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MEAN–FIELD CONTROL AND RICCATI EQUATIONS

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Abstract. We present a control approach for large systems of interacting agents based on the Riccati equation. If the agent dynamics enjoys a strong symmetry the arising high dimensional Riccati equation is simplified and the resulting coupled system allows for a formal mean–field limit. The steady–states of the kinetic equation of Boltzmann and Fokker Planck type can be studied analytically. In case of linear dynamics and quadratic objective function the presented approach is optimal and is compared to the model predictive control approach introduced in [2].

1. **Introduction.** Many mathematical models of self-organized systems of interacting agents have been introduced in the recent literature. Representative examples in biology, engineering, economy and sociology can be found in [7, 8, 14, 15, 16, 19, 27, 42, 32, 33, 6, 31]. We refer the reader to the recent surveys in [38, 39, 43], to the book [40] and the collection of papers [36] for an extensive introduction to the subject.

Most of the models start by describing a microscopic agents dynamic on the basis of a system of ordinary differential equations. Due to the possibly large number of interacting agents a mesoscopic and even macroscopic description can be derived to study the qualitative behavior as well as to present efficient numerical algorithms. The mesoscopic and macroscopic models are partial differential equations of kinetic and fluid–dynamic type, respectively. The relation between

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the microscopic dynamics and the partial differential equations through the corresponding kinetic and hydrodynamic description has been studied for example in [4, 6, 11, 19, 21, 24, 28, 29, 32, 31, 35, 42, 17].

Control mechanisms of self-organized systems have been studied for macroscopic models in [12, 13] and for kinetic and hydrodynamic models in [4, 22, 33]. Recently, the control of emergent behaviors in multiagent systems has been studied in [15, 26, 25] where the authors develop the idea of sparse optimization at the microscopic and kinetic level. We refer also to [9] for general results concerning the control of mean-field systems. A different approach has been proposed in [20] where by time–scale separation local mean–field games enter a macroscopic dynamics. In [2] a control problem for an opinion formation model has been studied, where the control is obtained by an instantaneous control algorithm. This is a special case of model predictive control strategy which allows to explicitly express the control in terms of the state dynamics and to derive the corresponding kinetic description. Model predictive control for Fokker-Planck equations has been applied recently also in [23]. In the recent papers [20, 18, 2] on control strategies for the microscopic particle interactions typically the control strategy is obtained solving an auxiliary problem (implicit or explicit) and the resulting expression for the control is substituted back into the original dynamics leading to a possibly modified and new dynamics. The later is studied using either a Boltzmann or a macroscopic approximation. This requires the action of the control to be *local* in time. This is an approximation to the optimal control that in general is obtained solving backwards in time the corresponding Hamilton–Jacobi Bellmann equation (HJB). The limit for large number of agents of the (HJB) equation is studied in [34] for Nash games.

Here, we want to combine the advantage in terms of computational costs and analysis of longtime behavior of kinetic and macroscopic equations for controlled dynamics where the control is *not* localized in time. The applied control still contains information on the backwards dynamics. In general, it is not possible to obtain explicit formulas in this case.

More precisely, we investigate problems where the collective behavior corresponds to the process of alignment, like in the opinion consensus dynamic [28, 42], in a constrained setting. In the case of a single population, the external intervention is introduced as an additional control subject to certain bounds, representing the limitations, in terms of economic resources, media availability, etc., of the opinion maker [2, 26, 25]. In the presence of multiple populations the control can be introduced as hierarchical leaderships where one population of leaders aims at controlling the populations of followers through a suitable cost function which characterizes the leaders' strategy in trying to influence the followers [5, 10].

We start the investigation using a simplified constrained alignment model with a single population. In order to decrease the complexity of the model when the number of agents is large, we follow the idea to rely on a kinetic description for the fully controlled process extending the existing approach [2] where the control action is only local. To this end the backwards dynamics have to be analyzed and restated in terms where a kinetic description is applicable. We derive the corresponding mean-field and Boltzmann description together with a Riccati equation. Further, we derive the Fokker-Planck model in the so called quasi-invariant opinion limit and show that it corresponds to the mean-field limit of the system. Several numerical results confirm the robustness of the present approach.

2. **Constrained models for alignment dynamics.** The starting point is a general framework which embeds several types of models describing the process of alignment. We consider the evolution of N agents, each of which is identified by its state $p_i = p_i(t) \in \mathbb{R}^d$, $d \geq 1$, at time t. The state p_i may account for opinion, wealth, velocity, or other attributes of agent i. Each agent adjusts its state according to the states of his neighbors [38, 40]

$$\frac{dp_i}{dt} = \frac{\alpha}{N} \sum_{j=1}^{N} a_{ij}(p_j - p_i), \quad a_{ij} \geq 0. \tag{1}$$

Here, $\alpha > 0$ is a scaling parameter and the coefficients a_{ij}, named *communication rates*, quantify the way the agents influence each other, independently of their total number N. In particular, the a_{ij}'s themselves are allowed to depend on the states of the agents i and j through a given function $\Phi(p_i, p_j)$. Most models for self-organized dynamics in social, biological and physical science assume that the dependence of a_{ij} decreases as a function of the distance $|p_i - p_j|$, where $|\cdot|$ is a problem-dependent metric, reflecting the common tendency to align with those who think or act alike.

The model (1) is also known as Krause–Hegselmann model and has been studied for example in [30].

