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# STABILITY OF CONDUCTIVITIES IN AN INVERSE PROBLEM IN THE REACTION-DIFFUSION SYSTEM IN ELECTROCARDIOLOGY

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Abstract. In this paper, we study the stability result for the conductivities diffusion coefficients to a strongly reaction-diffusion system modeling electrical activity in the heart. To study the problem, we establish a Carleman estimate for our system. The proof is based on the combination of a Carleman estimate and certain weight energy estimates for parabolic systems.

1. **Introduction.** Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and $\omega$ be a small nonempty subset of $\Omega$. We will denote $(0,T) \times \Omega$ by $Q_T$ and $(0,T) \times \partial\Omega$ by $\Sigma_T$.

To state the model of the cardiac electric activity in $\Omega$ ($\Omega \subset \mathbb{R}^3$ being the natural domain of the heart), we set $u_i = u_i(t,x)$ and $u_e = u_e(t,x)$ to represent the spacial cellular and location $x \in \Omega$ of the intracellular and extracellular electric potentials respectively. Their difference $v = u_i - u_e$ is the transmembrane potential. The anisotropic properties of the two media are modeled by intracellular and extracellular conductivity tensors $M_i(x)$ and $M_e(x)$. The surface capacitance of the membrane is represented by the constant $c_m > 0$. The transmembrane ionic current is represented by a nonlinear function $h(v)$.

The equations governing the cardiac electric activity are given by the coupled reaction-diffusion system:

$$\begin{cases} c_m \partial_t v - div(M_i(x)\nabla u_i) + h(v) = f\chi_\omega, \ in \ Q_T, \\ c_m \partial_t v + div(M_e(x)\nabla u_e) + h(v) = g\chi_\omega, \ in \ Q_T, \end{cases} \tag{1}$$

where $f$ and $g$ are stimulation currents applied to $\Omega$. We complete this model with Dirichlet boundary conditions for the intra- and extracellular electric potentials

$$u_i = 0, \ u_e = 0, \ on \ \Sigma_T, \tag{2}$$

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and with initial data for the transmembrane potential

$$v(0,x) = v_0(x),\ x \in \Omega. \tag{3}$$

It is important to point out that realistic models describing electrical activities include a system of ODEs for computing the ionic current as a function of the transmembrane potential and a series of additional "gating variables", which aim to model the ionic transfer across the cell membrane.

Assume that the intra and extracellular stimulations are equal: $f\chi_\omega = g\chi_\omega$. If $M_i = \mu M_e$ for some constant $\mu \in \mathbb{R}$, then by multiplying the second equation in (1) by $\mu$ and adding it to the first equation in (1) one gets the first equation in the following parabolic-elliptic system:

$$\begin{cases} c_m\partial_t v - \frac{\mu}{\mu+1} div(M_e(x)\nabla v) = -h(v) + f\chi_\omega,\ in\ Q_T, \\ div(M(x)\nabla u_e) = div(M_i(x)\nabla v),\ in\ Q_T, \\ v(0,x) = v_0(x),\ u_e(0,x) = u_{e,0}(x),\ in\ \Omega, \\ v = 0,\ u_e = 0,\ on\ \Sigma_T. \end{cases} \tag{4}$$

The second equation is obtained by computing the difference of the two equation in (1). Here $M = M_i + M_e$. System (4) is known as the monodomain model.

We approximate the above model (4) by the following family of parabolic equations

$$\begin{cases} c_m\partial_t v^\varepsilon - \frac{\mu}{\mu+1} div(M_e(x)\nabla v^\varepsilon) = -h(v^\varepsilon) + f^\varepsilon\chi_\omega,\ in\ Q_T, \\ \varepsilon\partial_t u_e^\varepsilon - div(M(x)\nabla u_e^\varepsilon) = div(M_i(x)\nabla v^\varepsilon),\ in\ Q_T, \\ v^\varepsilon(0,x) = v_0(x),\ u_e^\varepsilon(0,x) = u_{e,0}(x),\ in\ \Omega, \\ v^\varepsilon = 0,\ u_e^\varepsilon = 0,\ on\ \Sigma_T, \end{cases} \tag{5}$$

$\varepsilon$ is a fixed small constant. Since $v = u_i - u_e$ in the bidomain model, it is natural decompose the initial condition $v_0$ as $v_0 = u_{i,0} - u_{e,0}$. Note that when $\varepsilon \to 0$ in (5), we obtain the classical monodomain model.

In this work, we study the stability result for the conductivities diffusion coefficients to the following linearized system of (5) with semi-initial conditions

$$\begin{cases} c_m\partial_t v^\varepsilon - \frac{\mu}{\mu+1} div(M_e(x)\nabla v^\varepsilon) = -a(t,x)v^\varepsilon + f^\varepsilon\chi_\omega,\ in\ Q_T, \\ \varepsilon\partial_t u_e^\varepsilon - div(M(x)\nabla u_e^\varepsilon) = div(M_i(x)\nabla v^\varepsilon),\ in\ Q_T, \\ v^\varepsilon(\theta,x) = v_\theta(x),\ u_e^\varepsilon(\theta,x) = u_{e,\theta}(x),\ in\ \Omega, \\ v^\varepsilon = 0,\ u_e^\varepsilon = 0,\ on\ \Sigma_T, \end{cases} \tag{6}$$

where $a(t,x)$ and its derivative with respect to $t$ exists and are bounded in $Q_T$. For some $\theta \in (0,T)$, the semi-initial conditions $v_\theta(x)$, $u_{e,\theta}(x)$ are sufficiently regular. The unknown conductivity tensors $M$ and $M_e$ are assumed to be sufficiently smooth and shall be kept independent of time $t$.

The existence of weak solutions of (1) is proved in [10] by the theory of evolution variational inequalities in Hilbert space. Then Bendahmane and Karlsen [2] proved the existence and uniqueness for a nonlinear version of the bidomain equations (1) by a uniformly parabolic regularization of the system and the Faedo-Galerkin method. Moreover, Bendahmane and Chaves-Silva [1] studied exact null controllability to (1) for each $\varepsilon > 0$ by establishing estimates for its dual system. To learn more about the cardiac problems, one can refer to the work of Bendahmane et al. [3, 4]. However, it is noted that there is no stability results for the inverse bidomain model.

Since the pioneer work du to A.L. Bukhgeim and M.V. Klibanov [6, 7, 8], who generalized the method of global Carleman estimates in the context of inverse problems , three fundamental issues have been successfully studied: uniqueness, stability in determining coefficients, and numerical methods [16], [13, 14, 17, 20, 23, 24, 15]).

The paper by Cristofol et al. [11] obtains the stability results for reaction-diffusion system of two equations with constant coefficients using a Carleman estimate. Then Sakthivel et al. [21] established the stability results for Lotka-Volterra competition-diffusion system of three equations with variable diffusion coefficients. Our inverse stability results are new because system (6) contains a strong coupling term. The technics we shall discuss are similar to the framework using Carleman estimates for inverse problems but the obtained estimates differs from those of [24], [21] because of the strongly coupled terms.

