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MODELING OPINION DYNAMICS: HOW THE NETWORK ENHANCES CONSENSUS

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Abstract. In this paper we analyze emergent collective phenomena in the evolution of opinions in a society structured into few interacting nodes of a network. The presented mathematical structure combines two dynamics: a first one on each single node and a second one among the nodes, i.e. in the network. The aim of the model is to analyze the effect of a network structure on a society with respect to opinion dynamics and we show some numerical solutions addressed in this direction, i.e. comparing the emergent behaviors of a consensus-dissent dynamic on a single node when the effect of the network is not considered, with respect to the emergent behaviors when the effect of a network structure linking few interacting nodes is considered. We adopt the framework of the Kinetic Theory for Active Particles (KTAP), deriving a general mathematical structure which allows to deal with nonlinear features of the interactions and representing the conceptual framework toward the derivation of specific models. A specific model is derived from the general mathematical structure by introducing a consensus-dissent dynamics of interactions and a qualitative analysis is given.

1. **Introduction.** The phenomenon of evolution of opinions in a society is a fascinating field of research for applied and theoretical mathematicians because of the intrinsic complexity of the system at hand and due to the fact that the principal actors are individuals, whose behavior is heterogeneous in a population and sometimes not fully rational [6]. The basic interactions in these phenomena are modeled as binary exchanges of opinions between individuals, although the access to internet and others global resources is an important phenomenon that cannot be disregarded. This last aspect determines the fact that individuals are often influenced by global trends in the opinions, in addition to the influence exerted by direct interactions with other individuals and this fact represents a source of nonlinearities

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for the representative mathematical model [7]. Many modeling approaches try to reduce the intrinsic structural complexity of systems like these composed of a large number of interacting individuals; e.g. agent-based models or microscopic ones. Most of the time, in sociological applications, the populations are modeled as homogeneously distributed in space, however many real world applications suggest to consider the populations distributed in networks, considering that the interactions between nodes can play an important role in the overall dynamics [5]. A conceptual framework toward a mesoscopic approach to opinion formation including the presence of a network can be found in [19].

To pursue the objective of modeling opinion formation among a large number of individuals structured into a network, we have applied in this paper the methods of the KTAP theory [2] and the percentage of randomness always connected to individual behavior is introduced by the fact that the possibility to change opinion, although regulated by well–defined interaction rules, is given in probability. The founding idea of the kinetic models is that a system composed of a sufficiently large number of agents may be described using the laws of statistical mechanics as it happens in a physical system composed of many interacting particles [22], however considering that the main actors are not particles but living beings and this is reflected into their interaction rules. Starting from the modelization of the microscopic dynamics, kinetic models can be derived and allows one to derive general informations on the model and its asymptotic behavior. For example in the case of a market, each trade is an interaction where a fraction of money goes from one agent to another one [12] while in opinion dynamic interactions consist in exchanging informations inducing to change, in probability, one's opinion. Many applications of the kinetic theory are related to the field of opinion formation (see, as an example [9, 11, 25]), while relations with others approaches and with the macroscopic limit can be found in [3, 10, 18]. As well explained in [2], a philosophical change is unfolding in the socio-economic field of study in order to depict emergent collective behaviors arising from individual interactions. It is a common opinion in the scientific literature that a key point is the revisitation of the concept of rational socio-economic behavior going beyond that of bounded rationality [24] by modeling an heterogeneous distribution of strategies among individuals; another key point is that of modeling nonlinear interactions in the sense that the outcome of interactions are not depending only on the microstate of the interacting individuals but also on the probability distribution function defined on the microstates. We considered these features in the proposed model.

The paper is composed of six more sections. Sec. 2 introduces the general mathematical structure modeling the opinion evolution in a network in which each node represents an homogeneous distributed population. The modeling assumption is that individuals interact each other within each node exchanging informations and may change their opinion due to these interactions; moreover individuals "feel" the mean opinion of other nodes and may change their opinion also due to this influence. These dynamics are sources of nonlinearities for the model. The output of the interactions is given in probability. The general mathematical structure allows both for continuous or discrete opinion distributions allowing for two possible pictures: discrete - continuous (for the nodes and for the opinion respectively) or discrete - discrete. In Sec. 3 and Sec. 4 we derive, from the early introduced general structure, a specific model of opinion formation based on a dynamic of consensus–dissent governed by two thresholds introduced on the distance between individual opinions within each node and on the distance between individual opinions and mean opinions on each node, respectively. It can be considered as a generalization of the bounded confidence model of statistical physics [9, 13, 26, 27]. In Sec. 5 a qualitative analysis of the model is presented. Some numerical results are showed in Sec. 6 regarding a discrete opinion variable and focusing on the influence of the network structure on the long time distributions. The influence of the network, inducing consensus toward a mean opinion value, is showed in two different cases. Finally, some research perspectives are discussed in Sec. 7.

2. **A mathematical representation of opinion formation in a network.** In this section we introduce a general mathematical structure modeling the evolution of opinion distributions in a large system of individuals with an heterogeneous distributed opinion about a certain issue, supposing that the opinion is expressible as a real number. Moreover, we suppose that the global population is clustered into different groups which will be considered as nodes of a network in our model and individuals are homogeneously distributed within each node. The same dynamic of interactions is assumed on each node, but the parameters and/or the initial conditions, may differ from one node to another. Each individual may change his opinion due to interactions with other individuals and due to the influence of sources of informations, such as social networks, represented by the mean opinion on the other nodes. We adopt a mesoscopic approach by means of the KTAP theory, that however may be related to a microscopic (individually based) approach (see e.g. [4, 14, 15, 20] and references therein). Before dealing with the representation of the overall system, it is useful providing some definitions related to the scaling problem. We are considering individuals (active particles in the following) whose microstate is characterized only by a variable representing individual's opinion due to the assumption of space homogeneity; microscopic interactions consist in exchanging opinion and the output consists in a new opinion state, given in probability. The general framework which we are going to introduce allows for both a continuous or discrete representation of the variable describing the opinion of the active particles. A probability distribution characterizes the mesoscopic system and the related moments are the macroscopic variables representing the mean opinion within the considered population (first moment) or, in general, other macroscopic quantities (higher moments). The mesoscopic approach allows one to deal with a tractable system of equations, although retaining the features of heterogeneity of opinion distribution on a population and the nonlinearity features consisting of dependances of individual's opinion on the mean one, or other macroscopic variables.

The nodes of the network are characterized by a discrete variable j, while the opinion is represented by the variable u. The discrete variable j and the opinion variable u take values in $\mathcal{J} \subset \mathbb{N}$ and $\mathcal{U} \subset \mathbb{R}^1$, respectively. The variable u itself can be either discrete (then $\mathcal{U}$ is a discrete set – a subset of $\mathbb{N}$) or continuous (then $\mathcal{U}$ is a measurable bounded set in $\mathbb{R}^1$); where $\inf \mathcal{U}$ and $\sup \mathcal{U}$ corresponds to the case of maximum disagreement and of maximum agreement regarding a certain issue, respectively.

