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A KINETIC MODEL FOR AN AGENT BASED MARKET SIMULATION

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Abstract. A kinetic model for a specific agent based simulation to generate the sales curves of successive generations of high-end computer chips is developed. The resulting continuum market model consists of transport equations in two variables, representing the availability of money and the desire to buy a new chip. In lieu of typical collision terms in the kinetic equations that discontinuously change the attributes of an agent, discontinuous changes are initiated via boundary conditions between sets of partial differential equations. A scaling analysis of the transport equations determines the different time scales that constitute the market forces, characterizing different sales scenarios. It is argued that the resulting model can be adjusted to generic markets of multi-generational technology products where the innovation time scale is an important driver of the market.

1. **Introduction.** Modeling the success and failure of new products or new technologies in the market place has a long tradition of mathematical modeling. One of the most cited papers in that context is [3] which categorized the potential adopters of a product into innovators and followers. Innovators' decision to buy a product are mostly driven by the properties of the product itself (its utility, its price, its features) whereas followers are in addition strongly motivated by the success of the product, i.e whether others have bought the product too. The resulting Bass model leads to a logistic type of equation for the time evolution of the market share. The derivative of that curve represents the time evolution of the sales rate, typically showing a small starting rate, a single well defined maximum and then an exponentially decaying tail.

The Bass model is one of a large number of aggregate models for the way consumers adopt a new product. These models are typically based on the intuition of the modeler. The standard approach at validation has been to perform statistical fits of the parameters of the aggregate model tying the parameters of the aggregate model characterizing the buying decisions of individuals to the parameters that describe the utility of the product for an individual customer.

An alternative approach to a purely statistical fit is to develop agent based models, whereby agents representing potential buyers are imbued with an individual

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utility measure whose evaluation in a changing environment will lead to a decision to buy a product or not. The changing environment may include the amount of information on the product available to the agent, the changing price and relative competitiveness of the product and the relative importance given to this factors over time by an agent. The resulting agent based simulation then represents such decisions processes for a large number of agents over a time sequence. It therefore represents a sample time series of a stochastic simulation that needs to be repeated many times allowing for a statistical evaluation of the process.

Recently [1] have developed such an agent based model for the main features of the market for high-performance computer chips. According to INTEL's sales division these chips are almost exclusively purchased by people playing competitive games on the Internet. We call this market the high-end gamers market and the participants in that market the high-end gamers. The agent based simulation model showed that the sales curves of 19 chips for two manufacturers (INTEL and AMD) over about 3 years can be qualitatively and quantitatively approximated using two main parameters representing the agents' income allocated to the gaming hobby and the gamers motivation to improve their computer hardware.

Using the same data set, [10] have developed a population-growth model characterizing multiple product generations. Using an iterative-descent method they obtained parameter estimates for the population model.

This paper will develop a third approach, complementing the other two, by developing a kinetic model and subsequent partial differential equations for the agent based simulations. In doing so, we will use the high-end-gamers data and simulation model as an inspiration for a proof of concept for kinetic marketing models. Following the approach that leads from the kinetic Boltzmann equations to hydrodynamic transport equations, the resulting partial differential equations will be transport equations for the number density of agents owning a particular chip moving in the conceptual spaces of affinity and ability to buy. We will show that the kinetic model captures many features ascribed to generic behavior of markets. We will discuss the distinct advantages of kinetic models over individualized agent based models and over heuristic aggregate compartmental models like in population dynamics:

- The resulting transport equations allow a scaling analysis identifying different market processes at different time scales.
- Such a scaling analysis generates a small number of dimensionless parameters that characterize the market dynamics. These dimensionless parameters are built from the larger number of original parameters of the agent based simulation.
- A kinetic model allows to analyze limiting behavior which are hard to simulate via agents validating intuition.
- Since the resulting transport equations are already based on averages, there is no need for multiple simulations, making those simulations fast and hence amenable to "what if" scenario explorations.

The rest of this paper is organized as follows: In Section 2 we discuss the market of the high end computer chips and its agent based simulation model. Section 3 discusses the kinetic model; first (Sec. 3.1) the general background and second the specific model (Sec 3.2 and Sec 3.3) for the high end computer market. In Section 4 we create dimensionless equations and analyze the resulting dimensionless parameters. Numerical simulations for specific scenarios are discussed in Section 5 and we conclude in Sec. 6 with a discussion of the generalizability of our model.

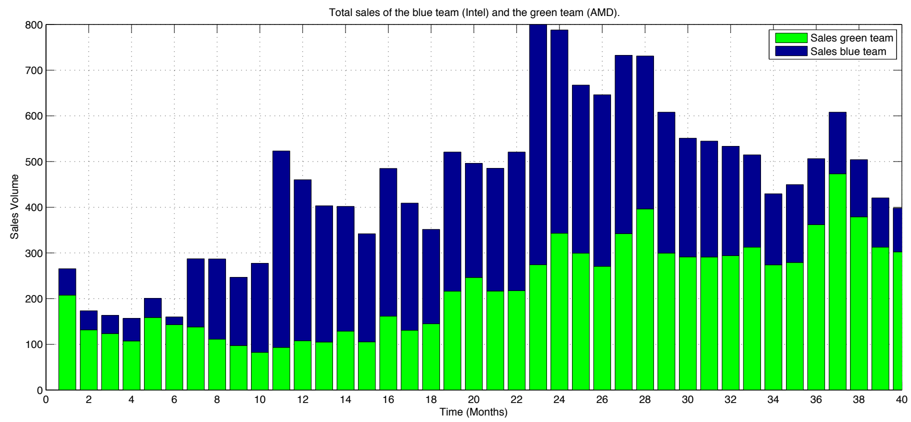


FIGURE 1. Total monthly sales data of the Blue and Green Team high-end chips (period: January 2006 until April 2009).

