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HOMOGENIZATION OF THE EVOLUTION STOKES EQUATION IN A PERFORATED DOMAIN WITH A STOCHASTIC FOURIER BOUNDARY CONDITION

Hakima Bessaih

University of Wyoming, Department of Mathematics
Dept. 3036, 1000 East University Avenue
Laramie WY 82071, USA

Yalchin Efendiev

Department of Mathematics, Texas A&M University
College Station, TX 77843, USA
and
Numerical Porous Media SRI Center, CEMSE Division
King Abdullah University of Science and Technology
Thuwal 23955-6900, Kingdom of Saudi Arabia

Florin Maris

Numerical Porous Media SRI Center, CEMSE Division
King Abdullah University of Science and Technology
Thuwal 23955-6900, Kingdom of Saudi Arabia

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Abstract. The evolution Stokes equation in a domain containing periodically distributed obstacles subject to Fourier boundary condition on the boundaries is considered. We assume that the dynamic is driven by a stochastic perturbation on the interior of the domain and another stochastic perturbation on the boundaries of the obstacles. We represent the solid obstacles by holes in the fluid domain. The macroscopic (homogenized) equation is derived as another stochastic partial differential equation, defined in the whole non perforated domain. Here, the initial stochastic perturbation on the boundary becomes part of the homogenized equation as another stochastic force. We use the two-scale convergence method after extending the solution with 0 in the holes to pass to the limit. By Itô stochastic calculus, we get uniform estimates on the solution in appropriate spaces. In order to pass to the limit on the boundary integrals, we rewrite them in terms of integrals in the whole domain. In particular, for the stochastic integral on the boundary, we combine the previous idea of rewriting it on the whole domain with the assumption that the Brownian motion is of trace class. Due to the particular boundary condition dealt with, we get that the solution of the stochastic homogenized equation is not divergence free. However, it is coupled with the cell problem that has a

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divergence free solution. This paper represents an extension of the results of Duan and Wang (Comm. Math. Phys. 275:1508–1527, 2007), where a reaction diffusion equation with a dynamical boundary condition with a noise source term on both the interior of the domain and on the boundary was studied, and through a tightness argument and a pointwise two scale convergence method the homogenized equation was derived.

1. **Introduction and formulation of the problem.** In this paper, we are interested in a fluid flow where the advective inertial forces are small compared with viscous forces. Starting from the evolution Stokes equations in domain containing periodically distributed obstacles or solid pores, with a dynamical boundary condition driven by a noise source on the solid pores, the homogenized dynamic is rigorously recovered by the use of two-scale convergence method. We represent the solid pores by holes in the fluid domain.

The homogenization of the Stokes problem in perforated domains goes back to Sánchez-Palencia in [18] where an asymptotic expansion method was used. Rigorous proofs were given later by Tartar in [20] by the energy method and by Allaire in [2] where two scale convergence method was used. The two scale convergence was first introduced by Nguetseng in 1989 in [12] and later developed by Allaire in [1]. The idea of this method was to give a rigorous justification to the asymptotic expansion method. A more general setting has been defined by Nguetseng in [13], [14] and later in [15]. The theory of the two scale convergence from the periodic to the stochastic setting has been extended by Bourgeat, A. Mikelić and Wright in [5], using techniques from ergodic theory. There is a vast literature for partial differential equations with random coefficients, where this method was used, however most of the tackled problems in this setting are not in perforated domains, (see [5], [4] and the references therein). Much less was done for homogenization of stochastic partial differential equations, in particular in perforated domains. We mention the paper [24] where a reaction diffusion equation with a dynamical boundary condition with a noise source term on both the interior of the domain and on the boundary was studied, and through a tightness argument and a pointwise two scale convergence method the homogenized equation was derived. A comprehensive theory for solving stochastic homogenization problems has been constructed recently in [26] and [17], where a Σ-convergence method adapted to stochastic processes was developed. An application of the method to the homogenization of a stochastic Navier-Stokes type equation with oscillating coefficients in a bounded domain (without holes) has been provided. In their setting, the Σ-convergence method implies the two-scale convergence.

For the deterministic Stokes or Navier Stokes equations in perforated domains we refer to [18], [20], [2], [10], [6], [3], [11]. In [6] the Stokes problem in a perforated domain with a nonhomogeneous Fourier boundary condition on the boundaries of the holes was studied while in [3] the same problem was studied with a slip boundary condition. As far as we know, the stochastic Stokes or Navier Stokes equation in a perforated domain has not been studied.

In this paper we consider a stochastic linear Navier Stokes equation in a periodically perforated domain with a noise source. On the boundaries of the holes, we consider a dynamical Fourier boundary condition driven by a noise. A boundary condition is called dynamical if it involves the time derivative. This problem modelises the flow of an incompressible viscous flow through a porous medium under the action of an external random perturbation. We will consider the scaling ε for the density on the boundary of the pores and 1 for the viscosity. Our boundary condition is very similar to the nonhomogenous boundary condition used by [6] where they used an electrical field as an external source. However, their model was stationary while ours takes into account time derivatives. For similar boundary conditions in a deterministic setting and for parabolic problems see [22].

Different scalings will be considered in a forthcoming paper like ε^2 for the viscosity that will lead to different limit problems.

We consider D, an open bounded Lipschitz domain of $\mathbb{R}^n$ with boundary ∂D, and let $Y = [0,1[^n$ the representative cell. Denote by O an open subset of Y with a smooth boundary ∂O, such that $\overline{O} \subset Y$, and let $Y^* = Y \setminus \overline{O}$. The elementary cell Y and the small cavity or hole O inside it are used to model small scale obstacles or heterogeneities in a physical medium D. Denote by $O^{\varepsilon,k}$ the translation of εO by εk, $k \in \mathbb{Z}^n$. We make the assumption that the holes do not intersect the boundary ∂D and we denote by $\mathcal{K}^\varepsilon$ the set of all $k \in \mathbb{Z}^n$ such that the cell $\varepsilon k + \varepsilon Y$ is strictly included in D. The set of all such holes will be denoted by O^ε, i.e.

$$O^\varepsilon := \bigcup_{k\in\mathcal{K}^\varepsilon} O^{\varepsilon,k} = \bigcup_{k\in\mathcal{K}^\varepsilon} \varepsilon(k+O),$$

and set

$$D^\varepsilon := D - \overline{O^\varepsilon}.$$

By this construction, D^ε is a periodically perforated domain with holes of size of the same order as the period. One of the difficulties of the homogenization in perforated domains consists in the fact that the ε- problems are defined in different domains. In [20], [6], [2], [3] suitable extensions for the velocity and for the pressure were defined to overcome this difficulty.

The evolution Stokes equation in the domain D^ε with a stochastic dynamical boundary condition on the boundaries of the holes is given by

$$\begin{cases} du^\varepsilon(t,x) &= [\nu\Delta u^\varepsilon(t,x) - \nabla p^\varepsilon(t,x) + f(t,x)]\,dt + g_1(t)dW_1(t) & \text{in } D^\varepsilon, \\ \operatorname{div} u^\varepsilon(t,x) &= 0 & \text{in } D^\varepsilon, \\ \varepsilon^2 du^\varepsilon(t,x) &= -\left[\nu\dfrac{\partial u^\varepsilon(t,x)}{\partial n} - p^\varepsilon(t,x)n + b\varepsilon u^\varepsilon(t,x)\right]dt + \varepsilon g_2^\varepsilon(t)dW_2(t) & \text{on } \partial O^\varepsilon, \\ u^\varepsilon(0,x) &= u_0^\varepsilon(x) & \text{in } D^\varepsilon, \\ u^\varepsilon(0,x) &= v_0^\varepsilon(x) & \text{on } \partial O^\varepsilon, \end{cases} \tag{1.1}$$

where u^ε is the velocity of the fluid and p^ε is the pressure.

Using Itô's formula and stochastic calculus we are able to prove some uniform estimates in some functional spaces that are ε- dependent. Our particular boundary condition makes it difficult to extend the velocity continuously in $H^1(D)^n$, so we chose to use the trivial extension by 0 of the velocity as well as of the gradient of the velocity. This extension is a continuous one in $L^2(D)$ for which the uniform estimates still hold. Our extension is not divergence free, so we cannot expect the homogenized solution to be divergence free. Hence we cannot use test functions that are divergence free in the variational formulation, which implies that the pressure has to be included. Because of the low regularity in time of the stochastic process, we have to define a more regular in time pressure $P(t) = \int_0^t p(s)ds$ in Theorem 3.3. We apply to it the same extension and we will recover in the limit the information into the coefficients of the homogenized equation.

For the convergence of the boundary integrals we use an idea from [6] to rewrite an integral on the boundary in terms of an integral on the whole domain. In this way, the integrals over the boundary ∂O^ε become in the limit integrals over the whole domain D. In particular, in the case of the stochastic integral on the boundaries of the holes, we combined this idea with the use of the decomposition of the Wiener process in terms of the basis, and that it is of trace class.

Our process depends on three variables: ω, t, and x but the oscillations appear only in x, hence we will use the two scale convergence method in the space variable but with a parameter $(\omega, t) \in \Omega \times [0,T]$. One of the condition to be able to use two scale convergence is the uniform boundedness with respect to $\varepsilon > 0$ of the L^2-norm. We were not able to show uniform bounds for $\omega \in \Omega$ or for any $t \in [0,T]$, but only in $L^2(\Omega \times [0,T] \times D)^n$, thus we decided using this type of convergence. Most of the results concerning the two scale convergence are being extended in a straighforward way to the results of our paper. We gather the results we use in Section 4. The convergence in two scale is a stronger type of convergence than the weak convergence (see Corollary 4.3), and so the convergence to the homogenized solution will be the weak one in $L^2(\Omega \times [0,T] \times D)^n$. This convergence although weak in the deterministic sense, is strong in the probability sense.