In the case where the individuals are assumed to freely interact, one can distinguish between two main classes of self-alignment models. In the global case, the rules of engagement are such that every agent is influenced by every other agent, $a_{ij} > \epsilon > 0$. The dynamics in this case is driven by global interactions. Global interactions which are sufficiently strong lead to unconditional consensus in the sense that all initial configurations of agents concentrate around an emerging limit state $p_i(t) \to p_\infty$. In more realistic models, however, interactions between agents are limited to their local neighbors, like in bounded confidence models in the case of opinion dynamics [28]. The behavior of local models where some of the a_{ij} may vanish, is more difficult to analyze. In the general scenario for such local models, agents tend to concentrate into one or more separate clusters. The particular case in which agents concentrate into one cluster, that is the emergence of a consensus, depends on the propagation of uniform connectivity of the underling (weighted) graph associated with the adjacency matrix $\{a_{ij}\}$ [38].

Different to the classical approach, we are interested in such problems in the constrained case. The general setting consists of a control problem where both the evolution of the state and the objective functional of each agent are influenced by the collective behavior of all other agents. In our setting agents obey the dynamical system

$$\frac{dp_i}{dt} = \frac{\alpha}{N} \sum_{j=1}^{N} a_{ij}(p_j - p_i) + u, \quad a_{ij} \geq 0, \tag{2}$$

where the control $u = u(t)$ is given as the solution of the following optimal control problem

$$u = \operatorname{argmin} \left\{ J(u, p_1, \ldots, p_N) \right\}, \tag{3}$$

for J being a suitable objective functional.

This can be used to study the exterior influence of the system dynamics to enforce emergence of non spontaneous desired asymptotic states. Classical examples are given by persuading voters to vote for a specific candidate, by influencing buyers towards a given good or asset or by forcing animals to follow a specific path or to reach a desired zone.

In order to derive non–local optimal controls we simplify the problem first to allow for explicit computations. In particular, we focus on the case where the desired state is a solution to the alignment dynamics in the un-controlled case, i.e., $u \equiv 0$. Then, the problem (3) corresponds to a control action which under minimal costs drives the system towards the desired state for arbitrary initial configuration of the agents. This approach may also be viewed as a stabilization action of the control since for a suitable cost functional J any initial perturbation of the desired state will be damped over time. The extension of the presented approach to problems with general desired states and nonlinear dynamics are detailed in Section 5.

To exemplify, let us consider the dynamic of N agents where each agent has an opinion $w_i = w_i(t) \in \mathcal{I}$, $\mathcal{I} = [-1, 1]$, $i = 1, \dots, N$ and this opinion can change over time according to

$$\frac{dw_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} P(w_i, w_j)(w_j - w_i) + u, \qquad w_i(0) = w_{0i}, \tag{4}$$

where the control $u = u(t)$ is given by the minimization of the cost functional over a certain time horizon T

$$u = \operatorname{argmin} \int_0^T \frac{1}{N}\sum_{j=1}^{N} \left(\frac{1}{2}(w_j - w_d)^2 + \frac{\nu}{2}u^2\right) ds. \tag{5}$$

In the formulation (5) the value w_d is the desired state and $\nu > 0$ is a regularization parameter. Typically, the function $P(w, v)$ is such that $0 \le P(w, v) \le 1$ and represents a measure of the inclination of the agents to change their opinion.

As already outlined, we are interested in the study of the behavior of the system where the number of agents is large, namely in the limit $N \to \infty$. To this goal we simplify problem (4) by assuming further

$$P(w, v) = P \text{ with } 0 < P < 1 \text{ and } \nu > 0. \tag{6}$$

Under assumption (6) problem (4) may be rewritten as

$$\frac{d}{dt}\vec{w} = A\vec{w} + Bu,\ A \in \mathbb{R}^{n\times n},\ A_{ij} = \begin{pmatrix} a_o & i \neq j \\ a_d & i = j \end{pmatrix},\ B \in \mathbb{R}^{N\times 1} \tag{7}$$

where $\vec{w} = (w_i)_{i=1}^N$ and the cost functional in problem (5) is given by

$$J := \int_0^T \left(\frac{1}{2}(\vec{w} - \vec{w}_d)^T M(\vec{w} - \vec{w}_d) + \frac{\nu}{2}u^2\right) ds,\ M \in \mathbb{R}^{N\times N}. \tag{8}$$

The coefficients are

$$a_o = \frac{P}{N}, \quad a_d = P\frac{1-N}{N}, \quad B_{i,1} = 1, \quad M_{ij} = \frac{1}{N}\delta_{ij}, \quad (\vec{w}_d)_i = w_d, \tag{9}$$

where δ_{ij} denotes the Kronecker Delta.

We are interested in the stabilization of the given dynamics (4) for a given desired state $\vec{w}_d$ as solution to the uncontrolled dynamics. According to equation (9) and equation (7) we obtain

$$(A\vec{w}_d)_i = w_d\left((N-1)a_o + a_d\right) = 0.$$

Therefore, in the uncontrolled case and for initial data $w_i(0) = w_d$ we obtain $w_i(t) = w_d$ for all $i = 1, \dots, N$. Thus, we assume in the following that the desired state w_d is constant

$$w_d = \text{ constant}. \tag{10}$$

Perturbations of the stable state w_d may be due to initial conditions. Under assumption (10) we may rewrite problem (7)–(8) introducing $\vec{x} = \vec{w} - \vec{w}_d$

$$\min \int_0^T \frac{1}{2}\vec{x}^{\,T} M \vec{x} + \frac{\nu}{2} u^2 ds \quad \text{subject to} \quad \frac{d}{dt}\vec{x} = A\vec{x} + Bu, \quad \vec{x}\,(0) = \vec{w}\,(0) - \vec{w}_d(0). \tag{11}$$

The minimization problem (11) is a convex quadratic problem since M is positive definite. Therefore, Pontryagins maximum principle is necessary and sufficient for optimality. Compared with general problems of this type the microscopic modeling introduces symmetry in the dynamics which is reflected in the special structure of the matrix A. Under suitable regularity assumptions on the solution it is known that the optimal control is given by

$$u(t) = -\frac{1}{\nu} B^T K(t) \vec{x}\,(t) \tag{12}$$

where $K(t) \in \mathbb{R}^{N\times N}$ fulfills the Riccati equation

$$-\frac{d}{dt}K = KA + A^T K - \frac{1}{\nu} KBB^T K + M, \quad K(T) = 0 \in \mathbb{R}^{N\times N}. \tag{13}$$

Hence, the controlled dynamics are given by (13) and

$$\frac{d}{dt}\vec{x} = A\vec{x} - \frac{1}{\nu} BB^T K(t) \vec{x}\,, \quad \vec{w} = \vec{x} + \vec{w}_d. \tag{14}$$

We are thus interested in considering the limit when $N \to \infty$. To this aim we need to analyze the structure of $BB^T K(t)$ and rewrite the previous equations in terms of (binary) particle interactions.

