NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2015.10.749
©American Institute of Mathematical Sciences
Volume **10**, Number **4**, December **2015** pp. **749–785**

ANALYSIS OF A SYSTEM OF NONLOCAL CONSERVATION LAWS FOR MULTI-COMMODITY FLOW ON NETWORKS

Martin Gugat, Alexander Keimer and Günter Leugering

Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
Department Mathematik, Chair of Applied Mathematics 2
Cauerstraße 11, 91058 Erlangen, Germany

Zhiqiang Wang

School of Mathematical Sciences and
Shanghai Key Laboratory for Contemporary Applied Mathematics
Fudan University
Shanghai 200433, China

(Communicated by Alfio Quarteroni)

Abstract. We consider a system of scalar nonlocal conservation laws on networks that model a highly re-entrant multi-commodity manufacturing system as encountered in semi-conductor production. Every single commodity is modeled by a nonlocal conservation law, and the corresponding PDEs are coupled via a collective load, the *work in progress*. We illustrate the dynamics for two commodities. In the applications, directed acyclic networks naturally occur, therefore this type of networks is considered. On every edge of the network we have a system of coupled conservation laws with nonlocal velocity. At the junctions the right hand side boundary data of the foregoing edges is passed as left hand side boundary data to the following edges and PDEs. For distributing junctions, where we have more than one outgoing edge, we impose time dependent distribution functions that guarantee conservation of mass. We provide results of regularity, existence and well-posedness of the multi-commodity network model for L^p-, BV- and $W^{1,p}$-data. Moreover, we define an L^2-tracking type objective and show the existence of minimizers that solve the corresponding optimal control problem.

1. **Introduction.** We consider a system of nonlocal conservations laws that models multi-commodity flow on networks. This generalizes a model for supply chains that has been introduced in [5] and – for unbounded space horizon – been studied in [10]. In [2] the authors consider systems of nonlocal conservation laws, where the "nonlocalness" is realized by a convolution, while in [8] scalar nonlocal conservation laws are investigated, arising particularly in traffic flow modelling. Thereby, the nonlocal modeling is again realized by a convolution over a neighborhood of a given position x. For finite space-time horizon [11] offers regularity results for the

2010 *Mathematics Subject Classification.* 49J20, 93C20, 49N60, 35F61, 35L65, 35L50.

Key words and phrases. Existence of minimizers, conservation laws, multi-commodity model, conservation laws on networks, nonlocal conservation laws, optimal nodal control, optimal control of conservation laws on networks, systems of hyperbolic PDEs.

Zhiqiang Wang was partially supported by the National Science Foundation of China (No. 11271082), the State Key Program of National Natural Science Foundation of China (No. 11331004).

underlying PDE and shows existence of an optimal control in an optimal control framework for a single commodity. In [20] the authors use a formal Lagrange approach to compute an optimal control numerically. For the named finite time-space horizon the model reads for $T \in \mathbb{R}_{>0}$ as

$$
\begin{aligned}
\dot{\rho}(t,x) &= -\lambda(W(t,\rho))\rho_x(t,x) && (t,x) \in (0,T) \times (0,1)\\
\rho(0,x) &= \rho_0(x) && x \in (0,1)\\
\lambda(W(t,\rho))\rho(t,1) &= a(t) && t \in (0,T)\\
W(t,\rho) &:= \int_0^1 \rho(t,s)\,\mathrm{d}s && t \in [0,T].
\end{aligned}
$$

Here, we have $\lambda \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}_{>0})$ and ρ_0 and a are given data of a certain regularity. A typical example for λ is $\lambda(W) = \frac{1}{1+W}$ (see [5],[20]). The function $W(t,\rho)$ is called *work in progress*.

$\dot{\rho}(t,x)$ denotes the partial derivative with respect to the first variable, which we call t and identify it canonically as time variable, while ρ_x denotes the partial derivative of ρ with respect to the second variable x, identified as space variable. The "nonlocalness" of the model is due to the *work in progress* $W(t,\rho)$ that takes into account the load of goods on the entire lane, parametrized by $x \in (0,1)$. We extend this model to a multi-commodity flow model on networks.

For the extension to networks we name [17, 18]. In [16] the authors consider the extension of stationary isothermal Euler equations on pipeline networks for gas transportation. Multi-commodity models come primary from linear min-cost-flow problems as given in [6, 12, 19, 4].

For first order informations and the (formal) Lagrange approach to compute optimality conditions and to obtain the adjoint PDEs we refer to [20] and [15], where the authors also study the regularity of the emerging adjoint PDE.

In Section 2 we introduce the named multi-commodity model and propose a detailed analysis of the regularity with respect to initial and boundary data (L^p, BV, $W^{1,p}$). We illustrate the dynamics and the introduced interdependence for two commodities in Section 2.4.

Furthermore, we show the well-posedness of the model in Section 3.3 if generalized to networks as performed in Section 3. Thereby, the distribution of the multi-commodity flow on dispersing vertices will be realized time-dependent (see Definition 3.2), while in the concerning literature the distribution is often constant in time [17]. Also the named distribution functions will be subject to optimization, such that we have a model, optimizing the inflow and outflow with respect to a given demand, and also the routing of goods in the supply chain. We refer to the objective functional in Definition 3.6.

To show existence of minimizers we have to choose the distribution functions from the set of functions with finite BV-norm as done in Section 3.4. In application, this set seems to be a good choice since it still allows jumps in the distribution functions with respect to time but not infinitely many – assuming the height of the jumps is not converging to zero sufficiently fast. The work is finalized by a conclusion in Section 4.

2. **The multi-commodity model.**

First, we will concentrate on one single edge and generalize the results to networks in Section 3, afterwards.

Definition 2.1 (Number of commodities)**.** In the following $N \in \mathbb{N}$ denotes the number of different commodities and $\mathfrak{N} := \{1, \dots, N\}$ the set of commodities.

Definition 2.2 (Assumptions on the velocity functions)**.** For $N \in \mathbb{N}$ let $\boldsymbol{\lambda} \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}^N_{>0})$ be a vector of velocities and $\boldsymbol{\Lambda} \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}^{N \times N}_{\geq 0})$ with

$$(\boldsymbol{\Lambda}(\cdot))_{i,j} :\equiv \delta_{i,j} \boldsymbol{\lambda}_i(\cdot) \quad \text{ for } i, j \in \mathfrak{N}$$

a diagonal matrix ($\delta_{i,j}$ represents the Kronecker delta).

Then we consider for $T \in \mathbb{R}_{>0}$ the initial boundary value problem

$$\dot{\boldsymbol{\rho}}(t,x) = -\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}_x(t,x) \qquad (t,x) \in (0,T) \times (0,1), \tag{1}$$

$$\boldsymbol{\rho}(0,x) = \boldsymbol{\rho}_0(x) \qquad x \in (0,1), \tag{2}$$

$$\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}(t,0) = \boldsymbol{a}(t) \qquad t \in (0,T), \tag{3}$$

using the integral expression, the so called generalized WIP (work in progress),

$$W(t,\boldsymbol{\rho}) := \sum_{i=1}^{N} \int_0^1 \boldsymbol{\rho}_i(t,s)\,\mathrm{d}s \qquad t \in [0,T]. \tag{4}$$

Thereby, the functions $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_0$ are vectors of dimension N, $\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}_x(t,x)$, $\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}(t,1)$ and $\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}(t,0)$ denote the usual matrix vector products.

Remark 1 (Invertibility of $\boldsymbol{\Lambda}(W(\cdot,\boldsymbol{\rho}))$)**.** Due to the assumption in Definition 2.2 we can invert $\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))$ and write

$$\boldsymbol{\rho}(t,0) = \boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))^{-1}\boldsymbol{a}(t) \qquad \forall t \in [0,T],$$

where $\boldsymbol{\Lambda}(W(\cdot,\boldsymbol{\rho}))^{-1}$ is given by

$$(\boldsymbol{\Lambda}(W(\cdot,\boldsymbol{\rho})))^{-1}_{i,j} \equiv \delta_{i,j} \frac{1}{\boldsymbol{\lambda}_i(W(\cdot,\boldsymbol{\rho}))} \quad i,j \in \mathfrak{N}.$$

Let us notice at this point that the PDE model is nonlinear, since $\boldsymbol{\lambda}(W(\cdot,\boldsymbol{\rho}))$ itself also depends on the solution of the PDEs system. Let us furthermore notice that the expression

$$W(t,\boldsymbol{\rho}) = \sum_{i \in \mathfrak{N}} \int_0^1 \boldsymbol{\rho}_i(t,s)\,\mathrm{d}s$$

collects the load of all commodities at time $t \in [0,T]$ and is still called WIP (work in progress). We refer also to Remark 3. The modeling is reasonable since one can expect from a processing unit, processing several goods simultaneously that the load of one single commodity influences the processing velocity of any remaining commodity in the processing unit. This behavior is the reason to call the dynamics nonlocal.

In the following we will propose a strict and rigorous analysis study concerning regularity of the system of PDEs with respect to the regularity of initial and boundary values. To this end we have to define the weak solution of the PDEs system and come up with the canonical definition as was for instance proposed in [11] for the single-commodity model.

2.1. **Weak solution for the multi-commodity model.**

Definition 2.3 (Weak solution of the PDE system)**.** Let $N \in \mathbb{N}$, $T \in \mathbb{R}_{>0}$, $\boldsymbol{\rho}_0 \in L^1((0,1);\mathbb{R}^N)$, $\boldsymbol{a} \in L^1((0,T);\mathbb{R}^N)$ and $\boldsymbol{\lambda} \in C^1(\mathbb{R}_{\geq 0};\mathbb{R}^N_{>0})$ be given. A weak solution to the system of PDEs given in Equation (1) - Equation (4) is a function $\boldsymbol{\rho} \in C([0,T];L^1((0,1);\mathbb{R}^N))$, satisfying the following condition:

$$\forall \tau \in [0,T],\ \forall \boldsymbol{\phi} \in \boldsymbol{\Phi}_\tau,\ \forall i \in \mathfrak{N}:$$

$$\int_0^\tau \int_0^1 \boldsymbol{\rho}_i(t,x)\left(\dot{\boldsymbol{\phi}}_i(t,x) + \boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}))\boldsymbol{\phi}_{ix}(t,x)\right)\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \boldsymbol{a}_i(t)\boldsymbol{\phi}_i(t,0)\,\mathrm{d}t + \int_0^1 \boldsymbol{\rho}_{0,i}(x)\boldsymbol{\phi}_i(0,x)\,\mathrm{d}x = 0\,.$$

Thereby,

$$\boldsymbol{\Phi}_\tau := \left\{\boldsymbol{\phi} \in C^1([0,\tau]\times[0,1])^N : \boldsymbol{\phi}(\tau,x) = \mathbf{0}\ \forall x \in [0,1] \text{ and } \boldsymbol{\phi}(t,1) = \mathbf{0}\ \forall t \in [0,\tau]\right\}$$

represents the space of test functions, depending on τ and

$$W(t,\boldsymbol{\rho}) := \sum_{i=1}^N \int_0^1 \boldsymbol{\rho}_i(t,x)\,\mathrm{d}x, \qquad t \in [0,T].$$

We prove existence of a solution for boundary and initial data of L^p regularity ($p \in [1,\infty)$). In the corresponding proof of Theorem 2.7 we construct the solution with the help of the method of characteristics. That solution can used to deduce regularity results for BV-data and even $W^{1,p}$-data (again for $p \in [1,\infty)$).

2.2. **Preparations for the proof of the main theorem.** For a better understanding and shorter notation we collect some definitions:

Definition 2.4 (Definitions regarding $\boldsymbol{\lambda}$)**.** Let $N \in \mathbb{N}$ and $\boldsymbol{\lambda}$ be given as in Definition 2.2 and $M \in \mathbb{R}_{>0}$. Then we define for $i \in \mathfrak{N}$

$$\begin{aligned} \overline{\boldsymbol{\lambda}}_i(M) &:= \sup_{0\leq W\leq M} \boldsymbol{\lambda}_i(W) & \underline{\boldsymbol{\lambda}}_i(M) &:= \inf_{0\leq W\leq M} \boldsymbol{\lambda}_i(W)\\ \overline{\boldsymbol{\lambda}}(M) &:= \max_{i\in\mathfrak{N}} \overline{\boldsymbol{\lambda}}_i(M) & \underline{\boldsymbol{\lambda}}(M) &:= \min_{i\in\mathfrak{N}} \underline{\boldsymbol{\lambda}}_i(M)\\ \boldsymbol{d}_i(M) &:= \sup_{0\leq W\leq M} \left|\boldsymbol{\lambda}_i{}'(W)\right| & \boldsymbol{d}(M) &:= \max_{i\in\mathfrak{N}} \boldsymbol{d}_i(M). \end{aligned}$$

For the proof of Theorem 2.7 we need the following Lemmata:

Lemma 2.5 (A property of the weak solution)**.** *If $\boldsymbol{\rho} \in C([0,T];L^1((0,1);\mathbb{R}^N))$ is a weak solution as defined in Definition 2.3 to the initial boundary value problem given in Equation (1) - Equation (4), we obtain for every $\tau \in [0,T]$, for every $\boldsymbol{\psi} \in \boldsymbol{\Psi}_\tau$ and for every $i \in \mathfrak{N}$*

$$\int_0^\tau \int_0^1 \boldsymbol{\rho}_i(t,x)\left(\dot{\boldsymbol{\psi}}_i(t,x) + \boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}))\boldsymbol{\psi}_{ix}(t,x)\right)\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \boldsymbol{a}_i(t)\boldsymbol{\psi}_i(t,0)\,\mathrm{d}t - \int_0^1 \boldsymbol{\rho}_i(\tau,x)\boldsymbol{\psi}_i(\tau,x)\,\mathrm{d}x + \int_0^1 \boldsymbol{\rho}_{0,i}(x)\boldsymbol{\psi}_i(0,x)\,\mathrm{d}x = 0,$$

where

$$\boldsymbol{\Psi}_\tau := \left\{\boldsymbol{\psi} \in C^1([0,\tau]\times[0,1])^N : \boldsymbol{\psi}(t,1) = \mathbf{0}\ \forall t \in [0,\tau]\right\}. \tag{5}$$

Proof. We refer to [11, Lemma 2.2]. Although we consider a system of PDEs coupled via the multi-commodity property the proof is almost identical. □

Lemma 2.6 (A property of L^p functions)**.** *Let $p \in [1,\infty), I := (a,b) \subseteq \mathbb{R}$ an open and bounded interval and $f \in L^p(I)$ be given. Then we have*

$$\forall \epsilon > 0,\ \epsilon < \tfrac{b-a}{2}:\ \lim_{h\searrow 0} \|f(\cdot + h) - f\|_{L^p(a,b-\epsilon)} = 0$$

Proof. The proof can easily be carried out by using the density result

$$\overline{C_c^\infty(a,b)}^{\|\cdot\|_{L^p(a,b)}} = L^p(a,b)$$

as for instance stated in [23, Theorem 6.3.15], and approximating f by compactly supported smooth functions. □

2.3. **Existence, uniqueness and regularity of solutions.** In this section we will prove the existence and uniqueness of a weak solution and will provide regularity results for L^p-data, BV-data and $W^{1,p}$-data, where $p \in [1,\infty)$ is given.

Theorem 2.7 (Existence, uniqueness and regularity for L^p-data)**.** *For $N \in \mathbb{N}$, $p \in [1,\infty)$ and $T \in \mathbb{R}_{>0}$ let $\boldsymbol{\rho}_0 \in L^p((0,1);\mathbb{R}^N_{\geq 0})$ and $\boldsymbol{a} \in L^p((0,T);\mathbb{R}^N_{\geq 0})$ be given. Then the initial boundary value problem defined in Equation (1) - Equation (4) admits a unique weak solution $\boldsymbol{\rho}$ as defined in Definition 2.3 with regularity*

$$\boldsymbol{\rho} \in C([0,T]; L^p((0,1);\mathbb{R}^N)) \cap C([0,1]; L^p((0,T);\mathbb{R}^N)).$$

Furthermore, the solution $\boldsymbol{\rho}$ is component-by-component nonnegative almost everywhere in $(0,T)\times(0,1)$.

Proof. Let us begin with citing the reference [11, Proof of Theorem 2.3]. Therein the claimed result is shown for $p = 1$ and $N = 1$, i.e. for only one commodity in the Banach space L^1.

Although the proof that we will provide for the multi-commodity case $N > 1$, has a similar structure and uses the same tools as in the given reference, there are substantial differences to consider:

- We will construct a solution of the initial boundary value problem with the methods of characteristics. Due to the nonlocal flux the characteristics are solutions of a fixed-point equation. For $N > 1$ the existence of a solution of the named fixed-point equation has to be shown simultaneously for all commodities, since the change of one characteristics influences every remaining characteristics. We solve this problem by considering the appropriate product space in Equation (12) for the system of fixed-point equations and by applying Banach fixed-point theorem in Equations (16),(17).
- Another difficulty consists in extending the results to an arbitrary time T as done on Page 760. For one commodity we have to iterate the solution scheme as soon as the characteristic ξ arrives at 1, since henceforward the solution is just dependent on the boundary value and independent on the initial value. For $N > 1$ we have N different commodities and for any of these commodities we have to deal with the problem that every corresponding characteristic – sooner or later – arrives at 1 for the first time. Once every characteristic has arrived at 1, the solution, henceforward, is again just dependent on the boundary values.

Existence of Solutions: The main strategy of proving existence relies on showing the existence of characteristics. This is done in the following. With the help of these characteristics it becomes quite easy to construct the solution for every commodity $i \in \mathfrak{N}$. We define for $\delta \in [0,T]$ and $N \in \mathbb{N}$

$$\boldsymbol{\Omega}_{\delta,M}^{N} := \Bigg\{ \boldsymbol{\xi} \in C([0,\delta])^{N} : \ \boldsymbol{\xi}(0) = \mathbf{0},\ \underline{\boldsymbol{\lambda}}_i(M) \leq \frac{\boldsymbol{\xi}_i(\mathfrak{t}) - \boldsymbol{\xi}_i(t)}{\mathfrak{t} - t} \leq \overline{\boldsymbol{\lambda}}_i(M) \quad \forall i \in \mathfrak{N},\ \forall \mathfrak{t}, t \in [0,\delta], \mathfrak{t} > t \Bigg\}, \tag{6}$$

where we refer to Definition 2.4. In the Definition of $\boldsymbol{\Omega}_{\delta,M}^{N}$ the M is given by

$$M := \sum_{i \in \mathfrak{N}} \left(\|\boldsymbol{a}_i\|_{L^1((0,T);\mathbb{R})} + \|\boldsymbol{\rho}_{0,i}\|_{L^1((0,1);\mathbb{R})} \right). \tag{7}$$

Let us point out that we will name $\boldsymbol{\xi}$ as characteristic curve. Furthermore, we define the following mapping, depending on initial and boundary data $\boldsymbol{\rho}_0$ and $\boldsymbol{a}$:

$$\boldsymbol{F} : \boldsymbol{\Omega}_{\delta,M}^{N} \to C([0,\delta])^{N},$$

$$\boldsymbol{F}_i(\boldsymbol{\xi})(t) := \int_0^t \boldsymbol{\lambda}_i \Bigg(\sum_{j=1}^{N} \Big(\int_0^s \boldsymbol{a}_j(\sigma)\,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x \Big) \Bigg) \mathrm{d}s,\ t \in [0,\delta],\ i \in \mathfrak{N}.$$

As one can observe, this equation will become a fixed-point equation in $\boldsymbol{\xi}$ (assuming that $\boldsymbol{F}$ will again map onto $\boldsymbol{\Omega}_{\delta,M}^{N}$ as a subset of $C([0,\delta])^{N}$). For $\mathfrak{t}, t \in [0,\delta]$ and $\mathfrak{t} > t$ we obtain by the Definition of $\boldsymbol{\Omega}_{\delta,M}^{N}$ in (6)

$$\begin{aligned}\boldsymbol{F}_i(\boldsymbol{\xi})(\mathfrak{t}) - \boldsymbol{F}_i(\boldsymbol{\xi})(t) &= \int_t^{\mathfrak{t}} \boldsymbol{\lambda}_i \Bigg(\sum_{j=1}^{N} \Big(\int_0^s \boldsymbol{a}_j(\sigma)\,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x \Big) \Bigg) \mathrm{d}s \\ &\leq \int_t^{\mathfrak{t}} \overline{\boldsymbol{\lambda}}_i(M)\,\mathrm{d}s = (\mathfrak{t} - t)\overline{\boldsymbol{\lambda}}_i(M)\end{aligned} \tag{8}$$

and

$$\begin{aligned}\boldsymbol{F}_i(\boldsymbol{\xi})(\mathfrak{t}) - \boldsymbol{F}_i(\boldsymbol{\xi})(t) &= \int_t^{\mathfrak{t}} \boldsymbol{\lambda}_i \Bigg(\sum_{j=1}^{N} \Big(\int_0^s \boldsymbol{a}_j(\sigma)\,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x \Big) \Bigg) \mathrm{d}s \\ &\geq \int_t^{\mathfrak{t}} \underline{\boldsymbol{\lambda}}_i(M)\,\mathrm{d}s = (\mathfrak{t} - t)\underline{\boldsymbol{\lambda}}_i(M),\end{aligned} \tag{9}$$

recalling Definition 2.4 and M as defined in Equation (7). The latter estimation is only valid if the expression $\int_0^{1-\boldsymbol{\xi}_j(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x$ is well defined, i.e. $1 - \boldsymbol{\xi}_j(s) \geq 0$. This is certain if $\delta < \frac{1}{\overline{\boldsymbol{\lambda}}_j(M)}$, but the afore-mentioned estimate has to hold for every $j \in \mathfrak{N}$, so we assume

$$\delta < \min_{j \in \mathfrak{N}} \tfrac{1}{\overline{\boldsymbol{\lambda}}_j(M)} \tag{10}$$

and, hence, we obtain by Inequality (8) and Inequality (9)

$$\underline{\boldsymbol{\lambda}}_i(M) \leq \frac{\boldsymbol{F}_i(\boldsymbol{\xi})(\mathfrak{t}) - \boldsymbol{F}_i(\boldsymbol{\xi})(t)}{\mathfrak{t} - t} \leq \overline{\boldsymbol{\lambda}}_i(M) \quad \text{for } i \in \mathfrak{N},$$

which results in

$$\boldsymbol{F} : \boldsymbol{\Omega}_{\delta,M}^N \to \boldsymbol{\Omega}_{\delta,M}^N$$

for $\delta \in (0, T]$ and

$$\delta < \min_{i \in \mathfrak{N}} \tfrac{1}{\overline{\boldsymbol{\lambda}}_i(M)}. \tag{11}$$

Note that such a $\delta > 0$ always exists since $\boldsymbol{\lambda}_i(M)$ is a strictly positive continuous function which attains its maximum and minimum over a compact set. Now we have $\boldsymbol{F}$ to be a self-map and it remains to show that $\boldsymbol{F}$ is a contraction map on $\boldsymbol{\Omega}_{\delta,M}^N$ with respect to a certain normed space. To this end we define a norm of the product space $C([0, \delta])^N$ as

$$\|\boldsymbol{\xi}\|_{C([0,\delta])^N} := \sum_{j=1}^{N} \|\boldsymbol{\xi}_j\|_{C([0,\delta])} = \sum_{j=1}^{N} \max_{t \in [0,\delta]} |\boldsymbol{\xi}_j(t)| \quad \text{for } \boldsymbol{\xi} \in C([0, \delta])^N. \tag{12}$$

This makes the space $C([0, \delta])^N$ a normed space, actually a Banach space.