The paper is organized as follows: in Section 2 we formulate more precisely our problem, set the functional setting and give the assumptions used throughout the paper. In Section 3 we study the problem (1.1) in the perforated domain and show the existence and uniqueness in Theorem 3.1. The estimates, needed for the two scale convergence method, are derived. We also introduce the pressure in Theorem 3.3 and setup the variational formulation to be used for the passage to the limit. In Section 4, we introduce the two scale convergence and give a number of results that we will use. In Section 5, we pass to the limit in the variational formulation (3.11) and get the a variational formulation for the two-scale limit. In section 6, we write the two-scale limit solution in terms of the cell problem (6.4) coupled with the homogenized system (6.12). We also, prove that the system (6.12) is well posed and we finish with some remarks related to the properties of the limit problem.

2. Preliminaries and assumptions.

In this section we introduce some functional spaces and assumptions in order to study the problem (1.1).

2.1. Functional setting.

Let us introduce the following Hilbert spaces

$$\mathbf{L}^2_\varepsilon := L^2(D^\varepsilon)^n \times L^2(\partial O^\varepsilon)^n, \tag{2.1}$$

$$\mathbf{H}^1_\varepsilon := H^1(D^\varepsilon)^n \times H^{\frac{1}{2}}(\partial O^\varepsilon)^n. \tag{2.2}$$

equipped respectively with the inner products

$$\langle \mathbf{U}, \mathbf{V} \rangle = \int_{D^\varepsilon} \big[u(x) \cdot v(x)\big] dx + \int_{\partial O^\varepsilon} \big[\overline{u}(x') \cdot \overline{v}(x')\big] d\sigma(x'),$$

and

$$((\mathbf{U}, \mathbf{V})) = \int_{D^\varepsilon} \big[\nabla u(x) \cdot \nabla v(x)\big] dx.$$

We introduce the bounded linear and surjective operator $\gamma^\varepsilon : H^1(D^\varepsilon)^n \mapsto H^{\frac{1}{2}}(\partial O^\varepsilon)^n$ such that $\gamma^\varepsilon u = u|_{\partial O^\varepsilon}$ for all $u \in C^\infty\left(\overline{D^\varepsilon}\right)^n$. γ^ε is the trace operator, (see [19], pp47). We denote by $H^{-\frac{1}{2}}(\partial O^\varepsilon)^n$ the dual space of $H^{\frac{1}{2}}(\partial O^\varepsilon)^n$.

We denote by $\mathbf{H}^\varepsilon$ the closure of $\mathcal{V}^\varepsilon$ in $\mathbf{L}^2_\varepsilon$, and by $\mathbf{V}^\varepsilon$ the closure of $\mathcal{V}^\varepsilon$ in $\mathbf{H}^1_\varepsilon$, where for $\mathbf{U} = (u, \overline{u}) \in C^\infty\left(\overline{D^\varepsilon}\right)^n \times \gamma^\varepsilon(C^\infty\left(\overline{D^\varepsilon}\right)^n)$

$$\mathcal{V}^\varepsilon := \left\{\mathbf{U} = (u, \overline{u}) \ | \ \operatorname{div} u = 0,\ \overline{u} = \varepsilon u \text{ on } \partial O^\varepsilon,\ u = 0 \text{ on } \partial D\right\}, \tag{2.3}$$

Let $\Pi^\varepsilon : \mathbf{L}^2_\varepsilon \mapsto L^2(D^\varepsilon)^n$ be the operator that represents the projection onto the first component, i.e. $\Pi^\varepsilon \mathbf{U} = u$, for every $\mathbf{U} = (u, \overline{u}) \in \mathbf{L}^2_\varepsilon$.

Let us also denote by H^ε and V^ε the images under the operator Π^ε of the spaces $\mathbf{H}^\varepsilon$ and $\mathbf{V}^\varepsilon$, with:

$$H^\varepsilon := \big\{u \in L^2(D^\varepsilon)^n \mid \text{div } u = 0 \text{ in } D^\varepsilon,\ u\cdot n = 0 \text{ on } \partial D,\ u\cdot n = \varepsilon\overline{u}\cdot n \text{ on } \partial O^\varepsilon\big\}$$

and for $\overline{u} \in L^2(\partial O^\varepsilon)^n$.

$$V^\varepsilon := \big\{u \in H^1(D^\varepsilon)^n \mid \text{div } u = 0 \text{ in } D^\varepsilon,\quad u = 0 \text{ on } \partial D\big\}.$$

$\mathbf{H}^\varepsilon$ and $\mathbf{V}^\varepsilon$ are separable Hilbert spaces with the inner products and norms inherited from $\mathbf{L}^2_\varepsilon$ and $\mathbf{H}^1_\varepsilon$ respectively:

$$\|\mathbf{U}\|^2_{\mathbf{H}^\varepsilon} = \langle \mathbf{U}, \mathbf{U}\rangle\,,$$

$$\|\mathbf{U}\|^2_{\mathbf{V}^\varepsilon} = ((\mathbf{U}, \mathbf{U}))\,,$$

and H^ε and V^ε are also separable Hilbert spaces with the norms induced by the projection Π^ε.

Denoting by $(\mathbf{H}^\varepsilon)'$ and $(\mathbf{V}^\varepsilon)'$ the dual spaces, if we identify $\mathbf{H}^\varepsilon$ with $(\mathbf{H}^\varepsilon)'$ then we have the Gelfand triple $\mathbf{V}^\varepsilon \subset \mathbf{H}^\varepsilon \subset (\mathbf{V}^\varepsilon)'$ with continuous injections.

We denote the dual pairing between $\mathbf{U} \in \mathbf{V}^\varepsilon$ and $\mathbf{V} \in (\mathbf{V}^\varepsilon)'$ by $\langle \mathbf{U}, \mathbf{V}\rangle_{\langle \mathbf{V}^\varepsilon, (\mathbf{V}^\varepsilon)'\rangle}$. When $\mathbf{U}, \mathbf{V} \in \mathbf{H}^\varepsilon$, we have $\langle \mathbf{U}, \mathbf{V}\rangle_{\langle \mathbf{V}^\varepsilon, (\mathbf{V}^\varepsilon)'\rangle} = \langle \mathbf{U}, \mathbf{V}\rangle$.

Assume that b is a strictly positive constant, and define the linear operator $\mathbf{A}^\varepsilon : D(\mathbf{A}^\varepsilon) \subset \mathbf{H}^\varepsilon \mapsto \mathbf{H}^\varepsilon$:

$$\mathbf{A}^\varepsilon \mathbf{U} = \mathbf{A}^\varepsilon \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = Proj_{\mathbf{H}^\varepsilon} \begin{pmatrix} -\nu\Delta u \\ \dfrac{\nu}{\varepsilon}\dfrac{\partial u}{\partial n} + \dfrac{b}{\varepsilon}\overline{u} \end{pmatrix}, \tag{2.4}$$

with

$$D(\mathbf{A}^\varepsilon) = \{\mathbf{U} \in \mathbf{H}^\varepsilon | -\Delta u \in L^2(D^\varepsilon)^n \text{ and } \frac{\partial u}{\partial n} \in L^2(\partial O^\varepsilon)^n\},$$

and

$$\langle \mathbf{A}^\varepsilon \mathbf{U}, \mathbf{V}\rangle = \int_{D^\varepsilon} \nu \nabla u \nabla v dx + \int_{\partial O^\varepsilon} \varepsilon b \gamma^\varepsilon(u)\gamma^\varepsilon(v) d\sigma.$$

Let $\mathbf{F}^\varepsilon \in L^2(0,T; L^2(D^\varepsilon)^n \times L^2(\partial O^\varepsilon)^n)$ be defined by

$$\mathbf{F}^\varepsilon(t,x) = \begin{pmatrix} f(t,x) \\ 0 \end{pmatrix}, \tag{2.5}$$

Let Q_1 and Q_2 be linear positive operators in $L^2(D)^n$ of trace class. Let $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ be two mutually independent $L^2(D)^n$- valued Wiener processes defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ and with the covariances Q_1 and Q_2. The expectation is denoted by $\mathbb{E}$. If K and H are two separable Hilbert spaces, then we will denote by $L_2(K, H)$ the space of bounded linear operators that are Hilbert-Schmidt from K in H. If Q is a linear positive operator in K of trace class, then we will denote by $L_Q(K, H)$, the space of bounded linear operators that are Hilbert-Schmidt from $Q^{\frac{1}{2}}K$ to H, and the norm will be denoted by $\|\cdot\|_Q$ and is defined as follows: $||g||^2_Q := \sum_{j=1}^\infty \lambda_j ||ge_j||^2_H$ where $\{e_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty$ are respectively the eigenvectors and eigenvalues of Q.

