\alpha(\alpha\phi^{\epsilon;\delta}+v)\right] = 0 \tag{4.35}$$

where $\|v_\alpha(\alpha = 0)\| \leq O(\epsilon^6)$ via Proposition 3.5 with $\mu = \epsilon\sqrt{2\delta}$. As $\|F_\alpha(v)\| \leq O(\epsilon^6)$, using the simplifications (4.32) and (4.33), (4.35) is then reduced to

$$L^{\epsilon;\delta}(v_{\alpha\delta}) + (I-E)_{\phi^{\epsilon;\delta}}\left[F_{\alpha\delta}(v)\right] = O(\epsilon^9).$$

Noting that $(-6U\phi)_\delta = -6\epsilon^3\left(u_2\phi + U\widetilde{\phi}_1\right)$, thus it follows $\|F_{\alpha\delta}(v)\| \leq O(\epsilon^9)$ and $\|v_{\alpha\delta}\| \leq O(\epsilon^9)$. □

4.2. **Derivation of the reduced bifurcation equation.** In this section, we obtain formulas for $B_n^\epsilon(\delta)$, $n = 0, 1, \cdots, 5$ which are then expanded as functions of the bifurcation parameter δ. The computation load for some of the coefficients is greatly reduced taking full advantages of the odd symmetry of the pitchfork spatial inhomogeneity $a(x,\beta) = -\dfrac{1}{2}x^3 + \beta x$.

Before proceeding, we recall the regime for μ and δ ($\mu = \epsilon\sqrt{2\delta}$) in this section:

$$\mu \ll O(\epsilon^{\frac{5}{2}}), \text{ or equivalently, } \delta \ll O(\epsilon^3). \tag{4.36}$$

We will also make use of Proposition 3.5 several times. We need v and related terms to be bounded by $O(\epsilon^6)$ (at $\alpha = 0$). This is true if

$$\epsilon^2\mu^4 \le O(\epsilon^6), \text{ i.e. } \mu \le O(\epsilon) \text{ or equivalently, } \delta \le O(1). \tag{4.37}$$

Note that the range (4.36) is included in (4.37).

With the above preparation, we first establish that

Lemma 4.4 (Even n)**.** *For any δ and $n = 0, 2, 4$, we have $B_n^\epsilon(\alpha = 0; \delta) = 0$.*

Proof. By the odd symmetry of a in t, $a(\epsilon t; \widetilde{\beta}_* + \delta) = -a(-\epsilon t; \widetilde{\beta}_* + \delta)$, the same holds for $U^{\epsilon;\delta}$ and $G^{\epsilon;\delta}(U^{\epsilon;\delta})$. On the other hand, by (4.8) and (4.17), the principle eigenfunction $\phi^{\epsilon;\delta}$ is even in t. By (2.13), this leads to that at $\alpha = 0$ and any δ, $v(\alpha = 0, \delta; \epsilon)$ and $F(v(\alpha = 0, \delta; \epsilon), \delta; \epsilon)$ are odd in t. Consequently, by (3.24), it holds that $B_0^\epsilon(\alpha = 0; \delta) \equiv 0$.

For the same reason, one has

$$v_{\alpha^{(n)}}(\alpha = 0, \delta; \epsilon)(t) = -v_{\alpha^{(n)}}(\alpha = 0, \delta; \epsilon)(-t) \text{ for } n = 2, 4$$

which then similarly leads to $B_2^\epsilon(\alpha = 0; \delta) = B_4^\epsilon(\alpha = 0; \delta) \equiv 0$. $\square$

Next, we analyze $B_1^\epsilon, B_3^\epsilon$ and B_5^ϵ. For notational simplicity, we supersede the dependence of $v, v_\alpha, \cdots$ on δ, ϵ, i.e., we write $v(t) = v(\alpha = 0, \delta; \epsilon)(t), v_\alpha(t) = v_\alpha(\alpha = 0, \delta; \epsilon)(t)$ and so forth.

The statement for B_1^ϵ is essentially that of Proposition 3.6 by substituting $\mu = \epsilon\sqrt{2\delta}$ into (3.43).

Lemma 4.5 (Asymptotics of $B_1^\epsilon(\alpha = 0; \delta)$)**.** *For $|\delta| \ll 1$, then*

$$B_1^\epsilon(\alpha = 0; \delta) = [N_{1,0}\epsilon^6 + o(1)\epsilon^6] + \left[K_1\epsilon^3\right]\delta + O(\epsilon^6\delta^2).$$

Again the most important conclusion is the positivity of K_1. Next we show the non-degeneracy of $B_3^\epsilon(\alpha = 0; \delta)$ at the order of $O(\epsilon^3)$.

Lemma 4.6 (Asymptotics of $B_3^\epsilon(\alpha = 0; \delta)$)**.** *For $|\delta| \ll 1$, then*

$$B_3^\epsilon(\alpha = 0; \delta) \quad = \quad \left[-K_2\epsilon^3 \, + \, O(\epsilon^4)\right] \, + \, B_{3,1}^\epsilon(\overline{\delta})\delta$$

where

$$K_2 = -\frac{\sqrt{2}}{6}\int \dot{u}_0^2\widetilde{a}_{sss}(s; \delta = 0)\, ds = \frac{1}{\sqrt{2}}\int \dot{u}_0^2\, ds > 0, \tag{4.38}$$

and $\left|B_{3,1}^\epsilon(\overline{\delta})\right| = O(\epsilon^3)$.

Note that the key conclusion of the above is that $K_2 > 0$. Together with Lemma 4.5, one recognizes that

$$B^\epsilon(\alpha; \beta) \approx B_1^\epsilon(\alpha = 0; \delta)\alpha + B_3^\epsilon(\alpha = 0; \delta)\alpha^3 \approx \epsilon^3\left(K_1\delta\alpha - K_2\alpha^3\right)$$

which resembles $a(x; \beta) = \beta x - \dfrac{1}{2}x^3$ with $\beta = \epsilon^2\left(\delta + \widetilde{\beta}_*\right)$. We infer that the desired non-degeneracy at order $O(\epsilon^3)$ of $B_{3,0}^\epsilon(\delta = 0)$ inherits from the cubic spatial inhomogeneity. Such an intrinsic connection between the spatial inhomogeneity and the reduced bifurcation equation explains why bifurcation of the interfacial problem and that of the associated transition layer problem behave similarly.

Proof. We first analyze $B^{\epsilon}_{3,0}(\delta = 0)$ which is done in several steps.

Step I - Simplification. Recalling that v, v_α at $\alpha = 0$ are bounded by $O(\epsilon^6)$ in (4.22), thus by identifying the dominating terms in formulas (3.26) and (3.21), we have,

$$\begin{aligned} B^{\epsilon}_3(\alpha = 0; \delta = 0) &= \frac{\left\langle (F(\mathfrak{q}))_{\alpha^{(3)}},\, \phi^{\epsilon;\delta=0} \right\rangle}{6\left\langle \phi^{\epsilon;\delta=0},\, \phi^{\epsilon;\delta=0} \right\rangle} \\ &= -\frac{\int [3U^{\epsilon;\delta=0} - a(\epsilon t; \widetilde{\beta}_*)](\phi^{\epsilon;\delta=0})^2 v_{\alpha\alpha} + (\phi^{\epsilon;\delta=0})^4}{\int (\phi^{\epsilon;\delta=0})^2} + O(\epsilon^6). \end{aligned} \tag{4.39}$$

Recall the following asymptotic expansions:

$$U^{\epsilon;\delta} = u_0 + \epsilon^3 \widetilde{\psi}, \quad a(\epsilon t; \widetilde{\beta}_* + \delta) = \epsilon^3 \widetilde{a}(t; \widetilde{\beta}_* + \delta), \quad \phi^{\epsilon;\delta=0} = \dot{u}_0 + \epsilon^3 \widetilde{\phi}_1 + O(\epsilon^6).$$

where $\widetilde{a}, \widetilde{\psi}, \widetilde{\phi}_1(s;\delta)$ are in (4.1, 4.2, 4.20) respectively. Now using (4.23) (for notational simplicity, we denote $g = g_2$ (4.24) in current proof), (4.39) becomes

$$\begin{aligned} &- B^{\epsilon}_3(\alpha = 0; \delta = 0) \\ &= \frac{\int [3(u_0 + \epsilon^3 \widetilde{\psi}) - \epsilon^3 \widetilde{a}(t; \widetilde{\beta}_* + \delta)] \left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^2 (\ddot{u}_0 + \epsilon^3 g) + \left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^4}{\int \left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^2} + O(\epsilon^6). \end{aligned} \tag{4.40}$$

The $O(1)$-term of the numerator is given by

$$\int \left[3u_0 \dot{u}_0^2 \ddot{u}_0 + \dot{u}_0^4\right] = \int \left[u_0 \frac{d}{dt} \dot{u}_0^3 + \dot{u}_0^4\right] = 0.$$

Hence, $B^{\epsilon}_3(\alpha = 0; \delta = 0) = -K_2 \epsilon^3 + O(\epsilon^6)$ for some constant K_2 which is analyzed next.