Let $(\tilde{v}^\varepsilon, \tilde{u}_e^\varepsilon)$ be a solution of system (6) with conductivity tensors $(\tilde{M}_e, \tilde{M})$ and semi-initial data $(\tilde{v}_\theta^\varepsilon, \tilde{u}_{e,\theta}^\varepsilon)$. Then setting $A_1 = v^\varepsilon - \tilde{v}^\varepsilon$, $A_2 = u_e^\varepsilon - \tilde{u}_e^\varepsilon$, $g_1 = M_e - \tilde{M}_e$ and $g_2 = M - \tilde{M}$, we obtain

$$\begin{cases} c_m \partial_t A_1 - \frac{\mu}{\mu+1} div(M_e(x)\nabla A_1(t,x)) = -a(t,x)A_1(t,x) + F(g_1, \nabla\tilde{v}^\varepsilon), \ in\ Q, \\ \varepsilon\partial_t A_2 - div(M(x)\nabla A_2) = div(M_i(x)\nabla A_1) + G(g_2, \nabla u_e^\varepsilon), \ in\ Q, \\ A_1(\theta,x) = A_1^\theta(x), \ A_2(\theta,x) = A_2^\theta(x), \ in\ \Omega, \\ A_1(t,x) = 0, \ A_2(t,x) = 0, \ on\ \Sigma, \end{cases} \tag{7}$$

where

$$F = \frac{\mu}{\mu+1} div(g_1(x)\nabla\tilde{v}^\varepsilon)$$

and

$$G = div(g_2(x)\nabla\tilde{u}_e^\varepsilon).$$

Throughout the paper, we make the following assumptions:

**Assumption 1.1.** *The conductivity tensors $M_e(x)$, $M_i(x)$ and $M(x)$ are $C^\infty$, bounded, symmetric, semi-definite, and elliptic matrixes (there exists $\beta > 0$ such that $\Sigma_{i,j}^3 M_{i,j}\xi_i\xi_j \geq \beta|\xi|^2$ for all $\xi \in \mathbb{R}^3$). All their derivatives up to the third order are respectively bounded by the positive constants $\gamma_1$, $\gamma_2$, $\gamma_3$.*

**Assumption 1.2.** *Assume the bounded measurements $\partial_t A_1$ and $\partial_t A_2$ in $(0,T)\times\omega$ are given. Also $A_i(\theta,x)$, $\nabla A_i(\theta,x)$, $\Delta A_i(\theta,x)$ and $\nabla(\Delta A_i(\theta,x))$ for some fixed $\theta \in (0,T)$, where $i = 1,2$ in $\Omega$ are given.*

Now the question of interest is whether we can determine the conductivity tensors $M_e$ and $M$ by the two measurements.

In details, let $(v^\varepsilon, u_e^\varepsilon)$ and $(\tilde{v}^\varepsilon, \tilde{u}_e^\varepsilon)$ be the solutions of the system (6) with two different conductivities. There exist a constant $C$ with $C(\Omega, \omega, T, \gamma_1, \gamma_2, \gamma_3) > 0$, such that the following estimate holds:

$$\begin{aligned} &\int_\Omega \Big(|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2\Big)dx \\ &\leq C\bigg(\int_{Q_\omega}\Big(|\partial_t A_1|^2 + |\partial_t A_2|^2\Big)dtdx + \int_\Omega |A_1^\theta|^2 + \sum_{j=1}^2\Big(|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2\Big)dx\bigg) \\ &\quad + C\int_{\tilde{\omega}}\Big(|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2\Big)dx. \end{aligned} \tag{8}$$

## 2. **A Carleman type estimate.**

In this section, we prove the Carleman estimate based on the standard technique for general parabolic equations. In order to frame a Carleman type estimate, we shall first introduce a particular type of weight functions.

2.1. **Weight functions.** First, we introduce weight functions for the parabolic equations given in [12].

Let $\tilde{\omega} \subset\subset \omega$ be a nonempty bounded set of $\Omega$, and $\psi \in C^2(\bar{\Omega})$ such that

$$\psi(x) > 0, \ for\ any\ x \in \Omega,$$

$$\psi(x) = 0, \ for\ any\ x \in \partial\Omega,$$

$$|\nabla\psi(x)| > 0, \ for\ any\ x \in \bar{\Omega} \setminus \tilde{\omega}.$$

Then we introduce another two weight functions:

$$\phi(t,x) = \frac{e^{\lambda\psi(x)}}{\beta(t)}, \tag{9}$$

$$\alpha(t,x) = \frac{e^{2\lambda\|\psi\|_{C(\bar{\Omega})}} - e^{\lambda\psi(x)}}{\beta(t)}, \tag{10}$$

where $\lambda > 1$, $t \in (0,T)$ and $\beta(t) = t(T-t)$. Note that the weight function $\alpha$ is positive, and blows up to $\infty$ as $t = 0$ or $t = T$. As a result, $e^{-2s\alpha}$ and $\phi e^{-2s\alpha}$ are smooth. Even they vanish when $t = 0$ or $t = T$. It can be seen that $\phi(t,x) \geq C > 0$ for all $(t,x) \in Q$, and $e^{-\epsilon\alpha}\phi^m \leq C < \infty$ for all $\epsilon > 0$ and $m \in \mathbb{R}$.

Before proving the main estimate, we give the following estimates for the two weight functions $\alpha$ and $\phi$. Note that throughout the paper we will denote $C$ as a generic positive constant. After some computations, we can obtain the following estimates:

$$\begin{cases} |\phi_t| = \frac{|2t-T|}{e^{\lambda\psi}}\phi^2 \leq CT\phi^2, \\ |\alpha_t| = \frac{|2t-T|}{\beta^2}(e^{2\lambda\|\psi\|_{C(\bar{\Omega})}} - e^{\lambda\psi}) \leq CT\phi^2, \\ |\alpha_{tt}| = \frac{2|T^2-3tT+3t^2|}{\beta^3}(e^{2\lambda\|\psi\|_{C(\bar{\Omega})}} - e^{\lambda\psi}) \leq CT\phi^3. \end{cases} \tag{11}$$

Furthermore, we also have

$$\begin{cases} \nabla\phi = \lambda\phi\nabla\psi, \\ \nabla\alpha = -\lambda\phi\nabla\psi, \\ \phi^{-1} \leq (\frac{T}{2})^2. \end{cases} \tag{12}$$

Refer to [12] for the details.

2.2. **Main proof of a Carleman type estimate.** Let us set $Q_\omega = (0,T)\times\omega$. For each positive integer $m$, we denote the Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives of order less than or equal to $m$ are also in $L^p(\Omega)$ with the norm denoted $\|\cdot\|_{L^p(\Omega)}$, by $W^{m,p}(\Omega)$ with $p > 1$ or $p = \infty$. When $p = 2$, we denote $W^{m,p}$ by $H^m(\Omega)$. Moreover, let $L^2(0,T;H^1(\Omega))$ be the space of all equivalent classes of square integrable functions from $(0,T)$ to $H^1(\Omega)$. For the space $L^2(0,T;L^\infty(\Omega))$, we define it in the same way.