Let us denote by

$$\mathbf{G}^\varepsilon(t) = \begin{pmatrix} g_1(t) & 0 \\ 0 & \varepsilon g_2^\varepsilon(t) \end{pmatrix}, \qquad \mathbf{W}(t) = (W_1(t), W_2(t)). \tag{2.6}$$

The system (1.1) can be rewritten:

$$\left\{ \begin{array}{ll} d\mathbf{U}^\varepsilon(t) & +\mathbf{A}^\varepsilon \mathbf{U}^\varepsilon(t)dt = \mathbf{F}(t)dt + \mathbf{G}^\varepsilon(t)d\mathbf{W}, \\ \mathbf{U}^\varepsilon(0) & = \mathbf{U}_0^\varepsilon = \left(\begin{array}{c} u_0^\varepsilon \\ \overline{u_0^\varepsilon} = \varepsilon v_0^\varepsilon \end{array} \right). \end{array} \right. \tag{2.7}$$

Lemma 2.1. *(Properties of the operator $\mathbf{A}^\varepsilon$) For every $\varepsilon > 0$, the linear operator $\mathbf{A}^\varepsilon$ is positive and self-adjoint in $\mathbf{H}^\varepsilon$. Moreover, it generates an analytic semigroup that we denote by $S_\varepsilon(t)$.*

Proof. We use Proposition A.10, page 389 from [7] prove the lemma. The operator is obviously symmetric, since for every $\mathbf{U},\ \mathbf{V} \in D(\mathbf{A}^\varepsilon)$:

$$\langle \mathbf{A}^\varepsilon \mathbf{U}, \mathbf{V} \rangle = \nu \int_{D^\varepsilon} \nabla u \nabla v dx + \int_{\partial O^\varepsilon} \varepsilon b \gamma^\varepsilon(u) \gamma^\varepsilon(v) d\sigma.$$

We only need to show now that $\mathbf{A}^\varepsilon$ is variational, according to the definition from page 388 from [7]. We consider the space $\mathbf{V}^\varepsilon$ densely embedded in $\mathbf{H}^\varepsilon$ and the continuous bilinear form $\mathbf{a}(\cdot,\cdot)$ on $\mathbf{V}^\varepsilon$, defined as:

$$\mathbf{a}(\mathbf{U}, \mathbf{V}) = \nu \int_{D^\varepsilon} \nabla u \nabla v dx + \int_{\partial O^\varepsilon} \varepsilon b \gamma^\varepsilon(u) \gamma^\varepsilon(v) d\sigma.$$

There exists a constant $c(\varepsilon)$ such that:

$$\mathbf{a}(\mathbf{U}, \mathbf{U}) \geq c(\varepsilon) \|\mathbf{U}\|^2_{\mathbf{V}^\varepsilon},$$

for every $\mathbf{U} \in \mathbf{V}^\varepsilon$. It will be enough to show that

$$D(\mathbf{A}^\varepsilon) = \{\mathbf{U} \in \mathbf{V}^\varepsilon \mid \langle \mathbf{A}^\varepsilon \mathbf{U}, \mathbf{V} \rangle \leq C||\mathbf{V}||_{\mathbf{H}^\varepsilon} \text{ for every } \mathbf{V} \in \mathbf{V}^\varepsilon\}.$$

But, $\langle \mathbf{A}^\varepsilon \mathbf{U}, \mathbf{V} \rangle \leq C||\mathbf{V}||_{\mathbf{H}^\varepsilon}$ for every $\mathbf{V} \in \mathbf{V}^\varepsilon$ is equivalent to

$$\nu \int_{D^\varepsilon} \nabla u \nabla v dx \leq C||v||_{L^2(D^\varepsilon)^n} + C||v||_{L^2(\partial O^\varepsilon)^n}\ \forall v \in V^\varepsilon \Longleftrightarrow$$

$$\nu \int_{D^\varepsilon} -\Delta u v dx + \nu \int_{\partial O^\varepsilon} \frac{\partial u}{\partial n} v d\sigma \leq C||v||_{L^2(D^\varepsilon)^n} + C||v||_{L^2(\partial O^\varepsilon)^n}\ \forall v \in V^\varepsilon \Longleftrightarrow \tag{2.8}$$

$\Delta u \in L^2(D^\varepsilon)^n$ and $\dfrac{\partial u}{\partial n} \in L^2(\partial O^\varepsilon)^n$, so $\mathbf{U} \in D(\mathbf{A}^\varepsilon)$.

Viceversa if $\mathbf{U} \in D(\mathbf{A}^\varepsilon)$, then it easy to see that $\langle \mathbf{A}^\varepsilon \mathbf{U}, \mathbf{V} \rangle \leq C||\mathbf{V}||_{\mathbf{H}^\varepsilon}$ for every $\mathbf{V} \in \mathbf{V}^\varepsilon$.

Now we use Proposition A.10, page 389 from [7] to infer that $\mathbf{A}^\varepsilon$ is self-adjoint and generates an analytic semigroup $S_\varepsilon(t)$. □

By continuity, the operator $\mathbf{A}^\varepsilon$ can be extended from $\mathbf{V}$ into $\mathbf{V}'$.

For $\alpha > 0$, let us denote by $(\mathbf{A}^\varepsilon)^\alpha$ the α-power of the operator $\mathbf{A}^\varepsilon$ and $D((\mathbf{A}^\varepsilon)^\alpha)$ its domain. In particular,

$$D((\mathbf{A}^\varepsilon)^0) = \mathbf{H}^\varepsilon, \quad \text{and} \quad D((\mathbf{A}^\varepsilon)^{1/2}) = \mathbf{V}^\varepsilon.$$

For more details, see [8], page 152.

Lemma 2.2. *Let $\phi \in H^1(D^\varepsilon)$. Then*

$$\sqrt{\varepsilon} \|\phi\|_{L^2(\partial O^\varepsilon)} \leq C \|\phi\|_{H^1(D^\varepsilon)}.$$

Proof. For each $k \in \mathcal{K}^\varepsilon$, let $\phi^{\varepsilon,k}$ be the function defined in Y^* by $\phi^{\varepsilon,k}(x) = \phi\left(\frac{x}{\varepsilon} - k\right)$. Then, we have

$$\|\phi^{\varepsilon,k}\|_{L^2(\partial O)} \leq C\left(\phi^{\varepsilon,k}\|_{L^2(Y^*)} + \|\nabla\phi^{\varepsilon,k}\|_{L^2(Y^*)^n}\right).$$

We use the change of variables $x = \varepsilon x + \varepsilon k$ and add over all $k \in \mathcal{K}^\varepsilon$ to obtain the result. □

In the next section, we will set up the assumptions.

2.2. **Assumptions.** We assume that we are given three maps

$$g_1 \in C([0,T]; L_{Q_1}(L^2(D)^n, L^2(D)^n),$$

$$g_{21} \in C([0,T]; L_{Q_2}(L^2(D)^n, H^1(D)^n)$$

and

$$g_{22} \in C([0,T]; L_{Q_2}(L^2(D)^n, L^2(\partial O)^n)$$

such that there exists a positive constant C_T

$$\begin{aligned} ||g_1(t)||^2_{Q_1} &:= \sum_{j=1}^{\infty} \lambda_{j1}||g_1(t)e_{j1}||^2_{L^2(D)^n} \leq C_T,\ t \in [0,T], \\ ||g_{21}(t)||^2_{Q_2} &:= \sum_{j=1}^{\infty} \lambda_{j2}||g_{21}(t)e_{j2}||^2_{H^1(D)^n} \leq C_T,\ t \in [0,T], \\ ||g_{22}(t)||^2_{Q_2} &:= \sum_{j=1}^{\infty} \lambda_{j2}||g_{22}(t)e_{j2}||^2_{L^2(\partial O)^n} \leq C_T,\ t \in [0,T], \end{aligned} \tag{2.9}$$

where $\{e_{j1}\}_{j=1}^\infty$ and $\{e_{j2}\}_{j=1}^\infty$ are respectively the eigenfunctions for Q_1 and Q_2, and $\{\lambda_{j1}\}_{j=1}^\infty$ and $\{\lambda_{j2}\}_{j=1}^\infty$ are the corresponding sequences of eigenvalues. For any element $h \in L^2(\partial O)$, we define the element $\mathcal{R}^\varepsilon h \in L^2(\partial O^\varepsilon)$ by

$$\mathcal{R}^\varepsilon h(x) = h\left(\frac{x}{\varepsilon}\right) \tag{2.10}$$

where h is considered $Y-$ periodic.

Lemma 2.3. *There exists a constant C independent of $\varepsilon > 0$ and $h \in L^2(\partial O)^n$) such that*

$$\sqrt{\varepsilon}\|\mathcal{R}^\varepsilon h\|_{L^2(\partial O^\varepsilon)^n} \leq C\|h\|_{L^2(\partial O)^n}.$$

Proof. We use the definition of $\mathcal{R}^\varepsilon h$ to obtain:

$$\|\mathcal{R}^\varepsilon h\|^2_{L^2(\partial O^\varepsilon)^n} = \int_{\partial O^\varepsilon} h^2\left(\frac{x'}{\varepsilon}\right) d\sigma(x') = \sum_{k\in\mathcal{K}^\varepsilon} \int_{\partial O^\varepsilon_k} h^2\left(\frac{x'}{\varepsilon}\right).$$

After a change of variables and given the periodicity of the function h, we obtain that

$$\begin{aligned} \|\mathcal{R}^\varepsilon h\|^2_{L^2(\partial O^\varepsilon)^n} &= \sum_{k\in\mathcal{K}^\varepsilon} \varepsilon^{n-1} \sum_{k\in\mathcal{K}^\varepsilon} \|h\|^2_{L^2(\partial O)^n} \\ &\leq \varepsilon^{n-1} C\varepsilon^{-n}\|h\|^2_{L^2(\partial O)^n} = C\varepsilon^{-1}\|h\|^2_{L^2(\partial O^\varepsilon)^n}, \end{aligned}$$

which gives the result. □

We define $g_2^\varepsilon(t)$ by

$$g_2^\varepsilon(t) = g_{21}(t) + \mathcal{R}^\varepsilon g_{22}(t),$$

where $\mathcal{R}^\varepsilon g_{22} \in C([0,T]; L_{Q_2}(L^2(D)^n, L^2(\partial O^\varepsilon)^n)$. Moreover, throughout the paper we will assume that $\mathbf{U}_0^\varepsilon = (u_0^\varepsilon, \overline{u_0^\varepsilon} = \varepsilon v_0^\varepsilon)$ is an $\mathcal{F}_0-$ measurable $\mathbf{H}^\varepsilon-$ valued random variable and there exists a constant C independent of ε, such that for every $\varepsilon > 0$:

$$\mathbb{E}\|u_0^\varepsilon\|^2_{L^2(D^\varepsilon)^n} + \varepsilon \mathbb{E}\|v_0^\varepsilon\|^2_{L^2(\partial O^\varepsilon)} \le C. \tag{2.11}$$

For any function z^ε defined in D^ε let us denote by $\widetilde{z^\varepsilon}$ the extension of z^ε to D by

$$\widetilde{z^\varepsilon} := \begin{cases} z^\varepsilon & \text{on } D^\varepsilon \\ 0 & \text{on } D \setminus D^\varepsilon \end{cases} \tag{2.12}$$

3. **The microscopic model.** This section is devoted to the study of system (1.1). We first prove the existence and uniqueness of mild and weak solutions and state some uniform estimates with respect to ε. Since we are using the two-scale convergence method for the passage to the limit on ε, a variational formulation is needed with test functions that are not divergence free . Hence, a variational formulation that contains a pressure term will be recovered. However, due to the stochastic integral and its lack of regularity in time, we will define a new process $U^\varepsilon(t) = \int_0^t u^\varepsilon(s)ds$ and recover a pressure term $P^\varepsilon(t) \in L^2(D^\varepsilon)$. Independently from the passage to the limit in ε, the results stated in this section are not classical. In particular, theorem 3.3 and the variational formulation (3.11) are new results.