In the following we will show that with respect to this norm the mapping $\boldsymbol{F}$ is contractive. We have the following estimate for $i \in \mathfrak{N}$, $\boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \in \boldsymbol{\Omega}_{\delta,M}^N$ and $t \in [0, \delta]$:

$$|\boldsymbol{F}_i(\boldsymbol{\xi}^2)(t) - \boldsymbol{F}_i(\boldsymbol{\xi}^1)(t)| =$$

$$\left| \int_0^t \boldsymbol{\lambda}_i\Big(\sum_{j=1}^{N} \int_0^s \boldsymbol{a}_j(\sigma) \,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j^2(s)} \boldsymbol{\rho}_{0,j}(x) \,\mathrm{d}x \Big) - \boldsymbol{\lambda}_i\Big(\sum_{j=1}^{N} \int_0^s \boldsymbol{a}_j(\sigma) \,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j^1(s)} \boldsymbol{\rho}_{0,j}(x) \,\mathrm{d}x \Big) \,\mathrm{d}s \right|$$

Using a Taylor expansion up to order 1 to linearize $\boldsymbol{\lambda}_i$, i.e. using the mean value theorem, the latter expression can be estimated as

$$|\boldsymbol{F}_i(\boldsymbol{\xi}^2)(t) - \boldsymbol{F}_i(\boldsymbol{\xi}^1)(t)|$$

$$\leq \left| \int_0^t \boldsymbol{\lambda}_i'\Big(\sum_{j=1}^{N} \int_0^s \boldsymbol{a}_j(\sigma) \,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j^2(s)} \boldsymbol{\rho}_{0,j}(x) \,\mathrm{d}x + \theta \Big) \sum_{j=1}^{N} \int_{1-\boldsymbol{\xi}_j^1(s)}^{1-\boldsymbol{\xi}_j^2(s)} \boldsymbol{\rho}_{0,j}(x) \,\mathrm{d}x \,\mathrm{d}s \right|,$$

where

$$\theta \in \left\langle 0, \sum_{j=1}^{N} \int_{1-\boldsymbol{\xi}_j^2(s)}^{1-\boldsymbol{\xi}_j^1(s)} \boldsymbol{\rho}_{0,j}(x) \,\mathrm{d}x \right\rangle \tag{13}$$

and $\langle \cdot, \cdot \rangle$ is defined as

$$\langle \cdot, \cdot \rangle := \begin{cases} \mathbb{R} \times \mathbb{R} & \to 2^{\mathbb{R}} \\ (a, b) & \mapsto (\min\{a, b\}, \max\{a, b\}). \end{cases}$$

Using the triangle inequality and pulling out the supremum of $\boldsymbol{\lambda}_i'$, we estimate:

$$|\boldsymbol{F}_i(\boldsymbol{\xi}^2)(t) - \boldsymbol{F}_i(\boldsymbol{\xi}^1)(t)| \leq \boldsymbol{d}_i(M) \int_0^t \left| \sum_{j=1}^{N} \int_{1-\boldsymbol{\xi}_j^1(s)}^{1-\boldsymbol{\xi}_j^2(s)} \boldsymbol{\rho}_{0,j}(x) \,\mathrm{d}x \right| \mathrm{d}s \tag{14}$$

Thereby, we recall Definition 2.4 and the definition of M in Equation (7) and remark that

$$\sum_{j=1}^{N}\Big(\int_0^s \boldsymbol{a}_j(\sigma)\,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j^2(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\Big) + \theta \le M$$

for every θ as given in Equation (13).

To eliminate the absolute value in the integral we define

$$\underline{\boldsymbol{\xi}}(t) := \min\{\boldsymbol{\xi}^1(t), \boldsymbol{\xi}^2(t)\} \text{ and } \overline{\boldsymbol{\xi}}(t) := \max\{\boldsymbol{\xi}^1(t), \boldsymbol{\xi}^2(t)\}.$$

Thereby max and min are meant component-wise and obviously $\underline{\boldsymbol{\xi}}, \overline{\boldsymbol{\xi}}$ are elements of $\boldsymbol{\Omega}_{\delta,M}^N$. This is due to the fact that the functions $(x,y)\mapsto \min\{x,y\}$ and $(x,y)\mapsto \max\{x,y\}$ are continuous itself and the restrictions in $\boldsymbol{\Omega}_{\delta,M}^N$ are also satisfied. We can furthermore estimate Inequality (14)

$$\begin{aligned}\boldsymbol{d}_i(M)\int_0^t \Big|\sum_{j=1}^{N}\int_{1-\boldsymbol{\xi}_j^1(s)}^{1-\boldsymbol{\xi}_j^2(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\Big|\,\mathrm{d}s &\le \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_0^t \Big|\int_{1-\boldsymbol{\xi}_j^1(s)}^{1-\boldsymbol{\xi}_j^2(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\Big|\,\mathrm{d}s\\ &\le \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_0^t \int_{1-\overline{\boldsymbol{\xi}}_j(s)}^{1-\underline{\boldsymbol{\xi}}_j(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

since $\boldsymbol{\rho}_0$ is nonnegative by assumption. Changing the order of integration and using that $\boldsymbol{\xi}_j^1, \boldsymbol{\xi}_j^2, \overline{\boldsymbol{\xi}}_j, \underline{\boldsymbol{\xi}}_j$ are strictly monotonically increasing for all $j \in \mathfrak{N}$ and, therefore, invertible, we obtain

$$\begin{aligned}&\boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_0^t \int_{1-\overline{\boldsymbol{\xi}}_j(s)}^{1-\underline{\boldsymbol{\xi}}_j(s)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\,\mathrm{d}s\\ &= \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\overline{\boldsymbol{\xi}}_j(t)}^{1-\underline{\boldsymbol{\xi}}_j(t)} \int_{\overline{\boldsymbol{\xi}}_j^{-1}(1-x)}^{t} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}s\,\mathrm{d}x + \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\underline{\boldsymbol{\xi}}_j(t)}^{1} \int_{\overline{\boldsymbol{\xi}}_j^{-1}(1-x)}^{\underline{\boldsymbol{\xi}}_j^{-1}(1-x)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}s\,\mathrm{d}x\\ &= \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\overline{\boldsymbol{\xi}}_j(t)}^{1-\underline{\boldsymbol{\xi}}_j(t)} \boldsymbol{\rho}_{0,j}(x)\left(t-\overline{\boldsymbol{\xi}}_j^{-1}(1-x)\right)\mathrm{d}x\\ &\quad + \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\underline{\boldsymbol{\xi}}_j(t)}^{1} \boldsymbol{\rho}_{0,j}(x)\left(\underline{\boldsymbol{\xi}}_j^{-1}(1-x)-\overline{\boldsymbol{\xi}}_j^{-1}(1-x)\right)\mathrm{d}x\\ &\le \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\overline{\boldsymbol{\xi}}_j(t)}^{1-\underline{\boldsymbol{\xi}}_j(t)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\cdot\left(\underline{\boldsymbol{\xi}}_j^{-1}(\underline{\boldsymbol{\xi}}_j(t))-\overline{\boldsymbol{\xi}}_j^{-1}(\underline{\boldsymbol{\xi}}_j(t))\right)\end{aligned}$$

$$
\begin{aligned}
&+ \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\underline{\boldsymbol{\xi}}_j(t)}^{1}\boldsymbol{\rho}_{0,j}(x)\left(\underline{\boldsymbol{\xi}}_j^{-1}(1-x)-\overline{\boldsymbol{\xi}}_j^{-1}(1-x)\right)\mathrm{d}x\\
\leq{}& \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\overline{\boldsymbol{\xi}}_j(t)}^{1-\underline{\boldsymbol{\xi}}_j(t)}\boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\cdot\left(\underline{\boldsymbol{\xi}}_j^{-1}(\underline{\boldsymbol{\xi}}_j(t))-\overline{\boldsymbol{\xi}}_j^{-1}(\underline{\boldsymbol{\xi}}_j(t))\right)\\
&+ \boldsymbol{d}_i(M)\sum_{j=1}^{N}\int_{1-\underline{\boldsymbol{\xi}}_j(t)}^{1}\boldsymbol{\rho}_{0,j}(x)\sup_{x\in[1-\underline{\boldsymbol{\xi}}_j(t),1]}\left(\underline{\boldsymbol{\xi}}_j^{-1}(1-x)-\overline{\boldsymbol{\xi}}_j^{-1}(1-x)\right)\mathrm{d}x\\
\leq{}& \boldsymbol{d}_i(M)\sum_{j=1}^{N}\sup_{y\in[0,\underline{\boldsymbol{\xi}}_j(t)]}\left(\underline{\boldsymbol{\xi}}_j^{-1}(y)-\overline{\boldsymbol{\xi}}_j^{-1}(y)\right)\int_{1-\overline{\boldsymbol{\xi}}_j(t)}^{1}\boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x.
\end{aligned}
$$

Thereby, we have used that $\overline{\boldsymbol{\xi}}_j^{-1}(1-x)$ is decreasing and $\overline{\boldsymbol{\xi}}_j^{-1}(1-x)\geq\overline{\boldsymbol{\xi}}_j^{-1}(\underline{\boldsymbol{\xi}}_j(t))\ \forall x\in[1-\overline{\boldsymbol{\xi}}_j(t),1-\underline{\boldsymbol{\xi}}_j(t)]$.

It remains to estimate the expression $\sup_{y\in[0,\underline{\boldsymbol{\xi}}_j(t)]}\left(\underline{\boldsymbol{\xi}}_j^{-1}(y)-\overline{\boldsymbol{\xi}}_j^{-1}(y)\right)$. We do this in the following way for given $t\in[0,\delta]$, $j\in\mathfrak{N}$ and $y\in[0,\underline{\boldsymbol{\xi}}_j(t)]$:

$$
0\leq\underline{\boldsymbol{\xi}}_j^{-1}(y)-\overline{\boldsymbol{\xi}}_j^{-1}(y)=\left(\underline{\boldsymbol{\xi}}_j^{-1}(y)-\tfrac{\underline{\boldsymbol{\xi}}_j^{-1}(y)+\overline{\boldsymbol{\xi}}_j^{-1}(y)}{2}\right)+\left(\tfrac{\underline{\boldsymbol{\xi}}_j^{-1}(y)+\overline{\boldsymbol{\xi}}_j^{-1}(y)}{2}-\overline{\boldsymbol{\xi}}_j^{-1}(y)\right)
$$

Using $\mathfrak{t}-t\leq\frac{1}{\underline{\boldsymbol{\lambda}}_j(M)}(\boldsymbol{\xi}_j(\mathfrak{t})-\boldsymbol{\xi}_j(t))$ for $\mathfrak{t}>t$ according to Definition (6), i.e. the definition of $\boldsymbol{\Omega}_{\delta,M}^N$, we furthermore estimate

$$
\begin{aligned}
&\left(\underline{\boldsymbol{\xi}}_j^{-1}(y)-\tfrac{\underline{\boldsymbol{\xi}}_j^{-1}(y)+\overline{\boldsymbol{\xi}}_j^{-1}(y)}{2}\right)+\left(\tfrac{\underline{\boldsymbol{\xi}}_j^{-1}(y)+\overline{\boldsymbol{\xi}}_j^{-1}(y)}{2}-\overline{\boldsymbol{\xi}}_j^{-1}(y)\right)\\
&=\tfrac{1}{\underline{\boldsymbol{\lambda}}_j(M)}\left(\overline{\boldsymbol{\xi}}_j\left(\tfrac{\underline{\boldsymbol{\xi}}_j^{-1}(y)+\overline{\boldsymbol{\xi}}_j^{-1}(y)}{2}\right)-\underline{\boldsymbol{\xi}}_j\left(\tfrac{\underline{\boldsymbol{\xi}}_j^{-1}(y)+\overline{\boldsymbol{\xi}}_j^{-1}(y)}{2}\right)\right)\\
&\leq\tfrac{1}{\underline{\boldsymbol{\lambda}}_j(M)}\|\boldsymbol{\xi}_j^1-\boldsymbol{\xi}_j^2\|_{C([0,\delta])}\leq\tfrac{1}{\underline{\boldsymbol{\lambda}}(M)}\|\boldsymbol{\xi}_j^1-\boldsymbol{\xi}_j^2\|_{C([0,\delta])}\,.
\end{aligned}
$$

For the latter inequality see Definition 2.4. Altogether we have for every $i\in\mathfrak{N}$ the estimate

$$
\begin{aligned}
|\boldsymbol{F}_i(\boldsymbol{\xi}^2)(t)-\boldsymbol{F}_i(\boldsymbol{\xi}^1)(t)|&\leq\tfrac{\boldsymbol{d}_i(M)}{\underline{\boldsymbol{\lambda}}(M)}\sum_{j=1}^{N}\|\boldsymbol{\xi}_j^1-\boldsymbol{\xi}_j^2\|_{C([0,\delta])}\int_{1-\overline{\boldsymbol{\xi}}_j(t)}^{1}\boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\\
&\leq\tfrac{\boldsymbol{d}(M)}{\underline{\boldsymbol{\lambda}}(M)}\sum_{j=1}^{N}\|\boldsymbol{\xi}_j^1-\boldsymbol{\xi}_j^2\|_{C([0,\delta])}\int_{1-\overline{\boldsymbol{\lambda}}(M)\delta}^{1}\boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\,,
\end{aligned}
$$

using Inequality (10). Since we have $\boldsymbol{\rho}_0\in L^p((0,1);\mathbb{R}^N)$ and $\boldsymbol{a}\in L^p((0,T);\mathbb{R}^N)$, we can choose $\delta\in\left(0,\frac{1}{\overline{\boldsymbol{\lambda}}(M)}\right)$ sufficiently small such that for every $j\in\mathfrak{N}$ and $\forall x\in[0,1]$

sufficiently large as well as for $t \in (0,T)$ sufficiently small we have

$$\int_{t}^{t+\frac{\overline{\boldsymbol{\lambda}}(M)}{\underline{\boldsymbol{\lambda}}(M)}\delta} \boldsymbol{a}_j(y)\,\mathrm{d}y + \int_{x-\overline{\boldsymbol{\lambda}}(M)\delta}^{x} \boldsymbol{\rho}_{0,j}(y)\,\mathrm{d}y \le \frac{\underline{\boldsymbol{\lambda}}(M)}{2N\boldsymbol{d}(M)} \tag{15}$$

The existence of such $\delta > 0$ uniformly in x and t is guaranteed by [25, Theorem 8.18]. Therein the uniform continuity of the Lebesgue integral with respect to the measure of integration is stated, i.e. for a function $f \in L^1(I;\mathbb{R})$ with $I \subset \mathbb{R}$ a real interval we have

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall \tilde{I} \subset I : \mu(\tilde{I}) \le \delta \implies \|f\|_{L^1(\tilde{I};\mathbb{R})} \le \epsilon,$$

where $\mu(\tilde{I})$ denotes the one dimensional Lebesgue measure of a measurable set $\tilde{I} \subset \mathbb{R}$. Therefore, choosing $\epsilon = \frac{\underline{\boldsymbol{\lambda}}(M)}{2N\boldsymbol{d}(M)}$ we obtain such a δ for $\boldsymbol{\rho}_{0,j}$ and also uniformly in j ($\mathfrak{N}$ is a finite set), since $\overline{\boldsymbol{\lambda}}(M)$ and $\underline{\boldsymbol{\lambda}}(M)$ are constant. The same holds true for $\boldsymbol{a}_j$ such that we can take the smallest $\delta > 0$, $\delta < \min_{i\in\mathfrak{N}} \frac{1}{\overline{\boldsymbol{\lambda}}_i}$, satisfying Inequality (15) for every $j \in \mathfrak{N}$.

By that inequality we obtain for $i \in \mathfrak{N}$ and $x = 1$

$$\|\boldsymbol{F}_i(\boldsymbol{\xi}^2) - \boldsymbol{F}_i(\boldsymbol{\xi}^1)\|_{C([0,\delta])} \le \tfrac{1}{2N} \sum_{j=1}^{N} \|\boldsymbol{\xi}_j^1 - \boldsymbol{\xi}_j^2\|_{C([0,\delta])} = \tfrac{1}{2N}\|\boldsymbol{\xi}^1 - \boldsymbol{\xi}^2\|_{C([0,\delta])^N} \tag{16}$$

and, thus,

$$\|\boldsymbol{F}(\boldsymbol{\xi}^2) - \boldsymbol{F}(\boldsymbol{\xi}^1)\|_{C([0,\delta])^N} \le \tfrac{1}{2}\|\boldsymbol{\xi}^1 - \boldsymbol{\xi}^2\|_{C([0,\delta])^N}\,. \tag{17}$$

Therefore, $\boldsymbol{F}$ is a contraction in $\boldsymbol{\Omega}_{\delta,M}^N$ and by the Banach fixed-point theorem and the closedness of $\boldsymbol{\Omega}_{\delta,M}^N$ we know that there exists a unique fixed-point $\boldsymbol{\xi}$ satisfying the equation

$$\boldsymbol{\xi} \equiv \boldsymbol{F}(\boldsymbol{\xi}).$$

Since $\boldsymbol{\xi} \in \boldsymbol{\Omega}_{\delta,M}^N$ we know furthermore that $\boldsymbol{\xi}(t)$ is component-wise increasing with respect to $t \in [0,\delta]$. Also, by the definition of $\boldsymbol{F}(\boldsymbol{\xi})(t)$, we can compute for $t \in [0,\delta]$ the derivative by the fundamental theorem of calculus as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F}(\boldsymbol{\xi})(t) = \boldsymbol{\lambda}\Bigg(\sum_{j=1}^{N}\Big(\int_0^t \boldsymbol{a}_j(\sigma)\,\mathrm{d}\sigma + \int_0^{1-\boldsymbol{\xi}_j(t)} \boldsymbol{\rho}_{0,j}(x)\,\mathrm{d}x\Big)\Bigg).$$

Obviously, this is a continuous function for $t \in [0,\delta]$ and, therefore, we can conclude that $\boldsymbol{\xi} \in C^1([0,\delta])^N$ and

$$\boldsymbol{\xi}'(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F}(\boldsymbol{\xi})(t) \quad \text{for } t \in [0,\delta].$$

As mentioned above, we are now able to give the explicit solution of the initial boundary value problem using the constructed characteristic, and state the solution for $i \in \mathfrak{N}$ as:

$$\boldsymbol{\rho}_i(t,x) = \begin{cases} \boldsymbol{\rho}_{0,i}(x - \boldsymbol{\xi}_i(t)) & \text{for } 0 \le \boldsymbol{\xi}_i(t) \le x \le 1,\ 0 \le t \le \delta \\ \frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))} & \text{for } 0 \le x \le \boldsymbol{\xi}_i(t) \le 1,\ 0 \le t \le \delta, \end{cases} \tag{18}$$

where

$$\boldsymbol{\xi}(t) = \int_0^t \boldsymbol{\lambda}(W(s,\boldsymbol{\rho}))\,\mathrm{d}s$$

is by the latter fixed-point results for sufficiently small $t \in [0,\delta]$ unique. For $t \in [0,\delta]$ we can compute $W(t,\boldsymbol{\rho})$ and upon substituting $\mathfrak{t} = \boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-s),\ \mathfrak{s} = s - \boldsymbol{\xi}_i(t)$ for $i \in \mathfrak{N}$ we estimate

$$\begin{aligned}
W(t,\boldsymbol{\rho}) &= \sum_{i=1}^{N} \int_0^1 \boldsymbol{\rho}_i(t,s)\,\mathrm{d}s = \sum_{i=1}^{N} \left(\int_0^{\boldsymbol{\xi}_i(t)} \frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-s))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-s))}\,\mathrm{d}s + \int_{\boldsymbol{\xi}_i(t)}^1 \boldsymbol{\rho}_{0,i}(s-\boldsymbol{\xi}_i(t))\,\mathrm{d}s \right) \\
&= \sum_{i=1}^{N} \left(\int_0^t \boldsymbol{a}_i(\mathfrak{t})\,\mathrm{d}\mathfrak{t} + \int_0^{1-\boldsymbol{\xi}_i(t)} \boldsymbol{\rho}_{0,i}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \right) && (19) \\
&\le \sum_{i=1}^{N} \left(\|\boldsymbol{a}_i\|_{L^1((0,T);\mathbb{R})} + \|\boldsymbol{\rho}_i\|_{L^1((0,1);\mathbb{R})} \right) = M. && (20)
\end{aligned}$$

Therefore, we have

$$0 < \underline{\boldsymbol{\lambda}}_i(M) \le \boldsymbol{\xi}_i'(t) = \boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho})) \le \overline{\boldsymbol{\lambda}}_i(M) \quad \forall t \in [0,\delta],\ i \in \mathfrak{N}.$$

The further steps to show that solution (18) is indeed a weak solution of postulated regularity will be omitted. Considering every single commodity on its own, this can be done exactly as in [11, Theorem 2.3].