Let $A_1$ be the solution of the first equation of (7) with help of using Assumption 1.1. We apply the Carleman estimate (see Theorem 6.1 in [1].) derived for the parabolic equations to the first equation in (7). For $\lambda > \lambda_0 \geq 1$, $s \leq s_0(T+T^2+T^4)$, there exists a constant $C$ depending on $\Omega$, $\omega$, $\psi$ and $\beta$ so that

$$\mathcal{I}(A_1) \leq C\Big(\int_Q e^{-2s\alpha}|F|^2dtdx + s^3\lambda^4\int_{Q_{\omega_1}}\phi^3e^{-2s\alpha}|A_1|^2dtdx\Big), \tag{13}$$

where $\tilde{\omega} \subset\subset \omega_1 \subset\subset \omega$, and

$$\mathcal{I}(A_1) = \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha}(|\partial_t A_1| + |\Delta A_1|^2) dtdx + \int_Q s\lambda^2 \phi e^{-2s\alpha} |\nabla A_1|^2 dtdx \\ + s^3\lambda^4 \int_Q e^{-2s\alpha}\phi^3 |A_1|^2 dtdx. \tag{14}$$

Similarly, for $\lambda > \lambda_0 \geq 1$, $s \geq s_0(T + T^2 + T^4)$, there exists a constant $C$ depending on $\Omega$, $\omega$, $\psi$ and $\beta$ satisfying

$$\begin{aligned} &\mathcal{I}(A_2) \\ \leq & C\left(\int_Q e^{-2s\alpha}\Big(|G|^2 |\nabla(M_i \nabla A_1)|^2\Big) dtdx + s^3\lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dtdx\right) \\ \leq & C\Big(\int_Q e^{-2s\alpha}|G|^2 dtdx + s^3\lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha}|A_1|^2 dtdx + \int_Q e^{-2s\alpha}|M_i \Delta A_1|^2 dtdx\Big) \\ & + C\int_Q e^{-2s\alpha} |\nabla M_i \nabla A_1|^2 dtdx, \end{aligned} \tag{15}$$

with

$$\begin{aligned} \mathcal{I}(A_2) \quad = \quad & \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha}\Big(|\partial_t A_2| + |\Delta A_2|^2\Big) dtdx + \int_Q s\lambda^2 \phi e^{-2s\alpha}|\nabla A_2|^2 dtdx \\ & + s^3\lambda^4 \int_Q e^{-2s\alpha}\phi^3 |A_2|^2 dtdx. \end{aligned} \tag{16}$$

Now coupling the above inequalities (13) and (15), we have

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) \quad \leq \quad & C\Big(\int_Q e^{-2s\alpha}(s|F|^2 + |G|^2) dtdx + s^4\lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha}|A_1|^2 dtdx \\ & + s^3\lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha}|A_2|^2 dtdx\Big) + C\int_Q e^{-2s\alpha}|M_i\Delta A_1|^2 dtdx \\ & + C\int_Q e^{-2s\alpha}|\nabla M_i \nabla A_1|^2 dtdx \end{aligned}$$

for sufficiently large $s \geq s_0(T + T^2 + T^4)$ and $\lambda \geq \lambda_0$. From the definition of $\mathcal{I}_1$, also $M_i$ and $\nabla M_i$ being bounded, we obtain

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) \quad \leq \quad & \tilde{C}\Big(\int_Q e^{-2s\alpha}(s|F|^2 + |G|^2) dtdx + s^4\lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha}|A_1|^2 dtdx \\ & + s^3\lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha}|A_2|^2 dtdx\Big). \end{aligned} \tag{17}$$

Then it can be summarized as our desired Carleman estimate as follows.

**Theorem 2.1.** *Let $\psi(x)$, $\phi(t,x)$ and $\alpha(t,x)$ be defined as in the above subsection, $a(t,x)$ is a bounded function. Moreover, Assumption 1.1 holds. Then there exist $\lambda_0$ and $s_0$ such that for all $\lambda > \lambda_0 \geq 1$ and sufficiently large enough $s > s_0$, the following inequality is true.*

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) \quad \leq \quad & \tilde{C}\Big(\int_Q e^{-2s\alpha}(s|F|^2 + |G|^2) dtdx + s^4\lambda^4 \int_{Q_{\omega}} \phi^3 e^{-2s\alpha}|A_1|^2 dtdx \\ & + s^3\lambda^4 \int_{Q_{\omega}} \phi^3 e^{-2s\alpha}|A_2|^2 dtdx\Big), \end{aligned}$$

*where $\tilde{C} > 0$ is a constant depending on $\Omega$, $T$, $\omega$, $\gamma_2$.*

3. **Stability of the conductivities.** In this section, we study the stability of the conductivity tensors $M_e$ and $M$. Then an inequality is established which estimates $g_1$, $g_2$, $\nabla g_1$, $\nabla g_2$ with an upper bound given by some Sobolev norms of the derivative of $A_1$ and $A_2$ over $Q_\omega$, certain spatial derivative of $A_j(\theta, \cdot)$, $j = 1, 2$, where $\theta \in (0, T)$ makes $\frac{1}{\beta(t)}$ attain its minimum value and the Sobolev norm of $g_1$, $g_2$, $\nabla g_1$, $\nabla g_2$ in a small space $\tilde{\omega}$.

First, we let $B_1 = \partial_t A_1$, $B_2 = \partial_t A_2$. Using this and (7), we get the following system:

$$\begin{cases} c_m \partial_t B_1 - \frac{\mu}{\mu+1} div(M_e(x)\nabla B_1(t,x)) = -\partial_t a(t,x) A_1(t,x) - a(t,x) B_1 \\ +F'(g_1, \nabla \tilde{v}^\varepsilon), \ in \ Q_T, \\ \varepsilon \partial_t B_2 - div(M(x)\nabla B_2) = div(M_i(x)\nabla B_1) + G'(g_2, \nabla u_e^\varepsilon), \ in \ Q_T, \\ c_m B_1(\theta, x) = H_1^\theta(x), \ B_2(\theta, x) = H_2^\theta(x), \ in \ \Omega, \\ B_1(t,x) = 0, \ B_2(t,x) = 0, \ on \ \Sigma_T, \end{cases} \tag{18}$$

where

$$F' = \frac{\mu}{\mu+1} div(g_1(x)\nabla(\partial_t \tilde{v}^\varepsilon)), \ G' = div(g_2(x)\nabla(\partial_t \tilde{u}_e^\varepsilon))$$

and

$$\begin{cases} c_m B_1(\theta, x) = \frac{\mu}{\mu+1} div(M_e(x)\nabla A_1(\theta, x)) - a(\theta, x) A_1(\theta, x) + F|_{t=\theta} = H_1^\theta, \ in \ Q_T, \\ \varepsilon B_2(\theta, x) = div(M(x)\nabla A_2(\theta, x)) + div(M_i(x)\nabla A_1(\theta, x)) + G|_{t=\theta} = H_2^\theta, \ in \ Q_T, \\ A_1(t,x) = A_1(0,x) + \int_0^t B_1(s,x) ds, \ in \ Q_T. \end{cases} \tag{19}$$

Indeed, to prove the main result here we need to impose some regularity properties as follows.

**Assumption 3.1.** *Suppose $v_\theta^\varepsilon$ and $u_{e,\theta}^\varepsilon$ are $C^3$ real valued functions. Then all their derivatives up to order three are bounded and satisfy $|\nabla\psi \cdot \nabla v_\theta^\varepsilon| \geq \delta > 0$, $|\nabla\psi \cdot \nabla u_{e,\theta}^\varepsilon| \geq \delta > 0$, on $\overline{\Omega \setminus \tilde{\omega}}$, where $\tilde{\omega} \subset\subset \omega \subset\subset \Omega$.*

**Assumption 3.2.** *Suppose $(|\Delta \tilde{v}^\varepsilon|, |\Delta \tilde{u}_e^\varepsilon|)$, $(|\nabla(\Delta \tilde{v}^\varepsilon)|, |\nabla(\Delta \tilde{u}_e^\varepsilon)|)$, $(|\nabla(\partial_t \tilde{v}^\varepsilon)|, |\nabla(\partial_t \tilde{u}_e^\varepsilon)|)$ and $(|\Delta(\partial_t \tilde{v}^\varepsilon)|, |\Delta(\partial_t \tilde{u}_e^\varepsilon)|)$ are bounded by a positive constant.*