Keeping this in mind we propose a general compact setting that may also serve for other various applications (cf. [4, 20]). Each individual (called here "*active particle*") is characterized by a variable

$$\mathbf{u} = (j, u) \in \mathbf{U} = \mathcal{J} \times \mathcal{U} \,,$$

describing the node to which the active particle belongs ($j \in \mathcal{J}$) and its opinion ($u \in \mathcal{U}$). In the general setting $(\mathbf{U}, \mu)$ is a space with a σ–finite measure μ; then $\mathbf{U}$ is a product of a discrete set $\mathcal{J}$ and a Lebesgue–measurable subset $\mathcal{U} \subset \mathbb{R}^1$ (a closed bounded interval in the discrete–continuous picture or a bounded discrete set in the discrete–discrete one) and the measure $\mu = \mu_1 \otimes \mu_2$ is a product of the counting measure μ_1 and the counting measure (in the discrete-discrete picture) or the Lebesgue measure (in the discrete-continuous picture) μ_2. The simulations that we perform on a specific example refer to the discrete–discrete case.

We are interested in the time evolution of the probability function

$$f = f(t, \mathbf{u}), \qquad f : \mathbb{R}^1_+ \cup \{0\} \times \mathbf{U} \to \mathbb{R}^1_+ \cup \{0\}. \tag{1}$$

Therefore f is a non-negative function, such that

$$\int_{\mathbf{U}} f(t, \mathbf{u})\, \mathrm{d}\mu(\mathbf{u}) = 1, \qquad \forall\, t \geq 0. \tag{2}$$

The time evolution of the probability function (1) is defined by the adequate kinetic equation which is derived later on.

According to the KTAP terminology [2], the *test active particle* is an entity representative of the system, whose micro–state is represented by the node to which it belongs j and his activity (here the opinion variable) u, i.e. $\mathbf{u} = (j, u)$, and his microstate is acquired, in probability, by the *candidate active particle*, represented by the variable $\mathbf{u}_* = (j_*, u_*)$. The *field active particle* is assumed to trigger the interactions and his micro–state is represented by the variable $\mathbf{u}^* = (j^*, u^*)$.

We model the effects on the candidate active particle due to the following phenomena:

1. binary interactions with field active particles within the node to which the candidate active particle belongs;
2. influence of the mean opinion of each node different to the one to which the candidate active particle belongs.

The phenomenon of migration among the nodes is not considered into the model.

The mathematical model is given by the kinetic equation

$$\partial_t f(t, \mathbf{u}) = J(f, f)(t, \mathbf{u}) + J_E[f](t, \mathbf{u}) \tag{3}$$

where each term on the right hand side represents the gain and loss in the microstates due to the two different phenomena listed above, respectively; in particular the bilinear operator J corresponds to interactions involving pairs of active particles whilst J_E is a nonlinear operator (square brackets are used to indicate a nonlinear dependance) corresponding to the influence of mean opinions on the candidate active particle. Following [2] the interactions are specified in relation to the particular dynamics involving the variables.

In order to derive explicit expressions for Eq. (3) we need to specify the following quantities:

- $\eta(\mathbf{u}_*, \mathbf{u}^*)$ — rate of interactions between active particles in a node;
- $B(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*, \mathbf{u}^*)$ — transition probability function defining the probability for a candidate active particle characterized by $\mathbf{u}_*$ to move to $\mathbf{u}$ due to an interaction with a field active particle characterized by $\mathbf{u}^*$ whithin the node to which the candidate active particle belongs;

- $\eta_E(\mathbf{u}_*; k, f, t)$ — rate of transitions of the active particles due to the influence of the mean opinion, at the instant t, on another node k, $k \neq j_*$, where $\mathbf{u}_* = (k_*, u_*)$;
- $B_E(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*; k, f, t)$ — transition probability function defining the probability for a candidate active particle characterized by $\mathbf{u}_*$ to move to $\mathbf{u}$ due to an influence of the mean opinion in another node k at the instant t.

The operators determining the gain and loss in the microstates in Eq. (3) assume the form

$$J(f,f)(t,\mathbf{u}) = \int_{\mathbf{U}^2} B(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*, \mathbf{u}^*)\, \eta(\mathbf{u}_*, \mathbf{u}^*) f(t,\mathbf{u}_*) f(t,\mathbf{u}^*)\, \mathrm{d}\mu(\mathbf{u}_*)\, \mathrm{d}\mu(\mathbf{u}^*)$$
$$- f(t,\mathbf{u}) \int_{\mathbf{U}} \eta(\mathbf{u},\mathbf{u}_*) f(t,\mathbf{u}_*)\, \mathrm{d}\mu(\mathbf{u}_*)\,,$$

$$J_E[f](t,\mathbf{u}) = \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \Big(\int_{\mathbf{U}} B_E(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*; k, f, t)\, \eta_E(\mathbf{u}_*; k, f, t) f(t,\mathbf{u}_*)\, \mathrm{d}\mu(\mathbf{u}_*)$$
$$- \eta_E(\mathbf{u}; k, f, t) f(t,\mathbf{u}) \Big)\,,$$

where $\mathbf{u} = (j, u)$.

Note that considering the fact that the solutions are *a priori* probability densities one may unify the terms $J(f,f)$ and $J_E[f]$ in one term $J[f]$ that, as in the general case of KTAP, refers not only to binary interactions but also to nonlinear dependence on some moments. Let

$$\mathcal{A}(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*, \mathbf{u}^*; f, t) =$$
$$= \begin{cases} \dfrac{1}{a(\mathbf{u}_*, \mathbf{u}^*; f, t)} \Big\{ B(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*, \mathbf{u}^*)\, \eta(\mathbf{u}_*, \mathbf{u}^*) \\ \quad + \displaystyle\sum_{\substack{k \in \mathcal{J} \\ k \neq j}} B_E(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*; k, f, t)\, \eta_E(\mathbf{u}_*; k, f, t) \Big\} & \text{if} \quad a(\mathbf{u}_*, \mathbf{u}^*; f, t) > 0 \\ 0 & \text{if} \quad a(\mathbf{u}_*, \mathbf{u}^*; f, t) = 0 \end{cases} \tag{4}$$

and

$$a(\mathbf{u}, \mathbf{u}_*; f, t) = \eta(\mathbf{u}, \mathbf{u}_*) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \eta_E(\mathbf{u}; k, f, t)\,,$$

with the natural assumptions that

$$\eta(\mathbf{u}, \mathbf{u}_*) \geq 0 \quad \text{and} \quad \eta_E(\mathbf{u}; k, f, t) \geq 0$$

for all $\mathbf{u} = (j, u)$, k, $\mathbf{u}_* = (j_*, u_*)$ and $t > 0$.