2. The market and its model.

2.1. The data. There are only two manufacturers of these high-end computer chips worldwide. From January 2006 until April 2009 the Blue Team (INTEL) released 10 high-end chips and the Green Team (AMD) released 9 high-end chips. The estimated total monthly sales data of these chips are shown in Figure 1 for each company. All sales data are relative and displayed in arbitrary units.

2.2. The agent based simulation model. We briefly review the setup for the agent based simulations for the high-end-gamers market. More details can be found in [1]. The agent based simulation involved a population of one to two thousand agents each playing 200 one-on-one games per month with random pairing against other agents. Games are reduced to a comparison of the performance of the current chip that an agent owns; the agent with the higher-performance chip wins with a high probability. Agents will keep a tally of their losses and are characterized by a loss threshold. Once the losses exceed the threshold, the agent decides that he/she needs a new chip and will buy the fastest chip that is on the market that one can afford. The simulation is non-autonomous since it is driven by the times that a particular chip with a particular performance and price will enter and exit the market. Those data were provided by INTEL and cover the full simulation time span of 40 months.

Notice that threshold variation will cover different skill levels for the agents: A top dog gamer presumably has a low threshold and few losses are enough to trigger the search for a new chip, whereas a newcomer to the game expects to lose and therefore has a larger threshold.

Figure 2 shows a typical result comparing the agent simulations to the sales data of multiple generations of products.

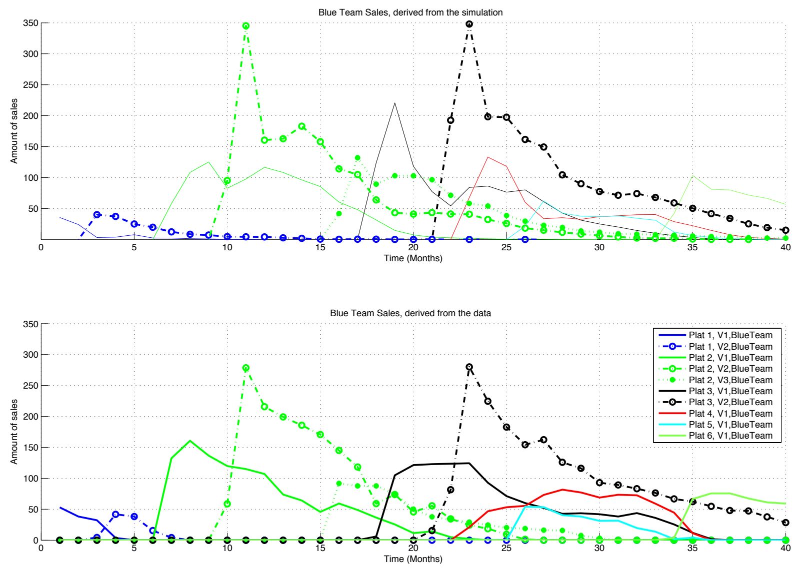


FIGURE 2. Sales for each chip of the Blue Team (top: agent based simulation results, bottom: data).

3. **The kinetic model.**

3.1. **Background.** This paper is concerned with the formulation of a kinetic model representing a continuum description of the agent model in section 2. The purpose of the model is to give a more detailed description than the simple Bass model, while at the same time allowing for an analytical investigation of features, such as steady states and time scale separation.

A general kinetic model is of the form

$$\partial_t f(q,t) + \nabla_q \Big(f \Phi_f(q,t) \Big) = \int P_f(q,q') \omega_f(q',t) f(q',t)\ dq' - \omega_f(q,t) f(q,t), \quad (1)$$

where $f(q,t)$ is the (number) density of agents, depending on the variable q which denotes a dynamic attribute of the agent. $\Phi_f(q,t)$ is a vector field describing the drift velocity of the density f in the space characterized by the variable q. The left hand side of Eq.(1) denotes a continuous change of the attribute q, i.e. q changes by $\Delta t \Phi(q,t)$ in an infinitesimal time interval Δt. The right hand side of Eq.(1) denotes a discontinuous random change of the attribute, i.e. the attribute switches from q' to q with probability $\Delta t \omega(q',t)$ in an infinitesimal time interval Δt and the new attribute q is chosen according to the probability distribution $P(q,q')\, dq$. Note, that the vector field Φ_f, the probability distribution P_f and the scattering frequency ω_f in Eq.(1) depend on the density f itself, making equation (1) nonlinear. In the case that the dynamics are given by binary interactions between agents, Φ_f and ω_f are linearly dependent on the the density f, i.e. $\Phi_f(q,t) = \int A(q,p) f(p,t)\ dp$ and $\omega_f(q,t) = \int W(q,p) f(p,t)\ dp$ holds. The formulation of the model Eq.(1) goes back to the seminal work of Boltzmann [4] and has successfully been employed in social and economical sciences, such as the areas of opinion formation and supply chains

[2],[8],[11],[12]. Separation of scales and entropies, which guarantee the convergence to a steady state, may be obtained by analytical methods, such as moment closures and Chapman -Enskog type expansion procedures [5],[6],[7].

The model presented in this paper has certain special structural features.