Uniqueness of the solution: Let us assume that we have another weak solution $\tilde{\boldsymbol{\rho}}$ to the initial boundary value problem stated in Equations (1)-(4). Then we can apply Lemma 2.5 to obtain the following integral equation for every $\tau \in [0,\delta]$, $\boldsymbol{\psi} \in \boldsymbol{\Psi}_\tau$ as defined in (5) and $i \in \mathfrak{N}$:

$$\begin{aligned}
&\int_0^\tau \int_0^1 \tilde{\boldsymbol{\rho}}_i(t,x) \left(\dot{\boldsymbol{\psi}}_i(t,x) + \boldsymbol{\lambda}_i(W(t,\tilde{\boldsymbol{\rho}}))\boldsymbol{\psi}_{ix}(t,x) \right) \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \boldsymbol{a}_i(t)\boldsymbol{\psi}_i(t,0)\,\mathrm{d}t \\
&\qquad - \int_0^1 \tilde{\boldsymbol{\rho}}_i(\tau,x)\boldsymbol{\psi}_i(\tau,x)\,\mathrm{d}x + \int_0^1 \boldsymbol{\rho}_{0,i}(x)\boldsymbol{\psi}_i(0,x)\,\mathrm{d}x = 0.
\end{aligned} \tag{21}$$

The approach to show uniqueness is to define the "right" test function $\boldsymbol{\psi} \in \boldsymbol{\Psi}_\tau$. Therefore, let $\tilde{\boldsymbol{\xi}}(t) := \int_0^t \boldsymbol{\lambda}(W(s,\tilde{\boldsymbol{\rho}}))\,\mathrm{d}s$ and $\boldsymbol{\psi}_0 \in C_{\mathrm{c}}^1((0,1))^N$. Furthermore, let for $i \in \mathfrak{N}$ and $0 \le t \le \tau$ the function $\boldsymbol{\psi} : [0,\tau] \times [0,1] \to \mathbb{R}^N$ be defined as

$$\boldsymbol{\psi}_i(t,x) := \begin{cases} \boldsymbol{\psi}_{0,i}\left(\tilde{\boldsymbol{\xi}}_i(\tau) - \tilde{\boldsymbol{\xi}}_i(t) + x \right) & \text{for } 0 \le x \le \tilde{\boldsymbol{\xi}}_i(t) - \tilde{\boldsymbol{\xi}}_i(\tau) + 1, \\ 0 & \text{for } 0 \le \tilde{\boldsymbol{\xi}}_i(t) - \tilde{\boldsymbol{\xi}}_i(\tau) + 1 \le x \le 1. \end{cases} \tag{22}$$

The function $\boldsymbol{\psi}_i$ obviously belongs for every $i \in \mathfrak{N}$ to $C^1([0,\tau]\times[0,1])$. However, it also belongs to $\boldsymbol{\Psi}_\tau$, since it also has the property $\boldsymbol{\psi}(0,x) = \mathbf{0}$ for every $x \in [0,1]$. So it is a suitable test function. Furthermore, this test function satisfies the following end boundary value problem,

$$\begin{aligned}
\dot{\boldsymbol{\psi}}(t,x) + \boldsymbol{\Lambda}(W(t,\tilde{\boldsymbol{\rho}}))\boldsymbol{\psi}_x(t,x) &= \mathbf{0} && (t,x) \in [0,\tau]\times[0,1] && (23) \\
\boldsymbol{\psi}(\tau,x) &= \boldsymbol{\psi}_0(x) && x \in [0,1]
\end{aligned}$$

$$\boldsymbol{\psi}(t,1)=\mathbf{0} \qquad\qquad t\in[0,\tau]$$

which can easily be verified. If we insert this function into Equation (21) as a test function, we obtain by integration by substitution (the first term vanishes, since $\boldsymbol{\psi}$ satisfies the afore-mentioned PDE (23)) and the definition of $\boldsymbol{\psi}_i$ in Equation (22) for $i\in\mathfrak{N}$

$$\begin{aligned}
\int_0^1 \tilde{\boldsymbol{\rho}}_i(\tau,x)\boldsymbol{\psi}_i(\tau,x)\,\mathrm{d}x &= \int_0^1 \tilde{\boldsymbol{\rho}}_i(\tau,x)\boldsymbol{\psi}_{0,i}(x)\,\mathrm{d}x\\
&= \int_0^\tau \boldsymbol{a}_i(t)\boldsymbol{\psi}_i(t,0)\,\mathrm{d}t + \int_0^1 \boldsymbol{\rho}_{0,i}(x)\boldsymbol{\psi}_i(0,x)\,\mathrm{d}x\\
&= \int_0^\tau \boldsymbol{a}_i(t)\boldsymbol{\psi}_{0,i}\Big(\tilde{\boldsymbol{\xi}}_i(\tau)-\tilde{\boldsymbol{\xi}}_i(t)\Big)\,\mathrm{d}t + \int_0^{1-\tilde{\boldsymbol{\xi}}_i(\tau)} \boldsymbol{\rho}_{0,i}(x)\boldsymbol{\psi}_{0,i}\Big(\tilde{\boldsymbol{\xi}}_i(\tau)+x\Big)\,\mathrm{d}x\\
&= \int_0^{\tilde{\boldsymbol{\xi}}_i(\tau)} \tfrac{\boldsymbol{a}_i(\tilde{\boldsymbol{\xi}}_i^{-1}(\tilde{\boldsymbol{\xi}}_i(\tau)-y))}{\tilde{\boldsymbol{\xi}}_i{}'(\tilde{\boldsymbol{\xi}}_i^{-1}(\tilde{\boldsymbol{\xi}}_i(\tau)-y))}\boldsymbol{\psi}_{0,i}(y)\,\mathrm{d}y + \int_{\tilde{\boldsymbol{\xi}}_i(\tau)}^1 \boldsymbol{\rho}_{0,i}(y-\tilde{\boldsymbol{\xi}}_i(\tau))\boldsymbol{\psi}_{0,i}(y)\,\mathrm{d}y\,.
\end{aligned}$$

Since this equation is true for every $\tau\in[0,\delta]$ and $\boldsymbol{\psi}_0\in C^1_{\mathrm{c}}((0,1))$, we can use the fundamental lemma of calculus of variations in [9, Corollary 4.24] and the fact that $C_0^\infty((0,T))\subset C^1_{\mathrm{c}}((0,1))$ to conclude that the solution $\tilde{\boldsymbol{\rho}}\in C([0,\delta];L^p((0,1);\mathbb{R}^N))$ can be stated for $i\in\mathfrak{N}$ and $t\in[0,\delta]$ as

$$\tilde{\boldsymbol{\rho}}_i(t,x)=\begin{cases}\boldsymbol{\rho}_{0,i}(x-\tilde{\boldsymbol{\xi}}_i(t)) & \text{for } 0\le\tilde{\boldsymbol{\xi}}_i(t)\le x\\ \frac{\boldsymbol{a}_i(\tilde{\boldsymbol{\xi}}_i^{-1}(\tilde{\boldsymbol{\xi}}_i(t)-x))}{\tilde{\boldsymbol{\xi}}_i{}'(\tilde{\boldsymbol{\xi}}_i^{-1}(\tilde{\boldsymbol{\xi}}_i(t)-x))} & \text{for } 0\le x\le\tilde{\boldsymbol{\xi}}_i(t).\end{cases} \tag{24}$$

Now the solution $\tilde{\boldsymbol{\rho}}$ can be used to compute $\tilde{\boldsymbol{\xi}}$, since

$$\begin{aligned}
\tilde{\boldsymbol{\xi}}(t) &= \int_0^t \boldsymbol{\lambda}(W(s,\tilde{\boldsymbol{\rho}}))\,\mathrm{d}s\\
&= \int_0^t \boldsymbol{\lambda}\bigg(\sum_{i=1}^N\Big(\int_0^{\tilde{\boldsymbol{\xi}}_i(s)} \tfrac{\boldsymbol{a}_i(\tilde{\boldsymbol{\xi}}_i^{-1}(\tilde{\boldsymbol{\xi}}_i(s)-x))}{\tilde{\boldsymbol{\xi}}_i{}'(\tilde{\boldsymbol{\xi}}_i^{-1}(\tilde{\boldsymbol{\xi}}_i(s)-x))}\,\mathrm{d}x + \int_{\tilde{\boldsymbol{\xi}}_i(s)}^1 \boldsymbol{\rho}_{0,i}(x-\tilde{\boldsymbol{\xi}}_i(s))\,\mathrm{d}x\Big)\bigg)\,\mathrm{d}s\\
&= \int_0^t \boldsymbol{\lambda}\bigg(\sum_{i=1}^N\Big(\int_0^s \boldsymbol{a}_i(\mathfrak{t})\,\mathrm{d}\mathfrak{t} + \int_0^{1-\tilde{\boldsymbol{\xi}}_i(s)} \boldsymbol{\rho}_{0,i}(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\Big)\bigg)\,\mathrm{d}s = \boldsymbol{F}(\tilde{\boldsymbol{\xi}})(t)\,.
\end{aligned}$$

Also $\tilde{\boldsymbol{\xi}}$ satisfies the fixed-point equation and $\tilde{\boldsymbol{\xi}}\in\boldsymbol{\Omega}^N_{\delta,M}$. This directly implies $\tilde{\boldsymbol{\xi}}\equiv\boldsymbol{\xi}$ since the Banach fixed-point theorem guarantees a unique fixed-point of $\boldsymbol{F}$ in $\boldsymbol{\Omega}^N_{\delta,M}$.

Therefore, we see by the construction of $\tilde{\boldsymbol{\rho}}$ that we have $\tilde{\boldsymbol{\rho}}\equiv\boldsymbol{\rho}$ in $(0,\delta)\times(0,1)$ for δ sufficiently small.

Extension to time $t\in[0,T]$**:** Since we know that for $\delta_1:=\delta>0$ small enough we have a unique weak solution $\boldsymbol{\rho}^1:=\boldsymbol{\rho}$ of regularity $C([0,\delta_1];L^1((0,1);\mathbb{R}^N))$, we can define another initial boundary value problem in $\boldsymbol{\rho}^2$

$$\dot{\boldsymbol{\rho}}^2(t,x)+\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}^2))\boldsymbol{\rho}^2_x(t,x)=\mathbf{0}$$

$$\begin{aligned}\boldsymbol{\rho}^2(0,x) &= \boldsymbol{\rho}^1(\delta_1,x)\\ \boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}^2))\boldsymbol{\rho}^2(t,0) &= \boldsymbol{a}(t+\delta_1).\end{aligned}$$

As a result of the regularity of $\boldsymbol{\rho}^1$ we have $\boldsymbol{\rho}^1(\delta_1,\cdot)\in L^1((0,1);\mathbb{R}^N)$ and also $\boldsymbol{a}(\cdot+\delta_1)\in L^1((0,T-\delta_1);\mathbb{R}^N)$. So the new initial boundary value problem in $\boldsymbol{\rho}^2$ with the afore-mentioned initial and boundary data is of the same type as the initial boundary value problem in $\boldsymbol{\rho}^1$. We again apply the fixed-point argument and can find $\delta_2>0$, such that

$$\boldsymbol{\rho}^2\in C([0,\delta_2];L^1((0,1);\mathbb{R}^N)).$$

Iterating this procedure, illustrated in Figure 1, we obtain a sequence $\boldsymbol{\rho}^k$ and $\delta_k>0$, $k\in\mathbb{N},\ k\geq 2$, satisfying the weak formulation in Definition 2.3 of

$$\begin{aligned}\dot{\boldsymbol{\rho}}^k(t,x)+\boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}^k))\boldsymbol{\rho}^k_x(t,x) &= \mathbf{0} && (t,x)\in[0,\delta_k]\times(0,1)\\ \boldsymbol{\rho}^k(0,x) &= \boldsymbol{\rho}^{k-1}(\delta_{k-1},x) && x\in(0,1)\\ \boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}^k))\boldsymbol{\rho}^k(t,0) &= \boldsymbol{a}\left(t+\sum_{j=1}^{k-1}\delta_j\right) && t\in[0,\delta_k]\end{aligned}$$

with regularity

$$\boldsymbol{\rho}^k\in C([0,\delta_k];L^1((0,1);\mathbb{R}^N)).$$

Of course, we stop the procedure if $\sum_{j=1}^{k}\delta_j>T$. The solution on the whole time horizon can then be written for $t\in\left[0,\sum_{j=1}^{k}\delta_j\right]$ as

$$\boldsymbol{\rho}(t,x):=\begin{cases}\boldsymbol{\rho}^1(t,x) & \text{for } (t,x)\in(0,\delta_1]\times(0,1)\\ \boldsymbol{\rho}^2(t-\delta_1,x) & \text{for } (t,x)\in(\delta_1,\delta_1+\delta_2]\times(0,1)\\ \quad\vdots & \quad\vdots\\ \boldsymbol{\rho}^k\left(t-\sum_{j=1}^{k-1}\delta_k,x\right) & \text{for } (t,x)\in\left(\sum_{j=1}^{k-1}\delta_j,\sum_{j=1}^{k}\delta_j\right]\times(0,1).\end{cases}$$

Therefore, we must make sure that we can exhaust the full time horizon, i.e. the sum $\sum_k\delta_k$ is unbounded.

To guarantee the named property we provide a proof using mathematical induction:

Considering $\boldsymbol{\rho}^2$, the initial value is for $x\in[0,1]$ and $i\in\mathfrak{N}$ piecewise defined as

$$\boldsymbol{\rho}^2_i(0,x)=\begin{cases}\boldsymbol{\rho}_{0,i}(x-\boldsymbol{\xi}_i(\delta)) & x\geq\boldsymbol{\xi}_i(\delta)\\ \frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\delta)-x))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\delta)-x))} & x\leq\boldsymbol{\xi}_i(\delta),\end{cases}\tag{25}$$

where $\boldsymbol{\xi}$ denotes the characteristic curve corresponding to the existence interval $[0,\delta]$ and $\boldsymbol{\rho}^1$. By Equation (25) and the estimate in (20) we obtain – whenever $1-\overline{\boldsymbol{\lambda}}(M)\delta_2<\boldsymbol{\xi}_i(\delta)$ – for $i\in\mathfrak{N}$

$$\int_{1-\overline{\boldsymbol{\lambda}}(M)\delta_2}^{1}\boldsymbol{\rho}^2_{0,i}(y)\,\mathrm{d}y=\int_{1-\overline{\boldsymbol{\lambda}}(M)\delta_2}^{\boldsymbol{\xi}_i(\delta)}\frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\delta)-x))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\delta)-x))}\,\mathrm{d}x+\int_{\boldsymbol{\xi}_i(\delta)}^{1}\boldsymbol{\rho}_{0,i}(x-\boldsymbol{\xi}_i(\delta))\,\mathrm{d}x$$

$$= \int_0^{\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\delta)-1+\overline{\boldsymbol{\lambda}}(M)\delta_2)} \boldsymbol{a}_i(y)\,\mathrm{d}y + \int_0^{1-\boldsymbol{\xi}_i(\delta)} \boldsymbol{\rho}_{0,i}(y)\,\mathrm{d}y. \tag{26}$$

If $1-\overline{\boldsymbol{\lambda}}(M)\delta_2 \geq \boldsymbol{\xi}_i(\delta)$ then $\boldsymbol{\rho}_i^2(0,x)$ is in a sufficiently large neighborhood around $x=1$ only dependent on the initial data and we can apply Inequality (15) with $\delta_2=\delta$.

Now we show that also for the case $1-\overline{\boldsymbol{\lambda}}(M)\delta_2 < \boldsymbol{\xi}_i(\delta)$ we can choose $\delta_2=\delta$. We estimate the area of integration and obtain the estimate

$$\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\delta)-1+\overline{\boldsymbol{\lambda}}(M)\delta_2) \leq \frac{\overline{\boldsymbol{\lambda}}(M)}{\underline{\boldsymbol{\lambda}}(M)}\delta_2,$$

since $\delta \in \left(0, \frac{1}{\overline{\boldsymbol{\lambda}}(M)}\right)$ and, therefore, $\boldsymbol{\xi}_i(\delta) \leq 1$. Thus, we can indeed set $\delta_2=\delta$. Let us furthermore mention that the area of integration in the second integral in Equation (26) is at most $\overline{\boldsymbol{\lambda}}(M)\delta$, such that we can apply Inequality (15) on Equation (26) and obtain for $i \in \mathfrak{N}$

$$\int_{1-\overline{\boldsymbol{\lambda}}(M)\delta}^{1} \boldsymbol{\rho}_i^2(0,x)\,\mathrm{d}x \leq \frac{\underline{\boldsymbol{\lambda}}(M)}{2N\boldsymbol{d}(M)}.$$

Therefore, we have shown that also in the second iteration for extending the time we have for $\delta_2=\delta$ a contraction as demanded in Inequality (17) with contraction factor $\frac{1}{2}$ and can prolongate the existence interval to 2δ.

If we do this for $\boldsymbol{\rho}^3$, we obtain as initial data for $i \in \mathfrak{N}$

$$\boldsymbol{\rho}_i^3(0,x) = \boldsymbol{\rho}_i^2(0,\delta) = \begin{cases} \boldsymbol{\rho}_{0,i}(x-\boldsymbol{\xi}_i^1(\delta)-\boldsymbol{\xi}_i^2(\delta)) & \boldsymbol{\xi}_i^1(\delta)+\boldsymbol{\xi}_i^2(\delta) \leq x \\ \frac{\boldsymbol{a}_i((\boldsymbol{\xi}_i^1)^{-1}(\boldsymbol{\xi}_i^1(\delta)+\boldsymbol{\xi}_i^2(\delta)-x))}{\boldsymbol{\xi}_i^{1\prime}((\boldsymbol{\xi}_i^1)^{-1}(\boldsymbol{\xi}_i^1(\delta)+\boldsymbol{\xi}_i^2(\delta)-x))} & \boldsymbol{\xi}_i^2(\delta) \leq x \leq \boldsymbol{\xi}_i^1(\delta)+\boldsymbol{\xi}_i^2(\delta) \\ \frac{\boldsymbol{a}_i(\delta+(\boldsymbol{\xi}_i^2)^{-1}(\boldsymbol{\xi}_i^2(\delta)-x))}{\boldsymbol{\xi}_i^{2\prime}((\boldsymbol{\xi}_i^2)^{-1}(\boldsymbol{\xi}_i^2(\delta)-x))} & \boldsymbol{\xi}_i^2(\delta) \geq x \end{cases}$$

where $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$ denote the characteristics on $[0,\delta]$, $[\delta,2\delta]$ respectively. Applying the fixed-point argument again, we end up with showing that the estimate

$$\int_{1-\overline{\boldsymbol{\lambda}}(M)\delta_3}^{1} \boldsymbol{\rho}_{0,i}^3(y)\,\mathrm{d}y \leq \frac{\underline{\boldsymbol{\lambda}}(M)}{2N\boldsymbol{d}(M)} \tag{27}$$

is true for sufficiently large $\delta_3>0$ to guarantee that we can exhaust the full time horizon with a sequence of these initial boundary value problems.

We have to distinguish two cases:

- $\boldsymbol{\xi}_i^1(\delta)+\boldsymbol{\xi}_i^2(\delta) \leq 1$: We obtain for Equation (27) the estimate

$$\int_{1-\overline{\boldsymbol{\lambda}}(M)\delta_3}^{1} \boldsymbol{\rho}_{0,i}^3(y)\,\mathrm{d}y \leq \int_0^{\overline{\boldsymbol{\lambda}}(M)\delta_3} \boldsymbol{\rho}_{0,i}(y)\,\mathrm{d}y + \int_0^{\frac{\overline{\boldsymbol{\lambda}}(M)}{\underline{\boldsymbol{\lambda}}(M)}\delta_3} \boldsymbol{a}_i(y)\,\mathrm{d}y \leq \frac{\underline{\boldsymbol{\lambda}}(M)}{2N\boldsymbol{d}(M)}, \tag{28}$$

  when choosing $\delta_3=\delta$ and applying Equation (15). Furthermore, we have used that $\boldsymbol{\xi}_i^{-1}$ is for every $i \in \mathfrak{N}$ Lipschitz-continuous with Lipschitz constant $\frac{1}{\underline{\boldsymbol{\lambda}}(M)}$.

- $\boldsymbol{\xi}_i^1(\delta)+\boldsymbol{\xi}_i^2(\delta)>1$: Then, the initial value $\boldsymbol{\rho}_i^3(0,x)$ is not explicitly dependent on $\boldsymbol{\rho}_0$ and we have just to perform the analogous estimate as in Inequality (28) omitting the first summand.

This procedure can be iterated arbitrarily and we can choose, by induction, $\delta_k=\delta$ for every $k\in\mathbb{N}$.