Theorem 3.1. *(Well posedness of the microscopic model) Assume that (2.9) holds, then for any $T > 0$ and any $\mathbf{U}_0^\varepsilon$ an $\mathbf{H}^\varepsilon-$ valued measurable random variable, the system (1.1) has a unique mild solution $\mathbf{U}^\varepsilon \in L^2(\Omega, C([0,T], \mathbf{H}^\varepsilon) \cap L^2(0,T; \mathbf{V}^\varepsilon))$,*

$$\mathbf{U}^\varepsilon(t) = S_\varepsilon(t)\mathbf{U}_0^\varepsilon + \int_0^t S_\varepsilon(t-s)\mathbf{F}(s)ds + \int_0^t S_\varepsilon(t-s)\mathbf{G}^\varepsilon(s)d\mathbf{W}, \quad t \in [0,T]. \tag{3.1}$$

The mild solution $\mathbf{U}^\varepsilon$ is also a weak solution, that is, $\mathbb{P}$-a.s.

$$\langle \mathbf{U}^\varepsilon(t), \boldsymbol{\phi}\rangle + \int_0^t \langle (\mathbf{A}^\varepsilon)^{1/2}\mathbf{U}^\varepsilon(s), (\mathbf{A}^\varepsilon)^{1/2}\boldsymbol{\phi}\rangle ds = \langle \mathbf{U}_0^\varepsilon, \boldsymbol{\phi}\rangle + \int_0^t \langle \mathbf{F}(s), \boldsymbol{\phi}\rangle ds + \int_0^t \langle \mathbf{G}^\varepsilon(s)d\mathbf{W}(s), \boldsymbol{\phi}\rangle \tag{3.2}$$

for $t \in [0,T]$ and $\boldsymbol{\phi} \in \mathbf{V}^\varepsilon$.

Moreover, if (2.11) holds then for every $\varepsilon > 0$

$$\mathbb{E}\|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} + \mathbb{E}\int_0^t \|\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{V}^\varepsilon} ds \le C_T(1 + \mathbb{E}\|\mathbf{U}_0^\varepsilon\|^2_{\mathbf{H}^\varepsilon}), \quad t \in [0,T] \tag{3.3}$$

and

$$\mathbb{E} \sup_{t\in[0,T]} \|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} \le C_T(1 + \mathbb{E}\|\mathbf{U}_0^\varepsilon\|^2_{\mathbf{H}^\varepsilon}). \tag{3.4}$$

Proof. Since the operator $\mathbf{A}^\varepsilon$ is the generator of a strongly continuous semigroup $S_\varepsilon(t)$, $t \ge 0$ in $\mathbf{H}^\varepsilon$ and using the assumption (2.9), then the existence and uniqueness of mild solutions in $\mathbf{H}^\varepsilon$ is a consequence of Theorem 7.4 of [7]. The regularity in $\mathbf{V}^\varepsilon$ is a consequence of the estimates below.

Applying the Itô formula to $\|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon}$ see [9], we get that

$$\begin{aligned} d\|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} &= 2\langle \mathbf{U}^\varepsilon(t), d\mathbf{U}^\varepsilon(t)\rangle dt + \|\mathbf{G}^\varepsilon(t)\|^2_Q dt \\ &= -2\langle \mathbf{A}^\varepsilon \mathbf{U}^\varepsilon(t), \mathbf{U}^\varepsilon(t)\rangle dt + 2\langle \mathbf{F}(s), \mathbf{U}^\varepsilon(t)\rangle dt \\ &\quad + 2\langle \mathbf{G}^\varepsilon(t) d\mathbf{W}, \mathbf{U}^\varepsilon(t)\rangle dt + \|\mathbf{G}^\varepsilon(t)\|^2_Q dt. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} + 2\int_0^t \|(\mathbf{A}^\varepsilon)^{1/2}\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{H}^\varepsilon} ds \le \|\mathbf{U}^\varepsilon_0\|^2_{\mathbf{H}^\varepsilon} + \int_0^t \|\mathbf{F}(s)\|^2_{\mathbf{H}^\varepsilon} ds \\ + \int_0^t \|\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{H}^\varepsilon} ds + 2\int_0^t \langle \mathbf{G}^\varepsilon(s) d\mathbf{W}, \mathbf{U}^\varepsilon(t)\rangle + \int_0^t \|\mathbf{G}^\varepsilon(s)\|^2_Q ds. \end{aligned} \tag{3.5}$$

Taking the expected value yields

$$\begin{aligned} \mathbb{E}\|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} \le \mathbb{E}\|\mathbf{U}^\varepsilon_0\|^2_{\mathbf{H}^\varepsilon} + \int_0^t \|\mathbf{F}(s)\|^2_{\mathbf{H}^\varepsilon} ds + \int_0^t \mathbb{E}\|\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{H}^\varepsilon} ds \\ + \int_0^t \|\mathbf{G}^\varepsilon(s)\|^2_Q ds \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \mathbb{E}\int_0^t \|\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{V}^\varepsilon} ds \le \mathbb{E}\|\mathbf{U}^\varepsilon_0\|^2_{\mathbf{H}^\varepsilon} + \int_0^t \|\mathbf{F}(s)\|^2_{\mathbf{H}^\varepsilon} ds + \int_0^t \mathbb{E}\|\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{H}^\varepsilon} ds \\ + \int_0^t \|\mathbf{G}^\varepsilon(s)\|^2_Q ds \end{aligned} \tag{3.7}$$

Now (3.3) follows from using Gronwall's lemma in (3.6).

On the other side, (3.5) implies that

$$\begin{aligned} \sup_{0\le t\le T} \|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} + 2\int_0^T \|\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{V}^\varepsilon} ds \le \|\mathbf{U}^\varepsilon_0\|^2_{\mathbf{H}^\varepsilon} + \int_0^T \|\mathbf{F}(s)\|^2_{\mathbf{H}^\varepsilon} ds \\ + \int_0^T \|\mathbf{U}^\varepsilon(s)\|^2_{\mathbf{H}^\varepsilon} ds + 2\sup_{0\le t\le T}\left|\int_0^t \langle \mathbf{G}^\varepsilon(s) d\mathbf{W}, \mathbf{U}^\varepsilon(t)\rangle\right| + \int_0^T \|\mathbf{G}^\varepsilon(s)\|^2_Q ds \end{aligned}$$

Moreover using the Burkholder-David-Gundy inequality and the Young inequality, we get that

$$\begin{aligned} \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \langle \mathbf{G}^\varepsilon(s) d\mathbf{W}, \mathbf{U}^\varepsilon(t)\rangle\right| &\le \mathbb{E}\left(\int_0^T |\mathbf{G}^\varepsilon(s)\mathbf{U}^\varepsilon(s)|^2\, ds\right)^{1/2} \\ &\le \frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} + C\int_0^T |\mathbf{G}^\varepsilon(s)|^2 ds \\ &\le \frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\|\mathbf{U}^\varepsilon(t)\|^2_{\mathbf{H}^\varepsilon} + C_T \end{aligned}$$

Now, plugging this estimate in the previous one and using Gronwall's lemma completes the proof of (3.4). □

Theorem 3.2. *Assume that the assumptions of Theorem 3.1 hold. Then, for every $t \in [0, T]$ and every $\varepsilon > 0$*

$$\begin{aligned} \mathbb{E}\|u^\varepsilon(t)\|^2_{H^\varepsilon} + \varepsilon^2\mathbb{E}\|\gamma_\varepsilon u^\varepsilon(t)\|^2_{L^2(\partial O^\varepsilon)^n} + \mathbb{E}\int_0^t \left(\|u^\varepsilon(s)\|^2_{V^\varepsilon} + \varepsilon^2\|\gamma_\varepsilon u^\varepsilon(s)\|^2_{H^{\frac{1}{2}}(\partial O^\varepsilon)^n}\right) ds \\ \le C_T(1 + \mathbb{E}\|\mathbf{U}^\varepsilon_0\|^2_{\mathbf{H}^\varepsilon}), \end{aligned} \tag{3.8}$$

and

$$\mathbb{E}\sup_{t\in[0,T]}\left(\|u^\varepsilon(t)\|^2_{H^\varepsilon}+\varepsilon^2\|\gamma_\varepsilon u^\varepsilon(t)\|^2_{L^2(\partial O^\varepsilon)^n}\right)\le C_T(1+\mathbb{E}\|\mathbf{U}_0^\varepsilon\|^2_{\mathbf{H}^\varepsilon}). \tag{3.9}$$

Moreover, for $q\ge 2$

$$\sup_{t\in[0,T]}\left(\mathbb{E}\|u^\varepsilon(t)\|^q_{H^\varepsilon}+\varepsilon^2\mathbb{E}\|\gamma_\varepsilon u^\varepsilon(t)\|^q_{L^2(\partial O^\varepsilon)^n}\right)\le C(q,T)(1+\mathbb{E}\|\mathbf{U}_0^\varepsilon\|^q_{\mathbf{H}^\varepsilon}), \tag{3.10}$$