Assuming the above notation, the equation

$$\partial_t f(t,\mathbf{u}) = J[f](t,\mathbf{u})\,, \qquad t > 0\,, \quad \mathbf{u} \in \mathbf{U}\,, \tag{5}$$

where

$$J[f](t,\mathbf{u}) = \int_{\mathbf{U}^2} \mathcal{A}(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*, \mathbf{u}^*; f, t) a(\mathbf{u}_*, \mathbf{u}^*; f, t) f(t,\mathbf{u}_*) f(t,\mathbf{u}^*)\,\mathrm{d}\mu(\mathbf{u}_*)\,\mathrm{d}\mu(\mathbf{u}^*)$$

$$-f(t,\mathbf{u}) \int_{\mathbf{U}} a(\mathbf{u},\mathbf{u}^*; f, t) f(t,\mathbf{u}^*)\,\mathrm{d}\mu(\mathbf{u}^*)\,,$$

is equivalent to Eq. (3) (if Eq. (2) is satisfied). Equations of the type (5) may be related to individually–based models — cf. Refs. [4, 20].

In the present paper we are going to consider only the case when the individuals do not change their nodes and the interactions between different nodes are only defined by the operator J_E. Therefore we assume that

$$B(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*, \mathbf{u}^*) = \delta_{j,j_*} \tilde{B}(u_* \to \mathbf{u} \,|\, u_*, u^*), \tag{6}$$

for all $\mathbf{u} = (j,u)$, $\mathbf{u}_* = (j_*, u_*)$, $\mathbf{u}^* = (j^*, u^*)$, where δ_{j,j_*} is the Kronecker delta;

$$B_E(\mathbf{u}_* \to \mathbf{u} \,|\, \mathbf{u}_*; k, f, t) = \delta_{j,j_*} \tilde{B}_E(u_* \to u \,|\, u_*; f(t,k,.)), \tag{7}$$

for all $\mathbf{u} = (j,u)$, $\mathbf{u}_* = (j_*, u_*)$, where $f(t,k,.)$ indicates the dependence on the moment of $f(t,k,.)$. Moreover, for simplicity we assume

$$\eta(\mathbf{u},\mathbf{u}_*) = \delta_{j,j_*}\tilde{\eta},$$

for all $\mathbf{u} = (j,u)$, $\mathbf{u}_* = (j_*, u_*)$, where $\tilde{\eta}$ is a positive constant;

$$\eta_E(\mathbf{u}; k, f, t) = \begin{cases} \eta_E(u; f(t,k,.)), & \text{if} \quad k \neq j \\ 0 & \text{if} \quad k = j \end{cases}$$

for all $\mathbf{u} = (j,u)$. Under these assumptions Eq. (3) can be rewritten in the form

$$\partial_t f(t,j,u) = J(f,f)(t,j,u) + J_E[f](t,j,u)\,, \tag{8}$$

where the operator

$$J(f,f)(t,j,u) = \tilde{\eta} \int_{\mathcal{U}^2} \tilde{B}(u_* \to u \,|\, u_*, u^*)\, f(t,j,u_*) f(t,j,u^*)\,\mathrm{d}\mu_2(u_*)\,\mathrm{d}\mu_2(u^*)$$

$$-\tilde{\eta}\, f(t,j,u) \int_{\mathcal{U}} f(t,j,u_*)\,\mathrm{d}\mu_2(u_*)$$

describes the interactions inside a fixed node $j \in \mathcal{J}$; whereas the operator

$$J_E[f](t,j,u) = \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \Big(\int_{\mathcal{U}} \tilde{B}_E(u_* \to u \,|\, u_*; f(t,k\,.))\, \eta_E(u_*; f(t,k,.))$$

$$f(t,j,u_*)\,\mathrm{d}\mu_2(u_*) - \eta_E(u; f(t,k,.))\, f(t,j,u) \Big)$$

describes the influence of different nodes (i.e. in the network). All functions depend only on the indicated variables. We assume that

$$\tilde{B} \geq 0, \quad \tilde{B}_E \geq 0 \tag{9}$$

and

$$\int_{\mathcal{U}} \tilde{B}(u_* \to u \,|\, u_*, u^*) d\mu_2(u) = 1, \tag{10}$$

for all u_*, u^* in $\mathcal{U}$;

$$\int_{\mathcal{U}} \tilde{B}_E(u_* \to u \,|\, u_*; f(t,k\,.))d\mu_2(u) = 1, \tag{11}$$

for all u_* in $\mathcal{U}$ and $f(t,k,.)$ integrable such that $\eta_E(u_*; f(t,k,.)) > 0$. Under these assumptions it is clear that not only $\int_{\mathbf{U}} f(t,\mathbf{u})d\mu(\mathbf{u})$ is (formally) conserved in time but also $\int_{\mathcal{U}} f(t,j,u)d\mu_2(u)$ is, for each $j \in \mathcal{J}$. Equation (8) defines a particular case of the general structure (5).

3. **Consensus-dissent dynamics in a separate node.** In this section we derive, from the general mathematical structure defined in the previous section, a class of opinion dynamic models based on specific assumptions regarding the dynamic of the interactions. In particular, we model a process based on a consensus-dissent dynamic, in the case in which active particles are influenced only by binary encounters with other active particles; a basic point is that this process is triggered by a threshold. In fact, the consensus dynamic happens when the opinion distance between the interacting active particles is below a given threshold and the dissent dynamic when the opinion distance between the interacting active particles is above it. In the consensus dynamic the post-interaction opinion distance is shorter then the pre-interaction one; basically, the active particles approach each other due to the interaction. In the dissent dynamic the post-interaction opinion distance is greater then the pre-interaction one; basically the active particles move away from one another. One may interpret this quantity in the sense that the greater is the threshold the greater is the propensity to reach a fair compromise in the population.

3.1. **Dynamic of interactions between pairs of active particles.** To avoid cumbersome notations, throughout the paper we adhere to the following convention: if we write an interval $[c_1, c_2] \subset \mathbb{R}^1$, $c_1 < c_2$, we have in mind $[c_1, c_2] \cap \mathcal{U}$.

A general model for the transition probability functions assuming a consensus–dissent interaction dynamic is introduced in the following.

1. In the **consensus dynamic**, the transition probability related to binary encounters between active particles represents the probability that the candidate active particle changes its opinion due to interactions with a field active particle, in order to reach a fair compromise; it applies when the opinion distance between candidate active particle and field one is below a given threshold d. Let $0 < d \leq \max \mathcal{U} - \min \mathcal{U}$; when $0 < |u_* - u^*| \leq d$, the post–interaction opinion of the candidate active particle is, in probability, nearer to the opinion of the field active particle, then it is in the interval between the candidate active particle opinion and the field active particle one. The transition probability characterizing this dynamics is as follows

$$\tilde{B}(u_* \to u \,|\, u_*, u^*)$$
$$= \begin{cases} \beta(u; u_*, u^*) & \text{if} \quad u \in \Big[\min\big\{u_*, u^*\big\}, \max\big\{u_*, u^*\big\}\Big] \\ 0 & \text{if} \quad u \in \mathcal{U} \setminus \Big[\min\big\{u_*, u^*\big\}, \max\big\{u_*, u^*\big\}\Big] \end{cases}, \tag{12}$$

where $\beta(u; u_*, u^*)$, for given u_* and u^*, is a given probability density (with respect to u) on the interval between $\min\big\{u_*, u^*\big\}$ and $\max\big\{u_*, u^*\big\}$. The most appropriate probability density $\beta(u; u_*, u^*)$ depends on the specific applications, the simplest one being the uniform probability density on the interval between $\min\{u_*, u^*\}$ and $\max\{u_*, u^*\}$, i.e.

$$\beta(u; u_*, u^*) = \frac{1}{\mu_2\Big(\big[\min\{u_*, u^*\}, \max\{u_*, u^*\}\big]\Big)}$$

for each $u \in \Big[\min\{u_*, u^*\}, \max\{u_*, u^*\}\Big]$. A rather more complicate form for the functions β_1 and β_2 could be that one modelling a decreasing effect with the opinion distance candidate-field active particles.

