

BOLTZMANN-TYPE MODELS FOR PRICE FORMATION IN THE PRESENCE OF BEHAVIORAL ASPECTS

CARLO BRUGNA

Department of Physics
Via Bassi, 6
27100 Pavia, Italy

GIUSEPPE TOSCANI

Department of Mathematics
Via Ferrata, 1
27100 Pavia, Italy

ABSTRACT. We introduce and discuss a new kinetic system for a financial market composed by agents that may belong to two different trader populations, whose behavior determines the price dynamic of a certain stock. Our mesoscopic description is based on the microscopic Lux–Marchesi model [16, 17], and share analogies with the recent kinetic model by Mالدarella and Pareschi [18], from which it differs in various points. In particular, it takes into account price acceleration, as well as a microscopic binary interaction for the exchange between the two populations of agents. Various numerical simulations show that the model can describe realistic situations, like regimes of boom and crashes, as well as the invariance of the large-time behavior with respect to the number of agents of the market.

1. Introduction. Agent-based models represent a broad class of mathematical models which have been recently considered to describe various phenomena of economic dynamics. It is nowadays clear that this description can reproduce various features of financial markets, like volatility clustering [14, 15, 16, 17, 18, 19, 26] and fat tails of returns [2, 3, 4, 5, 6, 7, 9]. This relatively new research field borrows several methods and tools from classical statistical mechanics, where the emerging complex behavior arises from relatively simple rules as a consequence of binary interactions among a large number of agents [21, 22].

Starting from the microscopic dynamics, kinetic models for trading can be derived with the tools of classical kinetic theory of fluids [7, 10, 11, 20, 24], where kinetic econophysics has been treated in the framework of Boltzmann-like equation for Maxwell-type molecules. In contrast with microscopic dynamics, where the large-time behavior of the system can be often studied only empirically through computer simulations, in many cases kinetic models based on integro-differential and/or partial differential equations allow to find analytically general information on the model and its asymptotic behavior.

In most of these market models, the population of agents shares the same rules with respect to trading. Only very recently, at the cost of increased complexity of

2010 *Mathematics Subject Classification.* Primary: 35B40, 91B60; Secondary: 82C40.

Key words and phrases. Kinetic models, markets and trades, multi-agent systems, opinion formation.

the model, the personal behavior of agents on trading has been introduced as a new parameter, to understand its effects on the long-time behavior of the system. In [23], the importance of personal knowledge in a system of traders has been studied by assuming that in a trade both the saving propensity and the risk could depend on the personal degree of knowledge. There, the Boltzmann-like evolution equation for binary trading is coupled with a kinetic equation for knowledge evolution. Also, the personal opinion and other psychological aspects have been considered by Maldarella and Pareschi in [18]. Their model is intended to describe a speculative market characterized by a single stock and a socio-economical interplay between two different types of traders. Starting from the microscopic Lux–Marchesi model for two categories of agents [15, 16, 17], the fundamentalists and chartists, and from the kinetic model of opinion formation introduced in [25], Maldarella and Pareschi derived a system of two kinetic equations for the two categories, which is particularly rich, taking into account also psychological and behavioral components of the agents, like the way they interact each other and perceive the risk. This is done by means of a suitable *value function* in agreement with the Prospect Theory by Kahneman and Tversky [12, 13].

The hypothesis adopted in [18], derived from the theory of [12, 13] implies an instantaneous reaction towards risks of the agents at a given time t , based on the ratio between the price and its derivative at time t . This assumption is somewhat controversial, and has been modified in many ways. Among others, Cristelli, Pietronero and A. Zaccaria [8] arrive to a different conclusion. The usual reaction of agents towards risks is generally based on a *sufficiently long time series*, which takes into account at least the recent history of the price. In [1], it is indeed pointed out that in price formation its first and second derivatives (velocity and acceleration respectively) are the primary objects of interest. The key remark in [1] is that the main modelling goal is to capture the dynamics of price, and in particular of how fast the price is changing or moving.

Since the attributes that are typically associated with a moving object are its velocity (or its speed) as well as its acceleration, following this dynamical example, we will assume as in [1] that the perception of the risk and its consequent influence on trading could depend, in addition to the actual price and its derivative, from the second derivative (the price acceleration). In terms of the realization of the numerical simulation, this corresponds to the knowledge of a two steps time history. While this assumption corresponds to a relatively small additional complexity of the model, it is enough to furnish the concavity (or convexity) of the price curve, thus giving to traders a more complete information on the expected behavior of the price.

It is important to remark that in the microscopic Lux–Marchesi model [15, 16, 17] and in its kinetic description [18], agents can move and change categories according to some (fixed) transition rules, which do not exclude the possibility that one class of agents could become empty. In particular, in [18] this exchange strategy is realized by adding to the kinetic equations for the classes of agents a suitable linear operator. As shown in [18] by various numerical experiments, this characteristics of the model plays an essential rule in the appearance of fluctuations of the price, as well as in the creation of boom and crash effects.

At difference with the model in [18], the transition rule between the two categories of agents will be here introduced as a microscopic binary interaction between agents of different classes. The result of this interaction will also depend on the personal opinion of agents. The advantage of our choice is evident. In agreement with the

classical kinetic theory [22], the whole evolution process is here based on microscopic binary interactions between agents. For this reason, the model will result as a useful prototype of collisional models in which there is more than one class of agents involved.

The choice of a microscopic transition law allows to understand (at a microscopic level) the importance of the opinion variable in the whole process of the price formation. Indeed, it is shown that a weak role of the opinion variable in the exchange between classes, drives the price towards upper and lower bounds, which can be explicitly given in terms of the initial state of the system. In other words, a weak role of the opinion variable is such that both boom and crashes in the market are not possible.