Altogether, we can extend the solution to a unique solution on the full time horizon $t\in[0,T]$ for finite but arbitrary $T>0$ and $i\in\mathfrak{N}$ to

$$\boldsymbol{\rho}_i(t,x)=\begin{cases}\boldsymbol{\rho}_{0,i}(x-\boldsymbol{\xi}_i(t)) & \text{for } 0\le\boldsymbol{\xi}_i(t)\le x\le 1,\ 0\le t\le T\\ \frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))} & \text{for } 0\le x\le\boldsymbol{\xi}_i(t)\le 1,\ 0\le t\le T\\ \frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))} & \text{for } 0\le x\le 1,\ \boldsymbol{\xi}_i^{-1}(1)\le t\le T.\end{cases}\tag{29}$$

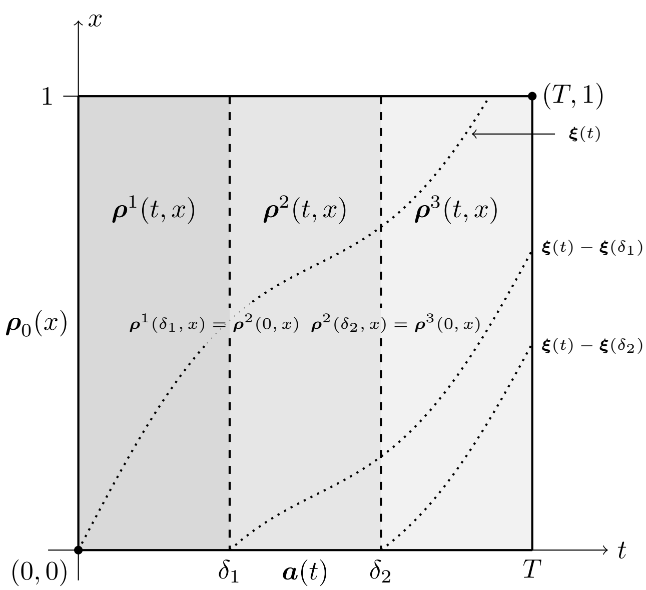

FIGURE 1. Decomposition of the PDE on the time horizon $[0,T]$ into a finite number (here 3) of initial boundary value problems on smaller time horizons

L^p-regularity and nonnegativity: Let us assume that $\boldsymbol{\rho}_0\in L^p((0,1);\mathbb{R}^N_{\ge 0})$ and $\boldsymbol{a}\in L^p((0,T);\mathbb{R}^N_{\ge 0})$ are given and let $t,\mathfrak{t}\in[0,T]$, $\mathfrak{t}>t$ be fixed. We obtain for $i\in\mathfrak{N}$, inserting the explicit solution for $\boldsymbol{\rho}$ as stated in Equation (29),

$$\begin{aligned}\|\boldsymbol{\rho}_i(\mathfrak{t},\cdot)-\boldsymbol{\rho}_i(t,\cdot)\|^p_{L^p((0,1);\mathbb{R})}&=\int_0^1|\boldsymbol{\rho}_i(\mathfrak{t},s)-\boldsymbol{\rho}_i(t,s)|^p\,\mathrm{d}s\\&=\int_0^{\boldsymbol{\xi}_i(t)}\left|\frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\mathfrak{t})-s))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(\mathfrak{t})-s))}-\frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-s))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-s))}\right|^p\mathrm{d}s\end{aligned}$$

$$+\int_{\boldsymbol{\xi}_i(t)}^{\boldsymbol{\xi}_i(\mathfrak{t})} |\boldsymbol{\rho}_i(\mathfrak{t},s)-\boldsymbol{\rho}_i(t,s)|^p \,\mathrm{d}s$$
$$+\int_{\boldsymbol{\xi}_i(\mathfrak{t})}^{1} \left|\boldsymbol{\rho}_{0,i}(s-\boldsymbol{\xi}_i(\mathfrak{t}))-\boldsymbol{\rho}_{0,i}(s-\boldsymbol{\xi}_i(t))\right|^p \,\mathrm{d}s\,.$$

Now, applying Lemma 2.6 and recalling that $\boldsymbol{\xi}$ is continuously differentiable and strictly monotone, therefore invertible, we can easily conclude that for $t \to \mathfrak{t}$ the first and third term converge to zero, while the second term $\int_{\boldsymbol{\xi}_i(t)}^{\boldsymbol{\xi}_i(\mathfrak{t})} |\boldsymbol{\rho}_i(\mathfrak{t},s)-\boldsymbol{\rho}_i(t,s)|^p \,\mathrm{d}s$ tends to zero for $t \to \mathfrak{t}$ due to the measure of integration tends to zero and the integrand is bounded in the L^p-norm.

Altogether, we obtain the claimed regularity for every good $i \in \mathfrak{N}$, and finally

$$\boldsymbol{\rho} \in C([0,T];L^p((0,1);\mathbb{R}^N)).$$

The result $\boldsymbol{\rho} \in C([0,1];L^p((0,T);\mathbb{R}^N))$ can analogously be proved. We omit the details. Finally, due to the explicit construction of the solution in Equation (29) by characteristics, it is easy to verify the nonnegativity of the solution. □

Remark 2 (Existence of a weak solution for arbitrary large time)**.** One might ask why we can expect global solution and not only local solutions for the pde system in Equation (1)-Equation (4) in time as it is, generally, for nonlinear conservation laws and why we do not need any smallness assumptions on initial data and boundary conditions. This is due to the fact that the velocity function $\boldsymbol{\lambda}$ is not dependent on the space domain and is - for finite W - strictly positive. Additionally, the work in progress $W(t,\boldsymbol{\rho})$ is not increasing over time as can be seen in Inequality (20). Both conditions prevent any blow-up and we can expect a global solution in time as stated in Theorem 2.7.

Remark 3 (The work in progress "WIP" $W(\cdot,*)$)**.** To be more precise about the definition of $W(\cdot,*)$ we can define it for $N \in \mathbb{N}$ and $T \in \mathbb{R}_{>0}$ as

$$W : \begin{cases} [0,T] \times C([0,T];L^1((0,1);\mathbb{R}^N_{\geq 0})) & \to C([0,T];\mathbb{R}_{\geq 0}) \\ (t,\boldsymbol{\rho}) & \mapsto \sum_{i=1}^{N} \int_0^1 \boldsymbol{\rho}_i(t,s)\,\mathrm{d}s. \end{cases} \tag{30}$$

Thereby, the necessary regularity in the Definition 2.3 for $W(\cdot,*)$ is justified by Theorem 2.9.

The following Theorem contains regularity results for BV-input data:

Theorem 2.8 (Regularity for BV-data)**.** *For $N \in \mathbb{N}$ and $T \in \mathbb{R}_{>0}$ let $\boldsymbol{\rho}_0 \in BV((0,1);\mathbb{R}^N_{\geq 0})$ and $\boldsymbol{a} \in BV((0,T);\mathbb{R}^N_{\geq 0})$ be given. Then the initial boundary value problem defined in Equation (1) - Equation (4) admits a unique weak solution $\boldsymbol{\rho}$ as defined in Definition 2.3 with the additional regularity*

$$\boldsymbol{\rho} \in L^\infty((0,T);BV((0,1);\mathbb{R}^N)) \cap L^\infty((0,1);BV((0,T);\mathbb{R}^N)).$$

Proof. Recall the definition of the BV-norm $\|\cdot\|_{BV(\Omega)} := \|\cdot\|_{L^1(\Omega)} + |\cdot|_{V(\Omega)}$, where $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ open and bounded and V is the total variation, i.e.

$$|f|_{V(\Omega)} := \sup_{\substack{\boldsymbol{\phi} \in C_c^1(\Omega;\mathbb{R}^d)\\ \|\boldsymbol{\phi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1}} \int_\Omega f(\boldsymbol{x}) \operatorname{div}(\boldsymbol{\phi}(\boldsymbol{x})) \,\mathrm{d}\boldsymbol{x}.$$

Note that $|\cdot|_{V(\Omega)}$ is only a semi-norm (take $f \equiv \text{const}$) and we denote it by $|\cdot|$ and not by $\|\cdot\|$. Coming back to our estimate, we only have $d = 1$ and to prove

$$\|\boldsymbol{\rho}\|_{L^\infty((0,T);BV((0,1);\mathbb{R}^N))} < \infty,$$

i.e.

$$\|\boldsymbol{\rho}\|_{L^\infty((0,T);L^1((0,1);\mathbb{R}^N))} + |\boldsymbol{\rho}|_{L^\infty((0,T);V((0,1);\mathbb{R}^N))} < \infty.$$

Since $BV(I) \hookrightarrow L^p(I)$ for all $p \in [1,\infty]$ and $I \subset \mathbb{R}$ an open bounded interval (see [3, Corollary 3.49]), and the trivial embedding

$$C([0,T];L^1((0,1);\mathbb{R}^N)) \hookrightarrow L^\infty((0,T);L^1((0,1);\mathbb{R}^N)),$$

we can easily conclude the regularity with respect to the $L^\infty((0,T);L^1(0,1);\mathbb{R}^N))$-topology by Theorem 2.7.

For the regularity with respect to $L^\infty((0,T);V((0,1);\mathbb{R}^N))$ let $t \in [0,T]$, $i \in \mathfrak{N}$ fixed, assume for simplicity and without loss of generality $T \leq \boldsymbol{\xi}_i^{-1}(1)$. Then, the following estimate yields

$$\begin{aligned}|\boldsymbol{\rho}_i(t,\cdot)|_{V(0,1)} \leq &\sup_{\substack{\phi \in C_c^1((0,1))\\ \|\phi\|_{L^\infty(0,1)} \leq 1}} \left(\int_0^{\xi(t)} \frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))} \phi'(x)\,\mathrm{d}x \right)\\ &+ \sup_{\substack{\phi \in C_c^1((0,1))\\ \|\phi\|_{L^\infty(0,1)} \leq 1}} \left(\int_{\xi(t)}^1 \boldsymbol{\rho}_{0,i}(x - \boldsymbol{\xi}_i(t))\phi'(x)\,\mathrm{d}x \right). \end{aligned} \tag{31}$$

To be able to estimate the second term by the variation, we first extend $\boldsymbol{\rho}_{0,i}$ to

$$\tilde{\boldsymbol{\rho}}_{0,i}(x) := \begin{cases} \boldsymbol{\rho}_{0,i}(x) & \text{for } x \geq 0\\ 0 & \text{for } x < 0. \end{cases}$$

and obtain

$$\begin{aligned}\sup_{\substack{\phi \in C_c^1((0,1))\\ \|\phi\|_{L^\infty(0,1)} \leq 1}} \int_{\boldsymbol{\xi}_i(t)}^1 \boldsymbol{\rho}_{0,i}(x - \boldsymbol{\xi}_i(t))\phi'(x)\,\mathrm{d}x &= \sup_{\substack{\phi \in C_c^1((0,1))\\ \|\phi\|_{L^\infty(0,1)} \leq 1}} \int_{-\boldsymbol{\xi}_i(t)}^{1-\boldsymbol{\xi}_i(t)} \tilde{\boldsymbol{\rho}}_{0,i}(y)\phi'(y + \boldsymbol{\xi}_i(t))\,\mathrm{d}y\\ &\leq \sup_{\substack{\psi \in C_c^1((-\boldsymbol{\xi}_i(t),1-\boldsymbol{\xi}_i(t)))\\ \|\psi\|_{L^\infty(-\boldsymbol{\xi}_i(t),1-\boldsymbol{\xi}_i(t))} \leq 1}} \int_{-\boldsymbol{\xi}_i(t)}^{1-\boldsymbol{\xi}_i(t)} \tilde{\boldsymbol{\rho}}_{0,i}(y)\psi'(y)\,\mathrm{d}y\\ &\leq |\boldsymbol{\rho}_{0,i}(0)| + |\boldsymbol{\rho}_{0,i}|_{V(0,1)} < \infty. \end{aligned} \tag{32}$$

Thereby, the boundary value $\boldsymbol{\rho}_{0,i}(0)$ can be defined for $i \in \mathfrak{N}$ as

$$\boldsymbol{\rho}_{0,i}(0) := \lim_{h \to 0} \tfrac{1}{h} \int_0^h \boldsymbol{\rho}_{0,i}(x)\,\mathrm{d}x \tag{33}$$

as done in [14, 2.10 Theorem]. For this definition, the behavior of the function in a neighborhood of 0 is crucial. A very similar approach can be used to estimate the first term in Inequality (31) to obtain the estimate

$$\sup_{\substack{\phi\in C_c^1((0,1))\\ \|\phi\|_{L^\infty(0,1)}\le 1}} \left(\int_0^{\xi(t)} \frac{\boldsymbol{a}_i(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))}{\boldsymbol{\xi}_i'(\boldsymbol{\xi}_i^{-1}(\boldsymbol{\xi}_i(t)-x))}\phi'(x)\,\mathrm{d}x\right) \le |\boldsymbol{a}_i(0)| + |\boldsymbol{a}_i|_{V(0,T)}. \tag{34}$$

Thereby, similar to Equation (33), the boundary value can be defined for $i \in \mathfrak{N}$ as (see again [14, 2.10 Theorem])

$$\boldsymbol{a}_i(0) := \lim_{h\to 0} \tfrac{1}{h} \int_0^h \boldsymbol{a}_i(t)\,\mathrm{d}t.$$

Since both estimates (32) and (34) are uniform in t, we finally have

$$\operatorname*{ess\,sup}_{t\in[0,T]} |\boldsymbol{\rho}_i(t,\cdot)|_{V(0,1)} \le |\boldsymbol{a}_i(0)| + |\boldsymbol{a}_i|_{V(0,T)} + |\boldsymbol{\rho}_{0,i}(0)| + |\boldsymbol{\rho}_{0,i}|_{V(0,1)}.$$

The boundedness of the right hand side for every $i \in \mathfrak{N}$ concludes the proof. Let us point out that we did not show the claimed regularity $L^\infty((0,1); BV((0,T);\mathbb{R}^N))$. However, the proof is very similar to the case of proofing $L^\infty((0,T); BV((0,1);\mathbb{R}^N))$ regularity and will be omitted. □

Remark 4 ($C([0,T]; BV(0,1))$ and $C([0,T]; L^\infty(0,1))$)**.** Assume for simplicity $N = 1$, although the following argumentation is still valid for the multi-commodity cases: One might ask why we cannot expect regularity of the solution in Theorem 2.7 for $p = \infty$. Since the L^p-norm for $p \in [1,\infty)$ smoothes jumps, the L^p-norm in space changes continuously in time. For $p = \infty$ this is not true. For instance, jumps in the space domain could vanish over time, such that the L^∞-norm would change discontinuously.

With the same argumentation one can explain – and give counterexamples – why we cannot expect $C([0,T]; BV(0,1))$ regularity but only $L^\infty((0,T); BV(0,1))$ regularity in Theorem 2.8.

Remark 5 (The evaluation of $\boldsymbol{\rho}$ for every $t \in [0,T]$, $x \in [0,1]$ respectively, BV)**.** Let us point out that the regularity result for BV data in Theorem 2.8 does not guarantee that taking **any** $x \in [0,1]$ the solution is BV regular with respect to time and – vice versa – taking **any** $t \in [0,T]$ the solution BV regular with respect to the space. To prove that, one can use the solution - constructed by the characteristics - and verify it explicitly, or one can use the following result:

Corollary 1 (BV regularity for **every** $x \in [0,1]$ and for **every** $t \in [0,T]$)**.** *For $N \in \mathbb{N}$, $T \in \mathbb{R}_{>0}$ let $q \in C([0,T]; L^1((0,1);\mathbb{R}^N)) \cap L^\infty((0,T); BV((0,1);\mathbb{R}^N))$ be given. Then we have*

$$q(t,\cdot) \in BV((0,1);\mathbb{R}^N) \quad \forall t \in [0,T].$$

Proof. Without loss of generality we chose $N = 1$. A generalization to arbitrary $N \in \mathbb{N}$ is evident. Therefore, let q be an element of $C([0,T]; L^1(0,1)) \cap L^\infty((0,T); BV(0,1))$. Then for every $t \in [0,T]$ there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset [0,T]$, $t_k \to t$ such that for every $k \in \mathbb{N}$ $q(t_k,\cdot) \in BV(0,1)$.

Since $q \in C([0,T]; L^1(0,1))$, the limit $\lim_{t_k \to t} q(t_k, \cdot)$ is an element of $L^1(0,1)$. Now using [3, Proposition 3.13], we obtain $q(t,\cdot) \in BV(0,1)$. The arbitrary choice of $t \in [0,T]$ concludes the proof. $\square$

For the sake of completeness we provide – as a last regularity result – regularity for $W^{1,p}$-data:

Theorem 2.9 (Regularity for $W^{1,p}$-data)**.** *For $N \in \mathbb{N}, p \in [1,\infty)$ and $T \in \mathbb{R}_{>0}$ let $\boldsymbol{\rho}_0 \in W^{1,p}((0,1); \mathbb{R}^N_{\geq 0})$ and $\boldsymbol{a} \in W^{1,p}((0,T); \mathbb{R}^N_{\geq 0})$ be given. Let furthermore the compatibility condition*

$$\boldsymbol{a}(0) = \boldsymbol{\Lambda}(W(0,\boldsymbol{\rho}))\boldsymbol{\rho}_0(0) \tag{35}$$

be satisfied. Then the initial boundary value problem defined in Equation (1) - Equation (4) admits a unique weak solution $\boldsymbol{\rho}$ as defined in Definition 2.3 with additional regularity

$$\boldsymbol{\rho} \in C([0,T]; W^{1,p}((0,1); \mathbb{R}^N)) \cap C([0,1]; W^{1,p}((0,T); \mathbb{R}^N)).$$

Proof. We know by the proof of Theorem 2.7 and Remark 3 that at least

$$W(\cdot, \boldsymbol{\rho}) \in C([0,T]; \mathbb{R}_{\geq 0})$$

and since $W^{1,p} \hookrightarrow L^p$ this is also true for Theorem 2.9. As in the proof of [15, Theorem 4.7] we can obtain for every $i \in \mathfrak{N}$ the postulated regularity result. $\square$

Remark 6 (The Compatibility Condition and the Continuity of the Solution)**.** Let us mention that the compatibility condition in Equation (35) is reasonable due to the Sobolev embedding theorem

$$W^{1,p}(I) \hookrightarrow C(\overline{I}) \text{ and } p \in [1,\infty]$$

in [1][Theorem 4.12, Part I and Part II, p. 85]) for $I \subset \mathbb{R}$ an open bounded interval.

Let us furthermore mention that the regularity of the solution $\boldsymbol{\rho}$ in the latter Theorem 2.9 allows to choose a continuous representative $\boldsymbol{\rho}$, such that the positivity of the solution is satisfied for every $(t,x) \in [0,T] \times [0,1]$.

2.4. **An example.** To get a better understanding of the proposed multi-commodity coupling and the resulting dynamics, we give an easy example. Let $N = 2$ and $T = 2$ and consider the two conservation laws

$$\begin{aligned} &\dot{\boldsymbol{\rho}}_1(t,x) = -\boldsymbol{\lambda}_1(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}_{x,1}(t,x) && \dot{\boldsymbol{\rho}}_2(t,x) = -\boldsymbol{\lambda}_2(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}_{x,2}(t,x)\\ &\boldsymbol{\rho}_{0,1}(x) = 0 && \boldsymbol{\rho}_{0,2}(x) = 0\\ &\boldsymbol{a}_1(t) = 1 && \boldsymbol{a}_2(t) = 2\\ &\boldsymbol{\lambda}_1(W) = \tfrac{1}{1+W} && \boldsymbol{\lambda}_2(W) = 1 + W \end{aligned}$$

coupled by the "work in progress"

$$W(t,\boldsymbol{\rho}) = \sum_{i=1}^{2} \int_0^1 \boldsymbol{\rho}_i(t,s)\,\mathrm{d}s = \int_0^1 \boldsymbol{\rho}_1(t,s) + \boldsymbol{\rho}_2(t,s)\,\mathrm{d}s.$$

Thereby, we define the outflow at the right hand side $x = 1$ as $\boldsymbol{y} \equiv \boldsymbol{\Lambda}(W(\cdot,\boldsymbol{\rho}))\boldsymbol{\rho}(\cdot,1)$.

Since we assumed $\boldsymbol{\rho}_0 \equiv 0$, we do not have to solve a fixed-point equation to state the solution and – using Equation (29) and the expression in Equation (20) for $W(t,\boldsymbol{\rho})$ – come easily up with $W(t,\boldsymbol{\rho}) = 3t$ and, thus,

$$\tfrac{1}{1+3t} = \boldsymbol{\lambda}_1(W(t,\boldsymbol{\rho})) \qquad\qquad 1 + 3t = \boldsymbol{\lambda}_2(W(t,\boldsymbol{\rho}))$$

$$\boldsymbol{\rho}_1(t,x) = \begin{cases} 0 & \text{for } x \geq \frac{\ln(1+3t)}{3} \\ \frac{1+3t}{\mathrm{e}^{3x}} & \text{else} \end{cases} \qquad \boldsymbol{\rho}_2(t,x) = \begin{cases} 0 & \text{for } x \geq \frac{(1+3t)^2-1}{6} \\ \frac{2}{\sqrt{(1+3t)^2-6x}} & \text{else} \end{cases}$$

$$\boldsymbol{y}_1(t) = 0 \qquad \boldsymbol{y}_2(t) = \begin{cases} 0 & \text{for } t \leq \frac{\sqrt{7}-1}{3} \\ \frac{2(1+3t)}{\sqrt{(1+3t)^2-6}} & \text{else.} \end{cases}$$

Figure 2 shows the graphs of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ on $(0,T) \times (0,1)$ for $T = 2$. As one can see, the more goods are passed on the supply chain the more the velocity for commodity 1 decreases, since we have chosen as velocity function $\frac{1}{1+W}$. The velocity becomes so slow that no good actually arrives at $x = 1$. This is completely different for commodity 2. Since we have as velocity function $1+W$ the more goods we have on the supply chain the higher the velocity becomes, such that for $t > \frac{\sqrt{7}-1}{3}$ goods arrive at $x = 1$.