Proof. (3.8), (3.9) and (3.10) are consequences of Theorem 3.1. ☐

Theorem 3.3. *Let us define* $U^\varepsilon(t)=\int_0^t u^\varepsilon(s)ds$, *where* u^ε *is given by the previous theorem, with* $U^\varepsilon\in C([0,T];V^\varepsilon)$. *Then, there exists a unique vector valued random variable* $P^\varepsilon:\Omega\mapsto C([0,T];L^2(D^\varepsilon))$, *such that for every* $t\in[0,T]$ *we have:*

$$\begin{aligned}&\int_{D^\varepsilon}\left(u^\varepsilon(t)-u_0^\varepsilon-\int_0^t f(s)-\int_0^t g_1(s)dW_1(s)\right)\phi dx+\\ &\int_{D^\varepsilon}\left(\nu\nabla U^\varepsilon(t)\nabla\phi-P^\varepsilon(t)\operatorname{div}\phi\right)dx=\\ &\int_{\partial O^\varepsilon}\left(\varepsilon^2u^\varepsilon(t)-\varepsilon^2v_0^\varepsilon+\int_0^t\varepsilon g_2^\varepsilon(s)dW_2(s)ds-\varepsilon bU^\varepsilon(t)\right)\phi d\sigma,\end{aligned} \tag{3.11}$$

$\mathbb{P}$*-a.s., and for every* $\phi\in H_0^1(D)^n$.

Moreover,

$$\sup_{0\le t\le T}\mathbb{E}\|P^\varepsilon(t)\|^2_{L^2(D^\varepsilon)^n}\le C(T). \tag{3.12}$$

Proof. We recall (3.2) which says that $\mathbf{U}^\varepsilon(t)=(u^\varepsilon,\varepsilon\gamma_\varepsilon(u^\varepsilon))$ is a variational solution in the following sense:

$$\langle\mathbf{U}^\varepsilon(t),\boldsymbol{\phi}\rangle+\int_0^t\langle(\mathbf{A}^\varepsilon)^{1/2}\mathbf{U}^\varepsilon(s),(\mathbf{A}^\varepsilon)^{1/2}\boldsymbol{\phi}\rangle ds=\langle\mathbf{U}_0^\varepsilon,\boldsymbol{\phi}\rangle+\int_0^t\langle\mathbf{F}(s),\boldsymbol{\phi}\rangle ds+\int_0^t\langle\mathbf{G}^\varepsilon(s)d\mathbf{W}(s),\boldsymbol{\phi}\rangle$$

$\mathbb{P}-$ a.s. for $t\in[0,T]$ and $\boldsymbol{\phi}\in\mathbf{V}^\varepsilon$.

In this variational formulation we consider the test function $\boldsymbol{\phi}=(\phi,0)$, where $\phi=\Pi^\varepsilon\boldsymbol{\phi}\in V^\varepsilon$, hence $\mathbb{P}$-a.s. we get

$$\langle u^\varepsilon(t)-u_0^\varepsilon-\nu\int_0^t\Delta u^\varepsilon(s)ds-\int_0^t f(s)-\int_0^t g_1(s)dW_1(s),\phi\rangle=0 \tag{3.13}$$

$\forall t\in[0,T],\quad\forall\phi\in V^\varepsilon\cap H_0^1(D^\varepsilon)^n$.

Using Proposition I.1.1 and I.1.2. of [21], we find for each $t\in[0,T]$, the existence of some $P^\varepsilon(t)\in L^2(D^\varepsilon)^n/\mathbb{R}$ such that $\mathbb{P}$-a.s.

$$u^\varepsilon(t)-u_0^\varepsilon-\nu\Delta U^\varepsilon(t)-\int_0^t f(s)-\int_0^t g_1(s)dW_1(s)=-\nabla P^\varepsilon(t), \tag{3.14}$$

where,

$$L^2(D^\varepsilon)^n/\mathbb{R}=\left\{P\in L^2(D^\varepsilon)^n,\quad\int_{D^\varepsilon}P(x)dx=0\right\}.$$

In particular, $\nabla P^\varepsilon\in C([0,T];H^{-1}(D^\varepsilon)^n)\quad\mathbb{P}-$ a.s..

The equation (3.14) may be written equivalently:

$$\begin{aligned}\int_{D^\varepsilon} (u^\varepsilon(t) - u_0^\varepsilon)\,\phi dx + \int_{D^\varepsilon} (\nu \nabla U^\varepsilon(t)\nabla\phi - P^\varepsilon(t)\operatorname{div}\phi)\,dx \\ = \int_{D^\varepsilon}\int_0^t f(s)\phi ds dx + \int_{D^\varepsilon}\int_0^t g_1(s)\phi dW_1(s)dx,\end{aligned} \tag{3.15}$$

for every $\phi \in H_0^1(D^\varepsilon)^n$.

The equation (3.14) gives that $\mathbb{P}$ a.s., $\operatorname{div}(\nu\nabla U^\varepsilon(t) - P^\varepsilon I)$ belongs to $L^2(D^\varepsilon)^n$, which implies that $\nu\dfrac{\partial U^\varepsilon(t)}{\partial n} - P^\varepsilon(t)n \in H^{-\frac{1}{2}}(\partial O^\varepsilon)^n$ (see [21]). Hence, for every $\phi \in H^1(D^\varepsilon)^n$, with $\phi_{|\partial D} = 0$ we have

$$\begin{aligned}\left\langle u^\varepsilon(t) - u_0^\varepsilon - \int_0^t f(s) - \int_0^t g_1(s)dW_1(s), \phi\right\rangle + \int_{D^\varepsilon} (\nu\nabla U^\varepsilon(t)\nabla\phi - P^\varepsilon(t)\operatorname{div}\phi)\,dx \\ = \left\langle \nu\frac{\partial U^\varepsilon(t)}{\partial n} - P^\varepsilon(t)n, \phi\right\rangle_{H^{-\frac{1}{2}}(\partial O^\varepsilon)^n, H^{\frac{1}{2}}(\partial O^\varepsilon)^n}.\end{aligned} \tag{3.16}$$

Any $\phi \in H^{\frac{1}{2}}(\partial O^\varepsilon)^n$ such that $\int_{\partial O^\varepsilon}\phi n d\sigma = 0$ is the trace on ∂O^ε of some function from V^ε that we will denote also by ϕ. We use such a function ϕ in (3.16) and get

$$\begin{aligned}\left\langle u^\varepsilon(t) - u_0^\varepsilon - \int_0^t f(s) - \int_0^t g_1(s)dW_1(s), \phi\right\rangle + \int_{D^\varepsilon} \nu\nabla U^\varepsilon(t)\nabla\phi dx = \\ \left\langle \nu\frac{\partial U^\varepsilon(t)}{\partial n} - P^\varepsilon(t)n, \phi\right\rangle_{H^{-\frac{1}{2}}(\partial O^\varepsilon)^n, H^{\frac{1}{2}}(\partial O^\varepsilon)^n}.\end{aligned}$$

Now, we use the variational formulation (3.2) and obtain that for every $\phi \in H^{\frac{1}{2}}(\partial O^\varepsilon)^n$ such that $\int_{\partial O^\varepsilon}\phi n d\sigma = 0$, we have

$$\begin{aligned}-\left\langle \nu\frac{\partial U^\varepsilon(t)}{\partial n} - P^\varepsilon(t)n, \phi\right\rangle_{H^{-\frac{1}{2}}(\partial O^\varepsilon)^n, H^{\frac{1}{2}}(\partial O^\varepsilon)^n} \\ = \int_{\partial O^\varepsilon}\left(\varepsilon^2 u^\varepsilon(t) - \varepsilon\overline{u_0^\varepsilon} + \varepsilon b U^\varepsilon(t) - \int_0^t g_2^\varepsilon(s)dW_2(s)\right)\phi d\sigma.\end{aligned} \tag{3.17}$$

But the orthogonal of the space of such functions ϕ is the space of functions of the type cn, where c is a real constant. So, there exists $c^\varepsilon \in C([0,T])$ such that if we re-denote $P^\varepsilon(t) = P^\varepsilon(t) + c^\varepsilon(t)$, (3.17) is satisfied for every $\phi \in H^{\frac{1}{2}}(\partial O^\varepsilon)^n$. Using again (3.2), we showed that there exists a unique $P^\varepsilon \in C([0,T], L^2(D^\varepsilon))$, such that for $t \in [0,T]$ (3.11) holds.