Step II - Representation of K_2. Note that the denominator in (4.40) is always positive for small ϵ. We will write the numerator of (4.40) as

$$B_{3,3}\epsilon^3 \int \dot{u}_0^2 = P + Q + O(\epsilon^6) \tag{4.41}$$

where

$$P := -\epsilon^3 \int \widetilde{a} \left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^2 (\ddot{u}_0 + \epsilon^3 g), \tag{4.42}$$

$$Q := \int 3\left(u_0 + \epsilon^3 \widetilde{\psi}\right)\left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^2 (\ddot{u}_0 + \epsilon^3 g) + \int \left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^4. \tag{4.43}$$

Note the parameter δ in functions $\widetilde{a}, \widetilde{\psi}, \widetilde{\phi}_1, g$ (in (4.24)) is evaluated at $\delta = 0$ starting from this point. We claim that Q can be transformed into the following form:

$$\int 3\left(u_0 + \epsilon^3 \widetilde{\psi}\right)\left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^2 \left[\epsilon^3 g_3 - \epsilon^3 \left(\widetilde{\phi}_1\right)_t\right] + \int \left(\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1\right)^3 \epsilon^3 E(t; \widetilde{\beta}_*), \tag{4.44}$$

where the function $E(t; \widetilde{\beta}_*)$ (which is even) uniquely solves

$$L_0(E) = -(1 - u_0^2)\, \widetilde{a}_t \quad \text{with} \int E(t; \widetilde{\beta}_*) \dot{u}_0 = 0. \tag{4.45}$$

To prove the claim (4.44), first note that

$$\widetilde{\phi}_1 = \widetilde{\psi}_t + E(t; \widetilde{\beta}_*), \text{ at } \delta = 0. \tag{4.46}$$

The above is obtained by taking derivative with respect to t of equation (4.2)

$$L_0\widetilde{\psi}_t = 6u_0\dot{u}_0\widetilde{\psi} - 2u_0\dot{u}_0\widetilde{a} + (1-u_0^2)\widetilde{a}_t = -\widetilde{q}_1\dot{u}_0 + (1-u_0^2)\widetilde{a}_t$$

and then comparing (4.20) for $\widetilde{\phi}_1$ at $\delta = 0$, i.e., (noting $\widetilde{\lambda}_1(\delta = 0) = 0$ in Proposition 4.1)

$$L_0(\widetilde{\phi}_1) = -\widetilde{q}_1\dot{u}_0 + \widetilde{\lambda}_1(\delta = 0)\dot{u}_0 = -\widetilde{q}_1\dot{u}_0.$$

With the introduction of the function E, we have,

$$\begin{aligned}
& \int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^4 \\
= & \int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^3 \left(\dot{u}_0 + \epsilon^3\widetilde{\psi}_t + \epsilon^3 E(t;\widetilde{\beta}_*)\right) \\
= & \int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^3 \left(\dot{u}_0 + \epsilon^3\widetilde{\psi}_t\right) + \epsilon^3 \int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^3 E(t;\widetilde{\beta}_*) \\
= & -3\int \left(u_0 + \epsilon^3\widetilde{\psi}\right)\left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^2\left(\ddot{u}_0 + \epsilon^3(\widetilde{\phi}_1)_t\right) + \epsilon^3\int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^3 E(t;\widetilde{\beta}_*),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\int 3\left(u_0 + \epsilon^3\widetilde{\psi}\right)\left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^2\left[\ddot{u}_0 + \epsilon^3(\widetilde{\phi}_1)_t\right] + \int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^4 \\
= \epsilon^3\int (\dot{u}_0 + \epsilon^3\widetilde{\phi}_1)^3 E(t;\widetilde{\beta}_*).
\end{aligned}$$

Now,

$$\begin{aligned}
Q = & \int 3\left(u_0 + \epsilon^3\widetilde{\psi}_t\right)\left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^2(\ddot{u}_0 + \epsilon^3 g) + \int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^4 \\
= & \int 3\left(u_0 + \epsilon^3\widetilde{\psi}_t\right)\left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^2\left(\epsilon^3 g - \epsilon^3(\widetilde{\phi}_1)_t\right) \\
& + \int 3\left(u_0 + \epsilon^3\widetilde{\psi}_t\right)\left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^2\left(\ddot{u}_0 + \epsilon^3(\widetilde{\phi}_1)_t\right) + \int \left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^4 \\
= & \int 3\left(u_0 + \epsilon^3\widetilde{\psi}_t\right)\left(\dot{u}_0 + \epsilon^3\widetilde{\phi}_1\right)^2\left(\epsilon^3 g - \epsilon^3(\widetilde{\phi}_1)_t\right) + \epsilon^3\int (\dot{u}_0 + \epsilon^3\widetilde{\phi}_1)^3 E(t;\widetilde{\beta}_*),
\end{aligned}$$

i.e., (4.44).

As a final simplification, since only $O(\epsilon^3)$-terms matter in the expression of P and Q, they can be further reduced to (recalling the function g in (4.24))

$$P = -\epsilon^3\int \widetilde{a}(t;\widetilde{\beta}_*)\dot{u}_0^2\ddot{u}_0, \tag{4.47}$$

$$Q = \epsilon^3\int 3u_0\dot{u}_0^2\left[g - \left(\widetilde{\phi}_1\right)_t\right] + \epsilon^3\int \dot{u}_0^3 E(t;\widetilde{\beta}_*). \tag{4.48}$$

Step III - Analysis of P and Q. For P, written as $\epsilon^3\widetilde{P}$, we have

$$\widetilde{P} = -\int \widetilde{a}(t;\widetilde{\beta}_*)\left(\frac{1}{3}\dot{u}_0^3(t)\right)_t = \frac{1}{3}\int \dot{u}_0^3\,\widetilde{a}_t\,dt. \tag{4.49}$$

For Q, we write it as

$$Q = \epsilon^3\int 3u_0\dot{u}_0^2\left[g - \left(\widetilde{\phi}_1\right)_t\right] + \epsilon^3\int \dot{u}_0^3 E(t;\widetilde{\beta}_*) := \epsilon^3\widetilde{Q}_1 + \epsilon^3\widetilde{Q}_2. \tag{4.50}$$

To analyze $\widetilde{Q}_1$, we make use of the fact that $L_0(\ddot{u}_0) = 6u_0\dot{u}_0^2$. Then,

$$\widetilde{Q}_1 = \frac{1}{2}\int L_0(\ddot{u}_0)\left[g(t;\widetilde{\beta}_*) - \left(\widetilde{\phi}_1\right)_t\right] = \frac{1}{2}\int \ddot{u}_0 L_0\left[g(t;\widetilde{\beta}_*) - \left(\widetilde{\phi}_1\right)_t\right]. \tag{4.51}$$

To continue, we need a formula for $L_0\left[g(t;\widetilde{\beta}_*) - \left(\widetilde{\phi}_1\right)_t\right]$. Taking derivative of equation (4.20) with respect to t at $\delta = 0$ gives

$$L_0(\widetilde{\phi}_1)_t \;=\; -\widetilde{q}_1\ddot{u}_0 - (\widetilde{q}_1)_t\,\dot{u}_0 + 6u_0\dot{u}_0\widetilde{\phi}_1.$$

Comparing the above with (4.24) for g leads to (using $(\widetilde{q}_1)_s = -(6\widetilde{\psi}_t - 2\widetilde{a}_t)u_0 - (6\widetilde{\psi} - 2\widetilde{a})\dot{u}_0$)

$$L_0\left(g - (\widetilde{\phi}_1)_t\right) \;=\; 6u_0\dot{u}_0\widetilde{\phi}_1 - \left(6\widetilde{\psi}_t - 2\widetilde{a}_t\right)u_0\dot{u}_0 \;=\; u_0\dot{u}_0\left\{6E(t;\widetilde{\beta}_*) + 2\widetilde{a}_t\right\}$$

so that

$$\widetilde{Q}_1 = \frac{1}{2}\int u_0\dot{u}_0\ddot{u}_0\left\{6E(t;\widetilde{\beta}_*) + 2\widetilde{a}_t\right\}. \tag{4.52}$$

For $\widetilde{Q}_2$, we compute

$$\int \dot{u}_0^3 E(t;\widetilde{\beta}_*) \;=\; -\int 2u_0\dot{u}_0\ddot{u}_0 E - \int u_0\dot{u}_0^2 E_t = -\int 2u_0\dot{u}_0\ddot{u}_0 E - \frac{1}{6}\int L(\ddot{u}_0)E_t$$

Differentiating the equation for E in (4.45), with respect to t gives

$$L_0(E_t) = 6u_0\dot{u}_0 E + 2u_0\dot{u}_0\widetilde{a}_t + (1-u_0^2)\widetilde{a}_{tt}.$$