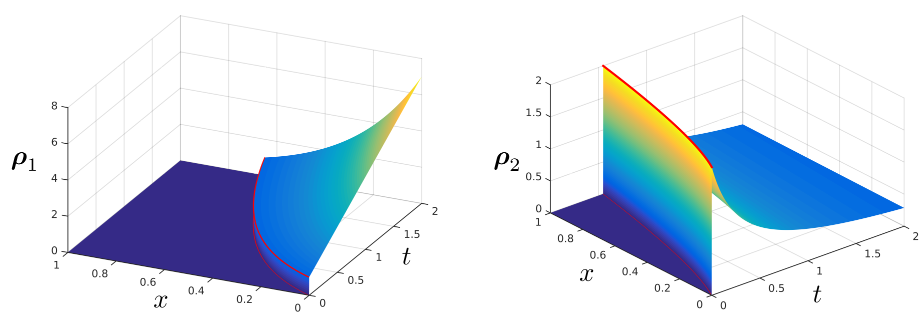

FIGURE 2. $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ for given inflow $\boldsymbol{a}_1 \equiv 1$ and $\boldsymbol{a}_2 \equiv 2$, $\boldsymbol{\rho}_0 \equiv \boldsymbol{0}$

Changing $\boldsymbol{a}_2 \equiv 0$, i.e. we do not have any goods for the commodity 2, also the dynamics for $\boldsymbol{\rho}_1$ changes due to the coupling in the WIP W and we obtain $W(t,\boldsymbol{\rho}) = t$ and, therefore,

$$\boldsymbol{\lambda}_1(W(t,\boldsymbol{\rho})) = \frac{1}{1+t} \qquad \boldsymbol{\lambda}_2(W(t,\boldsymbol{\rho})) = 1+t$$

$$\boldsymbol{\rho}_1(t,x) = \begin{cases} 0 & \text{for } x \geq \ln(1+t) \\ \frac{1+t}{\mathrm{e}^{x}} & \text{else} \end{cases} \qquad \boldsymbol{\rho}_2(t,x) = 0$$

$$\boldsymbol{y}_1(t) = \begin{cases} 0 & \text{for } t \leq \mathrm{e}-1 \\ \frac{1}{\mathrm{e}} & \text{else} \end{cases} \qquad \boldsymbol{y}_2(t) = 0.$$

Figure 3 illustrates these solutions. Since in comparison with the aforementioned example, illustrated in Figure 2 the amount of goods on the supply chain is significantly lower, the propagation speed of commodity 1 is higher, and goods arrive at the boundary $x = 1$ for $t > \mathrm{e} - 1$ as can be seen in Figure 3. The density of commodity 2 is identically zero, since initial and boundary data are zero.

3. The multi-commodity model on networks.

3.1. Some definitions regarding the network setting.

Definition 3.1 (Definition of the network)**.** A network is a directed acyclic graph $\mathfrak{G} = (\mathfrak{E}, \mathfrak{V})$ (DAG) as for instance defined in [13, Definition B.14] with a set of edges $\mathfrak{E}$ and a set of vertices $\mathfrak{V}$. The set of edges can be decomposed into

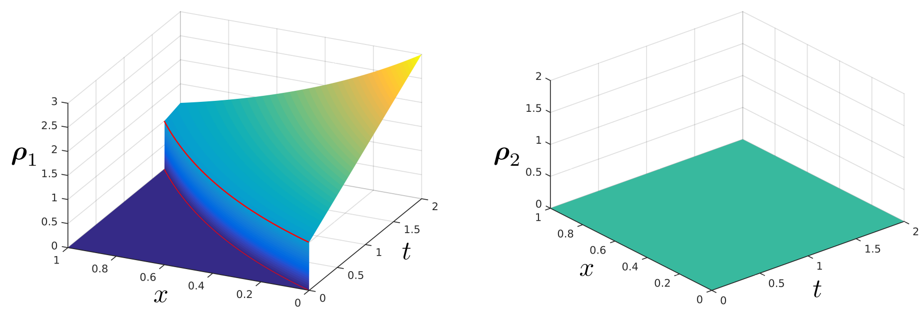

FIGURE 3. $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ for given inflow $\boldsymbol{a}_1 \equiv 1$ and $\boldsymbol{a}_2 \equiv 0$, $\boldsymbol{\rho}_0 \equiv \mathbf{0}$

$$\mathfrak{E} = \mathfrak{E}_{\mathrm{m}} \,\dot{\cup}\, \mathfrak{E}_{\mathrm{s}} \,\dot{\cup}\, \mathfrak{E}_{\mathrm{e}},$$

where $\mathfrak{E}_{\mathrm{s}}$ is the set of starting edges, $\mathfrak{E}_{\mathrm{e}}$ the set of ending edges, and $\mathfrak{E}_{\mathrm{m}}$ the set of edges which are connected at the corresponding start and end node to other edges.

Furthermore $\mathfrak{V}$ can be decomposed into

$$\mathfrak{V} = \mathfrak{V}_{\mathrm{s}} \,\dot{\cup}\, \mathfrak{V}_{\mathrm{m}},$$

where $\mathfrak{V}_{\mathrm{s}}$ are the vertices with only outgoing or only incoming edges, the start and end vertices (sources and sink), and $\mathfrak{V}_{\mathrm{m}}$ the set of vertices with at least one incoming and one outgoing edge.

The set $\mathfrak{V}_{\mathrm{d}}$ is the set of vertices with more than one outgoing edge. Let $\mathfrak{E}_{\mathrm{i}}(\mathfrak{v})$ be the set of edges which end in $\mathfrak{v}$ for given $\mathfrak{v} \in \mathfrak{V}$ and $\mathfrak{E}_{\mathrm{o}}(\mathfrak{v})$ the set of edges which start in $\mathfrak{v}$. We define

$$\mathfrak{V}_{\mathrm{d}} := \{\mathfrak{v} \in \mathfrak{V} : \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}) > 1\}$$

and, additionally,

$$\mathfrak{V}_{\mathrm{s}} := \{\mathfrak{v} \in \mathfrak{V} : \mathfrak{E}_{\mathrm{i}}(\mathfrak{v}) = \emptyset \vee \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}) = \emptyset\} \text{ and } \mathfrak{V}_{\mathrm{m}} := \{\mathfrak{v} \in \mathfrak{V} : |\mathfrak{E}_{\mathrm{i}}(\mathfrak{v})| \cdot |\mathfrak{E}_{\mathrm{o}}(\mathfrak{v})| \neq 0\}.$$

In Figure 4 we give an example of such a network.

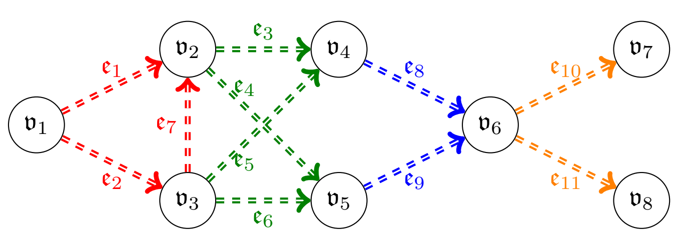

FIGURE 4. A typical supply network with two commodities, $\mathfrak{E} = \{\mathfrak{e}_i, i = 1, \ldots, 11\}$, $\mathfrak{V} = \{\mathfrak{v}_i, i = 1, \ldots, 8\}$, $\mathfrak{V}_{\mathrm{m}} = \mathfrak{V} \setminus \{\mathfrak{v}_1, \mathfrak{v}_7.\mathfrak{v}_8\}$, $\mathfrak{V}_{\mathrm{d}} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_6\}$, $\mathfrak{E}_{\mathrm{e}} = \{\mathfrak{e}_{10}, \mathfrak{e}_{11}\}, \mathfrak{E}_{\mathrm{s}} = \{\mathfrak{e}_1, \mathfrak{e}_2\}$, $\mathfrak{E}_{\mathrm{i}}(\mathfrak{v}_2) = \{\mathfrak{e}_1, \mathfrak{e}_7\}$, $\mathfrak{E}_{\mathrm{o}}(\mathfrak{v}_2) = \{\mathfrak{e}_3, \mathfrak{e}_4\}$, ...

Definition 3.2 (The set of distribution functions)**.** Let $N \in \mathbb{N}$, $T \in \mathbb{R}_{>0}$, a network as defined in Definition 3.1 be given and $D := \sum_{\mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}} |\mathfrak{E}_{\mathrm{o}}(\mathfrak{v})|$. Then we define the set of distribution functions as

$$\mathbf{\Sigma} := \Bigg\{ \boldsymbol{\sigma} \in L^\infty((0,T);\mathbb{R}^N_{\geq 0})^D : \boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}(t) \geq \mathbf{0}, \sum_{\mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v})} \boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}(t) = \mathbf{1} \quad \forall t \in (0,T) \text{ a.e.}, \mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}, \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}) \Bigg\}$$

and for an arbitrary subset $X \subseteq L^\infty((0,T);\mathbb{R}^N)$

$$\mathbf{\Sigma}_X := \Big\{ \boldsymbol{\sigma} \in \mathbf{\Sigma} : \boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}} \in X \ \forall \mathfrak{v} \in \mathfrak{V}_{\mathrm{d}},\ \forall \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}) \Big\}.$$

3.2. **The dynamics on the network.**

Definition 3.3 (The dynamics on the network)**.** Let $N \in \mathbb{N}$, $T \in \mathbb{R}_{>0}$ and a network as defined in Definition 3.1 with edges $\mathfrak{E}$ and vertices $\mathfrak{V}$ be given. For $\mathfrak{e} \in \mathfrak{E}$ let $\boldsymbol{\rho}_0^{\mathfrak{e}}$ as initial load on every edge be given as well as the velocity functions $\boldsymbol{\lambda}^{\mathfrak{e}} \in C^1(\mathbb{R}_{\geq 0};\mathbb{R}^N_{>0})$. Eventually, for $\mathfrak{e} \in \mathfrak{E}_{\mathrm{s}}$ let the inflow $\boldsymbol{a}^{\mathfrak{e}}$ be given. Then the dynamics on the network for $(t,x) \in (0,T) \times (0,1)$ can be stated as

$$\begin{aligned} \dot{\boldsymbol{\rho}}^{\mathfrak{e}}(t,x) &= -\boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}_x(t,x) && \mathfrak{e} \in \mathfrak{E} \\ \boldsymbol{\rho}^{\mathfrak{e}}(0,x) &= \boldsymbol{\rho}^{\mathfrak{e}}_0(x) && \mathfrak{e} \in \mathfrak{E} \\ \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t,0) &= \boldsymbol{a}^{\mathfrak{e}}(t) && \mathfrak{e} \in \mathfrak{E}_{\mathrm{s}} \end{aligned}$$

using the integral expression as defined in Equation (30) in Remark 3

$$W(t,\boldsymbol{\rho}^{\mathfrak{e}}) := \sum_{i=1}^N \int_0^1 \boldsymbol{\rho}^{\mathfrak{e}}_i(t,s)\,\mathrm{d}s \qquad \mathfrak{e} \in \mathfrak{E},$$

and the node conditions with $\mathbf{\Sigma}$ as defined in Definition 3.2

$$\begin{aligned} &\boldsymbol{\sigma} \in \mathbf{\Sigma} \\ \boldsymbol{\Lambda}^{\mathfrak{e}_{\mathrm{o}}(\mathfrak{v})}(W(t,\boldsymbol{\rho}^{\mathfrak{e}_{\mathrm{o}}(\mathfrak{v})}))\boldsymbol{\rho}^{\mathfrak{e}_{\mathrm{o}}(\mathfrak{v})}(t,0) &= \sum_{\mathfrak{e} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t,1) && \mathfrak{v} \in \mathfrak{V}_{\mathrm{m}} \setminus \mathfrak{V}_{\mathrm{d}} \\ \boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}(t) \odot \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(t,\boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(t,1) &= \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t,0) && \mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}, \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}) \end{aligned}$$

for $t \in [0,T]$ a.e.. For the abbreviations considering the network topology we again refer to Definition 3.1. The symbol $\odot$ denotes the component-wise multiplication and is for $N \in \mathbb{N}$ defined as

$$\odot : \begin{cases} \mathbb{R}^N \times \mathbb{R}^N & \to \mathbb{R}^N \\ (\boldsymbol{a},\boldsymbol{b}) & \mapsto \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{a}_N\boldsymbol{b}_N \end{pmatrix}. \end{cases}$$

For a better understanding of the dynamics in Definition 3.3 and Definition 3.2 we add an interpretation:

The set of vectors $\boldsymbol{\rho}_0^{\mathfrak{e}}$ gives the load of goods at the time $t = 0$ on every edge $\mathfrak{e} \in \mathfrak{E}$. Since in a general network there is most likely more than one inflow, $\mathfrak{E}_{\mathrm{s}}$ is the set of starting edges, and the corresponding vectors $\boldsymbol{a}^{\mathfrak{e}}$ represent for $\mathfrak{e} \in \mathfrak{E}_{\text{s}}$ the influx of goods into the network. The functions $\boldsymbol{a}^{\mathfrak{e}}$ will be subject to optimization in Section 3.4, for instance, to track a certain demand. Furthermore, the equation $\sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\text{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(t, \boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(t, 1) = \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t, \boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t, 0)\ \forall t \in [0, T]$ a.e. stands for the conservation of goods and is defined for every vertex which has ingoing edges and one outgoing edge (otherwise we would have to distribute the good in a certain way), i.e. for every $\mathfrak{v} \in \mathfrak{V}_{\text{m}} \setminus \mathfrak{V}_{\text{d}}$. Thus, the equation $\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}(t) \odot \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\text{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(t, \boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(t, 1) = \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t, \boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t, 0)\ \forall t \in [0, T]$ a.e. states with the help of the distribution vectors $\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}$ for every good which amount should be distributed and on which distributing edge. This equation is only defined for vertices, where there is more than one outgoing edge, $\mathfrak{v} \in \mathfrak{V}_{\text{m}}$. The distribution functions $\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}$ are nonnegative and summarize to $\mathbf{1}$ for every distributing edge $\mathfrak{v} \in \mathfrak{V}_{\text{d}}$ and for every good to ensure that every single good is also conserved in the case of distributing edges. Of course, the distribution functions are also subject to optimization in Section 3.4.

Altogether, the model simulates for a given network the flow of N (possible) different goods. The generality of the model allows distributing and collecting edges as well as N different goods. Coupling of the different goods *only* consists of the collective load $W(t, \boldsymbol{\rho}^{\mathfrak{e}})$ at time $t \in [0, T]$.

3.3. Well-posedness of the network-model for different spaces.

Theorem 3.4 (Well-posedness for L^p-data)**.** *Let $N \in \mathbb{N}$ be the number of commodities, $T \in \mathbb{R}_{>0}$, $p \in [1, \infty)$ and let a network with edges $\mathfrak{E}$ and vertices $\mathfrak{V}$ as defined in Definition 3.1 be given. For $\mathfrak{e} \in \mathfrak{E}_s$ let furthermore $\boldsymbol{a}^{\mathfrak{e}} \in L^p((0, T); \mathbb{R}^N_{\geq 0})$, for $\mathfrak{e} \in \mathfrak{E}$ the initial values $\boldsymbol{\rho}_0^{\mathfrak{e}} \in L^p((0, 1); \mathbb{R}^N_{\geq 0})$ and $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ as defined in Definition 3.2 be given. Then there exists a unique nonnegative solution $\boldsymbol{\rho} = (\boldsymbol{\rho}^{\mathfrak{e}})_{\mathfrak{e} \in \mathfrak{E}}$ on the network with regularity*

$$\boldsymbol{\rho}^{\mathfrak{e}} \in C([0, T]; L^p((0, 1); \mathbb{R}^N)) \cap C([0, 1]; L^p((0, T); \mathbb{R}^N)) \quad \forall \mathfrak{e} \in \mathfrak{E}.$$

Proof. This result can easily be established by induction over the set of edges. Therefore, since $\boldsymbol{a}^{\mathfrak{e}} \in L^p((0, T); \mathbb{R}^N_{\geq 0})$ and $\boldsymbol{\rho}_0^{\mathfrak{e}} \in L^p((0, 1); \mathbb{R}^N_{\geq 0})$, we have for $\mathfrak{e} \in \mathfrak{E}_{\text{s}}$ by Theorem 2.7

$$\boldsymbol{\rho}^{\mathfrak{e}} \in C([0, T]; L^p((0, 1); \mathbb{R}^N)) \cap C([0, 1]; L^p((0, T); \mathbb{R}^N)).$$

Furthermore, we have $\boldsymbol{\rho}^{\mathfrak{e}}(\cdot, 1) \in L^p((0, T); \mathbb{R}^N_{\geq 0})$ and thanks to the boundedness of $\boldsymbol{\Lambda}^{\mathfrak{e}}$ also $\boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot, \boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(\cdot, 1) \in L^p((0, T); \mathbb{R}^N_{\geq 0})$. These values are the boundary values for the following edges possibly weighted by the distribution functions $\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}$.

Since for $\mathfrak{v} \in \{\mathfrak{v} \in \mathfrak{V} : |\mathfrak{E}_{\text{i}}(\mathfrak{v})| > 1\}$ the l.h.s. boundary data for $\mathfrak{e} \in \mathfrak{E}_{\text{o}}(\mathfrak{v})$ consists of the sum

$$\sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\text{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(\cdot, \boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(\cdot, 1) \in L^p((0, T); \mathbb{R}^N)$$

– depending on the vertex – multiplied by $\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}} \in L^\infty((0, T); \mathbb{R}^N_{\geq 0})$, we can conclude that the boundary values of the following edges are also in $L^p((0, T); \mathbb{R}^N_{\geq 0})$. By induction (recall the Definition 3.1 of the network structure under consideration) this can be extended to the whole network. The solution is unique since the solution of every single edge is unique, again as a result of Theorem 2.7. □

Theorem 3.5 (Well-posedness for BV-data)**.** *Let $N \in \mathbb{N}$ be the number of commodities, $T \in \mathbb{R}_{>0}$, and let a network with edges $\mathfrak{E}$ and vertices $\mathfrak{V}$ as defined in*

Definition 3.1 be given. For $\mathfrak{e} \in \mathfrak{E}_s$ let furthermore $\boldsymbol{a}^{\mathfrak{e}} \in BV((0,T);\mathbb{R}^N_{\geq 0})$, for $\mathfrak{e} \in \mathfrak{E}$ the initial values $\boldsymbol{\rho}_0^{\mathfrak{e}} \in BV((0,1);\mathbb{R}^N_{\geq 0})$ and $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{BV}$ as defined in Definition 3.2 be given. Then there exists a unique nonnegative solution $\boldsymbol{\rho} = (\boldsymbol{\rho}^{\mathfrak{e}})_{\mathfrak{e}\in\mathfrak{E}}$ on the network with additional regularity

$$\boldsymbol{\rho}^{\mathfrak{e}} \in L^\infty((0,T);BV((0,1);\mathbb{R}^N)) \cap L^\infty((0,1);BV((0,T);\mathbb{R}^N)) \quad \forall \mathfrak{e} \in \mathfrak{E}.$$

Proof. For the BV regularity the proof can be carried out almost as in the L^p case in the proof of Theorem 3.4 using Remark 5. The only crucial question is when we have distributing edges. Thus, let $\mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}$ be given and $\mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v})$. We then have for the left hand side boundary data on the edge $\mathfrak{e}$ the equation

$$\boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t,0) = \boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}(t) \odot \sum_{\bar{\mathfrak{e}}\in\mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(t,\boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(t,1) \quad \text{for } t \in [0,T] \text{ a.e..}$$

By induction and Remark 5, for $\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})$ every summand $\boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(\cdot,\boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(\cdot,1)$ is BV regular and therefore also the sum. Furthermore, by assumption we have $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{BV}$, such that for every $\mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}, \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v})$ the functions $\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}$ are BV regular. Now, the product of two BV functions defined on the same domain is again in BV, which can easily been established by [21, Theorem 13.48 (2)]. Therein it is stated that for $I \subset \mathbb{R}$ open we have

$$u \in BV(I) \Longleftrightarrow u \in L^1(I) \quad \text{and} \quad \liminf_{h\searrow 0} \int_{\{x\in I: x+h\in I\}} \tfrac{|u(x+h)-u(x)|}{h}\,\mathrm{d}x < \infty. \tag{36}$$

So let $u, v \in BV(I)$ be given. Then we know by the embedding $BV(I) \hookrightarrow L^\infty(I)$ in [3, Corollary 3.49] that $u \cdot v \in L^1(I)$. Therefore, let us consider the second condition, fix h sufficiently small and estimate by adding a zero and by using the triangle inequality

$$\begin{aligned}\int_{\{x\in I:x+h\in I\}} \tfrac{|u(x+h)\cdot v(x+h)-u(x)v(x)|}{h}\,\mathrm{d}x \leq \|u\|_{L^\infty(I)} &\int_{\{x\in I:x+h\in I\}} \tfrac{|v(x+h)-v(x)|}{h}\,\mathrm{d}x\\ &+ \|v\|_{L^\infty(I)} \int_{\{x\in I:x+h\in I\}} \tfrac{|u(x)-u(x+h)|}{h}\,\mathrm{d}x.\end{aligned}$$

Applying the operation $\liminf_{h\searrow 0}$ on both sides gives the claimed result since the right hand side is bounded thanks to the characterization of BV functions in (36) (recall that we assumed $u, v \in BV(I)$). Moreover, by [21, Theorem 13.48 (1)] we also obtain the estimate for the variation of the product of two BV functions

$$|u \cdot v|_{V(I)} \leq \|u\|_{L^\infty(I)} \cdot |v|_{V(I)} + \|v\|_{L^\infty(I)} \cdot |u|_{V(I)}.$$

Altogether, the product of two BV functions is again BV regular, such that we have

$$\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}(\cdot) \odot \sum_{\bar{\mathfrak{e}}\in\mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(\cdot,\boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(\cdot,1) \in BV((0,T);\mathbb{R}^N)$$

and, thus,

$$\boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(\cdot,0) \in BV((0,T);\mathbb{R}^N) \quad \text{for } \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}), \mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}.$$

By mathematical induction, we can extend the BV-regularity to the entire network. ∎

Remark 7 (Problems with $W^{1,p}$-data)**.** A similar result cannot be deduced so easily for $W^{1,p}$-data, even if we assume $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{W^{1,\infty}}$, since in general the Compatibility Condition (35) in the interior of the network will not be satisfied.