Before start proving our main conclusion, we need to give the following Lemma 3.3, which will be useful in the following part. We define the following operators $P_0$ and $Q_0$ and the initial conditions on $\alpha$ and $\phi$ at $t = \theta$:

$$P_0 h = \nabla U_\theta \cdot \nabla h, \ Q_0(e^{-s\alpha^\theta} h) = e^{-s\alpha^\theta} P_0 h \text{ and } \varsigma(\theta, x) = \varsigma^\theta \text{ for } \varsigma = \alpha, \phi.$$

**Lemma 3.3.** *Consider the first order partial differential operator $P_0 h = \nabla U_\theta \cdot \nabla h$, where $U_\theta$ satisfies Assumption 3.1. Then there exists a constant $C > 0$, such that for sufficiently large enough $\lambda$ and $s$, the following result holds:*

$$s^2\lambda^2 \int_\Omega \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx \leq C\Big(\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0 h|^2 dx + s^2\lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx\Big),$$

*with $\theta \in (0, T)$ and $h \in H_0^1(\Omega)$.*

*Proof.* Let $B_1 = e^{-s\alpha^\theta} h$, we have

$$Q_0 B_1 = e^{-s\alpha^\theta} P_0(e^{s\alpha^\theta} B_1) = P_0 B_1 + s B_1 P_0 \alpha^\theta, \tag{20}$$

$h \in H_0^1(\Omega)$. Then we take the square of both sides in (20), multiply $\frac{1}{\phi^\theta}$ and integrate by parts with respect to space variable for both sides of (20) as follows:

$$
\begin{aligned}
& \int_\Omega \frac{1}{\phi^\theta}(Q_0 B_1)^2 dx \\
= & \int_\Omega \frac{1}{\phi^\theta}(P_0 B_1)^2 dx + \int_\Omega s^2 B_1^2 \frac{1}{\phi^\theta}(P_0 \alpha^\theta)^2 dx \\
& + \int_\Omega 2s \frac{1}{\phi^\theta} B_1 (P_0 B_1)(P_0 \alpha^\theta) dx \\
= & \int_\Omega \frac{1}{\phi^\theta}(P_0 B_1)^2 dx + \int_\Omega s^2 B_1^2 \frac{1}{\phi^\theta} \lambda^2 (\phi^\theta)^2 (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
& - \int_\Omega 2\lambda s \frac{1}{\phi^\theta} B_1 (\nabla U^\theta \cdot \nabla B_1)(\nabla U^\theta \cdot \phi^\theta \nabla \psi^\theta) dx \\
= & \int_\Omega \frac{1}{\phi^\theta}(P_0 B_1)^2 dx + \int_\Omega s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
& - \int_\Omega 2\lambda s B_1 (\nabla U^\theta \cdot \nabla B_1)(\nabla U^\theta \cdot \nabla \psi^\theta) dx \\
= & \int_\Omega \frac{1}{\phi^\theta}(P_0 B_1)^2 dx + \int_\Omega s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
& - \int_\Omega \lambda s P_0 \psi^\theta \nabla U^\theta \nabla (B_1^2) dx \\
= & \int_\Omega \frac{1}{\phi^\theta}(P_0 B_1)^2 dx + \int_\Omega s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
& + \int_\Omega \lambda s \nabla (P_0 \psi^\theta \nabla U^\theta) B_1^2 dx \\
\geq & \int_\Omega s^2 \lambda^2 B_1^2 \phi^\theta |\nabla U^\theta \cdot \nabla \psi^\theta|^2 dx + \int_\Omega s\lambda \nabla (P_0 \psi^\theta \nabla U^\theta) |B_1|^2 dx \\
\geq & \ s^2 \lambda^2 \delta^2 (\int_\Omega |B_1|^2 \phi^\theta dx - \int_{\tilde\omega} |B_1|^2 \phi^\theta dx) + \int_\Omega s\lambda \nabla (P_0 \psi^\theta \nabla U^\theta) |B_1|^2 dx,
\end{aligned}
$$

where we used Assumption 3.1 in the last step. Thus we obtain

$$
\begin{aligned}
& s^2 \lambda^2 \delta^2 \Big( \int_\Omega |B_1|^2 \phi^\theta dx - \int_{\tilde\omega} |B_1|^2 \phi^\theta dx \Big) \\
\leq & \int_\Omega \frac{1}{\phi^\theta} |Q_0 B_1|^2 dx + \int_\Omega s\lambda |\nabla (P_0 \psi^\theta \nabla U^\theta)| |B_1|^2 dx.
\end{aligned}
$$

From Assumption 3.2 and (12), we have

$$
\begin{aligned}
& s^2 \lambda^2 \delta^2 \int_\Omega e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx \\
\leq & \int_{\tilde\omega} s^2 \lambda^2 \delta^2 e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx + C_1 T^2 \int_\Omega s\lambda e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx \\
& + \int_\Omega e^{-2s\alpha^\theta} |P_0 h|^2 \frac{1}{\phi^\theta} dx. \qquad (21)
\end{aligned}
$$

Taking $\lambda \geq 1$ and $s \geq \frac{2C_1T^2}{\delta^2}$, we conclude the proof. □

With the help of the Lemma 3.3, we are proving the following proposition.

**Proposition 1.** *Let $(A_1, A_2)$ be the solution of (7), and $(B_1, B_2)$ be the solution of (18). Suppose all the conditions of Theorem 2.1 and Assumption 3.1 hold. Then there exists a constant $C = C(\gamma_1, \gamma_2, \delta) > 0$ such that for sufficiently large enough $s$ and $\lambda$ the following estimate is true.*

$$\begin{aligned} & s^2\lambda^2 \int_\Omega e^{-2s\alpha^\theta} \Big(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2\Big) dx \\ \leq \quad & C \sum_{j=1}^{9} E_j + C s^2 \lambda^2 \int_{\tilde{\omega}} e^{-2s\alpha^\theta} \Big(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2\Big) dx, \end{aligned} \tag{22}$$

*for any $g_1, g_2 \in H_0^2(\Omega)$, where the functions $E_j$, are given as follows:*

$$\begin{aligned} E_1 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_1^\theta|^2 dx, \\ E_2 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_2^\theta|^2 dx, \\ E_3 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_1^\theta|^2 dx, \\ E_4 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_2^\theta|^2 dx, \\ E_5 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \Big(|A_1^\theta|^2 + |\nabla A_1^\theta|^2 + |\Delta A_1^\theta|^2\Big) dx, \\ E_6 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \Big(|\nabla A_2^\theta|^2 + |\Delta A_2^\theta|^2\Big) dx, \\ E_7 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \Bigg(|A_1^\theta|^2 + \sum_{j=1}^{2} \Big(|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2\Big) \\ &\qquad + |\nabla(g_1 \Delta \tilde{v}_\theta^\varepsilon)|^2 + |\nabla(g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon)|\Bigg) dx, \\ E_8 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_1 \Delta \tilde{v}_\theta^\varepsilon|^2 dx, \\ E_9 &= \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon|^2 dx. \end{aligned}$$

*Proof.* Due to the value of the solutions satisfying the first equation in (18) at $t = \theta$, and $F = div(g_1(x)\nabla \tilde{v}^\varepsilon)$, from (19) we obtain

$$P_0 g_1 = \nabla \tilde{v}_\theta^\varepsilon \cdot \nabla g_1 = c_m B_1^\theta + a(\theta, x) A_1^\theta - \frac{\mu}{\mu+1} div(M_e \nabla A_1^\theta) - g_1 \Delta \tilde{v}_\theta^\varepsilon.$$