Hence,

$$\begin{aligned}
\widetilde{Q}_2 \;&=\; \int \dot{u}_0^3 E\,dt = -2\int u_0\dot{u}_0\ddot{u}_0 E - \frac{1}{6}\int \ddot{u}_0\left[6u_0\dot{u}_0 E + 2u_0\dot{u}_0\widetilde{a}_t + (1-u_0^2)\widetilde{a}_{tt}\right] \\
&=\; -3\int u_0\dot{u}_0\ddot{u}_0 E - \frac{1}{3}\int u_0\dot{u}_0\ddot{u}_0\widetilde{a}_t + \frac{1}{6}\int \widetilde{a}_{tt}\ddot{u}_0(1-u_0^2).
\end{aligned}$$

Combining the computations of $\widetilde{Q}_1$ and $\widetilde{Q}_2$, we have

$$\widetilde{Q}_1 + \widetilde{Q}_2 \;=\; \frac{2}{3}\int u_0\dot{u}_0\ddot{u}_0\widetilde{a}_t + \frac{1}{6}\int \widetilde{a}_{tt}\ddot{u}_0(1-u_0^2). \tag{4.53}$$

Step IV - Positivity of $B_{3,3}$. Putting (4.53) and (4.49) together, then

$$\widetilde{P} + \widetilde{Q}_1 + \widetilde{Q}_2 \;=\; \frac{1}{3}\int \dot{u}_0^3\,\widetilde{a}_t + \frac{2}{3}\int u_0\dot{u}_0\ddot{u}_0\widetilde{a}_t + \frac{1}{6}\int \widetilde{a}_{tt}\ddot{u}_0(1-u_0^2). \tag{4.54}$$

Clearly the non-degeneracy of $B_{3,3} := \widetilde{P} + \widetilde{Q}_1 + \widetilde{Q}_2$ relies on the spatial inhomogeneity $\widetilde{a}$. We verify that it is positive. Note the following computation,

$$\frac{2}{3}\int u_0\dot{u}_0\ddot{u}_0\widetilde{a}_t = -\frac{2}{3}\int \dot{u}_0\left(\dot{u}_0^2\widetilde{a}_t + u_0\ddot{u}_0\widetilde{a}_t + u_0\dot{u}_0\widetilde{a}_{tt}\right).$$

It leads to

$$\frac{2}{3}\int u_0\dot{u}_0\ddot{u}_0\widetilde{a}_t = -\frac{1}{3}\int \dot{u}_0^3\widetilde{a}_t - \frac{1}{3}\int u_0\dot{u}_0^2\widetilde{a}_{tt}.$$

Substituting it into (4.54),

$$\begin{aligned}
B_{3,3} =& \widetilde{P} + \widetilde{Q}_1 + \widetilde{Q}_2 \\
=& -\frac{1}{3}\int u_0 \dot{u}_0^2 \widetilde{a}_{tt} + \frac{1}{6}\int \widetilde{a}_{tt}\ddot{u}_0(1-u_0^2) \\
=& \frac{\sqrt{2}}{3}\int \dot{u}_0\ddot{u}_0\widetilde{a}_{tt} \quad (\text{we use } \ddot{u}_0 = -u_0(1-u_0^2) = -\sqrt{2}u_0\dot{u}_0, \text{ as } (1-u_0^2) = \sqrt{2}\dot{u}_0) \\
=& \frac{\sqrt{2}}{3}\int \left(\frac{\dot{u}_0^2}{2}\right)_t \widetilde{a}_{tt} = -\frac{\sqrt{2}}{6}\int \dot{u}_0^2 \widetilde{a}_{ttt} > 0
\end{aligned}$$

as $\widetilde{a}_{ttt} = -3 < 0$. We have proved (4.38).

It remains to show that $\left|B_{3,1}^{\epsilon}(\overline{\delta})\right| \le O(1;\overline{\delta})\epsilon^3$, where

$$\begin{aligned}
B_{3,1}^{\epsilon}(\delta) &= \frac{1}{6}\langle \partial_\delta F(v)_{\alpha^{(3)}} + \phi_\delta^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle \langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle^{-1} \\
&- \frac{1}{3}\langle F(v)_{\alpha^{(3)}}, \phi^{\epsilon;\delta}\rangle \langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle^{-2} \langle \phi_\delta^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle^{-1}.
\end{aligned}$$

Recall $F_{\alpha^{(3)}}$ in (3.21), it can be shown that at $\alpha = 0$ and for any small δ, then

$$\|\partial_\delta \left(F(v)_{\alpha^{(3)}}\right)\| \le O(\epsilon^3)$$

where we have used Propositions 4.3 and 4.2 in the estimates of $\|v, v_\alpha, v_\delta\|$, etc. In addition, we use $\|\phi_\delta^{\epsilon;\delta}\| \le O(\epsilon^3)$ in Proposition 4.1. Putting them together into the above expression of $B_{3,1}^{\epsilon}(\delta)$, therefore, $\left|B_{3,1}^{\epsilon}(\delta)\right| \le O(1;\delta)\epsilon^3$ as stated. □

The treatment for $B_5^{\epsilon}(\alpha;\delta)$ is slightly different where α is not necessarily zero. We give a uniform estimate on $B_5^{\epsilon}(\alpha;\delta)$, independent of small δ in the following sense.

Lemma 4.7 ($B_5^{\epsilon}(\alpha;\delta)$)**.** *We have* $|B_5^{\epsilon}(\alpha;\delta)| \le O(\epsilon + |\alpha|)$.

Proof. Note that it suffices to prove $|B_5^{\epsilon}(\alpha = 0;\delta)| \le O(\epsilon)$ as $B_5^{\epsilon}(\alpha;\delta)$ depends on α continuously and $\|v_\alpha(\alpha;\delta)\| \le O(|\alpha| + \epsilon^6)$ by (4.22).

Recall at $\alpha = 0$, $\|v(\alpha = 0, \delta; \epsilon)(t), v_\alpha(\alpha = 0, \delta; \epsilon)(t)\| \le O(\epsilon^6)$ in (4.22). Thus from (3.23), we arrive at a simplified formula for $B_5^{\epsilon}(\alpha = 0;\delta)$.

$$\begin{aligned}
5!\, B_5^{\epsilon}(\alpha = 0;\delta) &= \frac{\left\langle \left[-3U^{\epsilon;\delta} + a(\epsilon t; \widetilde{\beta}_* + \delta)\right]\left\{20 v_{\alpha\alpha} v_{\alpha^3} + 10\phi^{\epsilon;\delta} v_{\alpha^4}\right\}, \phi^{\epsilon;\delta}\right\rangle}{\langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle} \\
&\quad - \frac{\left\langle \left[90 v_{\alpha\alpha}^2 \phi^{\epsilon;\delta} + 60 v_{\alpha^3} (\phi^{\epsilon;\delta})^2\right], \phi^{\epsilon;\delta}\right\rangle}{\langle \phi^{\epsilon;\delta}, \phi^{\epsilon;\delta}\rangle} + O(\epsilon^6).
\end{aligned}$$

Next we recall the asymptotic forms (4.23), (4.25) and (4.26) for $v_{\alpha^{(n)}}(\alpha = 0;\delta)$ with $n = 2, 3, 4$.

Now collecting the $O(1)$-term in $B_5^{\epsilon}(\alpha = 0;\delta)$, it gives

$$5!\, B_{5,0} := \frac{\langle -60 u_0 g_2 g_3 - 30 u_0 \dot{u}_0 g_4, \dot{u}_0\rangle}{\langle \dot{u}_0, \dot{u}_0\rangle} + \frac{\langle -90\dot{u}_0 g_2^2 - 60\dot{u}_0^2 g_3, \dot{u}_0\rangle}{\langle \dot{u}_0, \dot{u}_0\rangle}.$$

Recall $g_2 = \ddot{u}_0$, then

$$5!\, B_{5,0}\int \dot{u}_0^2 = \int \left[-60 u_0 \dot{u}_0 \ddot{u}_0 g_3 - 30 u_0 \dot{u}_0^2 g_4 - 90\dot{u}_0^2\ddot{u}_0^2 - 60\dot{u}_0^3 g_3\right] dt. \qquad (4.55)$$

The proof of Lemma 4.7 is concluded if $B_{5,0} \equiv 0$. From (4.25) and (4.26), we observe that

$$g_3 = \dddot{u}_0 + \frac{2}{5}\dot{u}_0 \text{ and } g_4 = (g_3)_s + \frac{6}{5}\ddot{u}_0.$$

First from $g_4 = (g_3)_s + \frac{6}{5}\ddot{u}_0$, using integration by part, then

$$\int u_0\dot{u}_0^2 g_4 = -\int g_3\left[2u_0\dot{u}_0\ddot{u}_0 + \dot{u}_0^3\right] + \frac{6}{5}\int u_0\dot{u}_0^2\ddot{u}_0.$$

Substituting it into (4.55), it is left to show

$$5!B_{5,0}\int \dot{u}_0^2 = -30\left(\int g_3\dot{u}_0^3 + \int 3\dot{u}_0^2\ddot{u}_0^2 + \frac{6}{5}\int u_0\dot{u}_0^2\ddot{u}_0\right) = 0. \tag{4.56}$$

Next, we use $g_3 = \dddot{u}_0 + \frac{2}{5}\dot{u}_0$, it follows that

$$\int g_3\dot{u}_0^3 = \int \dot{u}_0^3 d(\ddot{u}_0 + \frac{2}{5}u_0) = -\int 3\dot{u}_0^2\ddot{u}_0^2 + \frac{6}{5}u_0\dot{u}_0^2\ddot{u}_0$$

Substituting it into (4.56), we then have

$$5!B_{5,0}\int \dot{u}_0^2 = 0,$$

which concludes our proof. □

4.3. **An alternative approach.** In general, for a bifurcation problem $g(x;\lambda) = 0$, it is often assumed that there is a trivial solution for any bifurcation parameter λ. For example, $g(x;\lambda) = -x^3 + \gamma x = 0$ always has a trivial solution $x = 0$. The same situation applies to our **Problem [G]**, when the spatial inhomogeneity is of pitchfork-type. i.e., $a(x;\beta) = -\frac{1}{2}x^3 + \epsilon^2\left(\widetilde{\beta}_* + \delta\right)x$. That is, there always exists a primary branch of transition layer u^ϵ (its existence is independent of the value of the bifurcation parameter), located at the middle (i.e., $u^\epsilon(x=0) = 0$).