2. In the **dissent dynamic**, the transition probability related to binary encounters between active particles represents the probability that the candidate active particle changes its opinion due to interactions with a field active particle, determining a kind of radicalization of its opinion in the sense that the post interaction opinion is nearer to the bound of the interval than the pre interaction one; it applies when the opinion distance between candidate active particle and field one is above the given threshold d.

   Then, when $|u_* - u^*| > d$, the post-interaction opinion of the candidate active particle is, in probability, farther from the opinion of the field active particle, then it is in the interval between the candidate active particle opinion and the bound of $\mathcal{U}$. A mathematical formulation for the transition probability characterizing this effect is the following

$$\tilde{B}(u_* \to u \,|\, u_*, u^*) = \begin{cases} \beta_1(u; u_*, u^*) & \text{if} \quad u \in [\min \mathcal{U}, u_*]\,, \quad \text{and} \quad u_* \le u^* \\ \beta_2(u; u_*, u^*) & \text{if} \quad u \in [u_*, \max \mathcal{U}]\,, \quad \text{and} \quad u_* > u^* \\ 0 & \qquad\qquad \text{otherwise} \end{cases}, \tag{13}$$

   where β_1 and β_2, for given u_* and u^*, are given probability densities on the intervals $[\min \mathcal{U}, u_*]$ (for $u_* \le u^*$) and $[u_*, \max \mathcal{U}]$ (for $u_* \ge u^*$), respectively and measure the attitude of the candidate active particle to increase the distance between its opinion and the field active particle one. The most appropriate probability densities β_1 and β_2 depend on the specific applications, the simplest ones being the uniform distribution on $[\min \mathcal{U}, u_*]$ for β_1 and the uniform distribution on $[u_*, \max \mathcal{U}]$ for β_2.

4. **Consensus–dissent dynamics in a network.** As already remarked in the introduction, the possibility to access informations on internet or using other sources, such as the media or social networks, determines the fact that individuals may be influenced by global opinion trends, giving rise to a stream effect. In the previous section we have modeled a consensus-dissent interaction dynamic in a separate node. In the present section we model the case in which a finite number of nodes interact by means of the mean opinion on each node. In this case, in addition to the consensus-dissent dynamic introduced in the previous section and regarding the interactions within a node, we consider an analogous consensus-dissent dynamic regarding the influence of the mean opinions of the nodes on the active particles in the network. Then, the outputs of the interactions depend now on two thresholds: the first threshold is related to the opinion distance between the candidate active

particle and the field one, and it is considered whenever interactions within each node are considered. The second threshold regards the distance between the candidate active particle's opinion and the mean opinion of a node and it is considered whenever interactions involving the network are considered. This threshold represents, in this case, a sort of propensity of the active particles to be influenced by the mean opinion of other populations. In the consensus dynamic the post-interaction opinion distance between the candidate active particle and the mean opinion on a node is shorter then the pre-interaction one, while in the dissent dynamic the the post-interaction opinion distance between the candidate active particle and the mean opinion on a node is greater then the pre-interaction one. Regarding the rate of transition, we assume here that η_E is a given positive constant.

Given the node, the mean opinion is the first moment with respect to the opinion variable is

$$\mathbb{E}_j(f,t) = \int_{\mathcal{U}} u f(t,j,u) \mathrm{d}\mu_2(u) \,. \tag{14}$$

1. In the **consensus dynamic** the transition probability related to the influence of the mean opinion of a node on the candidate active particle represents the probability that the candidate active particle changes its opinion due to this influence, in order to reach a fair compromise; it applies when the opinion distance between candidate active particle opinion and the mean opinion of the k-th node is below a given threshold D.

   Let $0 < D < \max\mathcal{U} - \min\mathcal{U}$. If $0 < |u_* - \mathbb{E}_k| \leq D$, for some $k \in \mathcal{J}$, the post-interaction opinion of the candidate active particle is, in probability, nearer to the mean opinion on a node and we assume

$$\tilde{B}_E(u_* \to u \,|\, u_*; f(t,k,.)) = \begin{cases} \gamma(u; u_*, \mathbb{E}_k) & \text{if} \quad u \in \Big[\min\big\{u_*, \mathbb{E}_k\big\}, \max\big\{u_*, \mathbb{E}_k\big\}\Big] \\ 0 & \text{if} \quad u \in \mathcal{U} \setminus \Big[\min\big\{u_*, \mathbb{E}_k\big\}, \max\big\{u_*, \mathbb{E}_k\big\}\Big] \end{cases} \tag{15}$$

   where $\gamma(u; u_*, \mathbb{E}_k)$, for given u_* and $\mathbb{E}_k$ is a probability density on $\Big[\min\big\{u_*, \mathbb{E}_k\big\}, \max\big\{u_*, \mathbb{E}_k\big\}\Big]$ and measures the attitude of the candidate active particle to decrease the distance between its opinion and the mean one of the k-th node. The most appropriate probability density $\gamma(u; u_*, \mathbb{E}_k)$ depends on the specific applications.
2. In the **dissent dynamic** the transition probability related to the influence of the mean opinion of a node on the candidate active particle represents the probability that the candidate active particle changes its opinion due to this influence, determining a kind of radicalization of its opinion; it applies when the opinion distance between the candidate active particle's opinion and the mean one on a node is above the given threshold D.

   Then, when $|u_* - \mathbb{E}_k| > D$ we model a dissent from the mean opinion on a node, introducing the following probability density

$$\tilde{B}_E(u_* \to u \,|\, u_*; f(t,k,.)) \quad , \tag{16}$$

$$
= \begin{cases} \gamma_1(u;u_*,\mathbb{E}_k) & \text{if} \quad u \in [\min \mathcal{U}, u_*]\,, \quad \text{and} \quad u_* \le \mathbb{E}_k \\ \gamma_2(u;u_*,\mathbb{E}_k) & \text{if} \quad u \in [u_*, \max \mathcal{U}]\,, \quad \text{and} \quad u_* > \mathbb{E}_k \\ 0 & \qquad \text{otherwise} \end{cases} \quad , \tag{17}
$$

where γ_1 and γ_2, for given u_* and u^*, are given probability densities on the intervals $[\min \mathcal{U}, u_*]$ (for $u_* \le \mathbb{E}_k$) and $[u_*, \max \mathcal{U}]$ (for $u_* > \mathbb{E}_k$), respectively, and measure the attitude of the candidate active particle to increase the distance between its opinion and the mean opinion of another node different to the one to which it belongs. Specific probability densities γ_1 and γ_2 depend on the problem at hand.