We discretize in time using a discrete time–step Δt and denote by $\vec{x}^{\,n} = \vec{x}\,(t^n)$ for $n = 1, \ldots, n_T$ and such that $T = n_T \Delta t$. The dynamics are then given by

$$\vec{x}^{\,n} = \vec{x}^{\,n-1} + \Delta t \left(A\vec{x}^{\,n-1} - \frac{1}{\nu} BB^T K^{n-1} \vec{x}^{\,n-1} \right) \tag{15}$$

and equation (13) yields

$$K^{n-1} = K^n + \Delta t \left(K^n A + A^T (K^n) - \frac{1}{\nu} K^n BB^T K^n + M \right), \; K^{n_T} = 0. \tag{16}$$

For the discrete dynamics (16) we have the following result.

Proposition 1 (Structure of $BB^T K^n$)**.** *Consider the iteration* (16) *for* $n = 0, \ldots, n_T$ *and assume the coefficients of* A, B *and* M *are given by* (9)*. Then, for any* n *the term* $(BB^T K^n)_{ij}$ *is independent of* i *and* j*, i.e.,*

$$(BB^T K^n)_{ij} = \mathcal{K}^n$$

Proof. Note that $(BB^T)_{ij} = 1$ for all i, j independent of n and $(BB^T M)_{ij} = \frac{1}{N}$ for all i, j. Next, we show that K^n has a similar structure as A, i.e., for all n we have $K^n = \begin{pmatrix} k_o^n & i \neq j \\ k_d^n & i = j \end{pmatrix}$. This is clearly fulfilled for $n = n_T$. Then by induction from n

to $n-1$:

$$
\begin{aligned}
R &:= (K^n BB^T K^n)_{ij} = ((N-1)k_o^n + k_d^n)^2 \ \ \forall i,j \\
K_{ij}^{n-1} &= K_{ij}^n + \frac{\Delta t}{N}\delta_{ij} + \Delta t\left(\sum_{m=1}^N K_{im}^n A_{mj} + A_{mi}K_{mj}^n\right) - \frac{\Delta t}{\nu}R \\
&= -\frac{\Delta t}{\nu}R + \begin{pmatrix} k_o^n + \Delta t\,(2(N-2)k_o^n a_o + 2k_o^n a_d + 2k_d^n a_o) & i \neq j \\ \\ k_d^n + \frac{\Delta t}{N} + \Delta t\,(2(N-1)k_o^n a_o + 2k_d^n a_d) & i = j \end{pmatrix}
\end{aligned}
$$

This proves that K^n has the same structure as A, but with time–dependent coefficients. A simple computation yields that

$$
\left(BB^T K^n\right)_{ij} = (N-1)k_o^n + k_d^n =: \mathcal{K}^n.
$$

This ends the proof. □

We rewrite the particle dynamics using the new variable $\mathcal{K}^n$. As a consequence of the proof of Proposition 1 we obtain

$$
\begin{aligned}
\mathcal{K}^{n-1} &= (N-1)k_o^{n-1} + k_d^{n-1} \\
&= -N\frac{\Delta t}{\nu}(\mathcal{K}^n)^2 + \mathcal{K}^n + \frac{\Delta t}{N} + 2\Delta t a_d \mathcal{K}^n + 2\Delta t(N-1)a_o\mathcal{K}^n \\
\mathcal{K}^{n-1} &= \mathcal{K}^n - N\frac{\Delta t}{\nu}(\mathcal{K}^n)^2 + \frac{\Delta t}{N} \qquad (17)
\end{aligned}
$$

The previous dynamics is an explicit (backwards) Euler discretization of the ordinary differential equation

$$
-\frac{d}{dt}\mathcal{K}(t) = \frac{1}{N} - \frac{N}{\nu}\mathcal{K}^2(t), \quad \mathcal{K}(T) = 0. \qquad (18)
$$

The above modified Riccati equation allows for an explicit solution given by

$$
\mathcal{K}(t) = \frac{\sqrt{\nu}}{N}\ \tanh\left(\frac{T-t}{\sqrt{\nu}}\right).
$$

The time evolution of $\mathcal{K}(t)$ is depicted in Figure 1.

Summarizing, the N particles model is given by equation (17) and (15), which written for each component i reads

$$
x_i^n = x_i^{n-1} + \frac{\Delta t}{N}\sum_{j=1}^N P(x_j^{n-1} - x_i^{n-1}) - \frac{\Delta t}{\nu}\mathcal{K}^{n-1}\sum_{j=1}^N x_j^{n-1}, \qquad (19)
$$

where $x_i \in \mathcal{I}_d = [-1-w_d, 1-w_d]$, $\forall\, i$. Several possibilities to derive a kinetic equation for a particle density $f(x,t)$ for the coupled system (17) and (19) exist and will be explored in the following sections.