Note that the exact transition layer solution $u_{\text{exact}}^{\epsilon;\delta_*^\epsilon+\tau}$ obtained in Theorem 3.11 with $(\alpha \equiv 0)$ is odd in the x-variable. In fact, for any value of δ, there is always a primary branch middle transition layer which takes the form:

$$u_{\text{exact}}^1 := U^{\epsilon;\delta}(x) + v(\alpha = 0, \delta; \epsilon)(x)$$

where $U^{\epsilon;\delta}(x)$ is given in (4.6). The function $v(\alpha = 0, \delta; \epsilon)(x)$ is odd in x and it solves equation (2.13) with $\alpha = 0$. The existence of $v(\alpha = 0, \delta; \epsilon)(x)$ is given in Lemma 2.4. Its oddness in x infers from the uniqueness and an application of reflection principle. Moreover, $\left\|u_{\text{exact}}^1 - U^\epsilon(x;\delta) = v(\alpha = 0, \delta; \epsilon)(x)\right\|_0 \leq O(\epsilon^6)$. As a notable instantiation, middle layer exists for the degenerate spatial inhomogeneity function $a(x) = -\frac{x^3}{2}$ (as $a_x(x_* = 0) = 0$).

In addition, the associated reduced bifurcation equation $B_e^\epsilon(\alpha;\delta) = 0$ (where the linearization of problem (2.2) is around u_{exact}^1) is odd in α. This follows from the symmetry properties of $v(\alpha = 0, \delta; \epsilon)$ and the principal eigenfunction $\phi^\epsilon(x;\delta)$, together with the odd symmetry of the approximate solution $U^{\epsilon;\delta}(x;\delta)$ and the perfect pitchfork spatial inhomogeneity $a(x;\beta)$.

Using the above exact middle transition layer, one can define a linearization of problem (2.2) around $u^1_{\text{exact}} = u^{\epsilon;\delta}_{\text{exact}}$ as follows.

$$L^{\epsilon;\delta}_e V(t) := V_{tt} + \left[1 - 3\left(u^{\epsilon;\delta}_{\text{exact}}\right)^2 + 2a(\epsilon t; \widetilde{\beta}_* + \delta)u^{\epsilon;\delta}_{\text{exact}}\right] V(t).$$

Let $\lambda^\epsilon_{1;e}, \phi^{\epsilon;\delta}_e$ be the associated principal eigenvalue and eigenfunction. Then there exists a δ^ϵ_* such that the only critical eigenvalue $\lambda^\epsilon_{1;e}(\delta)$ becomes zero at $\delta = \delta^\epsilon_*$, i.e., $\lambda^\epsilon_{1;e}(\delta^\epsilon_*) = 0$. Moreover, $|\delta^\epsilon_*| \leq O(\epsilon^3)$. From there, we can get a similar reduced bifurcation equation $B^\epsilon_e(\alpha;\tau)$ around $u^{\epsilon;\delta}_{\text{exact}}$ where $\tau = \delta - \delta^\epsilon_*$, except that $B^\epsilon_{1;e}(\alpha = 0; \tau = 0) \equiv 0$, i.e., the reduced bifurcation equation is the same as (3.54). In the bulk of this paper, we choose to use $U^{\epsilon;\delta}$ as sometimes it is not obvious or easy to have the primary branch solution readily at hand. But for the next section, we do take advantage of $u^{\epsilon;\delta}_{\text{exact}}$ as it simplifies some formulas.

5. **Imperfect pitchfork bifurcation.** In [8], we have considered Problem [**E**] (described in page 899) and analyzed the source of *imperfection* in the bifurcation diagram for the pitchfork bifurcation with higher-order perturbations of the form $\gamma\dfrac{x^n}{n}$ where $n \geq 0$ is an integer. By imperfection, we mean that the primary branch of layers is not connected to the other bifurcating branches of solutions. It is known there that adding even-order term $\gamma\dfrac{x^n}{n}$ where $n = 2k, \gamma \neq 0$ will lead to imperfection, while adding odd-terms, the perfection is preserved due to the odd symmetry of the spatial inhomogeneity. See [8] for a more comprehensive discussion of the relation between singular perturbation and bifurcation.

For concreteness, we consider $n = 4$, i.e., perturbed spatial pitchfork bifurcation of the form

$$a(x, \beta; \gamma) = -\frac{1}{2}x^3 + \beta x + \gamma x^4, \quad |\gamma| \ll 1. \tag{5.1}$$

In order to concentrate on the key issues, we will only explain how to modify Proposition 3.9 for the reduced bifurcation equation and skip the revised versions for Propositions 3.6 and 3.12 which can be done similarly. The adjustment for other values of n is similar and will be remarked at the end.

We start by using the exact middle transition layer solution $u^{\epsilon;\delta^\epsilon_*+\tau}_{\text{exact}}$ (obtained in Theorem 3.11 with $\alpha \equiv 0$ and also discussed in Section 4.3) as an approximate solution. We then construct an exact transition layer for (5.1) as a perturbation. Precisely, we construct solutions using the ansatz

$$u^{\epsilon;\tau}_{\text{exact}} + \alpha\phi^{\epsilon;\tau,\gamma} + v(\alpha;\epsilon;\tau,\gamma)(x).$$

Noting that when $\gamma = 0$, then both $\alpha = 0$ and $v = 0$. In principle, one could replace $u^{\epsilon;\tau}_{\text{exact}}$ above by $U^{\epsilon;\delta^\epsilon_*+\tau}(x)$ in (4.6), leading to nonzero small function $v(\gamma = 0)$ then. It is of advantage to use the exact middle layer in this section, as it is easier to control the microscopic term v, in the sense that all the estimates (cf. (5.9)) are controlled by γ.

In this section, we denote the reduced bifurcation equation for the unknown scalar α as

$$B^\epsilon(\alpha;\tau,\gamma) := \sum_{n=0}^{3} B^\epsilon_n(\alpha = 0;\tau,\gamma)\,\alpha^n + B^\epsilon_4(\overline{\alpha};\tau,\gamma)\,\alpha^4 = 0. \tag{5.2}$$

The derivations of $B_i^\epsilon(\alpha = 0; \tau, \gamma)$-terms are similar to those in (3.54), where τ is the bifurcation parameter and the bifurcation value is $\tau = 0$. In current section, we like to study the changes to the bifurcation (both bifurcation diagram and bifurcation value $\tau_*(\gamma)$) of (5.2) due to the additional perturbation parameter γ. We will see in Proposition 5.2 that, the resulting non-vanishing of $B_0^\epsilon(\tau; \gamma \neq 0)$ leads to *imperfection* in bifurcation diagram of (5.2). However, the ratio of $B_2^\epsilon(\tau; \gamma)$ and $B_0^\epsilon(\tau; \gamma)$ indicates that there is no *hysteresis* in the primary branch. See also [8] for a similar discussion of Problem **[E]**.

Similar to the perfect pitchfork case, the key is to study the linearized operator (denoted as $L^{\epsilon;\tau,\gamma}$) at $u^{\epsilon;\tau}_{\text{exact}}$.

$$L^{\epsilon;\tau,\gamma}[V] := V_{tt} + \left[1 - 3\left(u^{\epsilon;\tau}_{\text{exact}}\right)^2 + \left(2a(\epsilon t; \widetilde{\beta}_* + \delta^\epsilon_* + \tau) + 2\epsilon^4 t^4 \gamma\right) u^{\epsilon;\tau}_{\text{exact}}\right] V. \tag{5.3}$$

In the above, recall that $\epsilon^2(\widetilde{\beta}_* + \delta^\epsilon_*)$ ($|\delta^\epsilon_*| \leq O(\epsilon^3)$) is the bifurcation point for unperturbed problem (corresponding to $\tau = 0$). Now let $\lambda_1^{\epsilon;\gamma}(\tau)$ and $\phi^{\epsilon;\tau,\gamma}$ be the associated principal eigenvalue and eigenfunction pair which satisfy:

$$L^{\epsilon;\tau,\gamma}\phi^{\epsilon;\tau,\gamma} = \lambda_1^{\epsilon;\gamma}(\tau)\,\phi^{\epsilon;\tau,\gamma}, \qquad \int \phi^{\epsilon;\tau,\gamma}\dot{u}_0 = \int \dot{u}_0^2. \tag{5.4}$$

The next is the key result. See Proposition 4.1 for a comparison.