- First, we consider a discrete set of product levels with agents upgrading their equipment from one level to the next. Therefore a part of the attribute vector q in Eq.(1) will be discrete, the collision term describes only the upgrade and we are dealing with the evolution of a system of density functions. So Eq. (1) will be replaced by

$$\begin{aligned} \partial_t f_n(q,t) + \nabla_q \Big(f_n \Phi_{nf}(q,t) \Big) &= \sum_{m=1}^{N} \int P_{n,m,f}(q,q') \omega_{mf}(q',t) f_m(q',t)\ dq' \\ &- \omega_{nf}(q,t) f_n(q,t), \end{aligned} \tag{2}$$

  with the probability measure $P_{n,m,f}$ satisfying $\sum_{n=1}^{N} \int P_{n,m,f}(q,q')\ dq = 1,\ \forall m, q'$.
- We will make the decision whether to upgrade to the next product level dependent on a *degree of dissatisfaction* with the current product. This degree of dissatisfaction x is a part of the attribute vector q and grows continually with each negative outcome of an interaction. This allows us to replace the collision operator on the right hand side of (2) by a boundary term, i.e. agents go into a 'bin', once they reached a certain threshold $x \geq g$ and just wait to be able to upgrade, depending on the other elements of the attribute q (such as their finances and therefore their ability to upgrade).
- We will specify the model in more detail in the next section.

3.2. **Playing the game.** We consider agents with the following attributes:

- An agent owns a chip. The generations of chips are labeled by $n = 0 \dots N$.
- The number of losses that an agent has accumulated with the current chip is denoted by x. We assume $x \in \mathbb{R}$.
- The loss threshold variable for an agent is g.
- The accumulated capital of the agent is called μ and the rate of income for an agent is given by γ.

In addition there are a few more parameters that are relevant: A processor of type n has a cost of c_n and q is the frequency of games played (unit= $\frac{games}{time}$).

We define the two player density $F(X, M, \Gamma, G, N, t)$ as the density of agents 1 and 2 at time t that have lost $X = (x_1, x_2)$ games with M, Γ, G, N the two-vectors of capital, income rate, loss threshold, and chip numbers for the two players, respectively. A game is characterized by the fact that one player will win and another one will lose. Only the losers will keep track of their losses and increase the appropriate x value:

$$x_1 \to x_1 + r(N) q \Delta t, \quad x_2 \to x_2 + (1 - r(N)) q \Delta t,$$

$$\mathcal{P}[r(N) = 1] = w(N), \quad \mathcal{P}[r(N) = 0] = 1 - w(N).$$

where $w(n_1, n_2)$ is the probability that in a game between n_1 and n_2, the player with n_2 wins. Note: $w(n_1, n_2) + w(n_2, n_1) = 1$.

In the same time Δt both players increase their capital by the rate of income

$$\mu_j \to \mu_j + \Delta t \gamma, \quad j = 1, 2.$$

Hence, the evolution of the two agent density in the limit of $\Delta t \to 0$ becomes

$$\partial_t F = -w(N) q \partial_{x_1} F - [1 - w(N)] q \partial_{x_2} F - \Gamma \cdot \nabla_M F.$$

Making the usual assumptions about identical and statistically independent agents and factoring the two agent density as

$$F(X, M, \Gamma, G, N, t) = \prod_{j=1}^{2} f(x_j, \mu_j, \gamma_j, g_j, n_j, t),$$

we can derive the time evolution for the effective single agent density

$$\begin{aligned} \partial_t f(x, \mu, \gamma, g, n, t) + \partial_x \phi + \gamma \partial_\mu f &= 0, \\ \phi(x, \mu, \gamma, g, n, t) &= v_\rho(n, t) f(x, \mu, \gamma, g, n, t), \\ v_\rho(n, t) &= \frac{q}{\bar{f}(0)} \sum_m w(n, m) \rho(m, t), \\ \rho(n, t) &= \int f(x, \mu, \gamma, g, n, t)\ dx\mu\gamma g, \\ \bar{f}(0) &= \sum_n \rho(n, 0), \end{aligned} \tag{3}$$

where $\rho(n, t)$ is the number of agents at time t owning chip n, $\bar{f}(0)$ is the total number of agents in the system and hence $\frac{1}{\bar{f}(0)} \sum_m w(n, m) \rho(m, t)$ is the fraction of all the games for a player that has chip number n that lead to a loss.

3.3. **Buying a new chip.** Rewriting the number density in terms of games lost relative to thresholds, we set

$$f(x, \mu, \gamma, g, n, t) \to \frac{1}{g} f(\frac{x}{g}, \mu, \gamma, g, n, t), \quad v_\rho \to g v_\rho$$

giving

$$\begin{aligned} \partial_t f(x, \mu, \gamma, g, n, t) + \partial_x [v_\rho f] + \gamma \partial_\mu f &= 0, \\ v_\rho(n, g, t) &= \frac{q}{g \bar{f}(0)} \sum_m w(n, m) \rho(m, t). \end{aligned} \tag{4}$$

Hence in the scaled x variable, an agent whose loss tally exceeds his/her threshold is at $x > 1$. Such agents continue to play but they are not keeping track of their losses any more. Instead, their decision to upgrade to a new chip is driven entirely by their finances. The number of those agents are described by

$$u(\mu, \gamma, g, n, t) = \int_1^\infty f(x, \mu, \gamma, g, n, t)\ dx.$$

Once an agent decides to buy a new chip, a necessary condition for buying is that the available capital μ is higher than the sales price of a better chip than the currently owned one. Once the two necessary conditions, $x > 1$, $\mu > c_m$ for some chip $m > n$, are met, we assume that the agent buys one of the possible chips. Hence the only agents that go into the waiting bin are those that do not have enough money to buy at that moment and therefore the time evolution for the waiting bin becomes

$$\begin{aligned} \partial_t u(\mu, \gamma, g, n, t) + \gamma \partial_\mu u &= [1 - H(\mu - c_{n+1})] \phi_{bin}(\mu, \gamma, g, n, t), \\ \phi_{bin}(\mu, \gamma, g, n, t) &= v_\rho(n, t) f(x = 1, \mu, \gamma, g, n, t). \end{aligned}$$

where $H(x)$ is the Heaviside function with $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$.