Note that we replace $h$ by $g_1$ when choosing $U_\theta$ as $\tilde{v}_\theta^\varepsilon$. Therefore, inspired by Lemma 3.3, we get

$$\begin{aligned} & s^2\lambda^2 \int_\Omega \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx \\ \leq \quad & C\left(\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0 g_1|^2 dx + s^2\lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx\right) \end{aligned}$$

$$
\begin{aligned}
&\leq C\int_\Omega \frac{1}{\phi^\theta}e^{-2s\alpha^\theta}\Bigg(|B_1^\theta|^2+|A_1^\theta|^2+\Big(\frac{\mu}{\mu+1}\Big)^2\Big(|\nabla M_e|^2|\nabla A_1^\theta|^2\\
&\quad +|M_e|^2|\Delta A_1^\theta|^2\Big)+|g_1\Delta\tilde{v}_\theta^\varepsilon|^2\Bigg)dx+Cs^2\lambda^2\int_{\tilde{\omega}}\phi^\theta e^{-2s\alpha^\theta}|g_1|^2dx\\
&\leq C(\gamma_1)(E_1+E_5+E_8)+Cs^2\lambda^2\int_{\tilde{\omega}}\phi^\theta e^{-2s\alpha^\theta}|g_1|^2dx. \qquad (23)
\end{aligned}
$$

Similarly, from the value of the solutions satisfying the second equation in (18) at $t=\theta$, and $G=div(g_2(x)\nabla\tilde{u}_e^\varepsilon)$, we obtain

$$
P_0g_2=\nabla\tilde{u}_{e,\theta}^\varepsilon\cdot\nabla g_2=\varepsilon B_2^\theta-div(M\nabla A_2^\theta)-div(M_i\nabla A_1^\theta)-g_2\Delta\tilde{u}_{e,\theta}^\varepsilon.
$$

It leads to

$$
\begin{aligned}
&s^2\lambda^2\int_\Omega\phi^\theta e^{-2s\alpha^\theta}|g_2|^2dx\\
&\leq C\Bigg(\int_\Omega\frac{1}{\phi^\theta}e^{-2s\alpha^\theta}|P_0g_2|^2dx+s^2\lambda^2\int_{\tilde{\omega}}\phi^\theta e^{-2s\alpha^\theta}|g_2|^2dx\Bigg)\\
&\leq C\int_\Omega\frac{1}{\phi^\theta}e^{-2s\alpha^\theta}\Big(|B_2^\theta|^2+|\nabla M|^2|\nabla A_2^\theta|^2+|M|^2|\Delta A_2^\theta|^2\\
&\quad +|\nabla M_i|^2|\nabla A_1^\theta|^2+|M_i|^2|\Delta A_1^\theta|^2+|g_2\Delta\tilde{u}_{e,\theta}^\varepsilon|^2\Big)dx\\
&\quad +Cs^2\lambda^2\int_{\tilde{\omega}}\phi^\theta e^{-2s\alpha^\theta}|g_2|^2dx\\
&\leq C(\gamma_2,\gamma_3)(E_2+E_5+E_6+E_9)+Cs^2\lambda^2\int_{\tilde{\omega}}\phi^\theta e^{-2s\alpha^\theta}|g_2|^2dx. \qquad (24)
\end{aligned}
$$

On the other hand, from the expression of $P_0g_1$, we can see that,

$$
\begin{aligned}
P_0\nabla g_1 =&\nabla\tilde{v}_\theta^\varepsilon\cdot\nabla(\nabla g_1)=\nabla(\nabla\tilde{v}_\theta^\varepsilon\cdot\nabla g_1)-\nabla g_1\Delta\tilde{v}_\theta^\varepsilon\\
=&c_m\nabla B_1^\theta+\nabla a^\theta A_1^\theta+a^\theta\nabla A_1^\theta-\frac{\mu}{\mu+1}\Delta(M_e\nabla A_1^\theta)-\nabla(g_1\Delta\tilde{v}_\theta^\varepsilon)-\nabla g_1\Delta\tilde{v}_\theta^\varepsilon.
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
P_0\nabla g_2 &= \nabla\tilde{u}_{e,\theta}^\varepsilon\cdot\nabla(\nabla g_2)=\nabla(\nabla\tilde{u}_{e,\theta}^\varepsilon\cdot\nabla g_2)-\nabla g_2\Delta\tilde{u}_{e,\theta}^\varepsilon\\
&= \varepsilon\nabla B_2^\theta-\Delta(M\nabla A_2^\theta)-\Delta(M_i\nabla A_1^\theta)-\nabla(g_2\Delta\tilde{u}_{e,\theta}^\varepsilon)-\nabla g_2\Delta\tilde{u}_{e,\theta}^\varepsilon.
\end{aligned}
$$

Using the same method to preceding estimates and Lemma 3.3, it follows that

$$
\begin{aligned}
&s^2\lambda^2\int_\Omega\phi^\theta e^{-2s\alpha^\theta}|\nabla g_1|^2dx+s^2\lambda^2\int_\Omega\phi^\theta e^{-2s\alpha^\theta}|\nabla g_2|^2dx\\
\leq& C(\gamma_1,\gamma_2,\gamma_3)(E_3+E_4+E_7)+Cs^2\lambda^2\int_{\tilde{\omega}}\phi^\theta e^{-2s\alpha^\theta}\Big(|\nabla g_1|^2+|\nabla g_2|^2\Big)dx. \qquad (25)
\end{aligned}
$$

Combing the above three estimates (23), (24) and (25), the proof is complete. □

In order to prove the main conclusion, we need to get further estimations for $E_j$, $j=1,2,3,4$. The Carleman estimate in the previous section plays an important role in obtaining these estimations.

**Lemma 3.4.** *Assume all the conditions in Theorem 2.1 are satisfied. Then there exists a constant $C$ depending only on $\tilde{C}$, such that for any $\lambda \geq \lambda_0$ and $s \geq s_1(\Omega, T)$, the following inequality holds:*

$$E_1 + E_2 \leq C s \lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2), \tag{26}$$

*where $\mathcal{E}(g_1, g_2, B_1, B_2)$ is defined as follows*

$$\mathcal{E}(g_1, g_2, B_1, B_2) = \int_Q e^{-2s\alpha}\Big(|F'|^2 + |G'|^2\Big)dtdx + s^3\lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha}\Big(|B_1|^2 + |B_2|^2\Big)dtdx. \tag{27}$$

*Proof.* Note that $\alpha(0, x) = +\infty$. As a result of (11) and (12), we have

$$\begin{aligned}
&\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta}\Big(|B_1^\theta|^2 + |B_2^\theta|^2\Big)dx \\
=&\int_0^\theta \frac{\partial}{\partial t}\left(\int_\Omega \phi^{-1} e^{-2s\alpha}\Big(|B_1(t,x)|^2 + |B_2(t,x)|^2\Big)dx\right)dt \\
=&\int_{Q_\theta} \phi^{-1}(-2s)\partial_t e^{-2s\alpha}(|B_1|^2 + |B_2|^2)dtdx - \int_{Q_\theta} \phi^{-2}\partial_t \phi e^{-2s\alpha}(|B_1|^2 + |B_2|^2)dtdx \\
&+ 2\int_{Q_\theta} \phi^{-1} e^{-2s\alpha}(2B_1\partial_t B_1 + 2B_2 \partial_t B_2)dtdx \\
\leq& C(sT^5 + s\lambda T^8 + T^7)\int_Q \phi^3 e^{-2s\alpha}(|B_1(t,x)|^2 + |B_2(t,x)|^2)dtdx \\
&+ (s\lambda)^{-1}\int_Q \phi^{-1} e^{-2s\alpha}(|\partial_t B_1|^2 + |\partial_t B_2|^2)dtdx \\
\leq& C(s\mathcal{I}(B_1) + \mathcal{I}(B_2)),
\end{aligned}$$