Lemma 5.1. *In the above equation (5.4), for $|t| \leq \dfrac{1}{4\epsilon}$, the principal eigenfunction has the following form*

$$\phi^{\epsilon;\tau,\gamma} \;=\; \dot{u}_0 + \epsilon^3\widetilde{\phi}_1^\tau + \epsilon^4\gamma\,\widetilde{\phi}_\gamma + O(\epsilon^5), \tag{5.5}$$

where the functions $\widetilde{\phi}_1^\tau$, $\widetilde{\phi}_\gamma$ are orthogonal to $\dot{u}_0$ and they satisfy

$$L_0\left(\widetilde{\phi}_1^\tau\right) = 6u_0 u_1^\tau \dot{u}_0 + \left(t^3 - 2(\widetilde{\beta}_* + \tau)t\right) u_0\dot{u}_0, \quad \textit{and} \quad L_0\left(\widetilde{\phi}_\gamma\right) = -2t^4 u_0.$$

In the above, u_1^τ is the solution to

$$L_0\left(u_1^\tau\right) = [1 - u_0^2]\left(-\frac{1}{2}t^3 + (\widetilde{\beta}_* + \tau)t\right), \qquad \int u_1^\tau \dot{u}_0 = 0.$$

Moreover, the principal eigenvalue $\lambda_1^{\epsilon;\gamma}(\tau)$ satisfies

$$\lambda_1^{\epsilon;\gamma}(\tau) = \gamma^2\left[\epsilon^8\frac{\int t^4 u_0\dot{u}_0\,\widetilde{\phi}_\gamma}{\int \dot{u}_0^2} + O(\epsilon^9)\right] + \left[K_1\epsilon^3 + O(\epsilon^4)\right]\tau + \left[O(\gamma\epsilon^4) + O(\epsilon^6)\right]\tau^2. \tag{5.6}$$

Proof. The proof of asymptotic expansion (5.5) mimics that of (4.19) in Proposition 4.1. Noting that $\widetilde{\phi}_1^\tau$ in (5.5) is the same as $\widetilde{\phi}_1(\delta)$ in (4.20) as $|\delta^\epsilon_*| \leq O(\epsilon^3)$ in (5.4). The additional $\epsilon^4\gamma\widetilde{\phi}_\gamma$-term in (5.5) is due to the perturbed term γx^4. It is obtained by equating the terms at the order of ϵ^4 in equation (5.4).

Now we generalize the form (4.18) (at $\gamma = 0$) to nonzero γ in (5.6). Note

$$\lambda_1^{\epsilon;\gamma}(\tau)\int_{-\frac{1}{4\epsilon}}^{\frac{1}{4\epsilon}} \dot{u}_0\,\phi^{\epsilon;\tau,\gamma} \quad = \quad \int_{-\frac{1}{4\epsilon}}^{\frac{1}{4\epsilon}} [V^{\tau;\gamma}]\dot{u}_0\,\phi^{\epsilon;\tau,\gamma}$$

where

$$V^{\tau;\gamma} = \left[1 - \left(u^{\epsilon;\tau}_{\text{exact}}\right)^2 + 2a(\epsilon t; \widetilde{\beta}_* + \delta^\epsilon_* + \tau)u^{\epsilon;\tau}_{\text{exact}}\right] - \left[1 - 3u_0^2\right] + 2\gamma\epsilon^4 t^4 u^{\epsilon;\tau}_{\text{exact}}.$$

First we analyze $\lambda_1^{\epsilon;\gamma}(\tau)$ at $\tau = 0$. Note that $\lambda_1^{\epsilon;\gamma=0}(\tau = 0) = 0$. This is because at the bifurcation point $\tau = 0$ for the unperturbed spatial term, it holds that

$$\int_{-\frac{1}{4\epsilon}}^{\frac{1}{4\epsilon}} [V^{\tau=0;\gamma=0}]\dot{u}_0\, \phi^{\epsilon;\tau=0,\gamma=0} \equiv 0.$$

For nonzero γ, from the spatial inhomogeneity term

$$2\left(a\left(\epsilon t; \widetilde{\beta}_* + \delta_*^\epsilon\right) + \gamma\epsilon^4 t^4\right)\left(u_{\text{exact}}^{\epsilon;\tau=0}\right)\dot{u}_0 \phi^{\epsilon;\tau=0,\gamma}$$

and using (5.5), we identify a major term $\gamma^2\, \epsilon^8 \dfrac{\int t^4 u_0 \dot{u}_0\, \widetilde{\phi}_\gamma}{\int \dot{u}_0^2}$ in $\lambda_1^{\epsilon;\gamma}(\tau = 0)$. The leading factor γ^2 arises as the perturbed spatial inhomogeneity is linear in γ and $\phi^{\epsilon;\tau,\gamma}$ is at least linear in γ. We have also used the fact that $\dot{u}_0 + \epsilon^3 \widetilde{\phi}_1^\tau$ in (5.5) is even in t, and $u_{\text{exact}}^{\epsilon;\tau}$ is odd in t here.

Furthermore, we have that $\|u_{\text{exact}} - U^\epsilon(t;\tau) = v(\alpha = 0, \tau; \epsilon)(t)\|_0 \leq O(\epsilon^6)$ for the middle layer u_{exact}. Hence in order to find $\dfrac{d\lambda_1^{\epsilon;\gamma}(\tau)}{d\tau}$, it is sufficient to replace u_{exact} by $U^\epsilon(t;\tau)$. Recalling Proposition 4.1 with $\delta = \delta_*^\epsilon + \tau$ and noting that $\epsilon^4 t^4\gamma$-perturbation will not affect the rate of change of the $\lambda_1^{\epsilon;\gamma}(\tau)$ with respect to τ at the order of ϵ^3, we have

$$\frac{d\lambda_1^{\epsilon;\gamma}(\tau)}{d\tau} = K_1\epsilon^3 + O(\epsilon^4).$$

The $\epsilon^4 t^4\gamma$-perturbation brings at most a perturbation of $\gamma O(1; \epsilon, \overline{\tau})\epsilon^4$ to the rate $\dfrac{d\lambda_1^{\epsilon;\gamma}(\tau)}{d\tau}$, and that variation is absorbed into $O(\epsilon^4)$ in above. Similar to Proposition 4.1, one obtains $\left|\dfrac{d^2\lambda_1^{\epsilon;\gamma}(\tau)}{d\tau^2}\right| \leq \left[O(\gamma\epsilon^4) + O(\epsilon^6)\right]$, as stated in (5.6). $\square$

Now we can derive the coefficients $B_i^\epsilon(\alpha = 0; \tau, \gamma), i = 0, \cdots, 4$ in equation (5.2).

Proposition 5.2. *For $|\alpha, \tau| \ll 1$, then the coefficients of equation (5.2) read*

$$\begin{aligned}
B_0^\epsilon(\alpha = 0; \tau, \gamma) &= = -\frac{\int (1-u_0^2)t^4\dot{u}_0}{\int \dot{u}_0^2}\gamma\,\epsilon^4 + O(\epsilon^5|\gamma|);\\
B_1^\epsilon(\alpha = 0; \tau, \gamma) &= \gamma^2\left[B_{1,0}\epsilon^8 + O(\epsilon^9)\right] + \left[K_1\epsilon^3 + O(1;\epsilon,\gamma)\epsilon^4\right]\tau\\
&\quad + \left[O(1;\epsilon,\overline{\tau})\epsilon^6 + O(1;\epsilon,\overline{\tau})\epsilon^4\gamma\right]\tau^2.\\
|B_2^\epsilon(\alpha = 0; \tau, \gamma)| &\leq M|\gamma|\epsilon^4;\\
B_3^\epsilon(\alpha = 0; \tau, \gamma) &= \left[-B_{3,3}\epsilon^3 + \epsilon^4 O(1;\epsilon,\gamma)\right] + O(1;\gamma,\overline{\tau})\,\epsilon^3\,\tau;\\
|B_4^\epsilon(\alpha; \tau, \gamma)| &\leq O(1;\epsilon,\gamma,\tau)(|\gamma|\,\epsilon^4 + |\overline{\alpha}|).
\end{aligned}$$

Terms $O(1;,)$ are always bounded as the list of arguments approach zero. Variables $\overline{\tau}$ or $\overline{\alpha}$ (depending on ϵ, γ) lies between 0 and τ, or α respectively.