Remark 1. Proposition 1 yields the structure of K after discretization. It is also possible to prove directly that if $K(t) \in \mathbb{R}^{N\times N}$ is a solution to equation (13), then we have for (i,j) with $i,j = 1,\dots,N$

$$
(BB^T K(t))_{i,j} = \mathcal{K}(t),
$$

where $\mathcal{K}(t) \in \mathbb{R}$ is the solution to equation (18): Obviously, we have at terminal time $K(T) = 0$. We have $(BB^T)_{i,j} = 1$, $(BB^T BB^T)_{i,j} = N$, $A = A^T$ and $BB^T A = 0$. Multiplying equation (13) by BB^T we obtain

$$
-\frac{d}{dt}\left(BB^T K\right) = BB^T KA + 0 - \frac{1}{\nu}BB^T K\ BB^T\ K + \frac{1}{N}BB^T.
$$

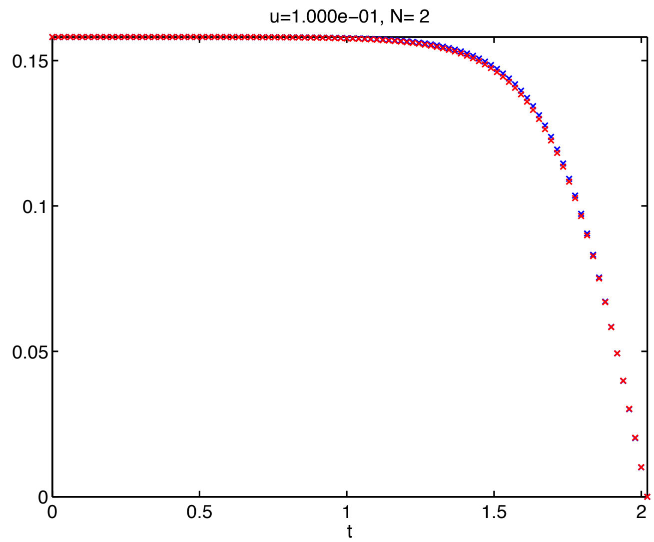

FIGURE 1. Analytical (red) and numerical solution (blue) of equation (18) for $\nu = \frac{1}{10}$ and $N = 2$ obtained by equation (17).

Hence, $BB^T K(t) = \mathcal{K}(t)BB^T$ is the solution to equation (13) provided that $\mathcal{K}$ fulfills

$$-\frac{d}{dt}\mathcal{K} = -\frac{1}{\nu}N\mathcal{K}^2 + \frac{1}{N}.$$

The previous equation coincides with equation (18).

3. **Mean-field Riccati description.** We are interested in a description of the meanfield corresponding to the controlled dynamics (19) and (17) for large number of agents. First, we consider the full dynamics (19) written in differential form. In order to obtain the limiting equation we introduce the new re–scaled Riccati variable

$$k(t) := N\mathcal{K}(t) \equiv \sqrt{\nu}\tanh\left(\frac{T-t}{\sqrt{\nu}}\right). \tag{20}$$

The equivalent system to (19) reads then

$$\begin{aligned} \frac{dx_i(t)}{dt} &= \frac{1}{N}\sum_{j=1}^{N} P(x_j(t) - x_i(t)) - \frac{1}{\nu N}k(t)\sum_{j=1}^{N} x_j(t), \\ -\frac{dk(t)}{dt} &= 1 - \frac{1}{\nu}k^2(t). \end{aligned} \tag{21}$$

To formulate the evolution of the probability density $f(x,t)$ with $\int f(x,t)dx = 1$ we introduce the empirical measure $f(x,t)$ of unit mass for each time t as

$$f(x,t) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i(t)),$$

we obtain in weak form for $\phi = \phi(x)$ sufficiently smooth

$$\begin{aligned}\partial_t \int_{\mathcal{I}_d} \phi(x) f(x,t) dx &= \frac{1}{N^2} \sum_{i,j} \left(\phi_x(x_i) P(x_j - x_i) - \frac{k(t)}{\nu} \phi_x(x_i) x_j \right) \\ &= \int_{\mathcal{I}_d} \int_{\mathcal{I}_d} P \phi_x(x)(y-x) f(x,t) f(y,t) dx dy \\ &\quad - \frac{k(t)}{\nu} \int_{\mathcal{I}_d} \int_{\mathcal{I}_d} \phi_x(x) y f(x,t) f(y,t) dx dy.\end{aligned}$$

Hence, the formal limiting equation for the evolution of the probability density $f(x,t)$ on $t \in [0,T]$ is given by the equation

$$\begin{aligned}&\partial_t f(x,t) + \partial_x \left(\int_{\mathcal{I}_d} P(y-x) f(x,t) f(y,t) dy \right) \\ &\quad - \partial_x \left(\frac{1}{\sqrt{\nu}} \tanh\left(\frac{T-t}{\sqrt{\nu}} \right) \int_{\mathcal{I}_d} y f(y,t) f(x,t) dy \right) = 0,\end{aligned} \tag{22}$$

where we replaced $k(t)$ by its analytical solution.

3.1. **Evolution of moments.** We consider moments of equation (22) by looking at the average deviation from the desired opinion. This is measured by $m(t) = \int x f(x,t) dx$ and we obtain

$$\frac{d}{dt} m(t) = -\frac{k(t)}{\nu} m(t) = -\frac{1}{\sqrt{\nu}} \tanh\left(\frac{T-t}{\sqrt{\nu}} \right) m(t). \tag{23}$$

The previous ordinary differential equation has the solution for $t \in [0,T]$ as

$$m(t) = \frac{m(0)}{\cosh\left(\frac{T}{\sqrt{\nu}}\right)} \cosh\left(\frac{T-t}{\sqrt{\nu}} \right), \tag{24}$$

with initial data $m(0)$. For $T >> 1$ sufficiently large we observe that $m(t)$ decays exponentially fast to $m(0)/\cosh(T/\sqrt{\nu})$. This latter term tends to zero for $T \to \infty$. The convergence rate is $1/\sqrt{\nu}$ where small values of ν correspond to a strong influence of the cost functional.