Buying instantly. For the players that have enough money to buy one or more chips, we define $B(\mu, m, n)$ to be the probability that, upon reaching the threshold, the player upgrades from n to m. That probability clearly depends on the current capital level μ. Once an agent upgrades, we deduct the cost of the chip from the capital and set the loss tally to zero. Hence the outflux of the n-level transport equation becomes an influx at $x = 0$ in the level m transport equation. Equivalently, the influx at $x = 0$ at the n_{th} level is the sum of all the outfluxes that had at least $\mu = c_n$ capital available:

$$v_\rho f(x=0,\mu,\gamma,g,n,t) = \sum_{m=1}^{n-1} B(\mu + c_n, n, m)\phi_{bin}(\mu + c_n, \gamma, g, m, t).$$

Buying after a waiting period. An agent in the waiting bin n that reaches a capital c_m decides to upgrade to level m with probability $A(\mu, m, n) = \delta(\mu - c_m)A(m, n)$, with $\delta(x)$ denoting the Dirac δ-function. Thus $A(m, n)$ reflects the policy of the agent to either pick the next possible chips $m = n + 1$ or wait for the capital to increase and buy a better chip $m > n+1$ at a later time. These upgrades lead to loss terms for the n^{th} level bin density u of the form

$$-\sum_{m=1}^{N} A(\mu, m, n)\gamma u(\mu, \gamma, g, n, t)$$

and an influx for the n^{th} level density f at the boundary corner $x = 0$ and $\mu = 0$ of the form

$$\gamma f(x=0,\mu=0,\gamma,g,n,t) = \sum_{m=1}^{n-1} A(\mu, n, m)\gamma u(c_n, \gamma, g, m, t).$$

To complete the boundary conditions, we have no flux boundaries at $\mu = 0$ for f and u: $\gamma f(x, \mu = 0, \gamma, g, n, t) = 0, \quad \gamma u(\mu = 0, \gamma, g, n, t) = 0$.

We assume that there is a maximal capital level μ at $\mu = M \geq c_N$ that a gamer might allocate to the game. Once a player reaches M the capital influx rate γ is set to zero. Setting $A(c_N, N, n) = 1, \forall n$, we do not have to worry about cutting off in the bin equation, since the term $A(c_N, N, n)\gamma u(c_N, \gamma, g, n, t)$ removes all agents. Therefore $u(\mu, \gamma, g, n, t) = 0$ for $\mu > c_N$ holds automatically.
Notice:

- The market is driven by the releases of new chips and the retirement from the market of older chips. Setting $c_n(t) = 10^6$ at times where the chip is not on the market, prevents anybody to buy them at these times.
- The sales at time t of chip n are given by

$$s(n,t) = \int v_\rho f(x=0,\mu,\gamma,g,n,t) + \gamma f(x=0,\mu=0,\gamma,g,n,t) d\mu\gamma g. \tag{5}$$

  This is the observable in the pde simulation that can be compared to the observable in the agent based simulations and to actual data.

We summarize the equations for the evolution of the number density for the gamers who are happy with their chips and the number density for the gamers who are unhappy and are waiting for their capital to increase such that they can buy a new chip:

Evolution equation

$$\begin{array}{rcl} \partial_t f(x,\mu,\gamma,g,n,t) & + & \partial_x \phi + \gamma \partial_\mu f = 0, \\ \phi(x,\mu,\gamma,g,n,t) & = & v_\rho(n,t) f(x,\mu,\gamma,g,n,t), \\ v_\rho(n,t) & = & \frac{q}{\bar{f}(0)} \sum_m w(n,m)\rho(m,t), \\ \rho(n,t) & = & \int f(x,\mu,\gamma,g,n,t)\ dx\mu\gamma g + \int u(\mu,\gamma,g,n,t)\ d\mu\gamma g, \\ \bar{f}(0) & = & \sum_n \rho(n,0). \end{array} \tag{6}$$

Boundary conditions

$$\begin{array}{rcl} \gamma f(x,\mu=0,\gamma,g,n,t) & = & 0, \\ v_\rho f(x=0,\mu,\gamma,g,n,t) & = & \sum_{m=1}^{n-1} B(\mu+c_n,n,m)\phi_{bin}(\mu+c_n,\gamma,g,m,t), \\ \gamma f(x=0,\mu=0,\gamma,g,n,t) & = & \sum_{m=1}^{n-1} A(\mu,n,m)\gamma u(c_n,\gamma,g,m,t). \end{array} \tag{7}$$

Waiting Bins

$$\begin{array}{rcl} \partial_t u(\mu,\gamma,g,n,t) + \gamma\partial_\mu u & = & \phi_{bin}(\mu,\gamma,g,n,t)[1-\sum_{m=n+1}^{N} B(\mu,m,n)] \\ & - & \sum_{m=n+1}^{N} A(\mu,m,n)\gamma u(\mu,\gamma,g,n,t), \\ \phi_{bin}(\mu,\gamma,g,n,t) & = & v_\rho(n,t) f(x=1,\mu,\gamma,g,n,t), \\ \gamma u(\mu=0,\gamma,g,n,t) & = & 0. \end{array} \tag{8}$$