Proof. The current case ($\gamma \neq 0$) is similar to (3.54), i.e., $\gamma = 0$. Due to the addition of γx^4, the symmetry of the spatial inhomogeneity is lost. Consequently, terms $B_0^\epsilon(\alpha = 0)$ and $B_2^\epsilon(\alpha = 0)$ are not vanishing. To continue the derivations, we need estimates of the solution $v(\alpha, \tau; \epsilon; \gamma)$ to the following equation

$$L^{\epsilon;\tau,\gamma}[v(\alpha,\tau;\epsilon;\gamma)] + (I-E)_{\phi^{\epsilon;\tau,\gamma}}\, G^{\tau;\epsilon;\gamma}(u_{\text{exact}}^{\epsilon;\tau}) + (I-E)_{\phi^{\epsilon;\tau,\gamma}}\{F\} = 0,$$

where the error of the middle exact layer $u^{\epsilon;\tau}_{\text{exact}}$ is

$$G^{\epsilon;\tau,\gamma}(u^{\epsilon;\tau}_{\text{exact}}) = -\gamma\epsilon^4 t^4 \left[1 - \left(u^{\epsilon;\tau}_{\text{exact}}\right)^2\right] \tag{5.7}$$

and

$$F(\alpha\phi^{\epsilon;\tau,\gamma} + v) := \left[-3u^{\epsilon;\tau}_{\text{exact}} + a(\epsilon t;\tau;\gamma)\right](\alpha\phi^{\tau;\epsilon;\gamma} + v)^2 - (\alpha\phi^{\tau;\epsilon;\gamma} + v)^3 . \tag{5.8}$$

Similar to the proof of Proposition 3.5, it follows then

$$\| v, v_\alpha|_{\alpha=0,\tau;\epsilon;\gamma} \|_0 \le O(\epsilon^4|\gamma|). \tag{5.9}$$

Now we proceed to the analysis of the B^ϵ_i's.

B^ϵ_0-term. Recall the formula (3.24), using the estimate (5.9) and the explicit form of (5.7), then $\frac{\int (1-u_0^2)t^4 \dot{u}_0}{\int \dot{u}_0^2}\gamma\,\epsilon^4$ dominates $B^\epsilon_0\,(\alpha = 0)$. Hence the error is at most $O(\epsilon^5|\gamma|)$.

B^ϵ_2-term. In the unperturbed case (3.54), $B^\epsilon_2\,(\alpha = 0; \tau = 0) = 0$ due to the odd symmetry of the spatial inhomogeneity. With $\gamma \ne 0$, then it is obvious that in B^ϵ_2 (see its formula (3.26) with $n = 2$), any part depending on γ must be accompanied by a minimal factor of $\epsilon^4\gamma$. Therefore, $|B^\epsilon_2\,(\alpha = 0; \tau,\gamma)| \le M|\gamma|\epsilon^4$ for sufficiently large constant M.

B^ϵ_3-term. As γx^4 (after rescaling $x = \epsilon t$) is sufficiently small, the term $B^\epsilon_3\,(\alpha = 0; \tau,\gamma)$ remains unchanged for all the terms at order of $O(\epsilon^3)$. The change is at most of the form $\epsilon^4 O(1; \epsilon,\gamma)$.

B^ϵ_4-term. Note that $B^\epsilon_4\,(\alpha = 0; \tau,\gamma = 0) \equiv 0$ for any τ. After the rescaling $x = \epsilon t$, we can treat $B^\epsilon_4\,(\alpha = 0; \tau,\gamma)$ as $B^\epsilon_4\left(\alpha = 0; \tau,\gamma\epsilon^4\right)$. Then its Taylor expansion with respect to the two variables $\alpha, \gamma\epsilon^4$ gives an upper bound for B^ϵ_4 which is valid for all small α.

B^ϵ_1-term. It remains to compute $B^\epsilon_1\,(\alpha = 0; \tau,\gamma)$, for which we need Lemma 5.1. Recall (3.25), i.e.,

$$\begin{aligned} B^\epsilon_1\,(\alpha = 0; \tau,\gamma) :=& \lambda^{\epsilon;\gamma}_1(\tau) - \frac{\left\langle \left[3v^2(\phi^{\epsilon;\tau,\gamma} + v_\alpha)\right], \phi^{\epsilon;\tau,\gamma}\right\rangle}{\langle \phi^{\epsilon;\tau,\gamma}, \phi^{\epsilon;\tau,\gamma}\rangle} \\ &+ \frac{\left\langle \left[-3u^{\epsilon;\tau}_{\text{exact}} + a(\epsilon t; \widetilde{\beta}_* + \delta^\epsilon_* + \tau;\gamma)\right]\left[2v(\phi^{\tau;\epsilon;\gamma} + v_\alpha)\right], \phi^{\epsilon;\tau,\gamma}\right\rangle}{\langle \phi^{\epsilon;\tau,\gamma}, \phi^{\epsilon;\tau,\gamma}\rangle}. \end{aligned}$$

From estimates (5.9), it follows that $\lambda^{\epsilon;\gamma}_1(\tau)$ dominates $B^\epsilon_1\,(\alpha = 0; \tau = 0,\gamma)$. Next recalling (5.6), we obtain the expression of $B^\epsilon_1\,(\alpha = 0; \tau = 0,\gamma)$, which is in the first bracket of the expression for$B^\epsilon_1\,(\alpha = 0; \tau,\gamma)$-term. For the rate of change $\frac{d}{d\tau}B^\epsilon_1\,(\alpha = 0; \tau,\gamma)$, taking derivative of the above expression for $B^\epsilon_1\,(\alpha = 0; \tau,\gamma)$ with respect to τ, and recalling both $\| v, v_\alpha|_{\alpha=0,\tau;\epsilon;\gamma} \|_0 \le O(\epsilon^4|\gamma|)$ and (5.6), we have

$$\frac{d}{d\tau}B^\epsilon_1\,(\alpha = 0; \tau,\gamma) = \frac{d}{d\tau}B^\epsilon_1\,(\alpha = 0; \tau,\gamma = 0) + O(\epsilon^4|\gamma|) = K_1\epsilon^3 + O(\epsilon^4|\gamma|),$$

as the contribution due to the perturbed term $\gamma\epsilon^4 t^4$ is always no greater than $O(\epsilon^4)$, leaving the $K_1\epsilon^3$-term unchanged. □

The main result in this section is a direct consequence of the above Proposition. Before stating it precisely, recall that we say there is a *hysteresis* point if the primary branch of solutions has a kink, as in Fig 3(2) or 3(4) in [8]. Otherwise, we say it does not have hysteresis. For convenience, we write $a(\epsilon) \propto b(\epsilon)$ meaning that $\liminf_{\epsilon\downarrow 0} \frac{a(\epsilon)}{b(\epsilon)} > 0$.

Theorem 5.3 (Imperfect bifurcation)**.** *With the perturbed spatial inhomogeneity (5.1), the following statements hold for $0 < \epsilon, |\gamma| \ll 1$,*

1. *The exact bifurcation point $(\tau_*^\epsilon(\gamma), \alpha_*^\epsilon(\gamma))$ of (5.2) is*
$$\tau_*^\epsilon(\gamma) \propto \epsilon^{\frac{2}{3}}\gamma^{\frac{2}{3}}, \quad \alpha_*^\epsilon(\gamma) \propto -\epsilon^{\frac{1}{3}}\gamma^{\frac{1}{3}}.$$
2. *(5.2) gives imperfect pitchfork bifurcation without hysteresis point. That is, for $\gamma \neq 0$, the resulting pitchfork is plotted in Fig.4. It is consistent with the result obtained in [8] for Problem [**E**] .*

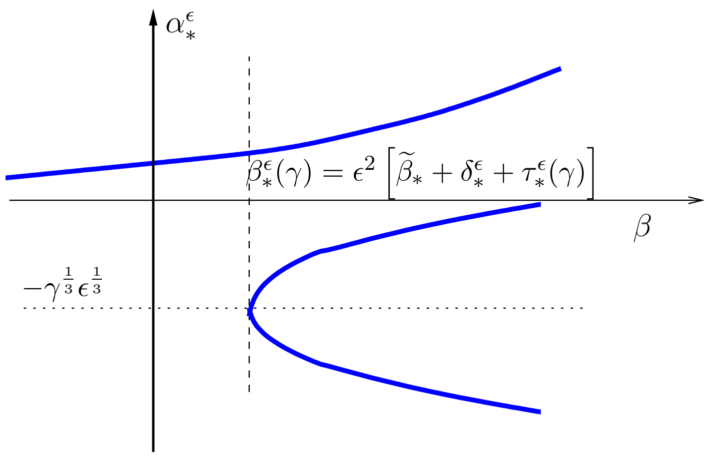


FIGURE 4. Imperfect Pitchfork Bifurcation due to even perturbation γx^4. Here the bifurcation diagram is depicted for $\gamma > 0$. In comparison, in Figure 1(b), it is depicted for negative perturbation parameter $c < 0$.