Keeping the above definitions in mind, we assume that $\eta_E = \eta_E(u; f(t,k,.))$ is a regular (at least Lipschitz continuous) function of $|u - \mathbb{E}_k|$ and such that

- η_E takes value 0 on the sets $|u - \mathbb{E}_k| \le \delta$ and $D - \delta \le |u - \mathbb{E}_k| \le D + \delta$;
- η_E takes a positive constant value $\tilde{\eta}_E$ on the sets $2\delta \le |u - \mathbb{E}_k| \le D - \delta$ and $D + \delta \le |u - \mathbb{E}_k|$, where δ is a small number.

Remark 1. In the discrete–discrete case, i.e. when $\mathcal{U}$ is a discrete set, we consider the following version of Eq. (8)

$$
\begin{aligned}
&\partial_t f(t,j,u) = \tilde{\eta} \sum_{u_*,u^* \in \mathcal{U}} \tilde{B}(u_* \to u | u_*, u^*)\, f(t,j,u_*) f(t,j,u^*) \\
&-\tilde{\eta}\, f(t,j,u) \sum_{u_* \in \mathcal{U}} f(t,j,u_*) \\
&+ \sum_{u_* \in \mathcal{U}} f(t,j,u_*) \sum_{k \ne j} \Big(\tilde{B}_E(u_* \to u | u_*; f(t,k,.)\, \eta_E(u_*; f(t,k,.)) \\
&- f(t,j,u)\, \eta_E(u_*; f(t,k,.)) \Big), \qquad j \in \mathcal{J},\, u \in \mathcal{U},
\end{aligned} \tag{18}
$$

where all the functions depend only on the indicated variables.

5. **Qualitative analysis.**

For a bounded set $\mathbf{U}$ we define the Banach space $L_1(\mu)$ equipped with the norm

$$
\|f\| = \int_{\mathbf{U}} |f(\mathbf{u})| d\mu(\mathbf{u}). \tag{19}
$$

Theorem 5.1. *Let $f_0 \ge 0$, $f_0 \in L_1(\mu)$ be initial data such data*

$$
\|f_0\| = 1. \tag{20}
$$

Assume that

- $\mathcal{A}(\mathbf{u}_* \to \mathbf{u} | \mathbf{u}_*, \mathbf{u}^*; f, t) \ge 0$*;*
- $\int_{\mathbf{U}} \mathcal{A}(\mathbf{u}_* \to \mathbf{u} | \mathbf{u}_*, \mathbf{u}^*; f, t) d\mu(\mathbf{u}) = 1$*, for all* $\mathbf{u}, \mathbf{u}_*, \mathbf{u}^* \in \mathbf{U}$*,* $f \in L_1(\mu)$ *and* $t > 0$*;*
- $a(\mathbf{u}, \mathbf{u}_*; t) \ge 0$ *for all* $\mathbf{u}, \mathbf{u}_* \in \mathbf{U}$ *and* $t > 0$*;*

- *$\mathcal{A}(\mathbf{u}_* \to \mathbf{u}|\mathbf{u}_*, \mathbf{u}^*; f, t)$ and $a(\mathbf{u}, \mathbf{u}_*; t)$ are continuous function of $t > 0$ for all $\mathbf{u}$, $\mathbf{u}_*$, $\mathbf{u}^* \in \mathbf{U}$ and $f \in L_1(\mu)$;*
- *$\mathcal{A}(\mathbf{u}_* \to \mathbf{u}|\mathbf{u}_*, \mathbf{u}^*; f, t)$ is locally Lipschitz-continuous functions of $f \in L_1(\mu)$ that is for each $f \in L_1(\mu)$ there exist a neighborhood $\mathcal{N}$ of f and a constant $c > 0$ such that*

$$\int_{\mathbf{U}} |\mathcal{A}(\mathbf{u}_* \to \mathbf{u}|\mathbf{u}_*, \mathbf{u}^*; f_1, t) - \mathcal{A}(\mathbf{u}_* \to \mathbf{u}|\mathbf{u}_*, \mathbf{u}^*; f_2, t)| d\mu(\mathbf{u}) \leq c\|f_1 - f_2\|$$

for all $f_i \in \mathcal{N}$, $i = 1, 2$. Then, for any $t > 0$, there exists a unique solution $f = f(t) \in L_1(\mu)$ of Eq. (5) with initial data f_0. Moreover

$$f(t) \geq 0\,, \tag{21}$$

and

$$\|f(t)\| = 1\,, \tag{22}$$

for any $t > 0$.

Proof. The result follows in a standard way. The local Lipschitz continuity provides the local (in time) existence and uniqueness. Moreover, the solution corresponding to the nonnegative initial data is nonnegative and satisfies (21). Therefore the solution may be prolonged to any $t > 0$ and (21), (22) are satisfied. □

Remark 2. From Theorem 5.1 the corresponding result for Eq. (8) follows. Under the Assumptions in Section 2., for the solution to Eq. (8) not only

$$\int_{\mathbf{U}} f(t, \mathbf{u})\, \mathrm{d}\mu(\mathbf{u}) = \int_{\mathbf{U}} f(0, \mathbf{u})\, \mathrm{d}\mu(\mathbf{u})\,, \qquad t > 0$$

is conserved, but also

$$\int_{\mathcal{U}} f(t, j, u)\, \mathrm{d}\mu_2(u) = \int_{\mathcal{U}} f(0, j, u)\, \mathrm{d}\mu_2(u)\,, \qquad t > 0,\ j \in \mathcal{J}.$$

Therefore we may refer to $f(t, j, \,.\,)$, for all $t > 0$, $j \in \mathcal{J}$, as the probability density under the suitable descaling. This approach is used in the numerical simulations.

6. **Simulations looking for asymptotic opinion trends.** In this section we present some numerical solutions of the system of equations regarding the model of consensus-dissent opinion dynamic introduced in the previous section, with the aim to investigate the following features:

- influence of the threshold on the opinion trends on each separate node whenever the network is not taken into consideration and comparison with the results obtained in the case in which the nodes are connected into a network;
- influence of the initial opinion distribution on the opinion trendson on each separate node whenever the network is not taken into consideration and comparison with the results obtained in the case in which a small network is considered;
- sensitivity of the model to the threshold in a separate node whenever the network is not taken into consideration and comparison with the results obtained in the case in which a small network is considered.

The aforesaid features are analyzed through three case studies. To fix ideas one can think that the nodes are geographical; as an example, each one could represent the distribution of opinions regarding a central issue in a population belonging to a country. In order to show the main features of the proposed class of models, in the numerical simulations we have adopted uniform probability distributions for both the binary interaction between active particles and for the influence of the mean opinion on active particles in the network. We have considered a network with three nodes. The opinion variable is taken as discrete and takes value in the set $\mathcal{U} = \{1, \ldots, 7\}$ for each node. In all simulations the encounter rate inside each node, i.e. for binary encounter between active particles, takes the constant value $\eta = 0.6$ and the encounter rate for the network, i.e. for the influence of the mean opinion on the active particles, takes the constant value $\eta_E^0 = 0.2$.