4. Scaling analysis.

4.1. Dimensionless equations.

We can scale all quantities to make the equations dimensionless: Scaling the losses x with the loss threshold g, the capital μ with the maximally accumulated capital M, the monthly income γ by the maximal monthly income Γ, the individual loss threshold g by the maximal loss threshold G, the number density ρ by the maximal number of potential buyers P, the velocity in loss space v_ρ by the frequency q of games played and time t by a scale T, we transform equations (6,7,8) into a dimensionless form:

Dimensionless evolution equation

$$\begin{array}{rcl} \partial_t f(x,\mu,\gamma,g,n,t) & + & \frac{qT}{G}\partial_x \phi + \frac{\Gamma T}{M}\partial_\mu \gamma f = 0, \\ \phi(x,\mu,\gamma,g,n,t) & = & v_\rho(n,t) f(x,\mu,\gamma,g,n,t), \\ v_\rho(n,t) & = & \frac{1}{g\bar{f}(0)} \sum_m w(n,m)\rho(m,t), \\ \rho(n,t) & = & \int f(x,\mu,\gamma,g,n,t)\ dx\mu\gamma g + \int u(\mu,\gamma,g,n,t)\ d\mu\gamma g, \\ \bar{f}(0) & = & \sum_n \rho(n,0). \end{array} \tag{9}$$

Dimensionless boundary conditions

$$\begin{array}{rcl} \gamma f(x,\mu=0,\gamma,g,n,t) & = & 0, \\ v_\rho f(x=0,\mu,\gamma,g,n,t) & = & \sum_{m=1}^{n-1} B(\mu+\frac{c_n}{M},n,m)\phi_{bin}(\mu+\frac{c_n}{M}),\gamma,g,m,t), \\ \gamma f(x=0,\mu=0,\gamma,g,n,t) & = & \sum_{m=1}^{n-1} A(\mu,n,m)\gamma u(\frac{c_n}{M},\gamma,g,m,t). \end{array} \tag{10}$$

Dimensionless waiting bins

$$\begin{array}{rcl} \partial_t u(\mu,\gamma,g,n,t) + \frac{T\Gamma}{M}\gamma\partial_\mu u & = & \frac{qT}{G}\phi_{bin}(\mu,\gamma,g,n,t)[1-\sum_{m=n+1}^{N} B(\mu,m,n)] \\ & - & \frac{T\Gamma}{M}\sum_{m=n+1}^{N} A(\mu,m,n)\gamma u(\mu,\gamma,g,n,t), \\ \phi_{bin}(\mu,\gamma,g,n,t) & = & v_\rho(n,t) f(x=1,\mu,\gamma,g,n,t), \\ \gamma u(\mu=0,\gamma,g,n,t) & = & 0. \end{array} \tag{11}$$

4.2. **Asymptotics.** There are two possible time scales that govern the market dynamics: The financial scale $T_F = \frac{M}{\Gamma}$ is the mean time it takes the monthly income to accumulate to the maximal capital allocated to gaming. This time scale is of the same order of magnitude as $\tilde{T}_F = \frac{\bar{c}_n}{\Gamma}$ which represents the mean time that it takes for the monthly income to reach the average cost of a chip.

We call the second time scale the gaming scale $T_G = \frac{G}{q}$ which is the mean time it takes a player to accumulate enough losses to trigger the buying threshold, given a frequency q of playing games. We define ϵ to be the dimensionless ratio

$$\epsilon = \frac{T_G}{T_F} = \frac{\Gamma G}{Mq}.$$

Case 1, the committed gamer. Consider $\epsilon \ll 1$, i.e. $T_G \ll T_F$ and measure time on the fast time scale $T = T_G$. Hence we have a high frequency of reaching the loss threshold relative to the frequency of replenishing the capital. Slow financial recovery may have two sources: not enough income or high processor cost. A high frequency of reaching the loss threshold also has two sources: High frequency of games or a low loss threshold. Hence this case is characteristic for the highly committed and proficient gamer who spends a lot of time on gaming and is very good at it.

In this case, for the gaming equation Eq. 9, transport in the x direction is much faster than transport in the μ direction. As a result agents will reach $x = 1$ on a timescale of $O(1)$ and become frustrated because their financial situation has not kept pace. In all of these cases the fast equation for $\epsilon = 0$ is given for the game evolution by

$$\partial_t f(x, \mu, \gamma, g, n, t) + \partial_x \phi = 0 \tag{12}$$

indicating a transport in x only with constant speed. Let us assume then that $A(c_{n+1}, n+1, n) = 1$, i.e. all agents will buy the new chip once they have the money to do so. Hence they begin gaming again at $x = 0, \mu = 0$. As a result, the flux into the bin equation on the fast timescale is given by $\phi_x(1, 0, \gamma, g, n, t) = v_\rho(n, t) f(1, 0, \gamma, g, n, t)$. Since on the fast time scale capital does not increase, no gamer has enough money to buy a new chip on that timescale and hence $B(m, n) = 0$. Thus the bin equation on the fast time scale becomes a trivial ODE accumulating all the gamers that reach the loss threshold,

$$\partial_t u(\mu, \gamma, g, n, t) = \phi_{bin}(\mu, \gamma, g, n, t).$$

On the slower financial timescale in the bin equation, there is a drift towards increasing budget until the budget has reached the cost of the next chip c_{n+1} at which time everybody buys and resets the dynamics again at the origin.