To show (3.12) we consider the extension to the whole domain of the pressure $\widetilde{P}^\varepsilon(t)$, and compute

$$\mathbb{E}\|\nabla\widetilde{P}^\varepsilon(t)\|^2_{H^{-1}(D)^n} = \mathbb{E}\sup_{\substack{\phi\in H_0^1(D)^n\\ \|\phi\|_{H_0^1(D)^n}\le 1}}\left|\int_D \widetilde{P}^\varepsilon(t)\operatorname{div}\phi dx\right|^2,$$

from (3.11). We show that it is bounded uniformly for $t \in [0,T]$, and then apply the result given by Proposition 1.2 from [21] to obtain (3.12). We estimate first

$\left|\int_D \widetilde{P}^\varepsilon(t)\,\mathrm{div}\,\phi dx\right|$ from (3.11) and obtain by applying Hölder's inequality:

$$\left|\int_D \widetilde{P}^\varepsilon(t)\,\mathrm{div}\,\phi dx\right| \le \|\phi\|_{H_0^1(D)^n}\left(\|u^\varepsilon(t)\|_{L^2(D^\varepsilon)^n} + \|u_0^\varepsilon\|_{L^2(D^\varepsilon)^n} + T\|f\|_{L^2([0,T]\times D)^n}\right)$$
$$+\|\phi\|_{H_0^1(D)^n}\left(\|\int_0^t g_1(s)dW_1(s)\|_{L^2(D^\varepsilon)^n} + \nu\|U^\varepsilon(t)\|_{H_0^1(D^\varepsilon)^n}\right)$$
$$+\|\phi\|_{L^2(\partial O^\varepsilon)^n}\left(\|\varepsilon^2 u^\varepsilon(t) + \varepsilon^2 v_0^\varepsilon + \varepsilon b U^\varepsilon(t) + \int_0^t g_2^\varepsilon(s)dW_2(s)\|_{L^2(\partial O^\varepsilon)^n}\right).$$

We use Lemma 2.2 and the estimates (3.8) for $u^\varepsilon(t)$ and (2.11) for the initial conditions to get

$$\mathbb{E}\|\nabla\widetilde{P}^\varepsilon(t)\|^2_{H^{-1}(D)^n} \le C + C\mathbb{E}\|\int_0^t g_1(s)dW_1\|^2_{L^2(D^\varepsilon)^n} + C\varepsilon\mathbb{E}\|u^\varepsilon(t)\|^2_{L^2(\partial O^\varepsilon)^n} +$$
$$\varepsilon\mathbb{E}\|\int_0^t g_{21}(s)dW_2(s)\|^2_{L^2(\partial O^\varepsilon)^n} + \varepsilon\mathbb{E}\|\int_0^t \mathcal{R}^\varepsilon g_{22}(s)dW_2(s)\|^2_{L^2(\partial O^\varepsilon)^n}.$$

Ito's isometry gives

$$\mathbb{E}\|\nabla\widetilde{P}^\varepsilon(t)\|^2_{H^{-1}(D)^n} \le C + C\int_0^t\sum_{i=1}^\infty \lambda_{i1}\|g_1(s)e_{i1}\|^2_{L^2(D^\varepsilon)^n}ds +$$
$$C\varepsilon\int_0^t\sum_{i=1}^\infty \lambda_{i2}\|g_{21}(s)e_{i2}\|^2_{L^2(\partial O^\varepsilon)^n}ds + C\varepsilon\int_0^t\sum_{i=1}^\infty \lambda_{i2}\|\mathcal{R}^\varepsilon g_{22}(s)e_{i2}\|^2_{L^2(\partial O^\varepsilon)^n}ds.$$

We use again Lemmas 2.2 and 2.3 and then property (2.9) to obtain that $\mathbb{E}\|\nabla\widetilde{P}^\varepsilon(t)\|^2_{H^{-1}(D)^n}$ is bounded uniformly for $t\in[0,T]$. □

4. **Two scale convergence.** We will summarize in this section several results about the two scale convergence that we will use throughout the paper. Although the results stated in theorems 4.2 and 4.4 are not new, they had to be adapted to our setting; in fact our processes depend on ω, t, x. On the other side, the two scale limits are used in the variational formulation 3.11 that contains terms of the form $U^\varepsilon(t) = \int_0^t u^\varepsilon(s)ds$. Hence, we had to include a two-scale convergence result stated in theorem 4.5 that takes into account these kind of terms. This last theorem was not found in the literature.

For the results stated without proofs, see [1] or [26]. First we establish some notations of spaces of periodic functions. We denote by $C^k_\#(Y)$ the space of functions from $C^k(\overline{Y})$, that have $Y-$ periodic boundary values. By $L^2_\#(Y)$ we understand the closure of $C_\#(Y)$ in $L^2(Y)$ and by $H^1_\#(Y)$ the closure of $C^1_\#(Y)$ in $H^1(Y)$.

Definition 4.1. We say that a sequence $u^\varepsilon \in L^2(\Omega\times[0,T]\times D)^n$ two-scale converges to $u\in L^2(\Omega\times[0,T]\times D\times Y)^n$, and denote this convergence by

$$u^\varepsilon \xrightarrow{2-s} u \quad \text{in } \Omega\times[0,T]\times D,$$

if for every $\Psi \in L^2(\Omega\times[0,T]\times D; C_\#(Y))^n$ we have

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_D u^\varepsilon(\omega,t,x)\Psi(\omega,t,x,\frac{x}{\varepsilon})dxdtd\mathbb{P}$$
$$=\int_\Omega\int_0^T\int_D\int_Y u(\omega,t,x,y)\Psi(\omega,t,x,y)dydxdtd\mathbb{P}.$$

Theorem 4.2. *Assume the sequence u^ε is uniformly bounded in $L^2(\Omega\times[0,T]\times D)^n$. Then there exists a subsequence $(u^{\varepsilon'})_{\varepsilon'>0}$ and $u^0 \in L^2(\Omega\times[0,T]\times D\times Y)^n$ such that $u^{\varepsilon'}$ two-scale converges to u^0 in $L^2(\Omega\times[0,T]\times D)^n$.*

Corollary 4.3. *Assume the sequence $u^\varepsilon \in L^2(\Omega\times[0,T]\times D)^n$, two-scale converges to $u^0 \in L^2(\Omega\times[0,T]\times D\times Y)^n$. Then, u^ε converges weakly in $L^2(\Omega\times[0,T]\times D)^n$ to $\int_Y u^0(\omega,t,x,y)dy$.*

Theorem 4.4. *Let for any $\varepsilon>0$, $u^\varepsilon \in L^2(\Omega\times[0,T];H^1(D^\varepsilon)^n)$ such that $u^\varepsilon=0$ on ∂D. Assume that u^ε is a sequence uniformly bounded with respect to $\varepsilon>0$, i.e.*

$$\sup_\varepsilon ||u^\varepsilon||_{L^2(\Omega\times[0,T];H^1(D^\varepsilon)^n)} < \infty.$$

Then, there exists $u^0 \in L^2(\Omega\times[0,T];H^1_0(D)^n)$ and $u^1 \in L^2(\Omega\times[0,T]\times D;H^1_\#(Y)^n)$, such that

$$\widetilde{u}^\varepsilon(\omega,t,x) \xrightarrow{2-s} u^0(\omega,t,x)\mathbb{1}_{Y^*}(y)$$

and

$$\widetilde{\nabla u}^\varepsilon(\omega,t,x) \xrightarrow{2-s} \left(\nabla_x u^0(\omega,t,x)+\nabla_y u^1(\omega,t,x,y)\right)\mathbb{1}_{Y^*}(y).$$

Theorem 4.5. *Assume that the sequence u^ε two scale converges to $u \in L^2(\Omega\times[0,T]\times D\times Y)^n$. Then the sequence $U^\varepsilon(\omega,t,x)$ defined by*

$$U^\varepsilon(\omega,t,x)=\int_0^t u^\varepsilon(\omega,s,x)ds$$

two scale converges to $\int_0^t u(\omega,s,x,y)ds$.

Proof. We have for the sequence u^ε that

$$\lim_{\varepsilon\to0}\int_\Omega\int_0^T\int_D u^\varepsilon(\omega,t,x)\Psi(\omega,t,x,\frac{x}{\varepsilon})dxdtd\mathbb{P}$$
$$=\int_\Omega\int_0^T\int_D\int_Y u(\omega,t,x,y)\Psi(\omega,t,x,y)dydxdtd\mathbb{P},$$

for every $\Psi \in L^2(\Omega\times[0,T]\times D;C_\#(Y))^n$. Now if we choose Ψ to be of the form

$$\Psi(\omega,s,x,y)=\mathbb{1}_{[0,t]}(s)\Psi_1(\omega,x,y),$$

we obtain that

$$\lim_{\varepsilon\to0}\int_\Omega\int_0^t\int_D u^\varepsilon(\omega,s,x)\Psi_1(\omega,x,\frac{x}{\varepsilon})dxd\mathbb{P}$$
$$=\int_\Omega\int_0^t\int_D\int_Y u(\omega,s,x,y)\Psi_1(\omega,x,y)dydxd\mathbb{P},$$

for any fixed $t\in[0,T]$ and any $\Psi_1 \in L^2(\Omega\times D;C_\#(Y))^n$. As a consequence, for $\Psi_2 \in L^2(\Omega\times[0,T]\times D;C_\#(Y))^n$ the sequence

$$h^\varepsilon(t)=\int_\Omega\int_0^t\int_D u^\varepsilon(\omega,s,x)\Psi_2(\omega,t,x,\frac{x}{\varepsilon})dxdsd\mathbb{P}$$
$$=\int_\Omega\int_D U^\varepsilon(\omega,t,x)\Psi_2(\omega,t,x,\frac{x}{\varepsilon})dxd\mathbb{P}$$

is convergent for almost every $t \in [0,T]$ to

$$\begin{aligned} h(t) &= \int_\Omega \int_0^t \int_D \int_Y u(\omega,s,x,y)\Psi_2(\omega,t,x,y)dydxdsd\mathbb{P} \\ &= \int_\Omega \int_D \int_Y U(\omega,t,x,y)\Psi_2(\omega,t,x,y)dydxd\mathbb{P}. \end{aligned}$$

By applying Hölder's inequality,

$$h^\varepsilon(t) \le ||u^\varepsilon||_{L^2(\Omega\times[0,T]\times D)^n}||\Psi_2(t)||_{L^2(\Omega\times D\times Y)^n}.$$

We use the dominated convergence theorem to deduce that

$$\int_0^T h^\varepsilon(t)dt \to \int_0^T h(t)dt.$$

□

5. **Passage to the limit in ε.** Now, we are able to pass to the limit in ε. Using the uniform estimates obtained previously and the two-scale convergence method, we will pass to the limit in ε in each term of the variational formulation (3.11).