where $Q_\theta = (0, \theta) \times \Omega$, $\mathcal{I}(B_j)|_{j=1,2}$ is defined in (14) and (16), for any $s \geq C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)$ and $\lambda \geq 1$. Then using the estimate given in Theorem 2.1 to the system (18), we obtain

$$\begin{aligned}
& s\mathcal{I}(B_1) + \mathcal{I}(B_2) \\
\leq\ & \tilde{C}(\int_Q e^{-2s\alpha}(s|F'|^2| + |G'|^2)dtdx + s^4\lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha}|B_1|^2 dtdx \\
& + s^3\lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha}|B_2|^2 dtdx + \int_Q s e^{-2s\alpha}|\partial_t a(t,x)\Big(\int_0^t B_1(s,x)ds \\
& + A_1(0,x))|^2 dtdx\Big) \\
\leq\ & \tilde{C}\Bigg(s\int_Q e^{-2s\alpha}(|F'|^2| + |G'|^2)dtdx + s^4\lambda^4\int_{Q_\omega} \phi^3 e^{-2s\alpha}|B_1|^2 dtdx \\
& + s^3\lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha}|B_2|^2 dtdx + C_1 T s\int_Q e^{-2s\alpha}|B_1|^2 dtdx\Bigg).
\end{aligned} \tag{28}$$

Due to this term $C_1 T \int_Q e^{-2s\alpha}|B_1|^2 dtdx$ can be absorbed by $\mathcal{I}(B_1)$, $\lambda > 1$ and $s$ being large enough, we have

$$\mathcal{I}(B_1) + \mathcal{I}(B_2) \leq \tilde{C}_1 s\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2).$$

Thus for $s \geq s_1 = max\{s_0, C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)\}$, the proof is complete. □

**Lemma 3.5.** *Let Assumption 3.1 be satisfied. Then there exists* $\lambda_1 = \max\{\lambda_0, C(\gamma_1,\gamma_2,\gamma_3)\}$ *and* $s_2 = \max\{s_1, C(\gamma_1,\gamma_2,\gamma_3)(T + T^2 + T^4)\}$ *for all* $\lambda \geq \lambda$, $s \geq s_2$, *the following inequality holds:*

$$E_3 + E_4 \leq Cs\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2),$$

*where* $\mathcal{E}(g_1, g_2, B_1, B_2)$ *is defined in* (27).

*Proof.* First, we define

$$\pi(B_1) := e^{-2s\alpha}\phi^{-1}\nabla(M_e\nabla B_1).$$

We multiply the first equation in (18) by $\pi(B_1)$ and we integrate over $Q_\theta$, the result is

$$\int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) = \int_{Q_\theta} \pi(B_1)(\frac{\mu}{\mu+1} div(M_e\nabla B_1) - \partial_t a A_1 - aB_1 + F'(g_1, \nabla\tilde{v}^\varepsilon))dtdx. \tag{29}$$

We divide (29) into left and right sides integrals to estimate separately. Firstly, we integrate the left side integral by parts, and get

$$\begin{aligned}
&-\int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) dtdx \\
=&-\int_{Q_\theta} c_m \partial_t B_1 e^{-2s\alpha}\phi^{-1}\nabla(M_e\nabla B_1)dtdx \\
=&\int_{Q_\theta} c_m \partial_t B_1 \nabla(e^{-2s\alpha}\phi^{-1})M_e\nabla B_1 dtdx + \frac{1}{2}\int_{Q_\theta} c_m \partial_t(|\nabla B_1|^2)e^{-2s\alpha}\phi^{-1}M_e dtdx \\
=&\mathcal{J}_1 + \mathcal{J}_2.
\end{aligned} \tag{30}$$

Note that $|\nabla(\phi^{-1}e^{-2s\alpha})| \leq s\lambda e^{-2s\alpha}$ for $s \geq CT^2$. Thus we have

$$\begin{aligned}
\mathcal{J}_1 \leq& s\lambda(C\|M_e\|^2_{L^\infty(\Omega)} s\lambda \int_{Q_\theta} \phi e^{-2s\alpha}|\nabla B_1|^2 dtdx + (s\lambda)^{-1}\int_{Q_\theta} \phi^{-1}e^{-2s\alpha}|\partial_t B_1|^2 dtdx) \\
\leq& s\lambda\mathcal{I}(B_1)
\end{aligned} \tag{31}$$

for any $s \geq C(\gamma_2)T^2$. Integrating by parts with respect to time in $\mathcal{J}_2$, we have

$$\begin{aligned}
\mathcal{J}_2 &= \frac{1}{2}\int_{Q_\theta} c_m \partial_t(|\nabla B_1|^2)e^{-2s\alpha}\phi^{-1}M_e dtdx \\
&= -\frac{1}{2}\int_{Q_\theta} c_m|\nabla B_1|^2\partial_t(e^{-2s\alpha}\phi^{-1})M_e dtdx \\
&\quad +\frac{1}{2}\int_\Omega (c_m|\nabla B_1|^2 e^{-2s\alpha}\phi^{-1}M_e)\ |_{t=\theta}\ dx.
\end{aligned} \tag{32}$$

Here,

$$\begin{aligned}
|\partial_t(e^{-2s\alpha}\phi^{-1})| &= |e^{-2s\alpha}\phi^{-2}\phi_t + e^{-2s\alpha}\phi^{-1}(-2s)\alpha_t| \\
&= |e^{-2s\alpha}\phi^{-1}(\phi^{-1}\phi_t - 2s\alpha_t)| \\
&\leq |e^{-2s\alpha}\phi^{-1}|(\frac{T^2}{4} + 2s)CT\phi^2 \\
&\leq Cs\lambda^2\phi e^{-2s\alpha},
\end{aligned}$$

for $\lambda > 1$ and $s \geq CT^3$. Therefore,

$$\mathcal{J}_2 \geq \frac{1}{2}\int_\Omega (c_m e^{-2s\alpha}\phi^{-1}M_e|\nabla B_1|^2)\,|_{t=\theta}\,dx - Cs\lambda^2\int_{Q_\theta} c_m\phi e^{-2s\alpha}M_e|\nabla B_1|^2 dtdx. \tag{33}$$

Now coming to the right side integrals of (29), we have

$$\begin{aligned}
&\int_{Q_\theta}\pi(B_1)\Big(\frac{\mu}{\mu+1}div(M_e\nabla B_1) - \partial_t aA_1 - aB_1 + F'\Big)dtdx\\
=&\int_{Q_\theta}\pi(B_1)F'dtdx + \int_{Q_\theta}\pi(B_1)\frac{\mu}{\mu+1}div(M_e\nabla B_1)dtdx\\
&-\int_{Q_\theta}\pi(B_1)\partial_t a\Big(\int_0^t B_1(s,x)ds + A_1(0,x)\Big)dtdx - \int_{Q_\theta}\pi(B_1)aB_1dtdx\\
=&\sum_{j=1}^4\mathcal{K}_j. 
\end{aligned}\tag{34}$$

Then we estimate the above integrals one by one. Applying the Cauchy inequality, we get the following estimates for $\mathcal{K}_{j=1,2}$.