The paper is organized as follows. The model will be introduced in Section 2. Here, the evolution equation for the price of the stock, as well as the evolution equations for the number densities of the two populations of agents are written in the form of a kinetic system of correlated equations. In particular, we will outline the main differences among the present model and the previous ones. In Section 3 we will discuss the basic role of the opinion variable, and its consequences on fluctuations of the price. Finally, Section 4 collects various numerical simulations, which describe how the choice of the relevant parameters of the system can lead to boom, crashes and fluctuations of the price.

2. The model. The aim of the pioneering Lux–Marchesi model [16, 17] was to show that the scaling laws that are observed in financial markets can arise from mutual interaction between agents. In particular, Lux and Marchesi stressed the clear difference between the statistical properties of the model input, i.e. the normal noise that makes the stock fundamental price evolve in time, and the output, that is the price dynamics produced by the operations between agents.

The agents are divided in two categories: the fundamentalists, who believe in the existence of a reference price P_F for the traded stocks (and so, they sell stocks if the price $P(t) > P_F$ and they buy if $P(t) < P_F$), and the chartists, that are noise traders whose behavior is dictated by herding and historical prices. While the total number of agents N remains fixed in time, the number of fundamentalists and chartists, n_F and n_C , are allowed to vary. This is done by assuming that at each time step each agent can change its own category with a given transition probability. In this way, an internal dynamics within the two classes of agents is established.

The effect of the two classes of agents on the price is very different. While fundamentalists have a stabilizing effect on the market, as their operations drive the price towards the reference one, chartists have a destabilizing effect and can create bubbles and crashes. Furthermore, chartists agents are divided in two subcategories, optimists, who believe that the price will rise and hence always buy stocks, and pessimists, who believe that the price will decrease and so, on the contrary, always sell stocks.

The main features of Lux–Marchesi model have been merged in [18] into the framework of kinetic theory, with the goal to introduce a kinetic description both for the behavior of the microscopic agents and for the price, and then to exploit the tools furnished by kinetic theory to get more insight about the way the microscopic dynamic of each trading agent can influence the evolution of the price. The model introduced in [18] is still able to describe several phenomena like the presence of booms, crashes, and cyclic oscillations of the market price. The equilibrium

behavior has been studied in a suitable asymptotic regime which originates from the Boltzmann equations a system of Fokker–Planck equations for the chartist’s opinion dynamics and the price formation. However, in reason of the intrinsic complexity of the interactions, various interesting aspects of the kinetic system remain not fully studied and understood.

In what follows, still resorting to the classical methods of kinetic theory, we introduce a model which is reminiscent of both Lux–Marchesi [16, 17] and Maldarella–Pareschi [18] ideas. At difference with the model introduced in [18], the resulting exchange mechanism between classes is here introduced at a microscopic level, by means of a binary interaction between agents. Moreover, we will resort to the analysis in [1, 8] to model the behavior of the chartists.

Assuming continuous parameters for the number of fundamentalists and chartists, let $f(w, t)$ denote the probability to have a number $w \in \mathbb{R}_+$ of fundamentalist agents at time $t > 0$, and let $c(v, x, t)$, with $v \in \mathbb{R}_+$, denote the distribution function of the number of chartist agents. Note that, while the density of fundamentalists depends only on the time, the density of chartists is also depending on the opinion variable $x \in [-1, 1]$ [25]. This opinion variable describes the individual propensity of agents to sell or buy. Conventionally, we will assume that the $x > 0$ corresponds to the buyer-state, and $x < 0$ to the seller-state. When $x = 0$ an agent is considered in a neutral-state.

According to the classical kinetic theory, the knowledge of the density functions $f(w, t)$ and $c(x, v, t)$ allows to compute all averaged quantities of interest. Among them, the mass densities of fundamentalists

$$m_F(t) = \int_{\mathbb{R}_+} f(w, t) dw, \quad (2.1)$$

and chartists

$$m_C(t) = \int_{-1}^{+1} \int_{\mathbb{R}_+} c(x, v, t) dv dx. \quad (2.2)$$

The mean number of fundamentalists agents (respectively chartists) is obtained by taking the mean with respect to the number variables

$$n_F(t) = \int_{\mathbb{R}_+} w f(w, t) dw, \quad (2.3)$$

and

$$n_C(t) = \int_{-1}^{+1} \int_{\mathbb{R}_+} v c(x, v, t) dv dx. \quad (2.4)$$

By taking the mean with respect to the opinion variable, we obtain the average investment propensity $X(t)$ of chartists, defined by

$$X(t) = \int_{-1}^{+1} \int_{\mathbb{R}_+} x c(x, v, t) dv dx. \quad (2.5)$$

Other mean quantities will result of interest in what follows. By taking the means with respect to the opinion variable (respectively the number variable) only, we obtain the marginal distribution densities of chartists

$$m_C(v, t) = \int_{-1}^{+1} c(x, v, t) dx, \quad (2.6)$$

and

$$M_C(x, t) = \int_{\mathbb{R}_+} c(x, v, t) dx. \tag{2.7}$$

Likewise, by taking the mean with respect to the number variable only, we obtain the local mean number of chartists

$$n_C(x, t) = \int_{\mathbb{R}_+} v c(x, v, t) dx. \tag{2.8}$$

The total mean number $n(t)$ of agents at time $t \geq 0$ results by summing up $n_C(t) + n_F(t)$. Then $\rho_F(t) = n_F(t)/n(t)$ and $\rho_C(t) = n_C(t)/n(t)$ gives the percentages of chartists and fundamentalists participating to the market. We will speak of *conservative in the mean population* a market in which $n(t) = n_0$ is constant.