Since $\widetilde{u}_0^\varepsilon$ satisfies $\mathbb{E}\|\widetilde{u}_0^\varepsilon\|^2_{L^2(D^\varepsilon)^n} \le C$ according to (2.11), then using Theorem 4.2, there exists $u_0 \in L^2(\Omega; L^2(D; L^2_\#(Y))^n$ such that

$$\widetilde{u}_0^\varepsilon(x) \xrightarrow{2-s} u_0(x,y)\mathbb{1}_{Y^*}(y) \quad \text{in } \Omega\times D. \tag{5.1}$$

We recall that the processes $\widetilde{u}^\varepsilon$ and $\widetilde{\nabla u}^\varepsilon$ defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ are uniformly bounded in $L^2(\Omega; C([0,T]; L^2(D)^n))$ as well as in $L^2(\Omega; L^2(0, T; L^2(D)^{n\times n}))$. So using Theorem 4.4 there exist $u \in L^2(\Omega; L^2(0,T; H^1_0(D)^n))$ and $u_1 \in L^2(\Omega; L^2(0,T; L^2(D; H^1_\#(Y))^n))$ such that

$$\widetilde{u}^\varepsilon(\omega,t,x) \xrightarrow{2-s} u(\omega,t,x)\mathbb{1}_{Y^*}(y) \quad \text{in } \Omega\times[0,T]\times D, \tag{5.2}$$

and

$$\widetilde{\nabla u}^\varepsilon(\omega,t,x) \xrightarrow{2-s} (\nabla_x u(\omega,t,x) + \nabla_y u_1(\omega,t,x,y))\,\mathbb{1}_{Y^*}(y) \quad \text{in } \Omega\times[0,T]\times D. \tag{5.3}$$

We use Theorem 4.5 to obtain as consequences of (5.2) and (5.3) the following two scale convergences:

$$\widetilde{U}^\varepsilon(\omega,t,x) \xrightarrow{2-s} U(\omega,t,x)\mathbb{1}_{Y^*}(y) \quad \text{in } \Omega\times[0,T]\times D, \tag{5.4}$$

and

$$\widetilde{\nabla U}^\varepsilon(\omega,t,x) \xrightarrow{2-s} (\nabla_x U(\omega,t,x) + \nabla_y U_1(\omega,t,x,y))\,\mathbb{1}_{Y^*}(y) \quad \text{in } \Omega\times[0,T]\times D, \tag{5.5}$$

where $U(\omega,t,x) = \int_0^t u(\omega,s,x)ds$ and $U_1(\omega,t,x,y) = \int_0^t u_1(\omega,s,x,y)ds$.

Also (3.12) gives us the existence of $P \in L^2(\Omega\times[0,T]\times D\times Y)^n$ such that:

$$\widetilde{P}^\varepsilon(\omega,t,x) \xrightarrow{2-s} P(\omega,t,x,y)\mathbb{1}_{Y^*}(y) \quad \text{in } \Omega\times[0,T]\times D. \tag{5.6}$$

Now we are able to pass to the limit in (3.11). We will take a test function of the form

$$\phi^\varepsilon(\omega,t,x) = \phi(\omega,t,x) + \varepsilon\psi(\omega,t,x,\frac{x}{\varepsilon}),$$

such that

$$\phi(\omega,t,x) = \phi_1(t)\phi_2(\omega)\phi_3(x), \tag{5.7}$$

where $\phi_1 \in C_0^\infty(0,T)$, $\phi_2 \in L^\infty(\Omega)$ and $\phi_3 \in C_0^\infty(D)^n$ and

$$\psi(\omega,t,x,y) = \psi_1(t)\psi_2(\omega)\psi_3(x)\psi_4(y), \tag{5.8}$$

where $\psi_1 \in C_0^\infty(0,T)$, $\psi_2 \in L^\infty(\Omega)$, $\psi_3 \in C_0^\infty(D)^n$ and $\psi_4 \in C_\#^\infty(Y)$. We obtain after integrating over $\Omega \times [0,T]$:

$$\int_\Omega \int_0^T \int_D \widetilde{u}^\varepsilon(\omega,t,x)\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} + \varepsilon^2 \int_\Omega \int_0^T \int_{\partial O^\varepsilon} u^\varepsilon(\omega,t,x')\phi^\varepsilon(\omega,t,x')d\sigma(x')dtd\mathbb{P} -$$
$$\int_\Omega \int_0^T \int_D \widetilde{u}_0^\varepsilon(\omega,x)\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} - \varepsilon^2 \int_\Omega \int_0^T \int_{\partial O^\varepsilon} v_0^\varepsilon(x')\phi^\varepsilon(\omega,t,x')d\sigma(x')dtd\mathbb{P} +$$
$$\int_\Omega \int_0^T \int_D \int_0^t \nu \widetilde{\nabla u}^\varepsilon(\omega,s,x)\nabla\phi^\varepsilon(\omega,t,x)dsdxdtd\mathbb{P} +$$
$$\int_\Omega \int_0^T \int_{\partial O^\varepsilon} \int_0^t \varepsilon b u^\varepsilon(\omega,s,x')\phi^\varepsilon(\omega,t,x')dsd\sigma(x')dtd\mathbb{P} =$$
$$\int_\Omega \int_0^T \int_D \widetilde{P}^\varepsilon(\omega,t,x)\operatorname{div}\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} + \int_\Omega \int_0^T \int_{D^\varepsilon} \int_0^t f(s,x)\phi^\varepsilon(\omega,t,x)dsdxdtd\mathbb{P} +$$
$$\int_\Omega \int_0^T \int_{D^\varepsilon} \int_0^t g_1(s)dW_1(s)\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} +$$
$$\int_\Omega \int_0^T \int_{\partial O^\varepsilon} \int_0^t \varepsilon g_2^\varepsilon(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')dtd\mathbb{P}.$$

We will compute the limit when $\varepsilon \to 0$ in (5.9) term by term:

As a consequence of (5.2) and (5.1) which implies the weak convergences in $L^2(\Omega\times[0,T]\times D)^n$ of the sequences $\widetilde{u}^\varepsilon(\omega,t,x)$ and $\widetilde{u}_0^\varepsilon(\omega,x)$ to $\int_Y u(\omega,t,x)\mathbb{1}_{Y^*}(y)dy$ and to $\int_Y u_0(\omega,x)\mathbb{1}_{Y^*}(y)dy$ we have:

$$\lim_{\varepsilon\to 0} \int_\Omega \int_0^T \int_D \widetilde{u}^\varepsilon(\omega,t,x)\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} = |Y^*| \int_\Omega \int_0^T \int_D u(\omega,t,x)\phi(\omega,t,x)dxdtd\mathbb{P}, \tag{5.9}$$

and

$$\lim_{\varepsilon\to 0} \int_\Omega \int_0^T \int_D \widetilde{u}_0^\varepsilon(\omega,x)\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} = |Y^*| \int_\Omega \int_0^T \int_D u_0(\omega,x)\phi(\omega,t,x)dxdtd\mathbb{P}. \tag{5.10}$$

The estimates (2.11), (3.8), (3.9) imply that:

$$\varepsilon^2 \int_\Omega \int_0^T \int_{\partial O^\varepsilon} u^\varepsilon(\omega,t,x)\phi^\varepsilon(\omega,t,x')d\sigma(x')d\mathbb{P} = = \varepsilon^2 \int_\Omega \int_0^T \int_{\partial O^\varepsilon} v_0^\varepsilon(x)\phi^\varepsilon(\omega,t,x')d\sigma(x')dtd\mathbb{P} = 0. \tag{5.11}$$

We use the two scale convergences (5.5) and (5.6) and that $\frac{\partial}{\partial x_i}\phi^\varepsilon = \frac{\partial \phi}{\partial x_i} + \varepsilon\frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i}$ to deduce that

$$\lim_{\varepsilon\to 0} \int_\Omega \int_0^T \int_D \nu \widetilde{\nabla U}^\varepsilon(\omega,t,x)\nabla\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} =$$
$$\int_\Omega \int_0^T \int_D \int_{Y^*} \nu \left(\nabla_x U(\omega,t,x) + \nabla_y U_1(\omega,t,x,y)\right)\left(\nabla_x\phi(\omega,t,x) + \nabla_y\psi(\omega,t,x,y)\right)dydxdtd\mathbb{P}, \tag{5.12}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_D \widetilde{P}^\varepsilon(\omega,t,x)\operatorname{div}\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} =$$

$$\int_\Omega\int_0^T\int_D\int_{Y^*} P(\omega,t,x)\left(\operatorname{div}_x\phi(\omega,t,x)+\operatorname{div}_y\psi(\omega,t,x,y)\right)dydxdtd\mathbb{P}. \tag{5.13}$$

Now, if we fix $\omega\in\Omega$ and $t\in[0,T]$ the sequence of integrals

$$\int_{D^\varepsilon}\int_0^t f(s,x)\phi^\varepsilon(\omega,t,x,\frac{x}{\varepsilon})dsdx = \int_D\int_0^t f(s,x)\phi^\varepsilon(\omega,t,x,\frac{x}{\varepsilon})\mathbb{1}_{D^\varepsilon}(x)dsdx$$

converges using formula (5.5) from [1] to

$$|Y^*|\int_D\int_0^t f(s,x)\phi(\omega,t,x)dsdx.$$

Hence, after applying Vitali's theorem we obtain:

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_{D^\varepsilon}\int_0^t f(s,x)\phi^\varepsilon(\omega,t,x)dsdxdtd\mathbb{P} =$$

$$|Y^*|\int_\Omega\int_0^T\int_D\int_0^t f(s,x)\phi(\omega,t,x)dsdxdtd\mathbb{P}. \tag{5.14}$$

Now we compute the limits of the stochastic integrals and of the integrals over the boundary ∂O^ε, and the results we obtain are shown in the next three Lemmas.

Lemma 5.1.