Hence, the fast timescale acts as a instantaneous reset map, moving agents from the point $x = 1, \mu = c_{n+1}$ to $x = 0, \mu = 0$. Thus the whole dynamics reduces to periodic buying behavior with a buying period of $T_{buy} = \frac{c_{n+1}}{\gamma}$. If γ is constant, every agent has the same period and the sales distribution is given by the initial distribution of capital. If there is a distribution of γ the resulting sales signal becomes a superposition of periodic signals which might become periodic with a very long period.

Case 2, the casual gamer. Consider $\epsilon \gg 1$, i.e. $T_G \gg T_F$ and measure time on the fast time scale $T = T_F$. Hence we have a low frequency of reaching the loss threshold relative to the frequency of replenishing the capital. Short T_F can be

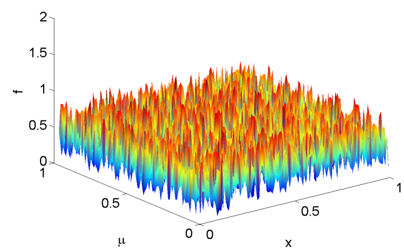


FIGURE 3. Random initial density distribution in (x, μ)-space.

generated by high income or low chip prices. Long T_G may come from infrequent gaming or a high loss tolerance both are characteristic for a casual gamer.

Here the fast equation for $\epsilon = 0$ is given for the game evolution by

$$\partial_t f(x, \mu, \gamma, g, n, t) + \gamma \partial_\mu \phi = 0 \tag{13}$$

indicating a transport in μ only with speed given by γ. That transport will stop once capital has reached the maximal value M. Hence no agent goes into the waiting bin and $u(\mu, \gamma, g, n, t) = 0, \forall t$. On the slower gaming timescale there is now a 1-d transport equation in x given by

$$\begin{aligned} \partial_t f(x, M, \gamma, g, n, t) &+ \epsilon \partial_x \phi = 0, \\ \phi(x, M, \gamma, g, n, t) &= v_\rho(n, t) f(x, M, \gamma, g, n, t), \\ v_\rho(n, t) &= \frac{1}{g\bar{f}(0)} \sum_m w(n, m) \rho(m, t). \end{aligned}$$

In this case the time to upgrade from chip n to chip $n+1$ will depend on the distribution of the number of chips at the higher level $m > n+1$. The more agents own chips at higher levels, the faster losses pile up at level n and hence the faster an agent upgrades to a new chip. As a consequence, agents that have bought early have a longer time between purchases than agents that bought late. Assuming a buying policy $B(M, m, n)$ that buys the best chip on the market, agents' buying behavior will become periodic and synchronize. The period will be of order $T = O(T_G)$ and they will buy the best chip on the market.

5. **Numerical simulations.**

Since Equations (9) and (11) are strictly hyperbolic, we simulate them with a simple upwind scheme [9]. In the player's equation (Eq. 9) this imposes a time dependent CFL condition

$$\frac{qT \max(v_\rho(n,t))}{G} \frac{\Delta t_f}{\Delta x} + \frac{T\Gamma \max(\gamma)}{M} \frac{\Delta t_f}{\Delta \mu} \leq 1$$

where Δt_f denotes the time step for the player's equation and Δx and $\Delta \mu$ are the discretizations of the loss variable x and the capital variable μ, respectively. Since the waiting bin equation (Eq. 11) is coupled through inflow and outflow with the player's equation, we use a time step $\Delta t_u = a \Delta t_f$ for the bin equation where $a \in \mathbb{Z}^+$ is chosen such that the CFL condition for the bin equation is satisfied

$$\frac{T\Gamma \max(\gamma)}{M} \frac{\Delta t_u}{\Delta \mu} = a \frac{T\Gamma \max(\gamma)}{M} \frac{\Delta t_f}{\Delta \mu} \leq 1.$$

Note that for each time step taken in the bin equation, a time steps are required for the player's equation. The CPU time for a full simulation is about 600 seconds with an Intel Core Duo i5-460M with 8GB of RAM.

We assume that at any given time there are four chips on the market with pricing $c_3 = 1, c_2 = 0.9, c_1 = .8$ and $c_0 = .7$, indicating pricing discounts according to the age of the chip, the oldest being the cheapest. At the time when a new chip is introduced, the prices for the old chips are adjusted downwards and the oldest chip leaves the market and is not available any more.

Whenever an agent reaches the threshold $x = 1$, a decision is made on buying a chip: Agents that have not enough money to upgrade will go into the waiting bin, agents that have enough money to upgrade by one chip generation will do so with 30% probability and go into the bin with 70% probability, agents that can upgrade two generations will do so with 70% probability and agents that have enough money to buy the most expensive chip on the market will do so with probability one. For agents in the waiting bin the same probabilities hold - when they do not buy, they continue to stay in the bin waiting for the accumulation of money.