6.1. **Case I: Effect of the threshold on the asymptotic opinion trend and the effect of the network.** We simulate the evolution of an initial uniform distribution of opinion in three separate nodes characterized by different values for the threshold regarding binary encounters between pairs of active particles. If one would follow the example of geographical nodes each one representing a country, this would mean that each country has a different propensity to reach a fair compromise, although they start with the same opinion distributions initially. In particular a low value for the threshold ($d_1 = 1$) on the first node is assumed, a medium value ($d_2 = 3.5$) on the second node and an high value ($d_3 = 5$) on the third node are assumed. A medium value for the threshold in the network, i.e. $D = 3.5$ is assumed. In the first set of simulations (Fig.1) we do not consider interactions among the nodes with the aim to compare the results with the second set of simulations (Fig.2) where on each node we take the same values for the parameters and the same initial conditions as the previous one apply but the nodes arenow interacting with a consensus-dissent dynamics.

The comparison of the results obtained in the case of separate nodes (Fig.1) with those obtained with the nodes of a network (Fig.2) show that the effect of the network is to induce consensus (compare Fig.1_b and Fig.2_b) toward the mean value for the opinion in the node with a medium value for the threshold. Furthermore, one can notice that Fig.1 refers to three separate nodes and the first moment is conserved whilst Fig.2 refers to the same nodes as in Fig.1 but which are now connected in a network; one can observe that in this case the first moment on each node is not conserved due to the effect of the network.

6.2. **Case II: Effect of the initial opinion profile on the asymptotic one and the effect of the network.** We simulate the evolution of an initial not uniform distribution of opinion in three nodes characterized by the same medium value ($d = 3.5$) on each of them for the threshold regarding binary encounters between pairs of active particles. If one would apply again the example of geographical nodes each one representing a country, this would mean that each country has in these simulations the same propensity to reach a fair compromise, although they have different opinion distributions initially. In the first set of simulations (Fig.3) we do not consider interactions among the nodes with the aim to compare the results with the second set of simulations (Fig.4) where the same conditions apply except for the fact that the nodes are now interacting and the network has the threshold $D = 3.5$.

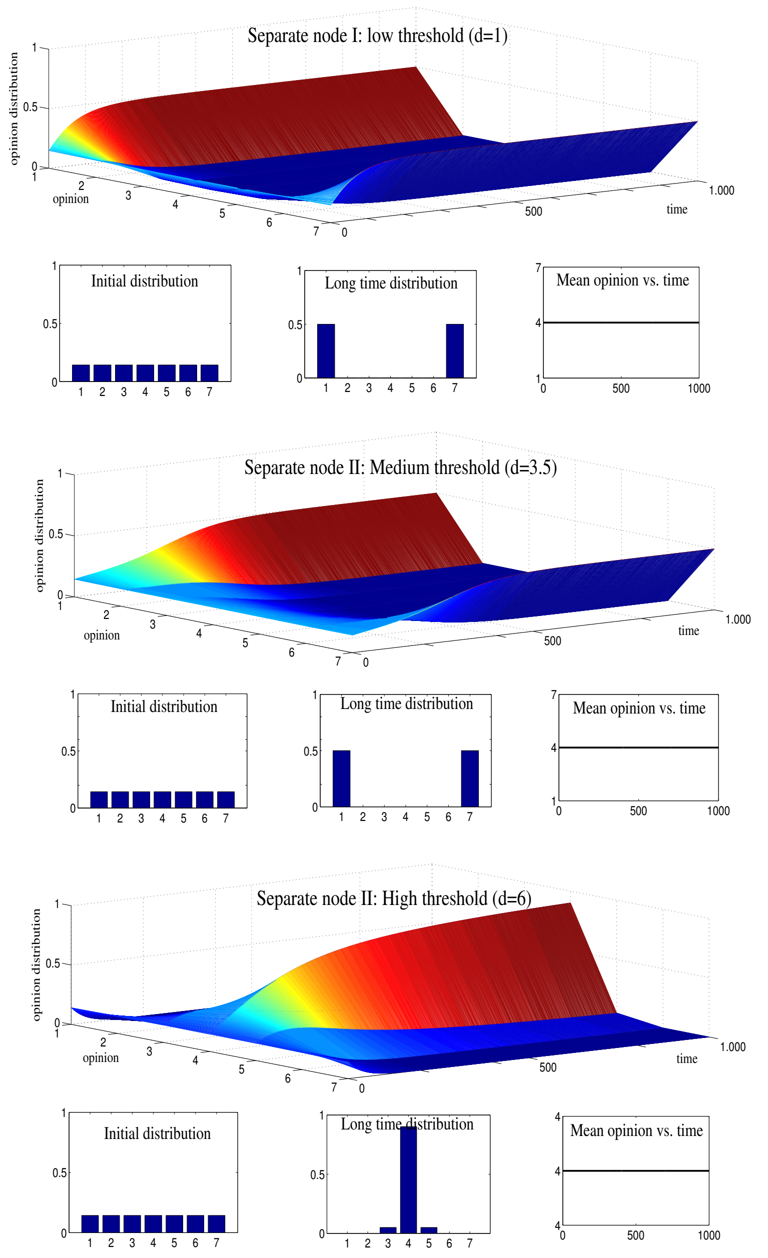

FIGURE 1. **Separate nodes.** Initial uniform distribution and different threshold for each separate node.

Fig.4 shows a consensus toward a medium value for the opinion distribution in all nodes, which is not present when the nodes are not interacting (see Fig.3). This phenomenon may be related to the one known in literature as *unconditonal consensus*, i.e. when all the initial configurations tend to an emerging limit state [21].