Together with the densities of agents, and given a certain stock, the principal object of investigation is the evolution of its price. To this aim, let us introduce the distribution function $h(p, t)$ of prices of this stock, which is such that $h(p, t)dp$ denotes the probability that the price belongs to the interval $(p, p + dp)$. Starting from $h(p, t)$ one can then define the effective market price $P(t)$. The effective market price is defined as the mean price of the stock

$$P(t) = \int_0^\infty p h(p, t) dp. \tag{2.9}$$

In order to write the kinetic system of equations for the distribution functions $f(w, t)$, $c(x, w, t)$ and $h(p, t)$ we have to introduce the rules of variations of the microscopic variables x, v, w, p . Following [17], we will assume that the microscopic variation of the price p is related both to the percentages of fundamentalists and chartists and to the mean investment propensity (2.5) by the formula

$$p^* = p + \beta(n_C \mu_C X) p + \gamma n_F (P_F - p) + \eta p. \tag{2.10}$$

In (2.10) the parameter β characterizes the price speed evaluation, while γ denotes the reaction strength of fundamentalists to deviations from the fundamental value P_F of the price. Moreover η is a random variable with zero mean and variance σ^2 . Thus, the term $\eta p(t)$ represents a risky component, which induces random deviations of the price. In (2.10) it is supposed that chartist agents buy or sell the same number of units, expressed by the parameter μ_C .

According to the kinetic theory of gases [22], the evolution in time of the density price, where the variation of p is given by (2.10), obeys to the linear in p kinetic equation (in weak form)

$$\frac{d}{dt} \int_{\mathbb{R}_+} \varphi(p) h(p, t) dp = \left\langle \int_{\mathbb{R}_+} (\varphi(p^*) - \varphi(p)) h(p, t) dp \right\rangle. \tag{2.11}$$

In (2.11) $\varphi(p)$ is a smooth function of p , and $\langle \cdot \rangle$ denotes mathematical expectation.

Choosing $\varphi(p) = p$ one obtains that the expected value for the price of the stock satisfies the differential equation [17]

$$\dot{P}(t) = \beta n_C(t) \mu_C X(t) P(t) + \gamma n_F(t) (P_F - P(t)). \tag{2.12}$$

The time variation of the price is linked to the percentages $n_C(t)$ and $n_F(t)$ of chartists and fundamentalists, as well as to the average investment propensity $X(t)$. Thus, the kinetic equation (2.11) needs to be coupled with the evolution equations of $f(w, t)$ and $c(x, v, t)$. As before, these evolution equations are constructed from

the microscopic dynamics of the number variables v and w , and of the opinion variable x .

In what follows we assume that these microscopic dynamics could depend on the expected value of the price $P(t)$ towards its first and second derivatives. Concerning the dynamics of the opinion variable of chartists, following [18, 25] we will assume the following law of variation

$$x^* = x + \alpha_1 H(x) [X(t) - x] + \alpha_2 [\Phi(\dot{P}(t), \ddot{P}(t)) - x] + \theta D(x). \quad (2.13)$$

In (2.13), the post-interaction opinion x^* is modified essentially by three different contributions. The first one is related to the distance of the individual opinion from the average investment propensity $X(t)$. Its intensity is measured by a parameter $\alpha_1 \in [0, 1]$, and by the function $H(x)$, which characterizes the herding behavior. The simplest choice is to fix $H(x) = 1$, which means that all chartists are influenced in the same way by the average investment propensity. A more realistic choice [18] is given by

$$H(x) = 1 - |x|. \quad (2.14)$$

Note that, according to [25], in this case extremal opinions for which $|x| = \pm 1$ are not influenced by the average investment propensity. The second contribution measures the variation of the individual opinion consequent to the market dynamical state. The size of this variation is expressed by the second parameter $\alpha_2 \in [0, 1]$. We will here assume that the function $\Phi(\dot{P}, \ddot{P})$ is defined by

$$\Phi(\dot{P}, \ddot{P}) = \begin{cases} 1 & \text{if } \dot{P} > 0, \ddot{P} > 0 \\ -1 & \text{if } \dot{P} < 0, \ddot{P} < 0 \\ 0 & \text{otherwise.} \end{cases} \quad (2.15)$$

Within this simple assumption, chartists agents are pushed to buy in presence of a local growing convex price $P(t)$, or to sale in presence of a local decreasing concave price. In the remaining cases, there is no variation of the opinion variable consequent to the variation of the price.

Last, the third contribution is related to a possible change of opinion due to external effects. This is represented by the random variable θ , assumed with zero mean and variance τ^2 . The function $D(x)$, defined by

$$D(x) = \sqrt{1 - x^2}, \quad (2.16)$$

describes the diffusive behavior [25]. We remark that, as in the case of the herding behavior, the choice (2.16) is such that the opinions close to be extremal are less influenced by external events than the neutral ones.

In order to guarantee that the new opinion x^* still belongs to the right interval $[-1, 1]$, the parameters α_1 , α_2 and θ are subjected to the constraints $\alpha_1 + \alpha_2 + |\theta| \leq 1$.

Other choices of functions are of course possible. Note however that in order to preserve the bounds for x it is essential that $D(x)$ vanishes in $x = \pm 1$.

The chartist's interaction (2.13) differs in a substantial way from the interaction in [18], where the mechanism of opinion formation for chartists was given as a function of the derivative of the logarithm of the price, and it is consistent with the remark of [1, 8], in which even chartists agents use their knowledge of a sufficiently long time history of the price to decide what to do. Clearly, our choice in (2.15) can be modified in many ways, including the possibility to generalize the function considered in [18] to include the acceleration of the logarithm of the price.