The second order moment $E(t) = \int \frac{1}{2} x^2 f(x,t) dx$ fulfills the following equation

$$\frac{d}{dt} E(t) = P\left(m^2(t) - 2E(t) \right) - \frac{k(t)}{\sqrt{\nu}} m^2(t).$$

From the previous computations we observe that $m(t)$ is bounded for T sufficiently large by $2m(0)\exp(-t/\sqrt{\nu})$. Therefore, also $E(t)$ tends to zero as t tends to infinity at rate $2P$ provided that $P > 0$ and T is sufficiently large. As a consequence, the large time behavior of the solution is the steady–state $f_\infty(x) = \delta_0(x)$. Hence, the average and the individual deviation from the desired opinion w_d tends to zero using the optimal control provided that $T >> 1$.

Proposition 2 (Long–term behavior)**.** *Assume $T >> 1$ and let $f_0(x)$ be some initial probability distribution of opinions. Let f be the solution to equation (22). Then, the average opinion $m(t) = \int x f(x,t) dx$ and the second order moment $E(t) = \int x^2 f(x,t) dx$ decay exponentially fast to zero, i.e., $f \to f_\infty = \delta(x)$.*

Remark 2. Due to the linear transformation between x and w introduced in problem (11) we also obtain convergence for the original dynamics (4). Consider the associated kinetic equation in $F(w,t)$ to problem (4) under assumptions (6) and

(10). Then, for any initial data $F(w,0)$ we obtain convergence for T sufficiently large towards $F_\infty(w) = \delta(w - w_d)$.

3.2. **Comparison with instantaneous control approaches.** We may compare the resulting control action with the instantaneous control approach in [2, Section 4]. In the Fokker–Planck equation therein we obtain an equation for the probability density $\tilde{f}$ as

$$\begin{aligned} \partial_t \tilde{f} + \partial_x \left(\int_{\mathcal{I}_d} P(y-x) \tilde{f}(x,t) \tilde{f}(y,t) dy \right) - \\ \partial_x \left(\frac{1}{\kappa} \int_{\mathcal{I}_d} \frac{x+y}{2} \tilde{f}(y,t) \tilde{f}(x,t) dy \right) = 0, \end{aligned} \tag{25}$$

where $\kappa = \nu/\epsilon$ is a scaled weight of the control and ϵ is the scaling of time, i.e., $t' = \epsilon t$. The instantaneous control is independent of the current time t and the terminal time T. The Riccati control of equation (21) acts only on the average deviation from the desired opinion, i.e., $\int y f(y,t) dy$, compared with an action on the average opinion $\int (x+y) f(y) dy/2$ in the other case. The corresponding average opinion $\tilde{m}(t) = \int x \tilde{f}(x,t) dx$ fulfills

$$\frac{d}{dt} \tilde{m}(t) = -\frac{4}{\kappa} \tilde{m}(t). \tag{26}$$

In the derivation of [2] the scaling ϵ corresponds to a discrete time–step Δt. Estimating the solution m to equation (23) by $2m(0) \exp(-t/\sqrt{\nu})$ we observe that the relation of the rates between instantaneous and Riccati are $-4\Delta t/\nu$ and $2/\sqrt{\nu}$, respectively. This implies that the action of the instantaneous control is weighted by Δt compared with the Riccati based control. We will exemplify this behavior in the numerical results where we simulate both control strategies on the same time–scale. Note that in the case $\nu \to \infty$ both kinetic equations coincide yielding the uncontrolled systems dynamics.

4. **Boltzmann-Riccati description.** In this section we discuss an approximation of equations (19) and (17) as binary interaction model. This is motivated by the fact that exchange of opinions is often considered as a binary process [11, 28, 42] and that, as shown in [3], this permits to obtain efficient numerical simulations algorithms for the original dynamic.

In order to derive the corresponding Boltzmann description we proceed as follows. Consider the discretized dynamics (17) and (19) and use as before $k = N\mathcal{K}$ to obtain

$$\begin{aligned} k^{n-1} &= k^n - \frac{\Delta t}{\nu} (k^{n-1})^2 + \Delta t, \\ x_i^n &= x_i^{n-1} + \frac{\Delta t}{N} \sum_{j=1}^N P(x_j^{n-1} - x_i^{n-1}) - \frac{\Delta t}{\nu} k^{n-1} \frac{1}{N} \sum_{j=1}^N x_j^{n-1}. \end{aligned}$$

Now, we consider two (out of N) agents i and j with states x_i and x_j and post interaction states x_i^* and x_j^*, respectively. We propose as interaction mechanism the following relation involving only two agents i and j as well as the control action k^n

$$\begin{aligned} x_i^* &= (1 - \tau P)\, x_i + \left(\tau P - \frac{\tau}{\nu} k^{n-1} \right) x_j, \\ x_j^* &= (1 - \tau P)\, x_j + \left(\tau P - \frac{\tau}{\nu} k^{n-1} \right) x_i, \end{aligned}$$

$$k^{n-1} = k^n - \frac{\Delta t}{\nu}(k^{n-1})^2 + \Delta t.$$

Here, $\tau = \Delta t$ is the interaction strength and we let the control act only on the other agent. If we let the control act on both particles then a similar term as in Section 3.2 is obtained, i.e., in the corresponding equation the control action is applied to the average of the opinions.