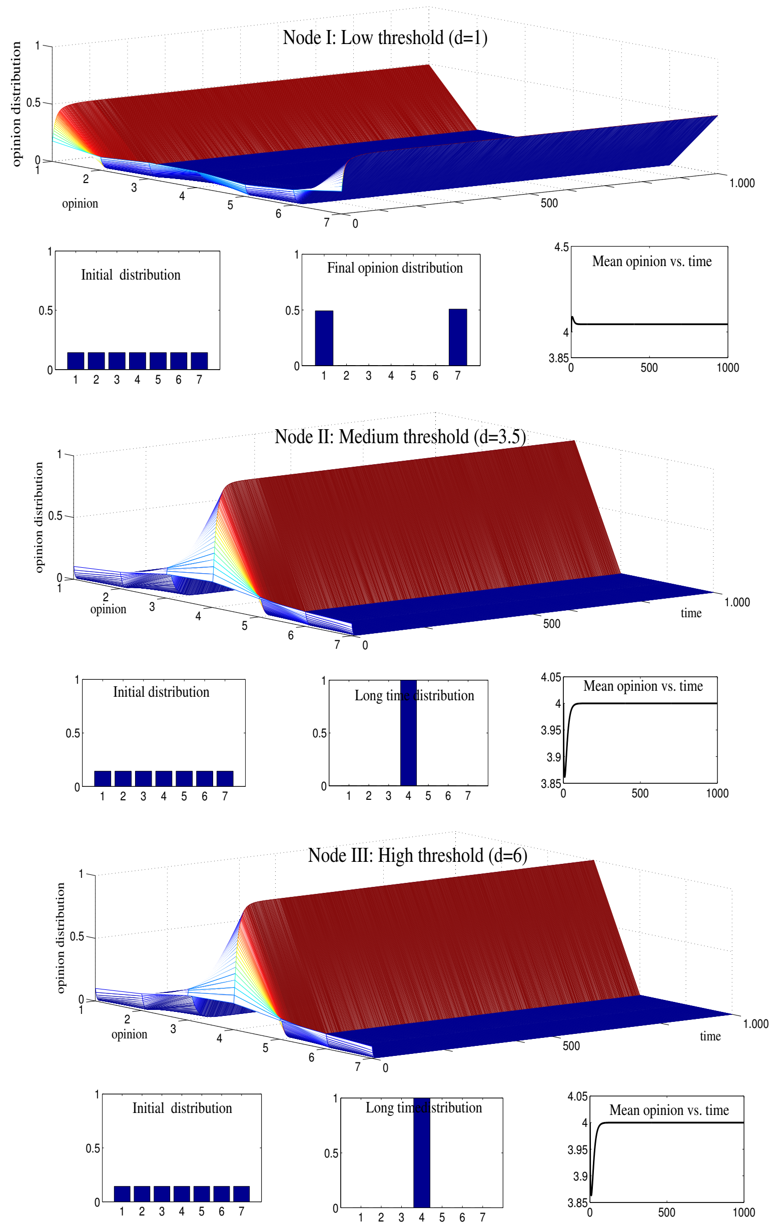

FIGURE 2. **Network.** Initial uniform distribution and different thresholds on each node **in a network.**

6.3. **Case III: Sensitivity of the long time distribution to the threshold.** In this last set of simulations we plot the long time distribution as function of the threshold varying in the interval $[0, 6]$. In Fig.5 the long time distribution obtained from an initial uniform distribution in a separate node is plotted as function of the threshold varying from the minimal value to the maximal one; one observes a transition from a polarization toward the opposite extreme values to a consensus

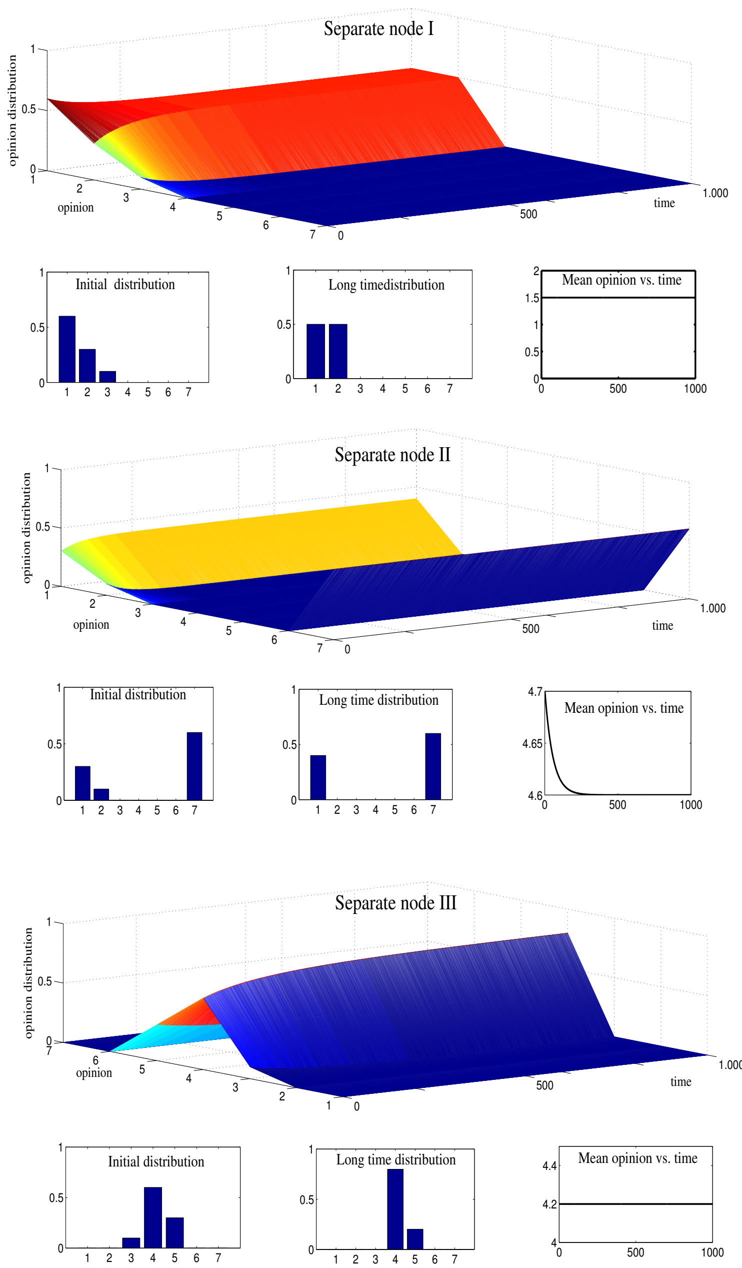

FIGURE 3. **Separate nodes.** Different initial distributions and medium threshold ($d = 3.5$) on each separate node.

toward a mean opinion value when the threshold approaches a medium value. In Fig.6. the long time opinion distribution is plotted, regarding a node of a network with three interacting nodes starting with the same initial opinion distributions.

As a result, in a separate node, when the threshold approaches a medium value, the long time distribution changes from polarization toward the opposite extreme