We are now in a position to define the microscopic law of variation of the number of chartists and fundamentalists. For any given pair (v, w) of numbers of the two classes of agents, we assume that the new pair (v^*, w^*) will depend both on the transition probabilities p_{CF} and p_{FC} to change class (from chartists to fundamentalists and viceversa), and on the local opinion of the chartists agents. The starting point of our rule of transition is based on the assumption that a chartist is close to become a fundamentalist when he has not a better strategy than a fundamentalist, and this eventually happens when his opinion is approximately neutral. Thus, changes from chartists to fundamentalists will be allowed only when their opinion is *not too far* from the neutral one. This *neutrality* will be expressed by a positive parameter $\lambda \leq 1$. Likewise, we will assume that the fundamentalists that change class will enter the class of chartists with opinion close to the neutral one. We will denote the measure of this interval by the parameter δ . Consequently, the general microscopic law of variation of the pair (v, w) reads

$$\begin{aligned} v^* &= v^*(x) = v - p_{CF}v(1 - \vartheta) I(|x| \leq \lambda) + p_{FC}w I(|x| \leq \delta), \\ w^* &= w^*(x) = w - p_{FC}w(1 - \tilde{\vartheta}) I(|x| \leq \delta) + p_{CF}v I(|x| \leq \lambda). \end{aligned} \tag{2.17}$$

In (2.17) the function $I(A)$ denotes the characteristic function of the set A , that is $I(a) = 1$ if $a \in A$, while $I(a) = 0$ if $a \notin A$. Similarly to formulas (2.10) and (2.13), (2.17) includes random fluctuations of the numbers of the two classes which maintain constant the mean number of the whole population. This is done in (2.17) by including in the first (respectively the second) relation in (2.17) a random term ϑ (respectively $\tilde{\vartheta}$), where ϑ and $\tilde{\vartheta}$ are independent random variables of mean zero and variance ζ^2 .

Note that (2.17) implies

$$\langle v^*(x) + w^*(x) \rangle = v + w, \tag{2.18}$$

namely the conservation (in the mean) of the total number of agents. Among the possible choices of the transition rates between the classes of agents, we will assume in the following that these rates could depend of the acceleration of the price. A possible law for this is the following

$$p_{CF} = \frac{|\ddot{P}|}{1 + |\ddot{P}|} \chi, \quad p_{FC} = \frac{1}{1 + |\ddot{P}|} \chi, \tag{2.19}$$

where $0 < \chi \leq 1$ is a statistical weight. Within this assumption, to mimic the resistance to move towards a risk strategy in a turbulent market, the probability of changing from fundamentalist to chartist decreases when the price acceleration increases.

Finally the time evolution of the distributions $c(x, v, t)$ of chartist agents and $f(w, t)$ of fundamentalists is given by the kinetic system (in weak form)

$$\begin{aligned} \frac{d}{dt} \int_{-1}^1 \int_{\mathbb{R}_+} \varphi(x, v) c(x, v, t) dv = \\ \left\langle \int_{-1}^1 \int_{\mathbb{R}_+^2} (\varphi(x^*, v^*) - \varphi(x, v)) f(w, t) c(x, v, t) dv dw dx \right\rangle \end{aligned} \tag{2.20}$$

$$\frac{d}{dt} \int_{\mathbb{R}_+} \varphi(w) f(w, t) dw = \left\langle \int_{-1}^1 \int_{\mathbb{R}_+^2} (\varphi(w^*) - \varphi(w)) f(w, t) c(x, v, t) dv dw dx \right\rangle. \quad (2.21)$$

We remark that, choosing the test functions $\varphi = 1$ in (2.20) and (2.21) shows that the densities $m_C(t)$ and $m_F(t)$ of chartists and fundamentalists remain probability densities at any subsequent time. Analogously, choosing $\varphi(x, v) = v$ in (2.20) and $\varphi(w) = w$ in (2.21) we obtain

$$\begin{aligned} \frac{d}{dt} n_C(t) &= p_{FC} n_F(t) \int_{\{|x| \leq \delta\}} M_C(x, t) dx - p_{CF} \int_{\{|x| \leq \lambda\}} n_C(x, t) dx \\ \frac{d}{dt} n_F(t) &= p_{CF} \int_{\{|x| \leq \lambda\}} n_C(x, t) dx - p_{FC} n_F(t) \int_{\{|x| \leq \delta\}} M_C(x, t) dx \end{aligned} \quad (2.22)$$

Hence, thanks to (2.18), we conclude that the total mean number of agents is conserved in time, so that $n_C(t) + n_F(t) = n_0$.

Furthermore, let us set $\varphi = x$ in equation (2.20). This allows to compute the evolution in time of the average investment propensity $X(t)$. In the case in which in (2.13) the herding function is assumed constant, $H = 1$, the evolution equation for $X(t)$ simplifies and takes the form

$$\frac{d}{dt} X(t) = \alpha_2 \left[\Phi(\dot{P}(t), \ddot{P}(t)) - X(t) \right]. \quad (2.23)$$

Equations (2.22), coupled with the evolution equation for $X(t)$, for example (2.23), can be used in the equation (2.12) for the price velocity to reckon an exact formula for the price acceleration.

3. The importance of the opinion of chartists. The kinetic system described in Section 2 aims to describe the time-evolution of the price of a stock in a system of agents which play according to different rules. In particular, the chartists strategy is deeply linked to the formation of a personal opinion which would help to decide whether or not it is opportune to buy the stock itself.

The basic hypothesis behind the construction of our model, with respect to the previous ones existing in the literature [16, 18] is the combined rules of the opinion formation of chartists, and the new idea of using acceleration of the price to measure both the change of opinion and the change of class (from chartists to fundamentalists and viceversa).

Despite our simple choices of interactions, which are described by (2.13) and (2.17), to obtain an analytic precise description of the evolution of the densities $h(p, t)$, $f(w, t)$ and $c(x, v, t)$ seems to be prohibitive.

Nevertheless, some insight into the time behavior of the price induced by the kinetic system can be done. The main result here will be to clarify the importance of the opinion of chartists in order to drive the system towards extremal situations, which are typically known as booms (the mean price increases unlimited) and crashes (the mean price collapses towards zero value).