Let us briefly sketch the derivation of the Boltzmann model [40]. For a rate γ we assume that there are $\gamma\Delta t$ interactions within the time interval $(t^{n-1}, t^{n-1}+\Delta t)$. Hence, the probability to change from any state $X = (x_1, \dots, x_N)$ to any state $Y = (y_1, \dots, y_N)$ within Δt is given by

$$\begin{aligned} \mathcal{P}(Y,X) =& (1-\gamma\Delta t)\delta(Y-X) \\ &+ \frac{\gamma\Delta t}{N(N-1)} \sum_{i,j=1,\, i\neq j}^{N} \left(\delta(y_i - x_i^*)\delta(y_j - x_j^*) \prod_{m=1, m\neq i,j}^{N} \delta(y_m - x_m) \right), \end{aligned} \tag{27}$$

where x_i^* and x_j^* are defined above and dependent on x_i and x_j. Clearly,

$$\int \mathcal{P}(Y,X)dY = 1$$

so that $\mathcal{P}$ is a probability distribution in Y. The Boltzmann equation for $f(X,t)$ is given by

$$f(X, t+\Delta t) = \int \mathcal{P}(X,Y)f(Y,t)dY. \tag{28}$$

If at time t we have $\int f(X,t)dX = 1$ we obtain due to the property of $\mathcal{P}$, that $\int f(X, t+\Delta t)dX = 1$. Therefore, f remains a probability distribution in the full space X. Under the chaos assumption we may decompose

$$f(X,t) = \prod_{i=1}^{N} f(x_i, t).$$

Upon integration against $dx_2 \dots dx_n$ and renaming x_1 to x, we obtain the dynamics of the single particle distribution $f(x,t)$. Carrying out the integration we obtain for a sufficiently smooth test function $\phi(x)$ the weak form of the Boltzmann-Riccati system

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{I}_d} \phi(x) f(x,t)dx = \frac{2\gamma}{N} \int_{\mathcal{I}_d}\int_{\mathcal{I}_d} \left(\phi(x^*) - \phi(x)\right) f(x,t) f(y,t) dx dy, \\ &-\frac{dk(t)}{dt} = 1 - \frac{1}{\nu}k^2(t), \quad k(T) = 0. \end{aligned} \tag{29}$$

Next, we determine an interaction rate γ such that (29) can be approximated by the corresponding Vlasov–Fokker–Planck model and show that this coincides with the mean field model (22). This asymptotic procedure is usually referred to as quasi-invariant opinion limit, we refer to [40, 42] for more details and here we report directly the result. Using Taylor expansion of the test function at state x we obtain

$$\phi(x^*) - \phi(x) = \phi_x(x) \left(\tau P(y-x) + \frac{\tau}{\nu} k(t) y \right) + O(\tau^2).$$

Substituting the above expression in (29) in strong form we get

$$\begin{aligned}\partial_t f(x,t) + \frac{2\gamma}{N}\tau\partial_x \Bigg(\Bigg(\int_{\mathcal{I}_d} P(y-x) f(x,t) f(y,t) dy \Bigg) \\ - \int_{\mathcal{I}_d} \frac{1}{\nu} k(t) y f(y) f(x) dy \Bigg) = O(\tau^2)\frac{2\gamma}{N}.\end{aligned} \tag{30}$$

Hence, if we scale

$$\gamma = \frac{N}{2\tau}$$

and let $\tau \to 0$ (and $N \to \infty$ such that $\gamma = 1$) we formally obtain again the strong form of the kinetic equation (22). Since $\tau = \Delta t$, this scaling corresponds to infinitesimal time scales for the discrete particle dynamics. Note that $k(t) = N\mathcal{K} = \sqrt{\nu}\tanh\left((T-t)/\sqrt{\nu}\right)$ is independent of N and therefore the previous limit is well–defined even in the limit $N \to \infty$.

5. **Extensions of the present approach.**

The presented approach may be extended to various other problems. We state some of the possible extensions below.

- The Riccati equation can only be derived in the case of linear dynamics and quadratic objective functions leading to strong assumptions on the underlying dynamics as present for example in assumption (6) and (10). In case of a nonlinear function $P(w,v)$ the feedback linearization [41, Definition 5.3.2] may be used, see also [41] for more details. Consider the general problem (4) and (5) and a given state w_d. Then, we linearize $P(w,v)(w-v)$ at (w_d, w_d) to get

  $$P(w,v)(w-v) = P(w_d,w_d)(w-w_d) - P(w_d,w_d)(v-w_d) + \dots$$

  where the dots denote higher order terms. This leads to the approximate dynamics

  $$\frac{dw_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} P(w_d,w_d)(w_j - w_i) + u, \quad w_i(0) = w_{0i}. \tag{31}$$

  Note that, the last equation is used for determining the control law only, but not for the agents interaction where the full dynamic (4) is solved. The derivation of the mean-field and Boltzmann Riccati descriptions follow straightforwardly and we omit the details.
- Backstepping can be employed if the dependence on the control is nonlinear in the system dynamics, e.g., if P in equation (4) additionally depends on the control variable u as $P = P(w_i, w_j, u)$. Then, backstepping control introduces an artificial new variable q and replaces $P = P(w_i, w_j, q)$ and adds the additional equation

  $$\frac{dq}{dt} = u.$$

  This changes the action of u to be an integral action, however, under general assumptions it can be shown that stabilization of the original and the extended system are equivalent. We refer to [41] for a general discussion.
- If we need to treat time–dependent desired states $w_d = w_d(t)$, then an approach similar to [2] may be used. This approach is called model–predictive control [37]. The time–horizon will be split into a finite number of slices where on each slice one assumes that w_d is constant. Then, the obtained control is

applied to the dynamics on this time slice only. When switching between different slices the initial position will be updated to accommodate for the changing desired state and the procedure is reapplied.