$$\begin{aligned}
\mathcal{K}_1 &= \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\nabla(M_e\nabla B_1)F'dtdx\\
&= \int_{Q_\theta} e^{-s\alpha}\phi^{-\frac{1}{2}}F'(e^{-s\alpha}\phi^{-\frac{1}{2}}\nabla M_e\nabla B_1 + e^{-s\alpha}\phi^{-\frac{1}{2}}M_e\Delta B_1)dtdx\\
&\leq \int_{Q_\theta} CT^2 e^{-2s\alpha}|F'|^2dtdx + \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|M_e|^2|\Delta B_1|^2dtdx\\
&\quad + \int_{Q_\theta} CT^4\phi e^{-2s\alpha}|\nabla M_e|^2|\nabla B_1|^2dtdx\\
&\leq s\lambda^2(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha}|F'|^2dtdx),
\end{aligned}\tag{35}$$

and

$$\begin{aligned}
\mathcal{K}_2 &= \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\nabla(M_e\nabla B_1)\frac{\mu}{\mu+1}div(M_e\nabla B_1)dtdx\\
&= \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\frac{\mu}{\mu+1}|\nabla(M_e\nabla B_1)|^2dtdx\\
&= \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\frac{\mu}{\mu+1}|\nabla M_e\nabla B_1 + M_e\Delta B_1|^2dtdx\\
&\leq \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\frac{\mu}{\mu+1}\Big(2|\nabla M_e|^2|\nabla B_1|^2 + 2|M_e|^2|\Delta B_1|^2\Big)dtdx\\
&\leq C\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|M_e|^2|\Delta B_1|^2dtdx + CT^4\int_{Q_\theta} e^{-2s\alpha}\phi|\nabla M_e|^2|\nabla B_1|^2dtdx\\
&\leq s\lambda^2\mathcal{I}(B_1),
\end{aligned}\tag{36}$$

where $\lambda \geq 1$ and $s \geq C(\gamma_1)T^4$. Next, we estimate the integral $\mathcal{K}_3$, and obtain

$$\mathcal{K}_3 = -\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\nabla(M_e\nabla B_1)\partial_t a(t,x)\Big(\int_0^t B_1(s,x)ds + A_1(0,x)\Big)dtdx$$

$$
\begin{aligned}
&\le C\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|M_e\Delta B_1+\nabla M_e\nabla B_1|^2 dtdx + CT^8\int_{Q_\theta} e^{-2s\alpha}\phi^3\int_0^t |B_1(s,x)|^2 dsdtdx\\
&\le C\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|M_e|^2|\Delta B_1|^2 dtdx + CT^4\int_{Q_\theta} e^{-2s\alpha}\phi|\nabla M_e|^2|\nabla B_1|^2 dtdx\\
&\quad + CT^8\int_{Q_\theta} e^{-2s\alpha}\phi^3\int_0^t |B_1(s,x)|^2 dsdtdx\\
&\le s\lambda^2\mathcal{I}(B_1).
\end{aligned} \tag{37}
$$

Similarly, we have

$$
\mathcal{K}_4 = -\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}a(t,x)B_1\nabla(M_e\nabla B_1 + M_e\Delta B_1)dtdx \le s\lambda^2\mathcal{I}(B_1). \tag{38}
$$

Using the assumptions on the conductivity $M_e$ and substituting the inequalities (35)-(38) into (29), we get

$$
-\mathcal{J}_1 - \mathcal{J}_2 \le \sum_{j=1}^4 \mathcal{K}_j \le s\lambda^2\Big(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha}|F'|^2 dtdx\Big),
$$

which means

$$
-\mathcal{J}_2 \le \sum_{j=1}^4 \mathcal{K}_j + \mathcal{J}_1 \le s\lambda^2\Big(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha}|F'|^2 dtdx\Big) + s\lambda\mathcal{I}(B_1).
$$

Substituting (33) to the above inequality, we have

$$
\begin{aligned}
&\Big|\int_\Omega (c_m e^{-2s\alpha}\phi^{-1}M_e|\nabla B_1|^2)\ |_{t=\theta}\ dx\Big| \\
\le\ & |\mathcal{J}_1| + \sum_{j=1}^4 \mathcal{K}_j\\
\le\ & s\lambda^2\Big(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha}|F'|^2 dtdx\Big) + s\lambda\mathcal{I}(B_1),
\end{aligned} \tag{39}
$$

which leads to

$$
\Big|\int_\Omega e^{-2s\alpha^\theta}(\phi^\theta)^{-1}|\nabla B_1^\theta|^2 dx\Big| \le s\lambda^2\Big(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha}|F'|^2 dtdx\Big). \tag{40}
$$

Next we multiply the second equation of (18) by $\xi(B_2) := e^{-2s\alpha}\phi^{-1}\nabla(M\nabla B_2)$, and integrate over $Q_\theta$ to get

$$
\begin{aligned}
&\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\nabla(M\nabla B_2)\varepsilon\partial_t B_2 dtdx\\
=\ & \int_{Q_\theta}\xi(B_2)\nabla(M\nabla B_2)dtdx + \int_{Q_\theta}\xi(B_2)\nabla(M_i\nabla B_1)dtdx + \int_{Q_\theta}\xi(B_2)G' dtdx\\
=\ & \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|\nabla(M\nabla B_2)|^2 dtdx + \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\nabla(M\nabla B_2)\nabla(M_i\nabla B_1)dtdx\\
&+\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\nabla(M\nabla B_2)G' dtdx.
\end{aligned}
$$

We estimate

$$
\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}\nabla(M\nabla B_2)\nabla(M_i\nabla B_1)dtdx
$$

$$
\begin{aligned}
= & \int_{Q_\theta} e^{-2s\alpha}\phi^{-1}(\nabla M\nabla B_2 + M\Delta B_2)\nabla(M_i\nabla B_1)dtdx \\
\le & \frac{1}{2}\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|\nabla M\nabla B_2 + M\Delta B_2|^2 dtdx \\
& \qquad\qquad +\frac{1}{2}\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|\nabla M_i\nabla B_1 + M_i\Delta B_1|^2 dtdx \\
\le & CT^4\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|\nabla M|^2|\nabla B_2|^2 dtdx + C\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|M|^2|\Delta B_2|^2 dtdx \\
& +CT^4\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|\nabla M_i|^2|\nabla B_1|^2 dtdx + C\int_{Q_\theta} e^{-2s\alpha}\phi^{-1}|M_i|^2|\Delta B_1|^2 dtdx \\
\le & s\lambda^2\mathcal{I}(B_1) + s\lambda^2\mathcal{I}(B_2). \qquad (41)
\end{aligned}
$$

Continuing the similar computation as the preceding estimates and using Assumption 1.1, we obtain

$$
\begin{aligned}
& |\int_\Omega e^{-2s\alpha^\theta}(\phi^\theta)^{-1}|\nabla B_2^\theta|^2 dx| \\
\le & s\lambda^2(\mathcal{I}(B_1) + \mathcal{I}(B_2) + \int_{Q_\theta} e^{-2s\alpha}(|F'|^2 + |G'|^2)dtdx) \\
\le & Cs\lambda^2(\int_{Q_\theta} e^{-2s\alpha}(|F'|^2 + |G'|^2)dtdx + s^3\lambda^4\int_{Q_\omega}\phi^3 e^{-2s\alpha}(|B_1|^2 + |B_2|^2)dtdx), \qquad (42)
\end{aligned}
$$

for any $s \ge C(\gamma_1,\gamma_2,\gamma_3)(T + T^2 + T^4)$ and $\lambda \ge C(\gamma_1,\gamma_2,\gamma_3)$. Thus combining the estimates (40) and (42), we obtain the conclusion. □

Now we shall give the main result of the stability estimate of the conductivities in (6) based on the preceding lemmas and proposition.