However, assuming for every $\mathfrak{e} \in \mathfrak{E}$ that $\boldsymbol{\rho}_0^{\mathfrak{e}} \equiv \mathbf{0}$ guarantees this condition on every edge except the starting ones, where we would have to impose additionally for every $\mathfrak{e} \in \mathfrak{E}_{\mathrm{s}}$ the condition $\boldsymbol{a}^{\mathfrak{e}}(0) = \mathbf{0}$.

But these assumptions on the initial values would restrict the model significantly, such that we will not study them furthermore. Of course, we could also give even more complicated conditions on which that compatibility conditions would be satisfied even for $\boldsymbol{\rho}_0^{\mathfrak{e}} \not\equiv \mathbf{0}$. However, this conditions would also involve the velocity functions on each edge as well as the distributing functions $\boldsymbol{\sigma}$ and would be even harder to be realized.

3.4. **Existence of a minimizer for a typical L^2-tracking type objective.** We define a typical L^2 objective tracking type functional as

Definition 3.6 (Definition of the objective)**.** For $N \in \mathbb{N}$ and a network as defined in Definition 3.1 we specify the objective functional as

$$\begin{aligned} J : L^2((0,T);\mathbb{R}^N)^{|\mathfrak{E}_{\mathrm{s}}|} \times \boldsymbol{\Sigma} \times L^2((0,T);\mathbb{R}^N)^{|\mathfrak{E}_{\mathrm{e}}|} \to \mathbb{R}_{\geq 0} \\ (\boldsymbol{a},\boldsymbol{\sigma},\boldsymbol{y}) \mapsto \sum_{\mathfrak{e}\in\mathfrak{E}_{\mathrm{s}}} \alpha_{\mathfrak{e}} \|\boldsymbol{a}^{\mathfrak{e}} - \boldsymbol{a}_{\mathrm{d}}^{\mathfrak{e}}\|_{L^2((0,T);\mathbb{R}^N)}^2 + \sum_{\mathfrak{e}\in\mathfrak{E}_{\mathrm{e}}} \gamma_{\mathfrak{e}} \|\boldsymbol{y}^{\mathfrak{e}} - \boldsymbol{y}_{\mathrm{d}}^{\mathfrak{e}}\|_{L^2((0,T);\mathbb{R}^N)}^2 \\ + \sum_{\mathfrak{v}\in\mathfrak{V}_{\mathrm{d}}} \sum_{\mathfrak{e}\in\mathfrak{E}_{\mathrm{o}}(\mathfrak{v})} s_{\mathfrak{v},\mathfrak{e}} \|\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}} - \boldsymbol{\sigma}_{\mathrm{d}}^{\mathfrak{e},\mathfrak{v}}\|_{L^2((0,T);\mathbb{R}^N)}^2. \end{aligned}$$

For $\mathfrak{e} \in \mathfrak{E}_{\mathrm{s}}$ we furthermore assume $\alpha_{\mathfrak{e}} \in \mathbb{R}_{>0}$, $\boldsymbol{a}_{\mathrm{d}}^{\mathfrak{e}} \in L^2((0,T);\mathbb{R}^N)$, for $\mathfrak{e} \in \mathfrak{E}_{\mathrm{e}}$ we assume $\gamma_{\mathfrak{e}} \in \mathbb{R}_{\geq 0}$, $\boldsymbol{y}_{\mathrm{d}}^{\mathfrak{e}} \in L^2((0,T);\mathbb{R}^N)$ and, eventually, for $\mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}, \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v})$ we assume $s_{\mathfrak{v},\mathfrak{e}} \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{\sigma}_{\mathrm{d}}^{\mathfrak{e},\mathfrak{v}} \in L^2((0,T);\mathbb{R}^N)$. For simplicity, let in the following $\alpha_{\mathfrak{e}} = \gamma_{\mathfrak{e}} = s_{\mathfrak{v},\mathfrak{e}} = 1$. Eventually, $\boldsymbol{y}$ is a vector, consisting of the outflow of the network and will be, therefore, uniquely determined by $\boldsymbol{a}$ and $\boldsymbol{\sigma}$, where $\boldsymbol{a} := (\boldsymbol{a}^{\mathfrak{e}})_{\mathfrak{e}\in\mathfrak{E}_{\mathrm{s}}}$ and $\boldsymbol{\sigma} := (\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}})_{\mathfrak{v}\in\mathfrak{V}_{\mathrm{d}},\mathfrak{e}\in\mathfrak{E}_{\mathrm{o}}(\mathfrak{v})}$.

For the proof of the main Theorem 3.10 in this Section 3.4, we need the following Lemmata:

Lemma 3.7 ($\boldsymbol{\Sigma}_{BV_M}$ weakly-$*$ closed)**.** *For a network as defined in Definition 3.1, $N \in \mathbb{N}$ commodities and $M \in \mathbb{R}_{\geq 0}$ let*

$$BV_M := \{f \in BV : \|f\|_{BV} \leq M\}, \tag{37}$$

i.e. the scaled BV-ball. Then, the set $\boldsymbol{\Sigma}_{BV_M}$, defined in Definition 3.2, is weakly-$*$ closed, in particular we have

$$\forall (\boldsymbol{\sigma}_k)_{k\in\mathbb{N}} \in \boldsymbol{\Sigma}_{BV_M}\ \exists (\boldsymbol{\sigma}_{n_k})_{k\in\mathbb{N}} \text{ and } \boldsymbol{\sigma}_\infty \in \boldsymbol{\Sigma}_{BV_M} : \boldsymbol{\sigma}_{n_k} \to \boldsymbol{\sigma}_\infty \text{ in } L^p,\ p \in [1,\infty).$$

Proof. The proof follows partially from [3, Theorem 3.23] or [7, Proposition 10.1.1]. Therein it is stated that for every sequence which admits a uniform BV estimation, we also have a limit in BV and by the lower semi-continuity of the variation, the variation of the limit is bounded by the same uniform BV bound M. The strong convergence of a subsequence in L^p can furthermore be deduced by the compact embedding ([3, Corollary 3.49])

$$BV(I) \overset{c}{\hookrightarrow} L^p(I) \text{ for } p \in [1,\infty) \text{ and } I \subset \mathbb{R} \text{ a bounded interval.}$$

We also refer to the Rellich-Kondrachov Theorem in [21, Theorem 13.32] for the strong convergence of a subsequence in BV with respect to the L^p-norm. It remains to show that the inequality constraints in $\boldsymbol{\Sigma}_{BV_K}$ are fulfilled when passing $\boldsymbol{\sigma}_{n_k}$ to the limit. This can easily been established by contradiction or by using the convexity of $\boldsymbol{\Sigma}_{BV_M}$ and will be omitted here. ☐

Lemma 3.8 (Weak convergence of the outflow for weak convergent inflow)**.** *As archetype of a network as illustrated in Figure 5 we only consider one edge. For a*

$$\boldsymbol{a}_n \rightharpoonup \boldsymbol{a}_\infty \xrightarrow{\quad \boldsymbol{\rho}_n, \boldsymbol{\rho}_\infty \quad} \boldsymbol{y}_n \rightharpoonup \boldsymbol{y}_\infty$$

FIGURE 5. Archetype of a network – weak convergence of $\boldsymbol{a}_n$ implies weak convergence of $\boldsymbol{y}_n$

sequence of inflows $(\boldsymbol{a}_n)_{n\in\mathbb{N}} \in L^2((0,T);\mathbb{R}^N)$*, weakly converging in* $L^2((0,T);\mathbb{R}^N)$ *to* $\boldsymbol{a}_\infty \in L^2((0,T);\mathbb{R}^N)$*, the outflow* $\boldsymbol{y}_n$*, defined by*

$$\boldsymbol{y}_n := \boldsymbol{\Lambda}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n(\cdot,1),$$

will also weakly converge to $\boldsymbol{y}_\infty$*, i.e.*

$$\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}_\infty \ \text{in } L^2((0,T);\mathbb{R}^N).$$

Thereby, $\boldsymbol{\rho}_n$ *and* $\boldsymbol{\rho}_\infty$*, respectively, are defined as the solution of the initial boundary value problems*

$$\begin{aligned} \dot{\boldsymbol{\rho}}(t,x) + \boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}_x(t,x) &= \boldsymbol{0} \\ \boldsymbol{\rho}(0,x) &= \boldsymbol{\rho}_0(x) \end{aligned}$$

and left hand side boundary data

$$\begin{aligned} \boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}(t,0) &= \boldsymbol{a}_n(t), \\ \boldsymbol{\Lambda}(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}(t,0) &= \boldsymbol{a}_\infty(t), \end{aligned}$$

respectively. $\boldsymbol{\rho}_0 \in L^2((0,1);\mathbb{R}^N)$ *is given data and*

$$\boldsymbol{y}_\infty \equiv \boldsymbol{\Lambda}(W(\cdot,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_\infty(\cdot,1).$$

Proof. We refer to [11, Theorem 3.1] and consider the characteristic $\boldsymbol{\xi}_n$ defined by

$$\boldsymbol{\xi}_n(t) := \int_0^t \boldsymbol{\lambda}(W(s,\boldsymbol{\rho}_n))\,\mathrm{d}s \tag{38}$$

and the generalized WIP as defined in Remark 3

$$W(t,\boldsymbol{\rho}_n) := \sum_{i=1}^N \int_0^1 \boldsymbol{\rho}_{i,n}(t,s)\,\mathrm{d}s.$$

Inserting the solution of $\boldsymbol{\rho}_{i,n}$ for $i \in \mathfrak{N}$ with the help of the characteristics stated in Equation (24)

$$\boldsymbol{\rho}_{i,n}(t,x) = \begin{cases} \boldsymbol{\rho}_{0,i}(x - \boldsymbol{\xi}_{i,n}(t)) & \text{for } 0 \le \boldsymbol{\xi}_{i,n}(t) \le x \\ \frac{\boldsymbol{a}_{i,n}(\boldsymbol{\xi}_{i,n}^{-1}(\boldsymbol{\xi}_{i,n}(t)-x))}{\boldsymbol{\xi}'_{i,n}(\boldsymbol{\xi}_{i,n}^{-1}(\boldsymbol{\xi}_{i,n}(t)-x))} & \text{for } 0 \le x \le \boldsymbol{\xi}_{i,n}(t), \end{cases}$$

we obtain for $t \leq \min_{i\in\mathfrak{N}} \min\{\boldsymbol{\xi}_{i,n}^{-1}(1), T\}$ – using Equation (20) for $W(\cdot, *)$ –

$$\boldsymbol{\xi}_n(t) = \int_0^t \boldsymbol{\lambda}\left(\sum_{i=1}^N \left(\int_0^s \boldsymbol{a}_{i,n}(\mathfrak{t})\,\mathrm{d}\mathfrak{t} + \int_0^{1-\boldsymbol{\xi}_{i,n}(s)} \boldsymbol{\rho}_{0,i}(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\right)\right)\mathrm{d}s. \tag{39}$$

Since by assumption $\boldsymbol{a}_n$ is weakly convergent in $L^2((0,T);\mathbb{R}^N)$, it is uniformly bounded. Furthermore, we know from Theorem 2.7 and Remark 3 that $W(t,\boldsymbol{\rho}_n)$ is continuous with respect to t. A Hölder estimate and Equation (20) yields

$$\begin{aligned}
|W(t,\boldsymbol{\rho}_n)| &\leq \sum_{i=1}^N \left(\int_0^t |\boldsymbol{a}_{i,n}(\mathfrak{t})|\,\mathrm{d}\mathfrak{t} + \int_0^{1-\boldsymbol{\xi}_{i,n}(t)} |\boldsymbol{\rho}_{0,i}(\mathfrak{s})|\,\mathrm{d}\mathfrak{s}\right)\\
&\leq \sum_{i=1}^N \left(\sqrt{t}\|\boldsymbol{a}_{i,n}\|_{L^2((0,t);\mathbb{R}))} + \sqrt{1-\boldsymbol{\xi}_{i,n}(t)}\|\boldsymbol{\rho}_{0,i}\|_{L^2((0,1-\boldsymbol{\xi}_{i,n}(t));\mathbb{R})}\right)\\
&\leq \sqrt{T}\|\boldsymbol{a}_n\|_{L^2((0,T);\mathbb{R}^N)} + \|\boldsymbol{\rho}_0\|_{L^2((0,1);\mathbb{R}^N)}.
\end{aligned}$$

The L^2 function $\boldsymbol{\rho}_0$ is given data, therefore, $\|\boldsymbol{\rho}_0\|_{L^2((0,1);\mathbb{R}^N)}$ is bounded and – taking into account the uniform boundedness of $\boldsymbol{a}_n$ in L^2 – we altogether obtain

$$\max_{t\in[0,T]} |W(t,\boldsymbol{\rho}_n)| \leq C$$

with a given constant C independent of n. By $C \in \mathbb{R}$ we denote several, possibly different, constants. Furthermore, we have for $i \in \mathfrak{N}$

$$\|\boldsymbol{\xi}_{i,n}\|_{C^1([0,T];\mathbb{R})} \leq C$$

by the uniform boundedness of $W(t,\boldsymbol{\rho}_n)$ and the boundedness of the continuous function $\boldsymbol{\lambda}(\cdot)$ on a bounded set such that also

$$\|\boldsymbol{\xi}_n\|_{C^1([0,T];\mathbb{R}^N)} \leq C \tag{40}$$

is uniformly bounded with respect to the C^1-norm. Now define the set

$$\boldsymbol{\Xi} := \{\boldsymbol{\xi}_n : n \in \mathbb{N} \text{ and } \boldsymbol{\xi}_n \text{ defined by Equation (38)}\} \tag{41}$$

and observe that $\boldsymbol{\Xi}$ is bounded with respect to the uniform topology (i.e. the C^0-topology) and equicontinuous. Thereby, the equicontinuity is a direct consequence of the uniform estimate in the C^1-norm as stated in Equation (40) and thus of the uniform (with respect to n) boundedness of the first derivative of $\boldsymbol{\xi}_n$,

$$|\boldsymbol{\xi}_n'(t)| = |\boldsymbol{\lambda}(W(t,\boldsymbol{\rho}_n))| \leq \overline{\boldsymbol{\lambda}}(M) \quad \text{with } M := \sum_{i=1}^N \left(\|\boldsymbol{a}_i\|_{L^1(0,T)} + \|\boldsymbol{\rho}_{0,i}\|_{L^1(0,1)}\right).$$

Thus, the sequence $(\boldsymbol{\xi}_n)_{n\in\mathbb{N}}$ is Lipschitz continuous and hence equicontinuous.

So we can apply Arzelà-Ascoli's Theorem (for instance see [9, Theorem 4.25]) which then states that $\boldsymbol{\Xi}$ in (41) is relatively compact in $C([0,T];\mathbb{R}^N)$. Thus,

$$\exists \bar{\boldsymbol{\xi}}_\infty \in C([0,T];\mathbb{R}^N)\ \exists\{\boldsymbol{\xi}_{n_l}\}_{l\in\mathbb{N}} : \boldsymbol{\xi}_{n_l} \to \bar{\boldsymbol{\xi}}_\infty \text{ in } C([0,T];\mathbb{R}^N), \tag{42}$$

i.e. there exists a subsequence of $\boldsymbol{\xi}_n$ and $\bar{\boldsymbol{\xi}}_\infty \in C([0,T];\mathbb{R}^N)$ such that the subsequence converges to $\bar{\boldsymbol{\xi}}_\infty$ in the C^0-topology.

Note that up to now we do not have that $\boldsymbol{\Xi}$ in (41) is closed so it is unclear if $\bar{\boldsymbol{\xi}}_\infty$ is the corresponding characteristic to the inflow $\boldsymbol{a}_\infty$. Let

$$\begin{aligned}\bar{C} &:= \sup_{n\in\mathbb{N}} \|\boldsymbol{a}_n\|_{L^1((0,T);\mathbb{R}^N)} + \|\boldsymbol{\rho}_0\|_{L^1((0,1);\mathbb{R}^N)} \\ &\leq \sqrt{T}\sup_{n\in\mathbb{N}} \|\boldsymbol{a}_n\|_{L^2((0,T);\mathbb{R}^N)} + \|\boldsymbol{\rho}_0\|_{L^2((0,1);\mathbb{R}^N)} < \infty,\end{aligned}$$

since $L^2(\Omega) \hookrightarrow L^1(\Omega)$ for $\Omega \subseteq \mathbb{R}^d$ open and bounded, $d \in \mathbb{N}$, by a simple Hölder estimate. Then we have for all $i \in \mathfrak{N}$ – recalling Definition 2.4 –

$$0 < \underline{\boldsymbol{\lambda}}_i(\bar{C}) \leq \boldsymbol{\xi}'_{n,i}(t) = \boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}_n)) \leq \overline{\boldsymbol{\lambda}}_i(\bar{C}) \quad \forall t\in[0,T],\ n\in\mathbb{N} \tag{43}$$

as well as for $i \in \mathfrak{N}$ – in the following denoting an appropriate subsequence again as sequence –

$$\begin{aligned}\underline{\boldsymbol{\lambda}}_i(\bar{C})|\bar{\boldsymbol{\xi}}^{-1}_{i,\infty}(x) - \boldsymbol{\xi}^{-1}_{i,n}(x)| &\leq \left|\boldsymbol{\xi}_{n,i}\left(\bar{\boldsymbol{\xi}}^{-1}_{i,\infty}(x)\right) - \boldsymbol{\xi}_{n,i}\left(\boldsymbol{\xi}^{-1}_{i,n}(x)\right)\right| \\ &= \left|\boldsymbol{\xi}_{n,i}\left(\bar{\boldsymbol{\xi}}^{-1}_{i,\infty}(x)\right) - \bar{\boldsymbol{\xi}}_{\infty,i}\left(\bar{\boldsymbol{\xi}}^{-1}_{i,\infty}(x)\right)\right| \\ &\longrightarrow 0 \text{ for } n\to\infty\end{aligned}$$

uniformly for $x \in [0,\bar{\boldsymbol{\xi}}^{-1}_{i,\infty}(T)]$ thanks to Equation (42). Therefore, we get the convergence – at least for a subsequence –

$$\boldsymbol{\xi}^{-1}_{i,n} \to \bar{\boldsymbol{\xi}}^{-1}_{i,\infty} \text{ in } C([0,x_0]). \tag{44}$$

Now, for $t \in [0, \min_{i\in\mathfrak{N}} \min\{\boldsymbol{\xi}^{-1}_{n,i}(1), T\}]$ in Equation (39) we pass $n\to\infty$ and obtain by the weak convergence of $\boldsymbol{a}_n$ and the uniform convergence of $\boldsymbol{\xi}_n$

$$\bar{\boldsymbol{\xi}}_\infty(t) = \int_0^t \boldsymbol{\lambda}\Big(\sum_{i=1}^N \big(\int_0^s \boldsymbol{a}_{i,\infty}(\mathfrak{t})\,\mathrm{d}\mathfrak{t} + \int_0^{1-\bar{\boldsymbol{\xi}}_{i,\infty}(s)} \boldsymbol{\rho}_{0,i}(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\big)\Big)\,\mathrm{d}s,\ t\in[0,\min_{i\in\mathfrak{N}}\min\{\bar{\boldsymbol{\xi}}^{-1}_{\infty,i}(1),T\}].$$

Of course by definition we also have for $\boldsymbol{\xi}_\infty$, i.e. the characteristic to $\boldsymbol{\rho}_\infty$, in the assumption (Lemma 3.8),

$$\boldsymbol{\xi}_\infty(t) = \int_0^t \boldsymbol{\lambda}\Big(\sum_{i=1}^N \big(\int_0^s \boldsymbol{a}_{i,\infty}(\mathfrak{t})\,\mathrm{d}\mathfrak{t} + \int_0^{1-\boldsymbol{\xi}_{i,\infty}(s)} \boldsymbol{\rho}_{0,i}(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\big)\Big)\,\mathrm{d}s,\ t\in[0,\min_{i\in\mathfrak{N}}\min\{\boldsymbol{\xi}^{-1}_{\infty,i}(1),T\}].$$

Now it is evident that $\boldsymbol{\rho}_\infty \equiv \bar{\boldsymbol{\rho}}_\infty$ for small t (for this see the proof of Theorem 2.7) and by the extension procedure in the proof of Theorem 2.7 this is also true for larger t. Thus, we have

$$\begin{aligned}\boldsymbol{\rho}_\infty &\equiv \bar{\boldsymbol{\rho}}_\infty && \text{for } (t,x)\in(0,T)\times(0,1),\\ \boldsymbol{\xi}'_n &\to \boldsymbol{\xi}'_\infty && \text{in } C([0,T];\mathbb{R}^N),\\ \boldsymbol{\xi}_n &\to \boldsymbol{\xi}_\infty && \text{in } C^1([0,T];\mathbb{R}^N).\end{aligned}$$

Finally, we show the postulated weak convergence

$$\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}_\infty \text{ in } L^2((0,T);\mathbb{R}^N).$$

By definition, this means we must prove

$$\lim_{n\to\infty}\int_0^T (\boldsymbol{y}_n(t) - \boldsymbol{y}_\infty(t))\,\boldsymbol{g}(t)\,\mathrm{d}t = 0 \quad \forall \boldsymbol{g}\in L^2((0,T);\mathbb{R}^N).$$

We divide the remaining proof into different cases and show without loss of generality the claimed result only for one commodity $i \in \mathfrak{N}$. Let $i \in \mathfrak{N}$ be given and $\boldsymbol{\xi}_{i,\infty}(T) < 1$. By uniform convergence there is n sufficiently large such that we have also $\boldsymbol{\xi}_{i,n}(T) < 1$ and obtain by Equation (24) for arbitrary $\boldsymbol{g} \in L^2((0,T);\mathbb{R}^N)$

$$\left| \int_0^T \big(\boldsymbol{y}_{i,n}(t) - \boldsymbol{y}_{i,\infty}(t)\big)\, \boldsymbol{g}_i(t)\,\mathrm{d}t \right|$$
$$= \left| \int_0^T \big(\boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}_n))\boldsymbol{\rho}_{0,i}(1-\boldsymbol{\xi}_{i,n}(t)) - \boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_{0,i}(1-\boldsymbol{\xi}_{i,\infty}(t))\big)\, \boldsymbol{g}_i(t)\,\mathrm{d}t \right|$$

i.e. the outflow only depends on the initial data (and through $W(t,\boldsymbol{\rho})$ also on the initial data of the remaining commodities as well as possibly on the inflow of these).