- Terminal costs may be added to the objective functional J to measure mismatch at time T and prevent end–time–effects in the control as for example present now since $u(T) = 0$. Provided we have symmetric terminal cost of the type $(c/2)\vec{x}^T\vec{x}$, the terminal condition for the evolution of $\mathcal{K}$ in equation (13) is modified to $\mathcal{K}(T) = c$.
- The problem discussed in Section 2 and used to illustrate the control approaches has a very particular structure. However, it also part of second order alignment models as for example Cucker–Smale type models. For a discussion of general second–order flocking models we refer to [38]. The prototype of such models are agents having two states (x_i, v_i) where x_i typically denotes the position and v_i the velocity. The resulting model is the given by

$$\dot{x}_i = v_i \text{ and } \dot{v}_i = \frac{1}{N}\sum_{j=1}^{N} H_a(x_i, x_j)(v_j - v_i) + u.$$

In the Cucker–Smale model the function H_a fulfills $H_a(x, y) \neq H_a(y, x)$. Assume that the objective function only depends on $(v_i)_{i=1}^N$ and some desired state v_d but *not* on $(x_i)_{i=1}^N$. Then, the introduced derivation of Riccati based control can be applied by considering the dynamics in v_i only treating x_i as a parameter. Thus obtaining a Riccati feedback control with parameter dependent function P. In the framework of model predictive controls this procedure has been applied in [1, Chapter 4.7.2] and [10].

6. **Numerical results.**

In this section we report some numerical test obtained by solving the kinetic equation (22). In the numerical simulations we use a Monte Carlo methods for the corresponding kinetic model as described in [40, Chapter 4]. In order to efficiently simulate equation (22) we consider the derivation using binary interactions leading to the equivalent equation (30) provided that $\gamma = N/(2\tau)$. Equation (30) is obtained from the weak form of the Boltzmann equation (29) repeated here for convenience

$$\begin{aligned}\partial_t \int_{\mathcal{I}_d} \phi(x) f(x,t)dx = \frac{2\gamma}{N}\int_{\mathcal{I}_d}\int_{\mathcal{I}_d} \left(\phi(x^*) - \phi(x)\right) f(x,t)f(y,t)dxdy,\\ x^* = (1-\tau P)x + (\tau P - \frac{\tau}{\nu}k(t))y.\end{aligned} \tag{32}$$

Using the definition of γ we rewrite the equation as

$$\partial_t \int_{\mathcal{I}_d} \phi(x) f(x,t)dx = \frac{1}{\tau}\left(\int_{\mathcal{I}_d}\int_{\mathcal{I}_d} \phi(x^*)f(x,t)f(y,t)dxdy - \int_{\mathcal{I}_d}\phi(x)f(x,t)dx\right). \tag{33}$$

Hence, in strong form equation (32) is equivalent to

$$\partial_t f(x,t) = \frac{1}{\tau}\left(Q^+(f,f)(x,t) - f(x,t)\right), \tag{34}$$

where the gain term of the interaction operator is defined by $\int \phi(x) Q^+(f,f) dx := \int\int \phi(x^*) f(x,t) f(y,t) dx dy$. An explicit forward Euler discretization in time of equation (34) with time–step Δt and by denoting $f(x,t^n) = f^n(x)$ for $t^n = n\,\Delta t$ gives

$$f^{n+1}(x) = \left(1 - \frac{\Delta t}{\tau}\right) f^n(x) + \frac{\Delta t}{\tau} Q^+(f^n, f^n)(x, t^n).$$

We note that $Q^+ \geq 0$ is also a probability density. Therefore, the previous formulation allows for a probabilistic interpretation for any $0 < \Delta t \leq \tau$. We sample $f^0(x)$ by a pointwise measure with discrete states x_i^0. Then, the change of state for the discrete sample occurs with probability $\Delta t/\tau$ and is described by the probability $\mathcal{P}$ given in equation (27). For any fixed x_i^n the new state x_i^{n+1} is obtained by evaluation of the interactions described in x_i^* and x_j^*. This implies at each time–step to randomly select two particles (x_i, x_j) and perform a single interaction on both of them. A simulation for a stochastic initial condition and different values of ν

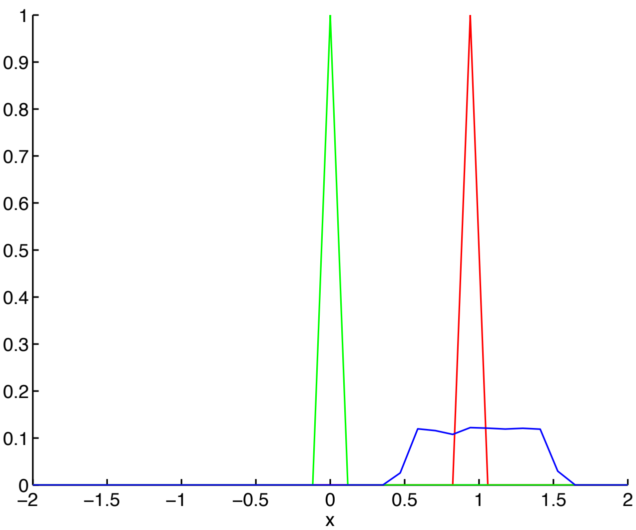

FIGURE 2. Numerical simulation of f given by equation (30) for an initial condition (blue) up to terminal time $T = 5$ for $\nu = 100$ (red) and $\nu = 1$ (green). The simulation uses $N = 5000$ particles.

is depicted in Figure 2. As expected the convergence depends on the control parameter ν where large values of ν correspond to the uncontrolled case. Due to the decay in second moment E, the solution in both cases (controlled and uncontrolled) converges to a Dirac distribution, however, in the uncontrolled case it concentrates at the mean of the initial distribution compared with the controlled case where convergence towards $x = 0$ is observed. The convergence of $m(t)$ and $E(t)$ and its dependence on ν is shown in Figure 3. Therein, we observe the expected exponential decay with rate depending on $\sqrt{\nu}$.