**Theorem 3.6.** *Let $(A_1, A_2)$ be the solution of (7). Suppose all the assumptions of Theorem 2.1 hold and $g_1, g_2 \in H_0^2(\Omega)$. In addition, suppose Assumption 3.1 and 3.2 are also satisfied. Then there exists a constant $C$ with $C(\Omega,\omega,T,\gamma_1,\gamma_2,\gamma_3) > 0$, such that for sufficiently large $\lambda \ge \lambda_0 \ge 1$ and $s \ge s_4$, the following estimate holds:*

$$
\begin{aligned}
& \int_\Omega\Big(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2\Big)dx \\
\le & C\Bigg(\int_{Q_\omega}\Big(|\partial_t A_1|^2 + |\partial_t A_2|^2\Big)dtdx + \int_\Omega |A_1^\theta|^2 + \sum_{j=1}^{2}\Big(|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2\Big)dx\Bigg) \\
& + C\int_{\tilde\omega}\Big(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2\Big)dx. \qquad (43)
\end{aligned}
$$

*Proof.* Substituting the results in Lemma 3.4 and Lemma 3.5 into the inequality in Proposition 1, one obtains

$$
\begin{aligned}
& s^2\lambda^2\int_\Omega \phi^\theta e^{-2s\alpha^\theta}\Big(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2\Big)dx \\
\le & Cs\lambda^2\mathcal{E}(g_1, g_2, B_1, B_2) + C\sum_{j=5}^{9} E_j(\theta) \\
\le & Cs\lambda^2\int_Q e^{-2s\alpha}\Big(|F'|^2 + |G'|^2\Big)dtdx + Cs^4\lambda^6\int_{Q_\omega}\phi^3 e^{-2s\alpha}\Big(|B_1|^2 + |B_2|^2\Big)dtdx
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta}\Big(|\nabla A_1^\theta|^2+|\Delta A_1^\theta|^2+|A_1^\theta|^2\Big)dx + C\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta}\Big(|\nabla A_2^\theta|^2+|\Delta A_2^\theta|^2\Big)dx\\
&+C\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta}\Big(|g_1\Delta\tilde{v}_\theta^\varepsilon|^2+|g_2\Delta\tilde{u}_{e,\theta}^\varepsilon|^2\Big)dx + C\int_{\tilde{\omega}}\Big(|g_1|^2+|g_2|^2+|\nabla g_1|^2+|\nabla g_2|^2\Big)dx\\
&+C\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta}\left(|A_1^\theta|^2+\sum_{j=1}^2\Big(|\Delta A_j^\theta|^2+|\nabla A_j^\theta|^2+|\nabla(\Delta A_j^\theta)|\Big)+|\nabla(g_1\Delta\tilde{v}_\theta^\varepsilon)|^2\right.\\
&\left.+|\nabla(g_2\Delta\tilde{u}_{e,\theta}^\varepsilon)|^2\right)dx\\
\le& Cs\lambda^2\int_Q e^{-2s\alpha}\Big(|F'|^2+|G'|^2\Big)dtdx + Cs^4\lambda^6\int_{Q_\omega}\phi^3 e^{-2s\alpha}\Big(|B_1|^2+|B_2|^2\Big)dtdx\\
&+C\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta}\left(|A_1^\theta|^2+\sum_{j=1}^2\Big(|\Delta A_j^\theta|^2+|\nabla A_j^\theta|^2+|\nabla(\Delta A_j^\theta)|\Big)\right)dx\\
&+C\int_{\tilde{\omega}}\Big(|g_1|^2+|g_2|^2+|\nabla g_1|^2+|\nabla g_2|^2\Big)dx,
\end{aligned}
$$

for large enough $s \ge s_3 = \max\{CT^2, s_2\}$ and $\lambda \ge \lambda_1$. Now for convenience, we set $R_1(t,x) = \nabla\tilde{v}^\varepsilon(t,x)$ and $R_2(t,x) = \nabla\tilde{u}_e^\varepsilon(t,x)$. Then from the regularity of the solutions $(\tilde{v}^\varepsilon(t,x), \tilde{u}_e^\varepsilon(t,x))$, we deduce that there exist $l_j \in L^2(0,T)$, $j = 1,2,3,4$,

$$|\partial_t R_j(t,x)| \le l_j(t)|R_j^\theta|,\ j=1,2,$$

$$|\partial_t \nabla R_1(t,x)| \le l_3(t)|\nabla R_1^\theta|,$$

$$|\partial_t \nabla R_2(t,x)| \le l_3(t)|\nabla R_2^\theta|,$$

for any $(t,x) \in Q$, and the functions $l_j \in L^2(0,T)$, implying $\int_0^T |l_j|^2 dt \le N < \infty$, $j = 1,2,3,4$. Then we show

$$F' = \partial_t(\nabla(g_1\nabla\tilde{v}^\varepsilon)) = \nabla g_1 \partial_t R_1 + g_1\partial_t\nabla R_1,$$

$$G' = \partial_t(\nabla(g_2\nabla\tilde{u}_e^\varepsilon)) = \nabla g_2 \partial_t R_2 + g_2\partial_t\nabla R_2.$$

Observe that from the definition of $\alpha$, we get easily $e^{-2s\alpha(t,x)} \le e^{-2s\alpha^\theta}$ for all $(t,x) \in Q$. This implies

$$
\begin{aligned}
&s^2\lambda^2\int_\Omega \phi^\theta e^{-2s\alpha^\theta}\Big(|g_1|^2+|g_2|^2+|\nabla g_1|^2+|\nabla g_2|^2\Big)dx\\
\le\ & Cs^4\lambda^6\int_{Q_\omega}\phi^3 e^{-2s\alpha}\Big(|B_1|^2+|B_2|^2\Big)dtdx + C\int_{\tilde{\omega}}\Big(|g_1|^2+|g_2|^2+|\nabla g_1|^2+|\nabla g_2|^2\Big)dx,\\
&+C\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta}\left(|A_1^\theta|^2+\sum_{j=1}^2\Big(|\Delta A_j^\theta|^2+|\nabla A_j^\theta|^2+|\nabla(\Delta A_j^\theta)|\Big)\right)dx,
\end{aligned}
$$

for sufficiently large $s \ge s_4 = \max\{CT^2N, s_3\}$. From the properties of $\alpha$ and $\phi$, there exist $e_0$ and $e_1$ such that

$$inf_{x\in\Omega}\Big(\frac{1}{\phi^\theta}e^{-2s\alpha^\theta}\Big) \ge e_0 > 0,$$

$$sup_{x\in\Omega}\Big(\frac{1}{\phi^\theta}e^{-2s\alpha^\theta}\Big) \le e_1 < \infty.$$

Furthermore, $e^{-\epsilon\alpha}\phi^m \le C < \infty$ for all $\epsilon > 0$ and $m \in \mathbb{R}$ in $Q_\omega$. Thus we obtain

$$\begin{aligned} & s^2\lambda^2 \int_\Omega \phi^\theta e^{-2s\alpha^\theta}\Big(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2\Big)dx \\ \le\ & Cs^4\lambda^6 \int_{Q_\omega}\Big(|B_1|^2 + |B_2|^2\Big)dtdx + C\int_{\tilde\omega}\Big(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2\Big)dx, \\ & + C\int_\Omega\left(|A_1^\theta|^2 + \sum_{j=1}^{2}\Big(|\Delta A_j^\theta|^2 + |\nabla A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|\Big)\right)dx, \end{aligned}$$

Then we fix the parameters $s$, $\lambda$ as $s = s_4$, $\lambda = \lambda_1$. This concludes the proof of the theorem. □

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