Proof. In (5.2), recall $B_0^\epsilon, B_2^\epsilon$ in Proposition 5.2. Then it holds that
$$|B_0^\epsilon\,(\alpha = 0; \tau, \gamma)| \gg |B_2^\epsilon\,(\alpha = 0, \tau, \gamma)|\,\alpha^2,$$
where α goes to zero as ϵ approaches zero (since the exact transition layers are considered as small perturbation of $u_{\text{exact}}^{\epsilon;\delta_*^\epsilon+\tau}$). In this sense, the quadratic term $B_2^\epsilon\,(\alpha = 0, \tau, \gamma)\,\alpha^2$ is always dominated by $B_0^\epsilon\,(\alpha = 0; \tau, \gamma)$ in equation (5.2). Thus to find the bifurcation point (denoted as $(\tau_*^\epsilon(\gamma), \alpha_*^\epsilon(\gamma))$), it suffices to consider
$$B^\epsilon(\alpha; \tau) = B_0^\epsilon + B_1^\epsilon\alpha + B_3^\epsilon\alpha^3 + B_4^\epsilon\alpha^4 \;= 0 \tag{5.10}$$
near $\alpha = 0, \tau = 0$ (ϵ, τ are parameters). Now $(\tau_*^\epsilon(\gamma), \alpha_*^\epsilon(\gamma))$ is a bifurcation point only if
$$B^\epsilon(\tau_*^\epsilon(\gamma), \alpha_*^\epsilon(\gamma)) = 0, \quad B_\alpha^\epsilon(\tau_*^\epsilon(\gamma), \alpha_*^\epsilon(\gamma)) = 0. \tag{5.11}$$
First, from the following equation
$$B_\alpha^\epsilon(\tau_*^\epsilon(\gamma), \alpha_*^\epsilon(\gamma)) = B_1^\epsilon + 3B_3^\epsilon(\alpha_*^\epsilon)^2 + 4B_4^\epsilon(\alpha_*^\epsilon)^3 = 0,$$
we thus look for bifurcation point satisfying
$$\tau_*^\epsilon(\gamma) \propto \left[\alpha_*^\epsilon(\gamma) + o(\alpha_*^\epsilon(\tau))\right]^2 \tag{5.12}$$
where $|o(\alpha_*^\epsilon(\tau))| \ll |\alpha_*^\epsilon(\gamma)|$. Substituting (5.12) into (5.10), it yields $\alpha_*^\epsilon(\gamma) \propto -\epsilon^{\frac{1}{3}}\gamma^{\frac{1}{3}}$ and hence $\tau_*^\epsilon(\gamma) \propto \epsilon^{\frac{2}{3}}\gamma^{\frac{2}{3}}$ in view of (5.12). To complete the existence

of $(\tau_*^\epsilon(\gamma), \alpha_*^\epsilon(\gamma))$, we proceed it similarly as in Theorem 3.11. More specifically, we consider

$$\alpha_*^\epsilon(\gamma) = -\widetilde{\alpha}_* \epsilon^{\frac{1}{3}} \gamma^{\frac{1}{3}}, \quad \tau_*^\epsilon(\gamma) = \widetilde{\tau}_* \epsilon^{\frac{2}{3}} \gamma^{\frac{2}{3}}. \tag{5.13}$$

Substituting them into equation (5.11), one then finds $\widetilde{\alpha}_*$ and $\widetilde{\tau}_*$ using the implicit function theorem.

Next, we show that (5.10) produces imperfect pitchfork-type bifurcation without hysteresis. We prove it for $\gamma > 0$. Recall from case **(B)** (page 901 and condition (1.9)) that the cubic equation

$$x^3 - \beta x + \alpha_1 + \alpha_2 x^2 = 0,$$

gives imperfect pitchfork bifurcation without hysteresis around $x = 0,\ \beta = 0$ if and only if

$$\alpha_1 > 0 \text{ and } \alpha_1 > \frac{\alpha_2^3}{27} \quad \text{ or } \quad \alpha_1 < 0 \text{ and } \alpha_1 < \frac{\alpha_2^3}{27}.$$

For convenience, we write equation (5.10) as

$$\alpha^3 - \frac{-B_1^\epsilon}{B_3^\epsilon}\alpha + \frac{-B_0^\epsilon}{B_3^\epsilon} + \frac{-B_2^\epsilon}{B_3^\epsilon}\alpha^2 = 0,$$

Further, we denote $\frac{-B_0^\epsilon}{B_3^\epsilon} = \alpha_1$, $\frac{-B_2^\epsilon}{B_3^\epsilon} = \alpha_2$. In case $\gamma > 0$, it follows that

$$\alpha_1 = \frac{-B_0^\epsilon}{B_3^\epsilon} < 0, \quad \alpha_1 = \frac{-B_0^\epsilon}{B_3^\epsilon} < \frac{1}{27}\left(\frac{-B_2^\epsilon}{B_3^\epsilon}\right)^3 = \frac{\alpha_2^3}{27}.$$

The second inequality holds as α_1 is strictly negative. This implies that (5.2) gives imperfect pitchfork bifurcation without hysteresis. ☐

Similar to the discussion for Problem [**E**] in [8], we can consider adding general perturbation $\gamma x^{2k}, k \geq 2$. As k increases, i.e. the perturbation term is weakened, the magnitude of $B_0^\epsilon(\alpha = 0; \tau, \gamma)$ becomes smaller, where $B_0^\epsilon(\alpha = 0; \tau, \gamma)$ measures the size of imperfection in the bifurcation. Moreover, it holds that $\left|B_2^\epsilon\left(\alpha = 0; \tau, \gamma x^{2k}\right)\right| \leq O(\epsilon^{2k})|\gamma|$. From the ratio, i.e., $\dfrac{B_0^\epsilon(\alpha = 0; \tau, \gamma)}{\left[B_2^\epsilon(\alpha = 0; \tau,)\right]^3}$ which measures the tendency for not having hysteresis, we thus verify that the resulting perturbed bifurcation has no hysteresis. This is the same as the study of Problem [**E**] in [8].

In conclusion, from the current work, we establish that bifurcation diagrams of transition layers to singularly perturbed equation (**RD**), restricted to certain type (regular or opposite), inherits *mostly* from that of the limiting interfacial problem. We also give an explicit example where subtle difference exists between them – the bifurcation of diffuse layers and that of the limiting interfaces behave differently in terms of (im)perfection. We emphasize that, due to the presence of singular perturbation, bifurcation point depends on ϵ and the form of spatial inhomogeneity in an interesting way. Moreover, bifurcation point may also depend on the spatial dimension. In higher dimensions, the curvature of the interface might come into play. This and other connections between bifurcation and singular perturbation will be deferred to future works.

Appendix A. **Asymptotic expansions for λ and ϕ^ϵ.** We first state a lemma which gives the exponential decay of solutions of $L_0 v = p$. The following version is from [9, Lemma 2.1] and [1, Lemma B.1].

Lemma A.1. *Consider the inhomogeneous equation*

$$L_0 v = p, \quad -\infty < s < \infty$$

and p satisfies $|p(s)| \le Ce^{-\gamma|s|}$ for some positive constants C and γ.

Then the above problem has a bounded solution on $\mathbb{R}$ if and only if

$$\int_{-\infty}^{\infty} p(s)\dot{u}_0(s)\, ds = 0.$$

Moreover, the solution satisfies,

$$\left| \frac{d^n}{ds^n} v(s) \right| \le C_n e^{-\gamma_n |s|}, \quad n = 0, 1, 2$$

for some positive constants C_n and γ_n.

The proof is omitted as it is elementary, essentially making use of the following explicit solution formula,

$$v(s) = A\dot{u}_0(s) + \dot{u}_0(s) \int_0^s \frac{1}{\dot{u}_0^2(r)} \int_{-\infty}^r p(\xi)\dot{u}_0(\xi)\, d\xi\, dr$$

where A is some constant.

Now we study the asymptotic expansion of the principal eigenvalue and eigenfunction for the eigenvalue problem (3.45).

Theorem A.2. *The principal eigenvalue $\lambda_1^\epsilon(\mu)$ of (3.45) satisfies*

$$\lambda_1^\epsilon(\mu) = \Lambda_1^\epsilon(\mu) + \Lambda_2^\epsilon(\mu) \tag{A.1}$$

where $|\Lambda_1^\epsilon(\mu)| \le O(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2)$ and $|\Lambda_2^\epsilon(\mu)| \le O(\epsilon^6 + \epsilon^2\mu^4)$.

The principal eigenfunction (3.45) has the following form

$$\phi^\epsilon(s;\mu) = \dot{u}_0(s) + \Phi_1^\epsilon(s;\mu) + \Phi_2^\epsilon(s;\mu) \tag{A.2}$$

with $\|\Phi_1^\epsilon(\cdot;\mu)\|_0 \le O(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2)$ and $\|\Phi_2^\epsilon(\cdot;\mu)\|_0 \le O(\epsilon^6 + \epsilon^2\mu^4)$. Both of them decay exponentially in s.