To start with, let us assume that the constants λ and δ which are responsible of the passage from chartists to fundamentalists and vice-versa are taken equal to unity. In this case, independently of their personal opinion, a percentage of chartists

is moved to the other class. In this case, equations (2.22) simplify and give

$$\frac{d}{dt}n_F(t) = -p_{FC}n_F(t) + p_{CF}n_C(t). \tag{3.1}$$

Note that, by definition (2.19), $p_{CF} = \chi - p_{FC}$, so that equation (3.1) can be rewritten as

$$\frac{d}{dt}n_F(t) = \chi n_C(t) - \frac{3}{2}p_{FC}n_F(t). \tag{3.2}$$

Therefore, when the right-hand side of equation (3.2) is non-negative, the mean density $n_F(t)$ starts to increase. On the other hand

$$\chi n_C(t) - \frac{3}{2}p_{FC}n_F(t) \geq 0$$

when

$$n_F(t) \leq \frac{2\chi}{3p_{FC}}n_C(t) = \frac{2}{3}(1 + |\ddot{P}|)n_C(t). \tag{3.3}$$

Condition (3.3) is automatically satisfied if $n_F(t) \leq n_C(t)/2$. Owing to the conservation law $n_F(t) + n_C(t) = n_0$ gives the lower bound

$$n_F(t) \geq \max \left\{ n_F(0), \frac{2}{5}n_0 \right\},$$

or, what is the same

$$\rho_F(t) \geq \max \left\{ \rho_F(0), \frac{2}{5} \right\} = \bar{\rho}.$$

Therefore, in case $\lambda = \delta = 1$, the percentage of fundamentalists is always greater than a fixed positive value $\bar{\rho}$.

Using this information on equation (2.12), and considering that the mean opinion $X(t) \geq -1$, we obtain that at any time $t > 0$, $P(t)$ can not be smaller than

$$P_{min}(t) = \frac{\gamma\rho_F(t)P_F}{\gamma\rho_F(t) + (1 - \rho_F(t))t_C} \geq \frac{\gamma\bar{\rho}P_F}{\gamma\bar{\rho} + (1 - \bar{\rho})t_C}.$$

Consequently, the crash of the price is prevented. Likewise, if the initial percentage of fundamentalists is bigger or equal than $2/5$, so that by the previous argument it remains bounded below by $2/5$, and $\gamma > 3t_C/2$, so that

$$\frac{\gamma}{\gamma + t_C} > \frac{3}{5},$$

it holds

$$\rho_C(t) < \frac{3}{5} \leq \frac{\gamma}{\gamma + t_C}. \tag{3.4}$$

Consequently, using that $X(t) \leq 1$ in (2.12) one obtains

$$\dot{P}(t) \leq \beta [((t_C + \gamma)\rho_C - \gamma)] P(t) + \rho_F\gamma P_F,$$

where the coefficient of $P(t)$, by (3.4) is negative. Therefore $P(t)$ remains always below the upper value

$$P_{max} = \frac{\gamma\bar{\rho}P_F}{\gamma\bar{\rho} - (1 - \bar{\rho})t_C},$$

and there is no possibility of an indefinite growth (boom-state) of the price.

This simple analysis enlightens the importance of the parameters λ and δ in the large-time behavior of the model, and at the same time the rule of the presence of fundamentalists, which guarantee that the price evolution is contained in a bounded interval, whit the lower value bounded away from zero.

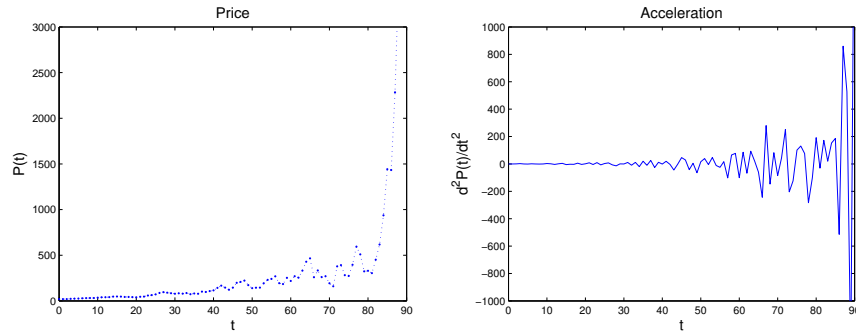


FIGURE 4.1. a) Average price $P(t)$ in the case of boom–state (figure left side). b) Acceleration $\ddot{P}(t)$ of the average price in the case of boom–state (figure right side).

4. Numerical experiments. To study the evolution of price, we have performed a series of kinetic Monte Carlo simulations for our Boltzmann-type system. In this rather basic simulations, known as direct simulation Monte Carlo (DSMC) method or Bird’s scheme, agents are randomly and non-exclusively selected for interactions, and modify opinion according to the respective trade rules. One *time step* corresponds to N such interactions, with N denoting the number of agents. In all our experiments the group is composed by $N = 10.000$ agents with an initial equal percentage of chartists and fundamentalists (i.e. $\rho_C = \rho_F = 0.5$). All agents possess initially a unit wealth, and all chartist agents possess initially a neutral opinion ($x = 0$).

The simulation steps are as follows. One agent is randomly selected. If the selected agent is a chartist, it changes the investment propensity according to the change of opinion rules (2.13). In the case in which the selected agent is a fundamentalist it buys ($x = 1$) when the average price increases ($\dot{P} > 0$) or sells ($x = -1$) when the price decreases ($\dot{P} < 0$) with respect to the fundamental price P_F .

To compute a stock approximation of the long-time behavior we noticed it is appropriate to carry out the simulation for about 10^3 time steps.

Owing to structure of the kinetics equations, and to the exact results obtained in Section 3, we tested the eventual formation of the boom and crash states, by acting on the exchange rules between chartists and fundamentalists, in such a way the number of density of chartists tends to dominate the number density of fundamentalists. A boom effect is then obtained by fixing the parameters $\beta = 4$, $t_C = 0.01$, $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\gamma = 0.1$, $\lambda = 0.1$, $\delta = 0.01$ and $\mu = 0.5$. The boom price behavior is displayed in Fig.(4.1).

Second, we fix the parameters $\beta = 2$, $t_C = 0.01$, $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\gamma = 0.1$, $\lambda = 0.8$, $\delta = 0.5$ and $\mu = 0.5$. Under this set of parameters, a crash price behavior is reported in Fig.(4.2).