The win-lose probability matrix for the four chips is given by

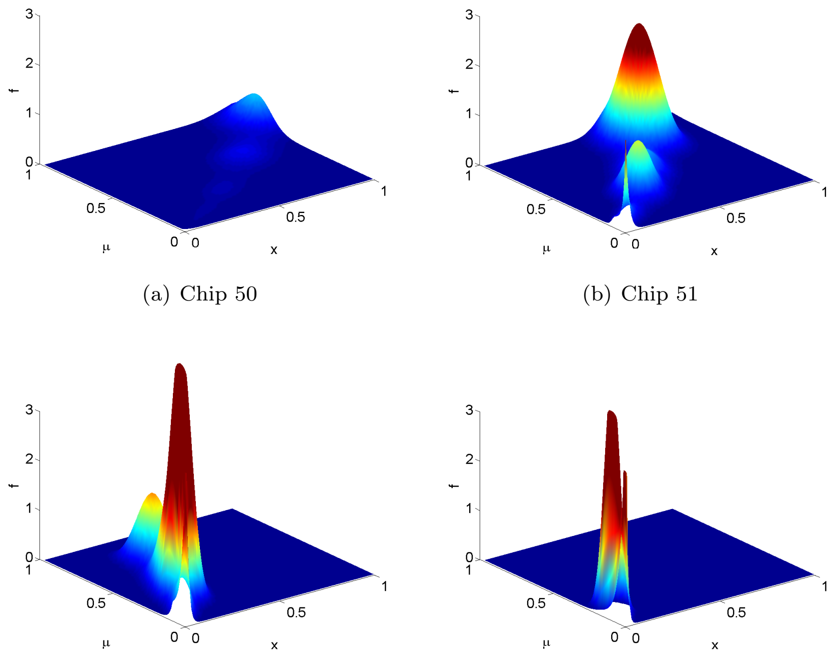
$$W = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} & \frac{7}{8} & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & \frac{7}{8} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \tag{14}$$

where $W_{i,j}$ denotes the probability that an agent with chip i will lose against an agent with chip j. In order to reduce the computational effort we fixed the values for g, γ and q to one and study the time evolution of the density function f in the space of losses and capital, i.e. $f(x, \mu)$.

The first major result of our numerical investigations is that for all parameter values we chose and for a random initial distribution (see Figure 3), we always get a coherent and smooth distribution as the long time behavior. Depending on the different timescales of the system, we find that at any given time we have between 1 and 4 chips that are owned by agents. Figure 4 shows densities in chip number 50 - 53. All the densities are smooth surfaces in (x, μ) space with at most 2 maxima. Close inspection of the surfaces in Figure 4 shows that the newest chip seems to move on a characteristic that has the slowest velocity in the x direction among the four chips, consistent with the fact that this chip is the best in the game and hence agents who have that chip will have fewer losses than agents with lower numbered chips.

A different view of the long term behavior is illustrated in Figure 5. We are sampling the density distributions for each chip at the time when the next chip is released. We find that the distributions are multimodal and identical, indicating periodic behavior driven by the periodic release times.

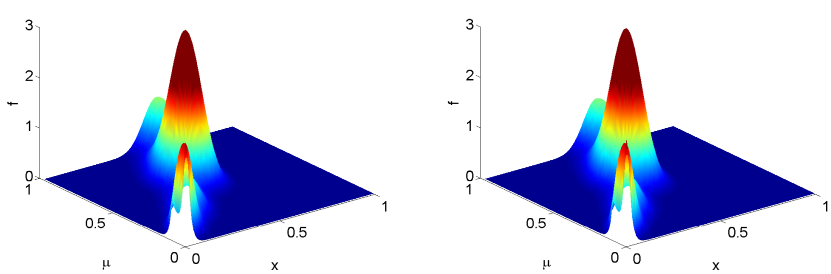
The next simulation illustrates the discussion about the asymptotic behavior for fast financial timescales. Assuming $T_F << T_R < T_G$ we find that all gamers very quickly end up with maximal capital $\mu = M$ and then evolve in loss space. At the time of the release time for the next chip number n, $t = T_{R_n}$ we can then register the distribution in x. Figure 6 shows these distributions at subsequent release times, suggesting a contraction whose fixed point will be the long term distribution similar to the one seen in Figure 4(b).



(a) Chip 50 (b) Chip 51

(c) Chip 52 (d) Chip 53

Figure 4. Density distributions for four simultaneously used chips.



(a) Chip 100 (b) Chip 103

Figure 5. Identical density distributions for all four simultaneously used chips. The figure shows the distribution for chip $n = 100$ at the release time for chip $n = 101$ and for $n = 103$ at the release time for chip $n = 104$.

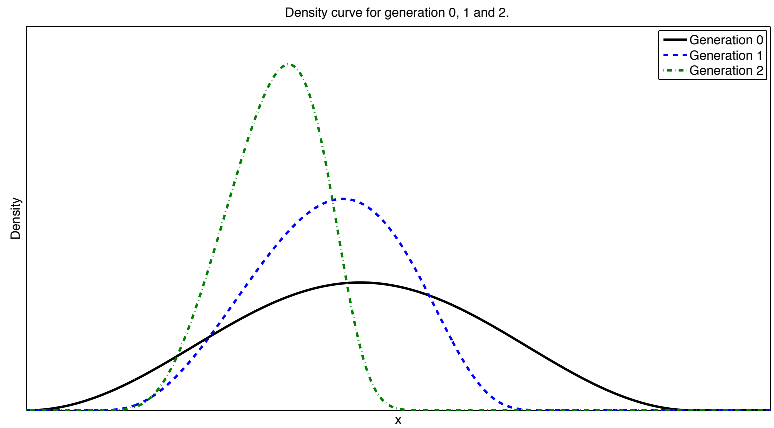
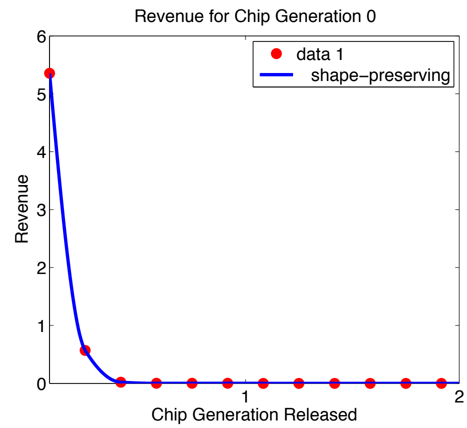


FIGURE 6. Density distribution in x at the introduction of subsequent new generations of chips.