Specifically, $\Lambda_1^\epsilon(\mu)$ and $\Phi_1^\epsilon(\mu)$ are defined as

$$\Lambda_1^\epsilon(\mu) = \left(\int \dot{u}_0^2 \right)^{-1} \int p_1 \dot{u}_0^2, \quad \text{and} \quad L_0 \Phi_1^\epsilon + p_1 \dot{u}_0 = \Lambda_1^\epsilon(\mu)\dot{u}_0, \qquad \int \Phi_1^\epsilon \dot{u}_0\, ds = 0. \tag{A.3}$$

Proof. **Step I**. This step shows that $|\phi^\epsilon(s;\mu) - \dot{u}_0(s)| \le Ce^{-\gamma|s|}$ for some positive constants C and γ (which do not depend on ϵ). We first write,

$$\phi^\epsilon(s;\mu) = \dot{u}_0 + \phi_r^\epsilon(s;\mu), \quad \text{where} \quad \int \phi_r^\epsilon(s;\mu)\dot{u}_0(s)\, ds = 0. \tag{A.4}$$

Substituting the above into (3.45) gives

$$L_0\phi_r^\epsilon(s;\mu) + (p_1 + p_2)\left(\dot{u}_0 + \phi_r^\epsilon(s;\mu)\right) = \lambda_1^\epsilon(\mu)\left(\dot{u}_0 + \phi_r^\epsilon(s;\mu)\right), \tag{A.5}$$

where p_1, p_2 are from (3.50). By Lemma 2.2, we know apriori that $\dot{u}_0(s) + \phi_r^\epsilon(s;\mu)$ (and also $\dot{u}_0(s)$) decays exponentially. In particular, we have for some constants C and γ that

$$|\dot{u}_0(s) + \phi_r^\epsilon(s)| \le C\,|\dot{u}_0(0) + \phi_r^\epsilon(0)|\, e^{-\gamma|s|} \quad \text{and} \quad |\dot{u}_0(s)| \le C\,|\dot{u}_0(0)|\, e^{-\gamma|s|}$$

which lead to that

$$|\phi_r^\epsilon(s;\mu)| \le C\Big(1 + |\phi_r^\epsilon(0)|\Big)e^{-\gamma|s|}. \tag{A.6}$$

Now we define

$$f(s) := \lambda_1^\epsilon \dot{u}_0 + \lambda_1^\epsilon \phi_r^\epsilon - (p_1+p_2)\dot{u}_0 - (p_1+p_2)\phi_r^\epsilon.$$

Then we have

$$|f(s)| \le C\Big[|\lambda_1^\epsilon| + |p_1(s)| + |p_2(s)|\Big]\Big(1+|\phi_r^\epsilon(0)|\Big)e^{-\gamma|s|}.$$

Since

$$\|p_1 e^{-\frac{\gamma}{2}|s|}\|_0 \le O(\epsilon\mu^2+\epsilon^2\mu+\epsilon^3) \quad \text{and} \quad \|p_2\|_0 \le O(\epsilon^6+\epsilon^2\mu^4), \tag{A.7}$$

it follows then

$$|f(s)| \le C\left[|\lambda_1^\epsilon| + \left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right)\right](1+|\phi_r^\epsilon(0)|)e^{-\frac{\gamma}{2}|s|}.$$

Now applying Lemma A.1 to equation (A.5) leads to for another constant γ_2 that,

$$|\phi_r^\epsilon(s)| \le C\left[|\lambda_1^\epsilon| + \left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right)\right](1+|\phi_r^\epsilon(0)|)e^{-\gamma_2|s|}.$$

Setting $s=0$ into the above, we have

$$|\phi_r^\epsilon(0)| \le C\left[|\lambda_1^\epsilon| + \left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right)\right](1+|\phi_r^\epsilon(0)|).$$

By Lemma 2.2, $\lambda_1^\epsilon(\mu)$ is critical, Hence $\left[|\lambda_1^\epsilon| + \left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right)\right] \ll 1$. So we have

$$|\phi_r^\epsilon(0)| \quad \le \quad C\left[|\lambda_1^\epsilon| + \left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right)\right] \tag{A.8}$$

$$\text{and hence} \quad |\phi_r^\epsilon(s)| \quad \le \quad C\left[|\lambda_1^\epsilon| + \left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right)\right]e^{-\gamma_2|s|}. \tag{A.9}$$

Step II. In this step, we prove

$$|\lambda_1^\epsilon(\mu)| \le O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2) \quad \text{and} \quad \|\phi_r^\epsilon(s;\mu)\|_0 \le O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2). \tag{A.10}$$

This is achieved by multiplying (A.5) by $\dot{u}_0$ and integrating. Then we arrive at

$$\lambda_1^\epsilon(\mu) = \left(\int \dot{u}_0^2\right)^{-1}\int (p_1+p_2)\left(\dot{u}_0 + \phi_r^\epsilon(s;\mu)\right)\dot{u}_0.$$

By (A.7) and (A.9), we have

$$\left|\int (p_1+p_2)\left(\dot{u}_0 + \phi_r^\epsilon(s;\mu)\right)\dot{u}_0\right| \le O\left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right),$$

leading to

$$|\lambda_1^\epsilon(\mu)| \le O\left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right).$$

The above, together with (A.9), gives for some γ that

$$\|\phi_r^\epsilon(s;\mu)\|_0 \le \|C\left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right)\, e^{-\gamma|s|}\|_0 \le C\left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right) \tag{A.11}$$

which is the desired statement.

Step III. Here we will decompose $\lambda_1^\epsilon(\mu), \phi_1^\epsilon(s;\mu)$ as

$$\lambda_1^\epsilon(\mu) \quad = \quad \Lambda_1^\epsilon(\mu) + \Lambda_2^\epsilon(\mu), \tag{A.12}$$

$$\phi^\epsilon(t;\mu) \quad = \quad \dot{u}_0(s) + \Phi_1^\epsilon(s;\mu) + \Phi_2^\epsilon(s;\mu), \tag{A.13}$$

and prove that

$$|\Lambda_1^\epsilon(\mu)|,\ \|\Phi_1^\epsilon(s;\mu)\|_0 \quad \le \quad O\left(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2\right), \tag{A.14}$$

$$\text{and} \quad |\Lambda_2^\epsilon(\mu)|,\ \|\Phi_2^\epsilon(s;\mu)\|_0 \quad \le \quad O\left(\epsilon^6+\epsilon^2\mu^4\right). \tag{A.15}$$

Recalling the definition given in (A.3) and making use of the estimate for p_1 together with Lemma A.1, we automatically have

$$|\Lambda_1^\epsilon(\mu)|,\ \|\Phi_1^\epsilon\|_0 \le O(\epsilon^3+\epsilon^2\mu+\epsilon\mu^2). \tag{A.16}$$

Next subtracting the equation for Φ_1^ϵ in (A.3) from (A.5) gives

$$L_0(\phi_r^\epsilon - \Phi_1^\epsilon) + p_1\phi_r^\epsilon + p_2(\dot{u}_0 + \phi_r^\epsilon) = (\lambda_1^\epsilon - \Lambda_1^\epsilon)\dot{u}_0 + \lambda_1^\epsilon\phi_r^\epsilon. \tag{A.17}$$

Multiplying the above by $\dot{u}_0$ and integrate gives

$$(\lambda_1^\epsilon - \Lambda_1^\epsilon) = \left(\int \dot{u}_0^2\right)^{-1}\left(\int \left[p_1\phi_r^\epsilon + p_2(\dot{u}_0 + \phi_r^\epsilon)\right]\dot{u}_0\right)$$

Making use of the estimates (A.7) and (A.11) for p_1, p_2, and ϕ_r^ϵ, we get that

$$|\lambda_1^\epsilon - \Lambda_1^\epsilon| \le O(\epsilon^6 + \epsilon^2\mu^4). \tag{A.18}$$

Now the inhomogeneous term of equation (A.17) satisfies:

$$\begin{aligned} & \|p_1\phi_r^\epsilon + p_2(\dot{u}_0 + \phi_r^\epsilon)\|_0 + \|(\lambda_1^\epsilon - \Lambda_1^\epsilon)\dot{u}_0\|_0 + \|\lambda_1^\epsilon\phi_r^\epsilon\|_0 \\ \le\ & \|p_1\phi_r^\epsilon\|_0 + \|p_2\|_0 + |\lambda_1^\epsilon - \Lambda_1^\epsilon| + |\lambda_1^\epsilon|\,\|\phi_r^\epsilon\|_0 \\ \le\ & O(\epsilon^3 + \epsilon^2\mu + \epsilon\mu^2)^2 \le O(\epsilon^6 + \epsilon^2\mu^4). \end{aligned}$$

Then applying Lemma A.1 to (A.17) gives

$$\|\phi_r^\epsilon - \Phi_1^\epsilon\|_0 \le O(\epsilon^6 + \epsilon^2\mu^4).$$

We obtain the conclusion of this step by defining

$$\Lambda_2^\epsilon(\mu) = \lambda_1^\epsilon - \Lambda_1^\epsilon(\mu), \quad \text{and} \quad \Phi_2^\epsilon(s;\mu) = \phi_r^\epsilon(s;\mu) - \dot{u}_0 - \Phi_1^\epsilon(s;\mu).$$

□

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