The last experiment refers to the case of predominance of the fundamentalist agents. After the discussion of Section 3 it is known that this situation would prevent the formation of critical situations. Indeed, the time evolution of the average price is shown to produce damped oscillations which converge towards the fundamental price value P_F . In Fig.(4.3) the average price and the density of the chartist agents are reported. In this case the choice of the parameters is $\beta = 0.1$, $t_C = 1$, $\alpha_1 = 0.1$,

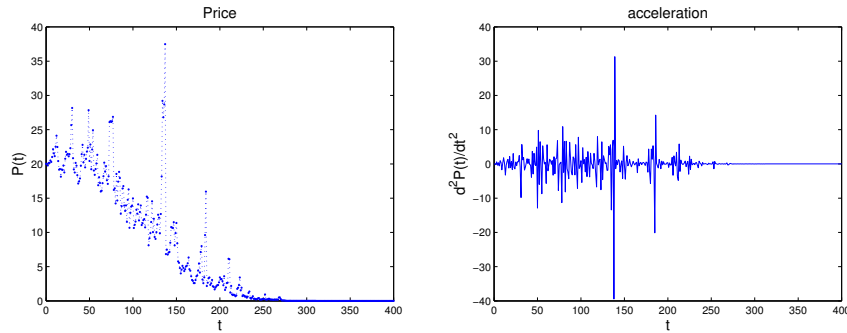


FIGURE 4.2. a) Average price $P(t)$ in the crash-state (figure left side). b) Acceleration $\ddot{P}(t)$ of the average price in the crash-state (figure right side).

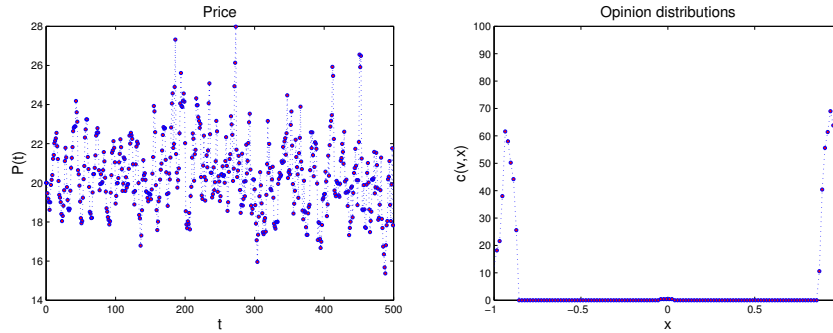


FIGURE 4.3. a) Average price $P(t)$ in the case of damped oscillations (figure left side). b) Density of the investment propensity (figure right side).

$\alpha_2 = 0.1$, $\gamma = 1.1$, $\lambda = 0.8$, $\delta = 0.8$ and $\mu = 0.5$. Fig.(4.4) shows the acceleration $\ddot{P}(t)$ and the returns market.

5. Conclusions. In this paper we analyzed a new kinetic version of the multi-agent system introduced by Maldarella and Pareschi [18] in agreement with the model introduced by Lux and Marchesi [15, 16, 17] to study the evolution of price in presence of two different human behaviors, represented by fundamentalists and chartists reactions to the market evolution. At difference with the kinetic description of the opinion formation considered by Maldarella and Pareschi [18], based on price velocity, we analyzed the effects of price acceleration. Also, a microscopic description of the migration from one population to the other has been introduced. It is show both theoretically and numerically that the formation of critical situations is linked to the opinion variable which characterizes the behavior of chartists agents, and to the parameters of migration, which depends on price acceleration. In the case in which migration towards chartists agent is predominant, one shows that the extremal phenomena of boom and crash can be observed for suitable values of the parameters characterizing the model itself. On the contrary, when migration towards chartists is not predominant, so that the percentage of fundamentalists

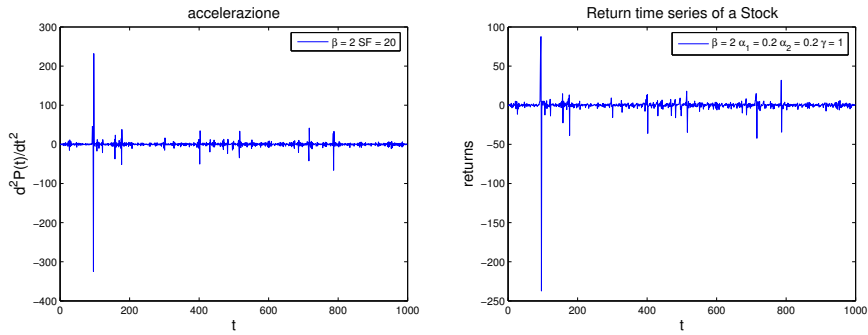


FIGURE 4.4. a) Acceleration of the average price $\ddot{P}(t)$ in the case of damped oscillations (figure left side). b) Returns of the average price in the case of damped oscillations (figure right side).

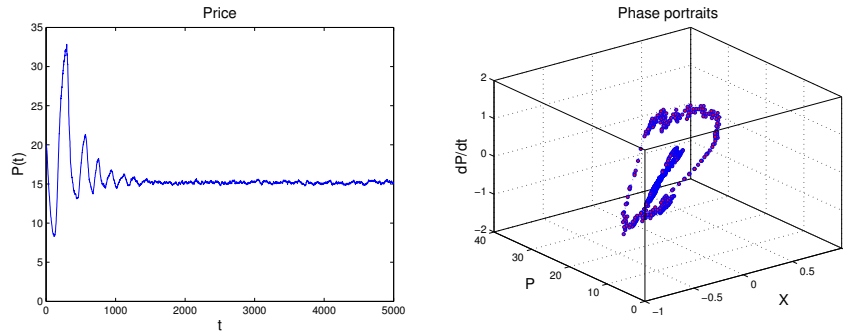


FIGURE 4.5. a) The average price $P(t)$ in the case of damped oscillations (figure left side). b) Phase portraits in the case of damped oscillations (figure right side).