Next we compare the results with a Monte Carlo simulation of the model predictive approach described by equation (25). Here, we choose $\kappa = \nu$ and set $w_d = 0$ to present the convergence of $m(t)$ and $E(t)$ in Figure 4. Compared with the Riccati based approach the expected rate using the same parameters is smaller since the control in the model predictive framework of [2] acts on a different time scale. For

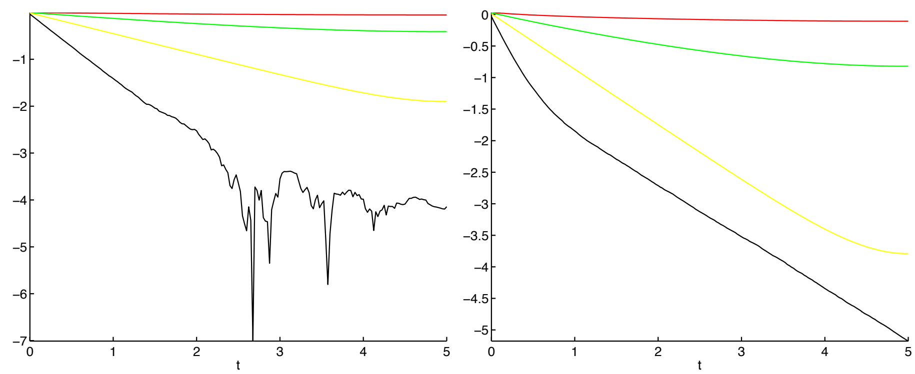

FIGURE 3. Numerical simulation of f given by equation (30) for an initial condition up to terminal time $T = 5$ for different values of $\nu = 100$, $\nu = 10$, $\nu = 1$ and $\nu = 0.1$ in the colors red, green, yellow and black. The simulation uses $N = 5000$ particles. Left: Time evolution of the mean $m(t)$. Right: Time evolution of the second moment $E(t)$. All plots are in log −scale in y.

$w_d = 0$ the control obtained in [2, Equation 3.6] at any binary interaction between particle i and j at time t is given by

$$\tilde{u}(t) = -\frac{1}{2}\frac{(\Delta t)^2}{\kappa + (\Delta t)^2}\left(x_i(t) + x_j(t)\right),$$

whereas in the present approach the control is given by

$$u(t) = -\frac{\Delta t}{\kappa}k(t)x_i(t).$$

The associated cost in objective functional is given by $\int_0^T \frac{\nu}{2}\tilde{u}^2 ds$ and $\int_0^T \frac{\nu}{2}u^2 ds$, respectively. The total objective function includes the additional term

$$\frac{1}{2}\int \sum_{i=1}^{N} x_i^2(s)ds$$

for both cases. We presented for varying ν the total control costs associated with the simulations for Figures 3 and 4. The results are given in Table 1. The costs of control using the model-predictive framework in [2] scales as $\approx 1/\nu$ and compared with the approach based on Riccati equations. However, the latter costs scale like $\tanh((T-t)/\nu)$. Hence, even if both control strategies converge the Riccati approach yields a lower value of the objective functional.

7. **Conclusions.** We derived a control approach for large systems of interacting multi–agents based on the Riccati equation. We investigated problems where the collective behavior corresponds to the process of alignment, like in the opinion consensus dynamic, in a constrained setting. In contrast to [2] where an instantaneous control strategy has been used, here we derived mean field and Boltzmann description when the control is *not* localized in time. In order to do this we restrict our analysis to linear agent interactions and derived the corresponding kinetic descriptions together with a Riccati equation. Extensions of the method to more general nonlinear interactions are also discussed. Several numerical results confirm

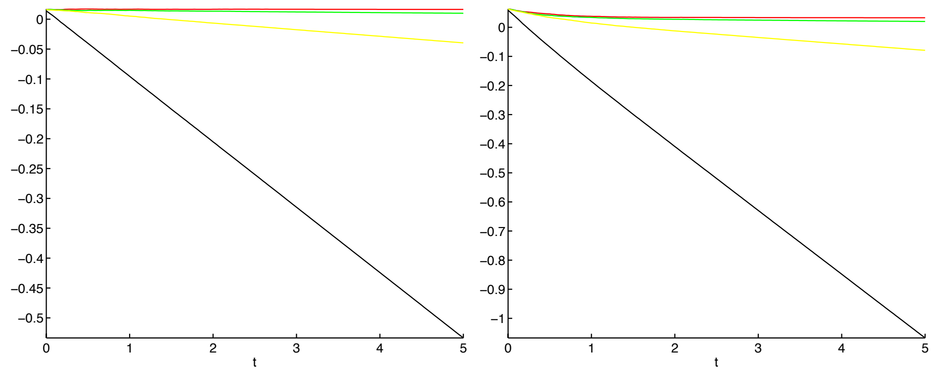

FIGURE 4. Numerical simulation of f given by equation (25) for an initial condition up to terminal time $T = 5$ for different values of $\nu = 100$, $\nu = 10$, $\nu = 1$ and $\nu = 0.1$ in the colors red, green, yellow and black. The simulation uses $N = 5000$ particles. Left: Time evolution of the mean $m(t)$. Right: Time evolution of the second moment $E(t)$. All plots are in log −scale in y. Compared with Figure 3 the y-axis ranges here only to $10^{-0.5}$.

ν	Riccati (control)	Riccati (total)	MPC (control)	MPC (total)
100.00	0.00	445.45	0.00	515.57
10.00	0.00	201.58	0.00	503.53
1.00	0.04	59.21	0.04	454.18
0.10	0.14	20.95	0.14	189.48

TABLE 1. Values of the objective functional (8) denoted by (total) for the control action based on the Riccati equation (12) and the model predictive control framework (MPC) presented in [2]. The part of the cost related to the control part is denoted by (control) in both cases. Other parameters are as in results associated with Figures 4 and Figure 1, respectively.

the optimality of the present approach for linear dynamics and quadratic objective functionals.

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