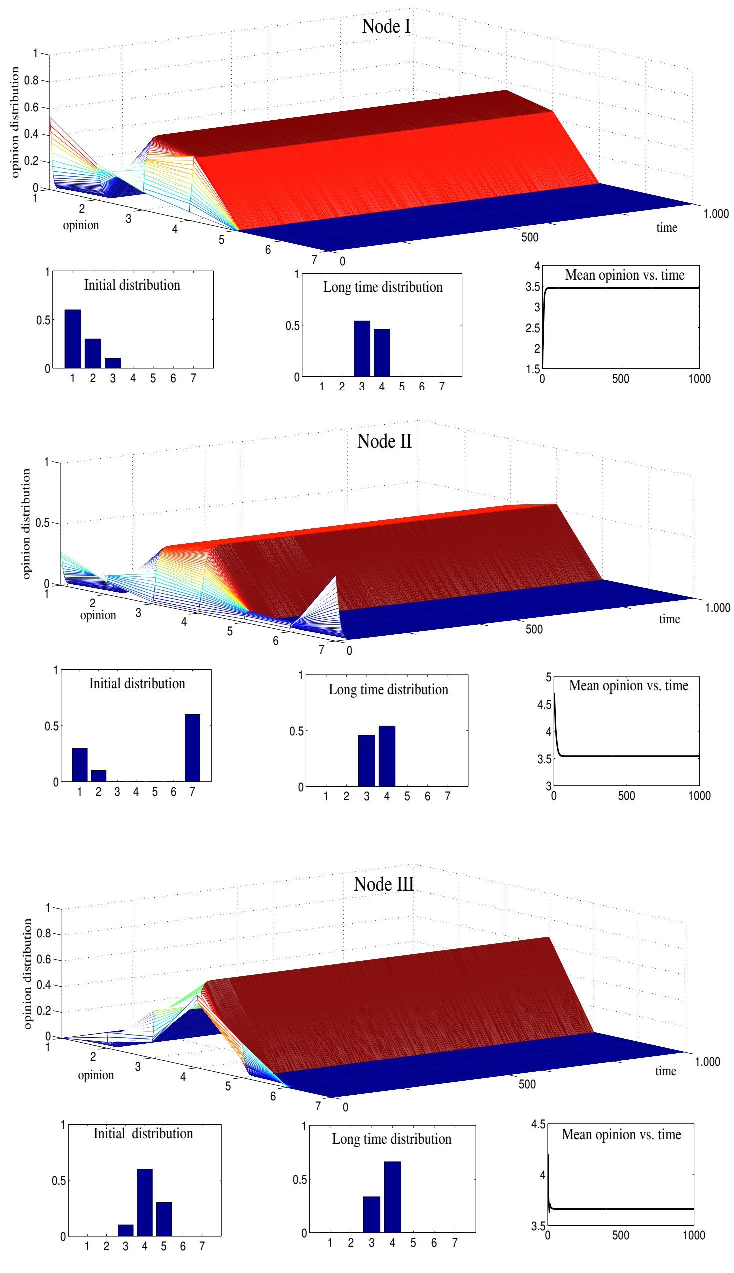

FIGURE 4. **Network.** Different initial distributions and medium threshold on each node ($d = 3.5$) and in the network ($D = 3.5$).

values of the opinion, to consensus toward the mean value of the opinion (Fig.5). In a node of a network, the long time distribution changes from polarization toward the opposite extreme values of the opinion to consensus toward the mean value of the opinion in correspondence of a smaller value for the threshold (Fig.6) with respect to the one previously obtained (Fig.5). From this simulation one can argue

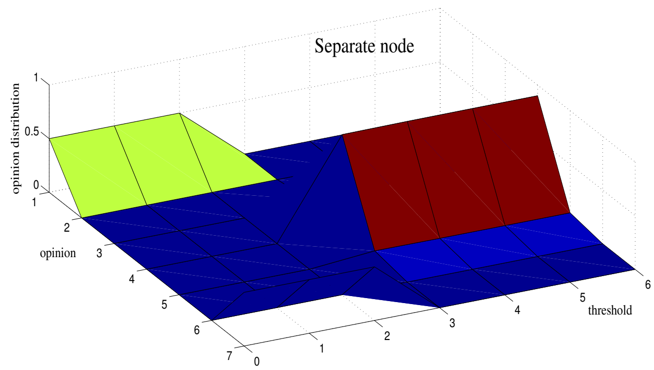

FIGURE 5. Asymptotic opinion trend as function of the threshold **in a separate node.**

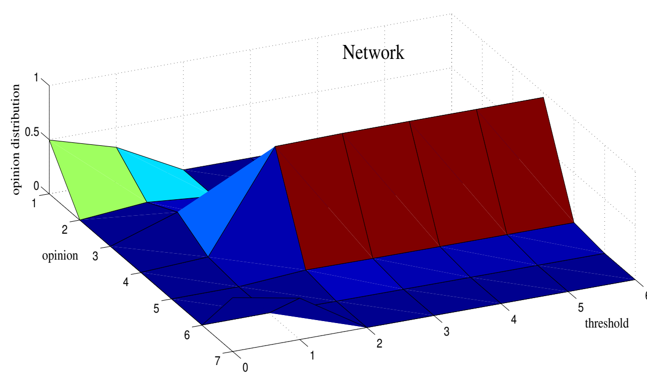

FIGURE 6. Asymptotic opinion trend as function of the threshold **in a network.**

that in the model the effect of the network is such that the polarization toward a mean opinion value is reached for a lower value of the threshold with respect to a separate node with the same dynamic of interactions. We remember here that the threshold represents the propensity of the population to reach a fair compromise and one can argue that this propensity is enhanced when the nodes are interacting in a network.

7. **Conclusions and research perspectives.** The particular conditions determining different trends for the long time distributions are presented in the tables at the end of the paper summarizing the results of the showed simulations.

Some perspectives on the research presented in the paper regard both analytical and modeling problems. With respect to the first viewpoint, we think that the analysis of the asymptotic behavior of the solutions including the identification of possible equilibrium states deserves attention and may open interesting mathematical problems.

As documented in [17], an active research field regards the distributions of opinions with respect to the public consensus on the European Union (EU) within and among the member states; on this issue the change in the shape of the distribution with respect to the simple analysis of a change in the mean value of the distribution is claimed, due also to the consideration that different distribution shape may have different political implications. In our framework the long time opinion distribution is influenced also by the shape of the opinion distribution on other nodes of a network which may be representative of the different union member states.

Another point of interest may be that of the analysis of the interactions of different dynamics, e.g. opinion dynamics and economic one. The modeling of interactions among different group of interests, e.g citizens, political institutions and competing groups [1], is an active research field in socio-economic studies (see [1] and references therein) and our opinion is that the mathematical framework proposed may be of interest in the development of new models enlighting the role of the network in these socio-economic contexts [16].

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Case I – *Effect of the threshold on the long time distribution.*

Low threshold **with initial uniform distribution**	**SEPARATE NODE**	**Polarization** **toward the extreme values**
	NETWORK	
Medium threshold **with initial uniform distribution**	**SEPARATE NODE**	**Polarization** **toward the extreme values**
	NETWORK	**Consensus** **toward the medium value**
High threshold **with initial uniform distribution**	**SEPARATE NODE**	**Consensus** **toward the medium value**
	NETWORK	

Case II – *Effect of the initial distribution on the long time one.*

Initial distribution of disagreement	**SEPARATE NODE**	**Long time distribution of disagreement**
	NETWORK	**Consensus toward the medium value**
Initial distribution polarized toward the extremes	**SEPARATE NODE**	**Long time distribution polarized toward the extremes**
	NETWORK	**Consensus toward the medium value**
Initial distribution of consensus toward the medium value	**SEPARATE NODE**	**Long time distribution of agreement**
	NETWORK	**Consensus toward the medium value**

Case III – *Bifurcation of the long time distribution.*

Bifurcation value	**SEPARATE NODE**	**Medium**
	NETWORK	**Smaller than medium**

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