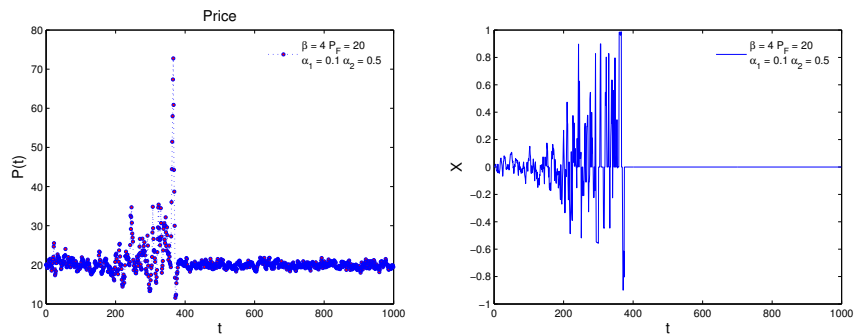


FIGURE 4.6. a) Average price $P(t)$ in the case of the predominance of fundamentalist agents. The relaxation behavior is characterized by damped oscillations toward the fundamental price P_F (figure left side). b) Average propensity of investment $x(t)$ in the case of the predominance of fundamentalist agents (figure right side).

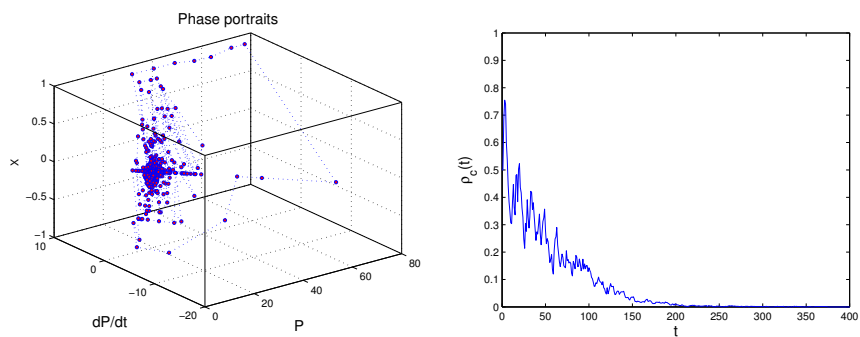


FIGURE 4.7. a) Phase portraits parameterized by the average propensity of investment $X(t)$ in the case of the predominance of fundamentalist agents (figure left side). b) Density of the chartist agents (figure right side).

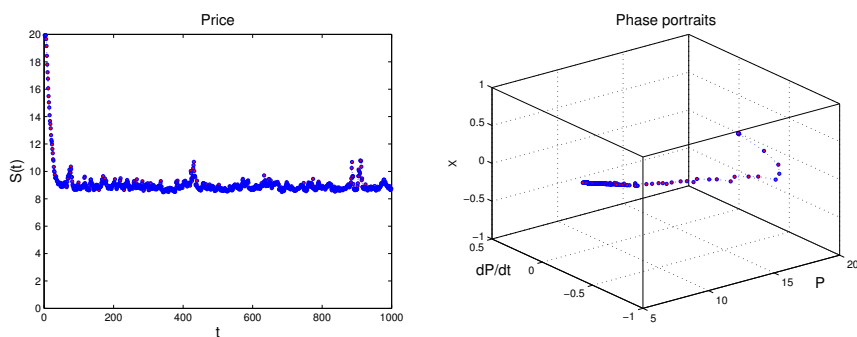


FIGURE 4.8. a) Average price $P(t)$ in the crash-state (figure left side). b) Acceleration $\dot{P}(t)$ of the average price in the crash-state (figure right side).

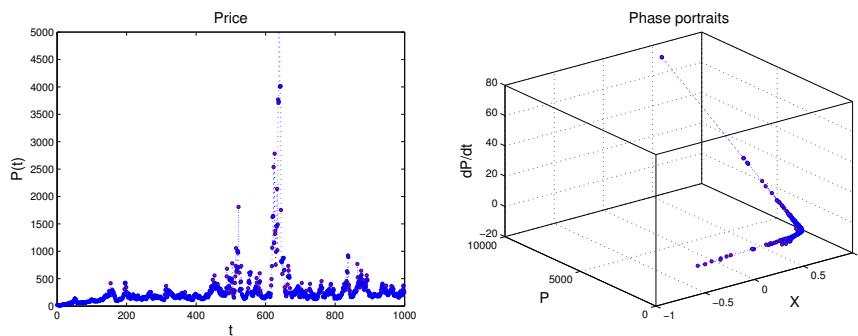


FIGURE 4.9. a) Average price $P(t)$ in the boom-state (dotted red line). b) Phase portraits in the boom-state (blue line).

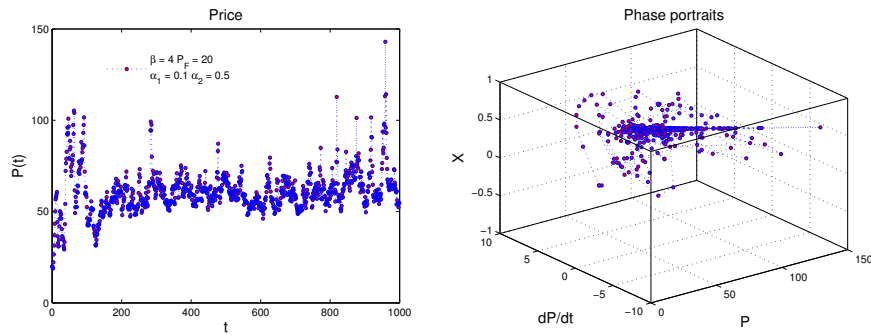


FIGURE 4.10. a) Average price $P(t)$ in the boom-state. b) Phase portraits parameterized by the average propensity of investment $X(t)$ in the boom-state.