By integration by substitution and the assumption $\boldsymbol{\xi}_{i,n}(T) \geq \boldsymbol{\xi}_{i,\infty}(T)$ we have

$$\left| \int_0^T \big(\boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}_n))\boldsymbol{\rho}_{0,i}(1-\boldsymbol{\xi}_{i,n}(t)) - \boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_{0,i}(1-\boldsymbol{\xi}_{i,\infty}(t))\big)\, \boldsymbol{g}_i(t)\,\mathrm{d}t \right|$$
$$= \left| \int_{1-\boldsymbol{\xi}_{i,n}(T)}^1 \boldsymbol{\rho}_{0,i}(y)\boldsymbol{g}_i(\boldsymbol{\xi}_{i,n}^{-1}(1-y))\,\mathrm{d}y - \int_{1-\boldsymbol{\xi}_{i,\infty}(T)}^1 \boldsymbol{\rho}_{0,i}(y)\boldsymbol{g}_i(\boldsymbol{\xi}_{i,\infty}^{-1}(1-y))\,\mathrm{d}y \right|$$
$$\leq \left| \int_{1-\boldsymbol{\xi}_{i,\infty}(T)}^1 \boldsymbol{\rho}_{0,i}(y)\left(\boldsymbol{g}_i(\boldsymbol{\xi}_{i,n}^{-1}(1-y)) - \boldsymbol{g}_i(\boldsymbol{\xi}_{i,\infty}^{-1}(1-y))\right)\,\mathrm{d}y \right| \tag{45}$$
$$+ \left| \int_{1-\boldsymbol{\xi}_{i,n}(T)}^{1-\boldsymbol{\xi}_{i,\infty}(T)} \boldsymbol{\rho}_{0,i}(y)\boldsymbol{g}_i(\boldsymbol{\xi}_{i,n}^{-1}(1-y))\,\mathrm{d}y \right|. \tag{46}$$

Now by Hölder's inequality we estimate Expression (45)

$$\left| \int_{1-\boldsymbol{\xi}_{i,\infty}(T)}^1 \boldsymbol{\rho}_{0,i}(y)\left(\boldsymbol{g}_i(\boldsymbol{\xi}_{i,n}^{-1}(1-y)) - \boldsymbol{g}_i(\boldsymbol{\xi}_{i,\infty}^{-1}(1-y))\right)\,\mathrm{d}y \right|$$
$$\leq \|\boldsymbol{\rho}_{0,i}\|_{L^2(0,1)} \|\boldsymbol{g}_i(\boldsymbol{\xi}_{i,n}^{-1}(1-\cdot)) - \boldsymbol{g}_i(\boldsymbol{\xi}_{i,\infty}^{-1}(1-\cdot))\|_{L^2(1-\boldsymbol{\xi}_{i,\infty}(T),1)},$$

which converges for $n \to \infty$ to zero by the uniform convergence of $\boldsymbol{\xi}_n$ to $\boldsymbol{\xi}_\infty$, the result in (44) and Lemma 2.6. The Expression (46) can also be estimated by Hölder's inequality to (assuming that $\boldsymbol{\xi}_{i,n}(T) \geq \boldsymbol{\xi}_{i,\infty}(T)$, otherwise we have to change the boundary of the integral)

$$\left| \int_{1-\boldsymbol{\xi}_{i,n}(T)}^{1-\boldsymbol{\xi}_{i,\infty}(T)} \boldsymbol{\rho}_{0,i}(y)\boldsymbol{g}_i(\boldsymbol{\xi}_{i,n}^{-1}(1-y))\,\mathrm{d}y \right|$$
$$\leq \|\boldsymbol{\rho}_{0,i}\|_{L^2(1-\boldsymbol{\xi}_{i,n}(T),1-\boldsymbol{\xi}_{i,\infty}(T))} \|\boldsymbol{\lambda}_i(W(\cdot,\boldsymbol{\rho}_n))\|_{L^\infty(0,T)} \|\boldsymbol{g}_i\|_{L^2(0,T)}.$$

For $n \to \infty$ the integration measure tends to zero in $\|\boldsymbol{\rho}_{0,i}\|_{L^2(1-\boldsymbol{\xi}_{i,n}(T),1-\boldsymbol{\xi}_{i,\infty}(T))}$ due to the uniform convergence of $\boldsymbol{\xi}_n \to \boldsymbol{\xi}_\infty$, while the remaining terms are bounded with respect to n (see Inequality (43)).

Now let for $i \in \mathfrak{N}$ fixed, $\boldsymbol{\xi}_{i,\infty}(T) \geq 1$. Then we also have for sufficiently large $n \in \mathbb{N}$ $\boldsymbol{\xi}_{i,n}(T) \geq 1$ and we can write by Equation (24) for sufficiently large t

$$\boldsymbol{y}_{i,\infty} \equiv \boldsymbol{\lambda}_i(W(\cdot,\boldsymbol{\rho}_\infty))\frac{\boldsymbol{a}_{i,\infty}(\boldsymbol{\xi}_{i,\infty}^{-1}(\boldsymbol{\xi}_{i,\infty}(\cdot)-1))}{\boldsymbol{\xi}_{i,\infty}'(\boldsymbol{\xi}_{i,\infty}^{-1}(\boldsymbol{\xi}_{i,\infty}(\cdot)-1))}, \quad \boldsymbol{y}_{i,n} \equiv \boldsymbol{\lambda}_i(W(\cdot,\boldsymbol{\rho}_n))\frac{\boldsymbol{a}_{i,n}(\boldsymbol{\xi}_{i,n}^{-1}(\boldsymbol{\xi}_{i,n}(\cdot)-1))}{\boldsymbol{\xi}_{i,n}'(\boldsymbol{\xi}_{i,n}^{-1}(\boldsymbol{\xi}_{i,n}(\cdot)-1))},$$

i.e. the outflow depends explicitly on the inflow (since the initial conditions have already passed the boundary) and implicitly on the remaining commodities. Similar estimations as already performed will thus show the weak convergence of the outflow $\boldsymbol{y}_n$ to $\boldsymbol{y}_\infty$ in $L^2((0,T);\mathbb{R}^N)$. We will not carry out these straightforward estimations. ☐

Lemma 3.9 (Weak and strong convergence and its product)**.** *As archetype of a network with distributing vertices we consider one vertex* $\mathfrak{v}$ *with one incoming edge* 0 *and* $m \in \mathbb{N}_{\geq 2}$ *outgoing edges* $\{1,\ldots,m\}$ *as illustrated in Figure 6. Therefore, let*

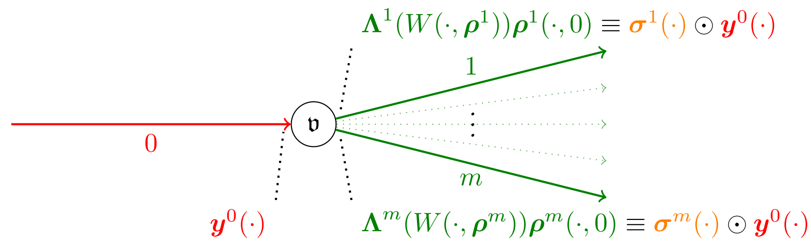

Figure 6. A diverging vertex, $\mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}$, with edges $\{0,1,\ldots,m\}$

$M \in \mathbb{R}_{\geq 0}$, $N \in \mathbb{N}$, $T \in \mathbb{R}_{>0}$ *be given and – for simplicity –*

$$\tilde{\boldsymbol{\Sigma}}_{BV_M} := \Big\{\boldsymbol{\sigma} \in BV((0,T);\mathbb{R}^N)^m : \boldsymbol{0} \leq \boldsymbol{\sigma}^{\mathfrak{e}}(t) \text{ a.e.}, \sum_{\mathfrak{e}=1}^m \boldsymbol{\sigma}^{\mathfrak{e}}(t) = \boldsymbol{1} \text{ a.e.} \\ \|\boldsymbol{\sigma}^{\mathfrak{e}}\|_{BV((0,T);\mathbb{R}^N)} \leq M \ \forall \mathfrak{e} \in \{1,\ldots,m\}\Big\} \tag{47}$$

with $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^m)$*. Let* $(\boldsymbol{y}_n)_{n\in\mathbb{N}} \subset L^2((0,T);\mathbb{R}^N)$ *be a weakly convergent sequence with weak limit* $\boldsymbol{y}_\infty \in L^2((0,T);\mathbb{R}^N)$ *and let* $(\boldsymbol{\sigma}_n^{\mathfrak{e}})_{n\in\mathbb{N}} \subset \tilde{\boldsymbol{\Sigma}}_{BV_M}$ *be a strongly convergent sequence in* $L^2((0,T);\mathbb{R}^N)$ *with limit* $\boldsymbol{\sigma}_\infty^{\mathfrak{e}} \in \tilde{\boldsymbol{\Sigma}}_{BV_M}$*. Then, we have for every* $\mathfrak{e} \in \{1,\ldots,m\}$

$$\boldsymbol{\sigma}_n^{\mathfrak{e}} \odot \boldsymbol{y}_n \rightharpoonup \boldsymbol{\sigma}_\infty^{\mathfrak{e}} \odot \boldsymbol{y}_\infty \text{ in } L^2((0,T);\mathbb{R}^N),$$

where $\odot$ *is the component-wise multiplication defined in Definition 3.3.*

Proof. We use the definition of weak convergence in $L^2((0,T);\mathbb{R}^N)$ and compute for given $\boldsymbol{g} \in L^2((0,T);\mathbb{R}^N)$, noticing that for any Hilbert space H, $\langle\,\cdot\,,*\,\rangle_H$ denotes the inner product,

$$\begin{aligned}
&\Big|\langle\, \boldsymbol{\sigma}_n^{\mathfrak{e}} \odot \boldsymbol{y}_n - \boldsymbol{\sigma}_\infty^{\mathfrak{e}} \odot \boldsymbol{y}_\infty \,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| \\
&= \Big|\langle\, \boldsymbol{\sigma}_n^{\mathfrak{e}} \odot \boldsymbol{y}_n - \boldsymbol{\sigma}_\infty^{\mathfrak{e}} \odot \boldsymbol{y}_n + \boldsymbol{\sigma}_\infty^{\mathfrak{e}} \odot \boldsymbol{y}_n - \boldsymbol{\sigma}_\infty^{\mathfrak{e}} \odot \boldsymbol{y}_\infty \,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| \\
&\leq \Big|\langle\, (\boldsymbol{\sigma}_n^{\mathfrak{e}} - \boldsymbol{\sigma}_\infty^{\mathfrak{e}}) \odot \boldsymbol{y}_n \,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| + \Big|\langle\, \boldsymbol{\sigma}_\infty^{\mathfrak{e}} \odot (\boldsymbol{y}_n - \boldsymbol{y}_\infty) \,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big|.
\end{aligned}$$

First, let us consider the second term, which can be transformed into

$$\Big|\langle\, \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty} \odot (\boldsymbol{y}_n - \boldsymbol{y}_{\infty})\,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| = \Big|\langle\, \boldsymbol{y}_n - \boldsymbol{y}_{\infty}\,,\, \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty} \odot \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big|.$$

Since $\boldsymbol{\sigma}^{\mathfrak{c}}_{\infty} \in L^{\infty}((0,T);\mathbb{R}^N)$ by the embedding [7, Theorem 10.1.3] and Definition 47, and $\boldsymbol{g} \in L^2((0,T);\mathbb{R}^N)$, the product $\boldsymbol{\sigma}^{\mathfrak{c}}_{\infty} \odot \boldsymbol{g}$ is again in $L^2((0,T);\mathbb{R}^N)$ such that we can use the weak convergence of $\boldsymbol{y}_n$ to $\boldsymbol{y}_{\infty}$ to conclude that

$$\lim_{n\to\infty}\Big|\langle\, \boldsymbol{y}_n - \boldsymbol{y}_{\infty}\,,\, \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty} \odot \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| = 0.$$

Second, let us consider the first term

$$\Big|\langle\, (\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}) \odot \boldsymbol{y}_n\,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big|. \tag{48}$$

Since we have the convergence of $\boldsymbol{\sigma}^{\mathfrak{c}}_{n} \to \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}$ only in $L^p((0,T);\mathbb{R}^N)$ for $p < \infty$ and $\boldsymbol{y}_n \in L^2((0,T);\mathbb{R}^N)$ as well as $\boldsymbol{g} \in L^2((0,T);\mathbb{R}^N)$, we cannot conclude directly that this term will also converge to zero. Thus, we use a smoothing argument as follows.

By [25, Theorem 17.12] we have for $T \in \mathbb{R}_{>0}$ the very famous result that

$$\text{for every } p \in [1,\infty):\ \overline{C([0,T])}^{\|\cdot\|_{L^p(0,T)}} = L^p(0,T),$$

i.e. the continuous functions on a bounded interval are dense in L^p and so also $C([0,T];\mathbb{R}^N)$ is dense in $L^p((0,T);\mathbb{R}^N)$. Therefore, we can find for every $\boldsymbol{g} \in L^2((0,T);\mathbb{R}^N)$ a sequence $(\boldsymbol{g}_m)_{m\in\mathbb{N}} \subset C([0,T];\mathbb{R}^N)$ such that

$$\lim_{m\to\infty}\|\boldsymbol{g} - \boldsymbol{g}_m\|_{L^2((0,T);\mathbb{R}^N)} = 0.$$

We can use this result in Expression (48) and obtain by applying Hölder's inequality

$$\begin{aligned}
&\Big|\langle\, (\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}) \odot \boldsymbol{y}_n\,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| \\
&= \Big|\langle\, (\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}) \odot \boldsymbol{y}_n\,,\, \boldsymbol{g} - \boldsymbol{g}_m + \boldsymbol{g}_m\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| \\
&\leq \Big|\langle\, (\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}) \odot \boldsymbol{y}_n\,,\, \boldsymbol{g} - \boldsymbol{g}_m\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| + \Big|\langle\, (\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}) \odot \boldsymbol{y}_n\,,\, \boldsymbol{g}_m\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| \\
&\leq \|\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}\|_{L^{\infty}((0,T);\mathbb{R}^N)}\|\boldsymbol{y}_n\|_{L^2((0,T);\mathbb{R}^N)}\|\boldsymbol{g} - \boldsymbol{g}_m\|_{L^2((0,T);\mathbb{R}^N)} \\
&\quad + \|\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}\|_{L^2((0,T);\mathbb{R}^N)}\|\boldsymbol{y}_n \odot \boldsymbol{g}_m\|_{L^2((0,T);\mathbb{R}^N)} \\
&\leq 2C\|\boldsymbol{g} - \boldsymbol{g}_m\|_{L^2((0,T);\mathbb{R}^N)} + C\|\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}\|_{L^2((0,T);\mathbb{R}^N)}\|\boldsymbol{g}_m\|_{L^{\infty}((0,T);\mathbb{R}^N)},
\end{aligned}$$

where $C = \sup_{n\in\mathbb{N}}\|\boldsymbol{y}_n\|_{L^2((0,T);\mathbb{R}^N)}$. C is bounded since $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $L^2((0,T);\mathbb{R}^N)$ and $\|\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}\|_{L^{\infty}((0,T);\mathbb{R}^N)}$ has been estimated by 2, since $\boldsymbol{\sigma}^{\mathfrak{c}}_{n}$ and $\boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}$ are by assumption in $\tilde{\boldsymbol{\Sigma}}_M$ and therefore essentially bounded by 1.

Then let $\epsilon > 0$ be given we obtain by the convergence and density results

$$\exists M_0(\epsilon) \in \mathbb{N} : \|\boldsymbol{g} - \boldsymbol{g}_{M_0}\|_{L^2((0,T);\mathbb{R}^N)} \leq \epsilon.$$

Furthermore, we choose $n \in \mathbb{N}$ sufficiently large to guarantee

$$\|\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}\|_{L^2((0,T);\mathbb{R}^N)}\|\boldsymbol{g}_{M_0}\|_{L^{\infty}((0,T);\mathbb{R}^N)} \leq \epsilon$$

and obtain for n sufficiently large altogether

$$\Big|\langle\, (\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}) \odot \boldsymbol{y}_n\,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| \leq 3C\epsilon.$$

Since $\epsilon > 0$ was arbitrary we can conclude that

$$\lim_{n\to\infty}\Big|\langle\, (\boldsymbol{\sigma}^{\mathfrak{c}}_{n} - \boldsymbol{\sigma}^{\mathfrak{c}}_{\infty}) \odot \boldsymbol{y}_n\,,\, \boldsymbol{g}\,\rangle_{L^2((0,T);\mathbb{R}^N)}\Big| = 0,$$

the missing link to show the weak convergence of the product of a strongly convergent sequence $(\boldsymbol{\sigma}_n^{\mathfrak{e}})_{n\in\mathbb{N}} \subset \tilde{\boldsymbol{\Sigma}}_M$ and a weak convergent sequence $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$. $\square$

Theorem 3.10 (Existence of a minimizer for $\boldsymbol{a} \in L^2$ and $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{BV_M}$). *Let $N \in \mathbb{N}$, $M \in \mathbb{R}_{\geq 0}$, $T \in \mathbb{R}_{>0}$ and a network as defined in Definition 3.1 be given. Consider the optimal control problem with objective as defined in Definition 3.6 subject to the dynamics on the network as defined in Definition 3.3 and replace the condition $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ as defined in Definition 3.2 by $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{BV_M}$ with BV_M as in (37). Then, there exists a minimizer*

$$\exists \boldsymbol{a}_\infty \in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_s|},\ \boldsymbol{\sigma}_\infty \in \boldsymbol{\Sigma}_{BV_M},\ \boldsymbol{y}_\infty \in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_e|}$$

such that

$$J(\boldsymbol{a}_\infty, \boldsymbol{\sigma}_\infty, \boldsymbol{y}_\infty) = \inf_{\substack{\boldsymbol{a}\in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_s|}\\ \boldsymbol{\sigma}\in\boldsymbol{\Sigma}_{BV_M}\\ \boldsymbol{y}\in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_e|}}} J(\boldsymbol{a}, \boldsymbol{\sigma}, \boldsymbol{y}),$$

i.e. $(\boldsymbol{a}_\infty, \boldsymbol{\sigma}_\infty, \boldsymbol{y}_\infty)$ solves the corresponding optimal control problem.

For $\mathfrak{e} \in \mathfrak{E}_e$ the outflow $\boldsymbol{y}^{\mathfrak{e}}$ is defined by

$$\boldsymbol{y}^{\mathfrak{e}} \equiv \boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot, \boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(\cdot, 1),$$

and is – as already mentioned – completely determined by $\boldsymbol{a}$ and $\boldsymbol{\sigma}$.

Proof. Since we have by Definition

$J(\boldsymbol{a}, \boldsymbol{\sigma}, \boldsymbol{y}) \geq 0$ for every $(\boldsymbol{a}, \boldsymbol{\sigma}, \boldsymbol{y}) \in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_s|} \times \boldsymbol{\Sigma}_{BV_M} \times L^2((0,T);\mathbb{R}^N)^{|\mathfrak{E}_e|}$,

we can find a minimizing sequence

$$(\boldsymbol{a}_n, \boldsymbol{\sigma}_n, \boldsymbol{y}_n)_{n\in\mathbb{N}} \subset L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_s|} \times \boldsymbol{\Sigma}_{BV_M} \times L^2((0,T);\mathbb{R}^N)^{|\mathfrak{E}_e|}$$

of J such that

$$\lim_{n\to\infty} J(\boldsymbol{a}_n, \boldsymbol{\sigma}_n, \boldsymbol{y}_n) = \inf_{\substack{\boldsymbol{a}\in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_s|}\\ \boldsymbol{\sigma}\in\boldsymbol{\Sigma}_{BV_M}\\ \boldsymbol{y}\in L^2((0,T);\mathbb{R}^N)^{|\mathfrak{E}_e|}}} J(\boldsymbol{a}, \boldsymbol{\sigma}, \boldsymbol{y}).$$

Therefore, there exists $C > 0$ so that

$$J(\boldsymbol{a}_n, \boldsymbol{\sigma}_n, \boldsymbol{y}_n) \leq C\ \forall n \in \mathbb{N}.$$

Recalling Definition 3.6 of J, we conclude in particular that for every $\mathfrak{e} \in \mathfrak{E}_s$

$$\|\boldsymbol{a}_n^{\mathfrak{e}}\|_{L^2((0,T);\mathbb{R}^N)} \leq C \quad \forall n \in \mathbb{N}$$

uniformly in n.

Since every uniformly bounded sequence in L^2 admits a weakly convergent subsequence there is $\boldsymbol{a}_\infty \in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_s|}$ such that

$$\boldsymbol{a}_{n_k} \rightharpoonup \boldsymbol{a}_\infty \text{ in } L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_s|} \text{ for } k \to \infty.$$

Furthermore by Definition 3.2 of $\boldsymbol{\Sigma}_{BV_M}$ and the results of Lemma 3.7, for every $\mathfrak{v} \in \mathfrak{V}_d,\ \mathfrak{e} \in \mathfrak{E}_o(\mathfrak{v})$ we can find a subsequence and $\boldsymbol{\sigma}_\infty^{\mathfrak{e},\mathfrak{v}} \in L^2((0,T);\mathbb{R}^N)$ such that

$$\boldsymbol{\sigma}_{n_l}^{\mathfrak{e},\mathfrak{v}} \to \boldsymbol{\sigma}_\infty^{\mathfrak{e},\mathfrak{v}} \text{ in } L^2((0,T);\mathbb{R}^N) \text{ for } l \to \infty.$$

As we will see, the strong convergence is crucial at this point. For simplicity we denote the subsequences again as sequences.