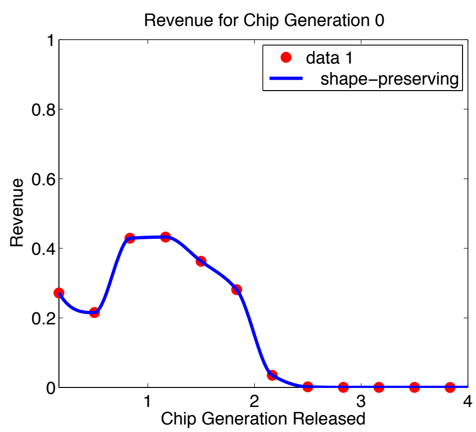
Figure 7 shows three typical sales curves: In all cases the gamer time scale is the fastest, i.e. the agents are waiting for a new chip to be released and for their budget to increase to the price of the new chip. In a) the financial time scale is approximately the release time scale. Hence everybody is ready and has the money to buy and therefore they will all buy the new chip. In b) the financial time scale is twice the release time scale. Hence, while everybody wants to buy, not everybody can afford the newest and most expensive chip; some will opt for the second to last release chip. As a result, the chip with label 0 will have initial sales in the time period 1 and even more sales in the time period 2 before the sales decline. Finally, in the third panel, we have that the financial time scale is four times the release time scale. Hence when an agent has financially recovered, there are four possible chips that she can buy on the market. As a result, the sales for the chip are slow and will last for a long time.

Note that the actual simulation data are the red points representing sales per time unit and the curves in Figure 7 are just possible sales curves that are consistent with these data points. Obviously the PDE simulation will allow for a much finer resolution of the sales data. However, the actual data that are typically available for real sales (cf. Figure 2) are coarse and on a comparable time scale to the ones presented in Figure 7. It is instructive to compare the sales curves in Figure 7 with the Bass model [3] or with some of the data and agent based simulations for the high-end gamers. In the Bass model, if the number of innovators is large, the maximum of the sales rate peak occurs at the release of the new product - the product is so compelling that everybody will buy it instantly. The Bass model will always have a single sales peak but the length of the tail can be adjusted by varying the model parameters.

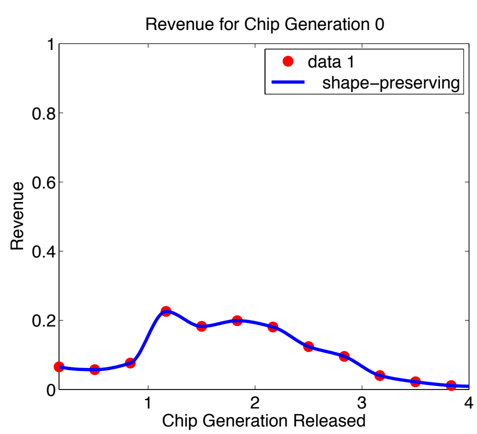
6. **Conclusions.** We have developed a kinetic model for a specific agent based simulation that has been used and parametrized to model the sales curves of successive generations of high-end computer chips. The continuum model equations are transport equations in two variables, representing the ability to buy, i.e. the availability of money, and the inclination to buy, i.e. the frustration with the current computer



(a) fast sales



(b) medium sales



(c) slow sales

FIGURE 7. Varying the financial timescale keeping $T_G = 0.2, T_R = 0.5$ and a) $T_F = 0.585$, b) $T_F = 1$ and c) $T_F = 2$.

chip. In lieu of typical collision terms in the kinetic equations that discontinuously change the attributes of an agent, we developed a model where the discontinuous changes are moderated via boundary conditions between multiple sets of partial differential equations.

While the details of the model are clearly tied to the semiconductor chip market, we believe that our kinetic model can be adjusted to become a fairly generic market model. Abstractly, our model deals with the interplay between market expectations, the time scale of innovation of multigenerational products and the financial time scale of capital increases. Specifically

- A market for multiple generations of a particular product is characterized by the fact that random individuals interact with each other and with the product.
- The interaction is either satisfactory (in our case, the agent wins) or not (the agent loses).
- Accumulated un-satisfactory use of a product leads to the desire for change.
- A fast innovation time scale generates expectations for the use of the product that increase with time, leading to an acceleration of the desire for change. This point is related to the observation of prospect theory [13] that the utility of a product depends on the reference point that a potential buyer is using. The utility of the old product decreases since innovation shifts the reference point.
- The financial time scale is characterized by the ability of a participant in the market to generate the necessary funds to act on his desire for change and serves as a constraint on the whole system.

Arguably this scenario is a caricature of the market for cars, TVs, smartphones and other technological gadgets.

We have discussed one particular feature of kinetic equations that makes them attractive to analyze agent based simulations: Those simulations typically have a large number of parameters characterizing the simulated scenarios and the particular agents that are observed. The interaction between these parameters is often very difficult to discern based on agent simulations alone. Reducing the kinetic equations to a non-dimensional form typically generates dimensionless parameters based on the characteristic scales in the system. Those dimensionless parameters become the control parameters of the system and their relationship to the original microscopic parameters is an important feature of the market mechanisms that are analyzed.

For the current model, the shape and timing of the sales curves is the major observable of interest. Case studies for limit scenarios show coherent and self-consistent results for the time evolution of those sales and allow us to easily generate typically observed sales curves.

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