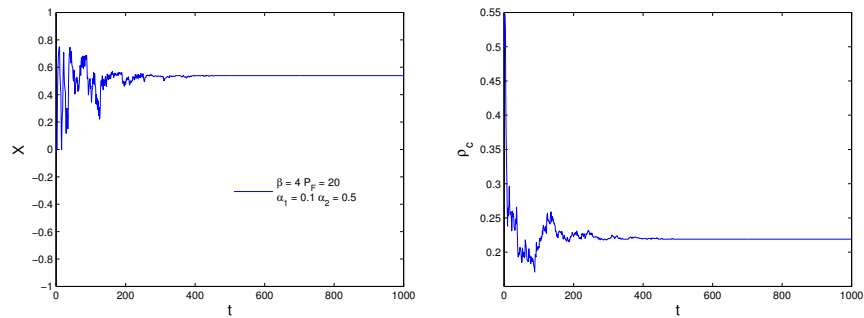


FIGURE 4.11. a) Average propensity of investment $X(t)$ in the boom-state (figure left side). b) Density of the chartist agents in the boom-state (figure right side).

remains bounded away from zero, these extremal phenomena are prevented. Numerical simulations enlighten the various behaviors of the price observed in the previous models, together with their stability with respect to the mean number of agents present in the market.

Acknowledgments. The authors acknowledge support by MIUR project “Optimal mass transportation, geometrical and functional inequalities with applications” and by the National Group of Mathematical Physics of INDAM. GT thanks L. Pareschi for enlightening discussions about the Lux–Marchesi model.

REFERENCES

- [1] R. Bapna, W. Jank and G. Shmueli, Price formation and its dynamics in online auctions, *Decision Support Systems*, **44** (2008), 641–656.
- [2] A. Chakraborti, [Distributions of money in models of market economy](#), *Int. J. Modern Phys. C*, **13** (2002), 1315–1321.
- [3] A. Chakraborti and B. K. Chakraborti, Statistical mechanics of money: Effects of saving propensity, *Eur. Phys. J. B*, **17** (2000), 167–170.
- [4] A. Chatterjee, B. K. Chakraborti and S. S. Manna, [Pareto law in a kinetic model of market with random saving propensity](#), *Physica A*, **335** (2004), 155–163.

- [5] A. Chatterjee, S. Yarlagadda and B. K. Chakrabarti, Eds., *Econophysics of Wealth Distributions*, New Economic Window Series, Springer-Verlag, Milan, 2005.
- [6] A. Chatterjee, B. K. Chakrabarti and R. B. Stinchcombe, [Master equation for a kinetic model of trading market and its analytic solution](#), *Phys. Rev. E*, **72** (2005), 026126.
- [7] S. Cordier, L. Pareschi and G. Toscani, [On a kinetic model for a simple market economy](#), *J. Stat. Phys.*, **120** (2005), 253–277.
- [8] M. Cristelli, L. Pietronero and A. Zaccaria, Critical overview of agent-based models for economics, in *Proceedings of the School of Physics E. Fermi*, course CLXXVI, Varenna, 2010. E-Print: [arXiv:1101.1847](#).
- [9] A. Drăgulescu and V. M. Yakovenko, Statistical mechanics of money, *Eur. Phys. Jour. B*, **17** (2000), 723–729.
- [10] B. Düring, D. Matthes and G. Toscani, [Kinetic equations modelling wealth redistribution: A comparison of approaches](#), *Phys. Rev. E*, **78** (2008), 056103.
- [11] B. Düring, D. Matthes and G. Toscani, A Boltzmann type approach to the formation of wealth distribution curves, *Riv. Mat. Univ. Parma*, **1** (2009), 199–261.
- [12] D. Kahneman and A. Tversky, [Prospect theory: An analysis of decision under risk](#), *Econometrica*, **47** (1979), 183–214.
- [13] D. Kahneman and A. Tversky, [Choices, values, and frames](#), *American Psychologist*, **39** (1984), 341–350.
- [14] M. Levy, H. Levy and S. Solomon, *Microscopic Simulation of Financial Markets: From Investor Behaviour to Market Phenomena*, Academic Press, San Diego, 2000.
- [15] T. Lux, [The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions](#), *Journal of Economic Behavior & Organization*, **33** (1998), 143–165.
- [16] T. Lux and M. Marchesi, [Volatility clustering in financial markets: A microscopic simulation of interacting agents](#), *International Journal of Theoretical and Applied Finance*, **3** (2000), 675–702.
- [17] T. Lux and M. Marchesi, Scaling and criticality in a stochastic multi-agent model of a financial market, *Nature*, **397** (1999), 498–500.
- [18] D. Maldarella and L. Pareschi, [Kinetic models for socio-economic dynamics of speculative markets](#), *Physica A*, **391** (2012), 715–730.
- [19] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics Correlations and Complexity in Finance*, Cambridge University Press, Cambridge, 2007.
- [20] D. Matthes and G. Toscani, [On steady distributions of kinetic models of conservative economies](#), *J. Stat. Phys.*, **130** (2008), 1087–1117.
- [21] G. Naldi, L. Pareschi and G. Toscani, Eds., *Mathematical Modelling of Collective Behavior in Socio-economic and Life Sciences*, Birkhäuser, Boston, 2010.
- [22] L. Pareschi and G. Toscani, *Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods*, Oxford University Press, Oxford, 2014.
- [23] L. Pareschi and G. Toscani, [Wealth distribution and collective knowledge. A Boltzmann approach](#), *Phil. Trans. R. Soc. A*, **372** (2014), 20130396, 15pp.
- [24] F. Slanina, [Inelastically scattering particles and wealth distribution in an open economy](#), *Phys. Rev. E*, **69** (2004), 046102.
- [25] G. Toscani, [Kinetic models of opinion formation](#), *Comm. Math. Scie.*, **4** (2006), 481–496.
- [26] J. Voit, *The Statistical Mechanics of Financial Markets*, Springer Verlag, Berlin, 2005.

Received August 2014; revised January 2015.

E-mail address: cabrugn@tin.it

E-mail address: giuseppe.toscani@unipv.it