For $\mathfrak{e} \in \mathfrak{E}$ let $\boldsymbol{\rho}_n^{\mathfrak{e}}$ be the weak solution of the initial boundary value problem stated in Definition 2.3 for $(t,x) \in (0,T) \times (0,1)$

$$\begin{aligned}\dot{\boldsymbol{\rho}}^{\mathfrak{e}}(t,x) + \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}_x^{\mathfrak{e}}(t,x) &= \mathbf{0} \\ \boldsymbol{\rho}^{\mathfrak{e}}(0,x) &= \boldsymbol{\rho}_0^{\mathfrak{e}}(x)\end{aligned}$$

complemented by the boundary value depending on the edges in the network

$$\begin{aligned}\boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t,0) &= \boldsymbol{a}_n^{\mathfrak{e}}(t) && \mathfrak{e} \in \mathfrak{E}_{\mathrm{s}} \\ \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(t,\boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(t,1) &= \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t,0) && \mathfrak{v} \in \mathfrak{V}_{\mathrm{m}} \setminus \mathfrak{V}_{\mathrm{d}}, \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}) \\ \boldsymbol{\sigma}_n^{\mathfrak{e},\mathfrak{v}}(t) \odot \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(t,\boldsymbol{\rho}^{\bar{\mathfrak{e}}}))\boldsymbol{\rho}^{\bar{\mathfrak{e}}}(t,1) &= \boldsymbol{\Lambda}^{\mathfrak{e}}(W(t,\boldsymbol{\rho}^{\mathfrak{e}}))\boldsymbol{\rho}^{\mathfrak{e}}(t,0) && \mathfrak{v} \in \mathfrak{V}_{\mathrm{d}}, \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v}).\end{aligned}$$

For $\mathfrak{e} \in \mathfrak{E}$ let $W(\cdot,\boldsymbol{\rho}_n^{\mathfrak{e}}) : [0,T] \to \mathbb{R}$ be defined as in Remark 3 and $\boldsymbol{\xi}_n^{\mathfrak{e}} : [0,T] \to \mathbb{R}^N$ as characteristic

$$W(t,\boldsymbol{\rho}_n^{\mathfrak{e}}) := \sum_{i=1}^{N} \int_0^1 \boldsymbol{\rho}_{i,n}^{\mathfrak{e}}(t,s)\,\mathrm{d}s, \quad \boldsymbol{\xi}_n^{\mathfrak{e}}(t) := \int_0^t \boldsymbol{\lambda}^{\mathfrak{e}}(W(s,\boldsymbol{\rho}_n^{\mathfrak{e}}))\,\mathrm{d}s.$$

In the proof we consider the different types of edges, separately, and use mathematical induction over the set of edges:
The initial or starting edges: For $\mathfrak{e} \in \mathfrak{E}_{\mathrm{s}}$ we can use Lemma 3.8 and obtain for $\boldsymbol{a}_n^{\mathfrak{e}} \rightharpoonup \boldsymbol{a}_\infty^{\mathfrak{e}}$ also

$$\boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\mathfrak{e}}(\cdot,1) \rightharpoonup \boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_\infty^{\mathfrak{e}}(\cdot,1) \quad \text{in } L^2((0,T);\mathbb{R}^N).$$

The non-distributing edges $\mathfrak{v} \in \mathfrak{V}_{\mathrm{m}} \setminus \mathfrak{V}_{\mathrm{d}},\ \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v})$: For those edges in the network we have as inflow the sum of some weakly convergent outflows, i.e.

$$\boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\mathfrak{e}}(\cdot,0) \equiv \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\bar{\mathfrak{e}}}(\cdot,1).$$

Assuming inductively that every inflow converges weakly, we obtain by Lemma 3.8

$$\sum_{\mathfrak{e} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\mathfrak{e}}(\cdot,1) \rightharpoonup \sum_{\mathfrak{e} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_\infty^{\mathfrak{e}}(\cdot,1),$$

and, thus,

$$\boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\mathfrak{e}}(\cdot,0) \rightharpoonup \boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_\infty^{\mathfrak{e}}(\cdot,0) \quad \text{in } L^2((0,T);\mathbb{R}^N).$$

The distribution edges $\mathfrak{v} \in \mathfrak{V}_{\mathrm{d}},\ \mathfrak{e} \in \mathfrak{E}_{\mathrm{o}}(\mathfrak{v})$: Here we have as inflow of a given edge

$$\boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\mathfrak{e}}(\cdot,0) \equiv \boldsymbol{\sigma}_n^{\mathfrak{e},\mathfrak{v}}(\cdot) \odot \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\bar{\mathfrak{e}}}(\cdot,1).$$

By Lemma 3.9 we obtain the weak convergence of

$$\boldsymbol{\sigma}_n^{\mathfrak{e},\mathfrak{v}}(\cdot) \odot \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\bar{\mathfrak{e}}}(\cdot,1) \rightharpoonup \boldsymbol{\sigma}_\infty^{\mathfrak{e},\mathfrak{v}}(\cdot) \odot \sum_{\bar{\mathfrak{e}} \in \mathfrak{E}_{\mathrm{i}}(\mathfrak{v})} \boldsymbol{\Lambda}^{\bar{\mathfrak{e}}}(W(\cdot,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_\infty^{\bar{\mathfrak{e}}}(\cdot,1)$$

such that by Lemma 3.8

$$\boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_n))\boldsymbol{\rho}_n^{\mathfrak{e}}(\cdot,0) \rightharpoonup \boldsymbol{\Lambda}^{\mathfrak{e}}(W(\cdot,\boldsymbol{\rho}_\infty))\boldsymbol{\rho}_\infty^{\mathfrak{e}}(\cdot,0) \quad \text{in } L^2((0,T);\mathbb{R}^N).$$

For the weak convergence of $\boldsymbol{y}_n$ to $\boldsymbol{y}_\infty$ we just note that $\boldsymbol{y}_n$ and $\boldsymbol{y}_\infty$ are uniquely determined by $\boldsymbol{a}_n, \boldsymbol{\sigma}_n,\ \boldsymbol{a}_\infty, \boldsymbol{\sigma}_\infty$ respectively. So we have constructed a minimizing sequence $(\boldsymbol{a}_n, \boldsymbol{\sigma}_n, \boldsymbol{y}_n)_{n \in \mathbb{N}}$ and have shown its (weak) convergence on the network.

As a result we obtain by the special structure of the cost functional J in Definition 3.6 and the weakly lower semi-continuity of J the inequality

$$\begin{aligned}
J(\boldsymbol{a}_\infty,\boldsymbol{\sigma}_\infty,\boldsymbol{y}_\infty) &= \sum_{\mathfrak{e}\in\mathfrak{E}_\mathrm{s}} \|\boldsymbol{a}_\infty^{\mathfrak{e}} - \boldsymbol{a}_\mathrm{d}^{\mathfrak{e}}\|_{L^2((0,T);\mathbb{R}^N)}^2 + \sum_{\mathfrak{e}\in\mathfrak{E}_\mathrm{e}} \|\boldsymbol{y}_\infty^{\mathfrak{e}} - \boldsymbol{y}_\mathrm{d}^{\mathfrak{e}}\|_{L^2((0,T);\mathbb{R}^N)}^2 \\
&\quad + \sum_{\mathfrak{v}\in\mathfrak{V}_\mathrm{d}} \sum_{\mathfrak{e}\in\mathfrak{E}_\mathrm{o}(\mathfrak{v})} \|\boldsymbol{\sigma}_\infty^{\mathfrak{e},\mathfrak{v}} - \boldsymbol{\sigma}_\mathrm{d}^{\mathfrak{e},\mathfrak{v}}\|_{L^2((0,T);\mathbb{R}^N)}^2 \\
&\leq \liminf_{n\to\infty} \sum_{\mathfrak{e}\in\mathfrak{E}_\mathrm{s}} \|\boldsymbol{a}_n^{\mathfrak{e}} - \boldsymbol{a}_\mathrm{d}^{\mathfrak{e}}\|_{L^2((0,T);\mathbb{R}^N)}^2 \\
&\quad + \liminf_{n\to\infty} \sum_{\mathfrak{e}\in\mathfrak{E}_\mathrm{e}} \|\boldsymbol{y}_n^{\mathfrak{e}} - \boldsymbol{y}_\mathrm{d}^{\mathfrak{e}}\|_{L^2((0,T);\mathbb{R}^N)}^2 \\
&\quad + \liminf_{n\to\infty} \sum_{\mathfrak{v}\in\mathfrak{V}_\mathrm{d}} \sum_{\mathfrak{e}\in\mathfrak{E}_\mathrm{o}(\mathfrak{v})} \|\boldsymbol{\sigma}_n^{\mathfrak{e},\mathfrak{v}} - \boldsymbol{\sigma}_\mathrm{d}^{\mathfrak{e},\mathfrak{v}}\|_{L^2((0,T);\mathbb{R}^N)}^2 \\
&\leq \liminf_{n\to\infty} J(\boldsymbol{a}_n,\boldsymbol{\sigma}_n,\boldsymbol{y}_n) = \inf_{\substack{\boldsymbol{a}\in L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_\mathrm{s}|}\\ \boldsymbol{\sigma}\in\boldsymbol{\Sigma}_{BV_M}\\ \boldsymbol{y}\in L^2((0,T);\mathbb{R}^N)^{|\mathfrak{E}_\mathrm{e}|}}} J(\boldsymbol{a},\boldsymbol{\sigma},\boldsymbol{y}).
\end{aligned}$$

Hence $(\boldsymbol{a}_\infty,\boldsymbol{\sigma}_\infty,\boldsymbol{y}_\infty)$ is indeed a minimizer of the cost functional and we also have

$$\liminf_{n\to\infty} J(\boldsymbol{a}_n,\boldsymbol{\sigma}_n,\boldsymbol{y}_n) = \lim_{n\to\infty} J(\boldsymbol{a}_n,\boldsymbol{\sigma}_n,\boldsymbol{y}_n).$$

Due to the fact that

$$\boldsymbol{a}_n \rightharpoonup \boldsymbol{a}_\infty \ \text{ in } L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_\mathrm{s}|}$$

and also

$$\|\boldsymbol{a}_n\|_{L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_\mathrm{s}|}} \to \|\boldsymbol{a}_\infty\|_{L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_\mathrm{s}|}},$$

we additionally have strong convergence of the chosen subsequence, i.e.

$$\boldsymbol{a}_n \to \boldsymbol{a}_\infty \text{ in } L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_\mathrm{s}|},$$

since convergence in the norm and weak convergence imply strong convergence in Hilbert spaces. The same argumentation is also valid for $\boldsymbol{y}_n$, such that we can also deduce for a chosen subsequence

$$\boldsymbol{y}_n \to \boldsymbol{y}_\infty \text{ in } L^2((0,T);\mathbb{R}_{\geq 0}^N)^{|\mathfrak{E}_\mathrm{s}|}.$$

□

Remark 8 (A modification of $\boldsymbol{\Sigma}_{BV_M}$)**.** One may ask why we do not optimize over the set $\boldsymbol{\Sigma}$ as the set of distribution functions in Theorem 3.10. In the proof of that theorem we needed for $\mathfrak{v}\in\mathfrak{V}_\mathrm{d}$ and $\mathfrak{e}\in\mathfrak{E}_\mathrm{o}(\mathfrak{v})$ the compactness of $\{\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}:\ \mathfrak{v}\in\mathfrak{V}_\mathrm{d}, \mathfrak{e}\in\mathfrak{E}_\mathrm{o}(\mathfrak{v}),\ \sum_{i=1}^n \boldsymbol{\sigma}_i^{\mathfrak{e},\mathfrak{v}}(t)=1,\ \boldsymbol{\sigma}(t)\geq \boldsymbol{0} \text{ a.e.}\}$ in $L^2((0,T);\mathbb{R}^N)$ to make a weak convergent sequence in $\boldsymbol{\Sigma}_M$ (i.e. in the BV ball) strong convergent in $L^2((0,T);\mathbb{R}^N)$ (see also Lemma 3.7 and Lemma 3.9).

This compact embedding is not true for $\boldsymbol{\Sigma}$ as defined in Definition 3.2 as the following theorem of Jacques Simon [22, Theorem 1] illustrates. Therein it is stated for a Banach space B that a subset M of $L^p((0,T);B)$ for $p\in[1,\infty)$ is relatively compact (i.e. $\overline{M}^{\|\cdot\|_{L^p((0,T);B)}}$ is compact) if and only if

- the set $\left\{\int_{t_1}^{t_2} f(t)\,\mathrm{d}t : f\in M\right\}$ is relatively compact in B for every $0<t_1<t_2<T$

- $\sup_{f\in M} \|f(\cdot+h)-f(\cdot)\|_{L^p((0,T);B)} \to 0$ for $h\to 0$.

In our case we have $B=\mathbb{R}^N,\ p=2$ and $M=\boldsymbol{\Sigma}$.

The first statement can easily be proved to be true since the functions belonging to $\boldsymbol{\Sigma}$ are essentially bounded and therefore the set $\left\{\int_{t_1}^{t_2} f(t)\,\mathrm{d}t : f\in M\right\}$ is bounded in B and by the Heine-Borel's Theorem the closure is compact.

Nevertheless the second statement is not true and in [24] one can find a counterexample.

Remark 9 (A change of the objective functional)**.** As one can easily see in the proof of the existence Theorem 3.10, we have for the distributing edges a product of distribution functions and weak convergent boundary values. To guarantee the weak convergence of the product it is crucial that we have the essential boundedness of the distribution functions $\boldsymbol{\sigma}$ and also the strong convergence in L^p, in particular in L^2 (see Lemma 3.7 and Lemma 3.9). This strong convergence is underwritten by the compactness of BV_M in L^p, i.e. by assuming $\boldsymbol{\sigma}\in\boldsymbol{\Sigma}_{BV_M}$ as defined in Definition 3.2. Therefore, we could also change the objective in Definition 3.6 to

$$\tilde{J}(\boldsymbol{a},\boldsymbol{y}) := \sum_{\mathfrak{e}\in\mathfrak{E}_{\mathrm{s}}} \alpha_{\mathfrak{e}}\|\boldsymbol{a}^{\mathfrak{e}}-\boldsymbol{a}_{\mathrm{d}}^{\mathfrak{e}}\|^2_{L^2((0,T);\mathbb{R}^N)} + \sum_{\mathfrak{e}\in\mathfrak{E}_{\mathrm{e}}} \gamma_{\mathfrak{e}}\|\boldsymbol{y}^{\mathfrak{e}}-\boldsymbol{y}_{\mathrm{d}}^{\mathfrak{e}}\|^2_{L^2((0,T);\mathbb{R}^N)},$$

even with the choice $\gamma_{\mathfrak{e}}=0$ for all $\mathfrak{e}\in\mathfrak{E}_{\mathrm{e}}$. In either case Theorem 3.10 would still hold.

We have already mentioned that the compactness of BV is essential in the proof of Theorem 3.10. The uniform boundedness of the set $\boldsymbol{\Sigma}_{BV_M}$ with respect to the BV-norm could also be achieved by measuring the distribution functions in the objective with respect to the BV-norm, i.e. by changing the objective – for instance – to

$$\hat{J}(\boldsymbol{a},\boldsymbol{\sigma},\boldsymbol{y}) := \tilde{J}(\boldsymbol{a},\boldsymbol{y}) + \sum_{\mathfrak{v}\in\mathfrak{V}_{\mathrm{d}}}\sum_{\mathfrak{e}\in\mathfrak{E}_{\mathrm{o}}(\mathfrak{v})} s_{\mathfrak{v},\mathfrak{e}}\|\boldsymbol{\sigma}^{\mathfrak{e},\mathfrak{v}}-\boldsymbol{\sigma}_{\mathrm{d}}^{\mathfrak{e},\mathfrak{v}}\|^2_{BV((0,T);\mathbb{R}^N)}.$$

Then, we could drop the assumption $\boldsymbol{\sigma}\in\boldsymbol{\Sigma}_{BV_M}$ in Theorem 3.10 and could choose as set to optimize the distribution functions $\boldsymbol{\Sigma}_{BV}$ (Definition 3.2).

Remark 10 (A change of the objective functional from L^2 to L^p with $p\in[1,\infty)$)**.** Lemma 3.8 will still hold, if we replace the objective in Definition 3.6 by the corresponding L^p-norms for $p\in[1,\infty)$ and assume that $\boldsymbol{\sigma}\in\boldsymbol{\Sigma}_{BV_M}$ as defined in Definition 3.2, since the key ingredient to prove the result is convergence of the characteristics in the uniform topology, which is – due to the smoothing of the generalized WIP – true for every $p\in[1,\infty)$.

Also Lemma 3.9 holds for $p\in[1,\infty)$ and, thus, also Theorem 3.10. The only difference for $p\neq 2$ is that we cannot prove the strong convergence of $\boldsymbol{a}_n\to\boldsymbol{a}_\infty$ in $L^p((0,T);\mathbb{R}^N_{\geq 0})^{|\mathfrak{E}_{\mathrm{s}}|}$, since convergence of the norm and weak convergence does not imply strong convergence in Banach spaces.

4. **Conclusions.**

In this paper we have shown the well-posedness of a PDE model for multi-commodity flow on networks. Furthermore, we have proven existence of an optimal control in a suitable analytical framework.

Acknowledgments. We want to thank the referee for the very valuable comments, in particular for the suggestion to add Remark 10 and for indicating that there was a gap in the proof of Theorem 2.7 when we extended the solution from a small time interval $[0, \delta]$ to $[0, T]$. This work was supported by DFG in the framework of the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, project C03 and project A05.

REFERENCES

[1] R. A. Adams and J. J. Fournier, *Sobolev Spaces*, vol. 140 of Pure and Applied Mathematics (Amsterdam), 2nd edition, Elsevier/Academic Press, Amsterdam, 2003.

[2] A. Aggarwal, R. M. Colombo and P. Goatin, Nonlocal systems of conservation laws in several space dimensions, *SIAM Journal on Numerical Analysis,* **53** (2015), 963–983.

[3] L. Ambrosio, N. Fusco and D. Pallara, *Functions Of Bounded Variation And Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[4] D. Armbruster, P. Degond and C. Ringhofer, A model for the dynamics of large queuing networks and supply chains, *SIAM J. Appl. Math.*, **66** (2006), 896–920.

[5] D. Armbruster, D. E. Marthaler, C. A. Ringhofer, K. G. Kempf and T.-C. Jo, A continuum model for a re-entrant factory, *Operations Research*, **54** (2006), 933–950.

[6] A. A. Assad, Multicommodity network flows – a survey, *Networks*, **8** (1978), 37–91.

[7] H. Attouch, G. Buttazzo and G. Michaille, *Variational Analysis in Sobolev and BV Spaces*, vol. 6 of MPS/SIAM Series on Optimization, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006.

[8] S. Blandin and P. Goatin, Well-posedness of a conservation law with non-local flux arising in traffic flow modeling, *Numerische Mathematik,* Springer Berlin Heidelberg, (2015), 1–25.

[9] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[10] R. M. Colombo, M. Herty and M. Mercier, Control of the continuity equation with a non local flow, *ESAIM Control Optim. Calc. Var.*, **17** (2011), 353–379.

[11] J.-M. Coron, M. Kawski and Z. Wang, Analysis of a conservation law modeling a highly re-entrant manufacturing system, *Discrete Contin. Dyn. Syst. Ser. B*, **14** (2010), 1337–1359.

[12] L. R. Ford Jr. and D. R. Fulkerson, *Flows in Networks*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1962.

[13] A. Freno and E. Trentin, *Hybrid Random Fields: A Scalable Approach to Structure and Parameter Learning in Probabilistic Graphical Models*, Intelligent Systems Reference Library, Springer, 2011.

[14] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Vol. 80 of Monographs in Mathematics, Birkhäuser Boston, 1984.

[15] M. Gröschel, A. Keimer, G. Leugering and Z. Wang, Regularity theory and adjoint based optimality conditions for a nonlinear transport equation with nonlocal velocity, *SIAM J. Control Optim.,* **52** (2014), 2141–2163.

[16] M. Gugat, F. M. Hante, M. Hirsch-Dick and G. Leugering, Stationary states in gas networks, *Networks and Heterogeneous Media,* **10** (2015), 295–320.

[17] M. Gugat, M. Herty, A. Klar and G. Leugering, Optimal control for traffic flow networks, *Journal of Optimization Theory and Applications*, **126** (2005), 589–616.

[18] M. Gugat, M. Herty, A. Klar, G. Leugering and V. Schleper, Well-posedness of networked hyperbolic systems of balance laws, in *Constrained optimization and optimal control for partial differential equations*, vol. 160 of Internat. Ser. Numer. Math., Birkhäuser/Springer Basel AG, Basel, 2012, 123–146.

[19] J. L. Kennington, A survey of linear cost multicommodity network flows, *Operations Res.*, **26** (1978), 209–236.

[20] M. La Marca, D. Armbruster, M. Herty and C. Ringhofer, Control of continuum models of production systems, *IEEE Trans. Automat. Contr.*, **55** (2010), 2511–2526.

[21] G. Leoni, *A First Course in Sobolev Spaces*, vol. 105 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2009.

[22] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl. (4)*, **146** (1987), 65–96

[23] D. W. Stroock, *Essentials of Integration Theory for Analysis*, vol. 262, Springer, 2011.

[24] W. W.-Y. Wong, Compactness in L^2, 2013, Personal Communication., http://math.stackexchange.com/questions/329253/compactness-in-l2.

[25] J. J. Yeh, *Lectures On Real Analysis*, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

Received October 2014; revised May 2015.

E-mail address: martin.gugat@fau.de
E-mail address: alexander.keimer@fau.de
E-mail address: guenter.leugering@fau.de
E-mail address: wzq@fudan.edu.cn