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_{D^\varepsilon}\int_0^t g_1(s)dW_1(s)\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P} =$$

$$|Y^*|\int_\Omega\int_0^T\int_D\int_0^t g_1(s)dW_1(s)\phi(\omega,t,x)dxdtd\mathbb{P}. \tag{5.15}$$

Proof. We substitute $\phi^\varepsilon(\omega,t,x)=\phi(\omega,t,x)+\varepsilon\psi(\omega,t,x,\frac{x}{\varepsilon})$ and use the decomposition (5.7) of ϕ^ε and we get that (5.15) can be rewritten as

$$\int_\Omega\phi_1(\omega)\int_0^T\phi_2(t)\int_D\int_0^t g_1(s)dW_1(s)\left(\phi_3(x)(\mathbb{1}_{D^\varepsilon}(x)-|Y^*|)+\varepsilon\psi(\omega,t,x,\frac{x}{\varepsilon})\right)dxdtd\mathbb{P}$$

$$\longrightarrow_{\varepsilon\to 0} 0.$$

Now using the Hölder's inequality, and the fact that $||\psi(\omega,t,x,\frac{x}{\varepsilon})||_{L^2(\Omega\times[0,T]\times D)^n}$ is a sequence uniformly bounded in ε by $||\psi(\omega,t,x,y)||_{L^2(\Omega\times[0,T]\times D;C_\#(Y))^n}$, we obtain:

$$\lim_{\varepsilon\to 0}\left|\int_\Omega\int_0^T\int_D\int_0^t g_1(s)dW_1(s)\varepsilon\psi(\omega,t,x,\frac{x}{\varepsilon})dxdtd\mathbb{P}\right| \le$$

$$\lim_{\varepsilon\to 0}\varepsilon T^2||\psi(\omega,t,x,\frac{x}{\varepsilon})||_{L^2(\Omega\times[0,T]\times D)^n}\sup_{t\in[0,T]}||g_1(t)||^2_{Q_1}=0.$$

Using Hölder's inequality again:

$$\lim_{\varepsilon\to 0}\left|\int_\Omega\phi_1(\omega)\int_0^T\phi_2(t)\int_D\int_0^t g_1(s)dW_1(s)\phi_3(x)(\mathbb{1}_{D^\varepsilon}(x)-|Y^*|)dxdtd\mathbb{P}\right| \le$$

$$\lim_{\varepsilon\to 0}||\phi_1||_{L^2(0,T)}||\phi_2||_{L^2(\Omega)}\int_\Omega\int_0^T\left|\int_D\int_0^t g_1(s)dW_1(s)\phi_3(x)(\mathbb{1}_{D^\varepsilon}(x)-|Y^*|)dx\right|^2 dtd\mathbb{P}.$$

We rewrite the integral as follows:

$$\int_\Omega \int_0^T |\int_D \sum_{i=1}^\infty \sqrt{\lambda_{i1}} \int_0^t g_1(s) e_{i1}(x) d\beta_i(s) \phi_3(x)(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|)dx|^2 dt d\mathbb{P}.$$

where $(\beta_i)_{i=1}^\infty$ is a sequence of real valued independent Brownian motions, and $(e_{i1})_{i=1}^\infty$ and $(\lambda_{i1})_{i=1}^\infty$ are previously defined in (2.9).

We apply the stochastic Fubini theorem and the Itô's isometry and the sum becomes

$$\sum_{i=1}^\infty \lambda_{i1} \int_\Omega \int_0^T |\int_0^t d\beta_i(s) \int_D g_1(s) e_{i1}(x)(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|)\phi_2(x)dx|^2 dt d\mathbb{P} =$$
$$\sum_{i=1}^\infty \lambda_{i1} \int_0^T |\int_0^t ds \int_D g_1(s) e_{i1}(x)(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|)\phi_2(x)dx|^2 dt \leq$$
$$C\sum_{i=1}^N \lambda_{i1}|\int_0^t ds \int_D g_1(s) e_{i1}(x)(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|)\phi_2(x)dx|^2 +$$
$$C\sum_{i=N}^\infty \lambda_{i1}||g_1(s)e_{i1}(x)||^2_{L^2(D)^n},$$

for any $N \in \mathbb{Z}_+^*$, and for a constant C independent of N. The first sum goes to 0 from dominated convergence theorem and the weak convergence to 0 in $L^2(D)$ of $(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|)\phi_2(x)$. The second term goes to 0 when $N \to \infty$ because of (2.9). Hence, (5.15) follows. □

We compute the limits that involve integrals over the boundaries, and we will make use of the techniques from [6] that give a way of transforming integrals over the surface in integrals over the volume.

Lemma 5.2.

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega \int_0^T \int_{\partial O^\varepsilon} \int_0^t \varepsilon b u^\varepsilon(\omega, s, x')\phi^\varepsilon(\omega, t, x')ds d\sigma(x') dt d\mathbb{P} = \\ |\partial O| \int_\Omega \int_0^T \int_D \int_0^t b u(\omega, s, x)\phi(\omega, t, x) ds dx dt d\mathbb{P}.\end{aligned} \tag{5.16}$$

Proof. We will define as in [6] the solution of the following system:

$$\begin{cases} -\Delta w_1 &= -\dfrac{|\partial O|}{|Y^*|} & \text{in } \ Y^*, \\ \dfrac{\partial w_1}{\partial n} &= 1 & \text{on } \ \partial O, \\ \displaystyle\int_{Y^*} w_1 &= 0, & \\ w_1 & -Y-periodic, & \end{cases} \tag{5.17}$$

which, according to Remark 4.3 from [6] belongs to $W^{1,\infty}(Y^*)$. We define $w_1^\varepsilon(x) = \varepsilon^2 w_1(\frac{x}{\varepsilon})$, and straightforward calculations show that w_1^ε satisfies:

$$\begin{cases} -\Delta w_1^\varepsilon &= -\dfrac{|\partial O|}{|Y^*|} & \text{in } \ D^\varepsilon, \\ \dfrac{\partial w_1^\varepsilon}{\partial n} &= \varepsilon & \text{on } \ \partial O^\varepsilon. \end{cases} \tag{5.18}$$

For fixed $\omega \in \Omega$ and $t \in [0,T]$, the sequence $v^\varepsilon(w,t,\cdot) = b\int_0^t u^\varepsilon(\omega,s,\cdot)\phi^\varepsilon(\omega,t,\cdot)ds$ belongs to $W^{1,1}(D^\varepsilon)$ and we have:

$$\int_{\partial O^\varepsilon} \varepsilon v^\varepsilon(\omega,t,x')d\sigma(x') = \int_{\partial O^\varepsilon} \frac{\partial w_1^\varepsilon}{\partial n}(x')v^\varepsilon(\omega,t,x')d\sigma(x') = \int_{D^\varepsilon} \left(\Delta w_1^\varepsilon(x)v^\varepsilon(\omega,t,x) + \nabla w_1^\varepsilon(x)\nabla v^\varepsilon(\omega,t,x)\right)dx.$$

But from (3.8) and the definitions of w_1^ε and v^ε,

$$|\int_\Omega\int_0^T\int_{D^\varepsilon} \nabla w_1^\varepsilon(x)\nabla v^\varepsilon(\omega,t,x)dxdtd\mathbb{P}| \to 0,$$

and

$$\int_\Omega\int_0^T\int_{D^\varepsilon} v^\varepsilon(\omega,t,x)dxdtd\mathbb{P} \to |Y^*|b\int_\Omega\int_D\int_0^t u(\omega,s,x)\phi(\omega,t,x)dsdxdtd\mathbb{P},$$

which implies (5.16). □

Lemma 5.3.

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_{\partial O^\varepsilon}\int_0^t \varepsilon g_2^\varepsilon(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')dtd\mathbb{P} = |\partial O|\int_\Omega\int_0^T\int_D\int_0^t g_{21}(s)dW_2(s)\phi(\omega,t,x)dxdtd\mathbb{P} + \int_\Omega\int_0^T\int_{\partial O}\int_0^t g_{22}(s)dW_2(s)d\sigma\int_D \phi(\omega,t,x)dxdtd\mathbb{P}.$$

Proof. Similar arguments discussed in (5.16) and (5.15) will be used here. We will prove:

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_{\partial O^\varepsilon}\int_0^t \varepsilon g_{21}(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')dtd\mathbb{P} = |\partial O|\int_\Omega\int_0^T\int_D\int_0^t g_{21}(s)dW_2(s)\phi(\omega,t,x)dxdtd\mathbb{P}, \tag{5.19}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_{\partial O^\varepsilon}\int_0^t \varepsilon\mathcal{R}^\varepsilon g_{22}(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')dtd\mathbb{P} = \int_\Omega\int_0^T\int_{\partial O}\int_0^t g_{22}(s)dW_2(s)d\sigma\int_D \phi(\omega,t,x)dxdtd\mathbb{P}. \tag{5.20}$$

To show (5.19) we use the functions w_1 and w_1^ε defined in (5.17) and (5.18):

$$\begin{aligned}&\int_{\partial O^\varepsilon}\int_0^t \varepsilon g_{21}(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')\\ &= \int_{\partial O^\varepsilon}\frac{\partial w_1^\varepsilon}{\partial n}\int_0^t g_{21}(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')\\ &= \int_{D^\varepsilon}\Delta w_1^\varepsilon(x)\int_0^t g_{21}(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')\end{aligned}$$

$$+\int_{D^\varepsilon} \nabla w_1^\varepsilon(x) \int_0^t g_{21}(s)dW_2(s)\nabla_x\phi^\varepsilon(\omega,t,x)dx$$
$$=\frac{|\partial O|}{|Y^*|}\int_{D^\varepsilon}\int_0^t g_{21}(s)dW_2(s)\phi^\varepsilon(\omega,t,x')d\sigma(x')$$
$$+\int_{D^\varepsilon} \nabla w_1^\varepsilon(x) \int_0^t g_{21}(s)dW_2(s)\nabla_x\phi^\varepsilon(\omega,t,x)dx.$$

But
$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_{D^\varepsilon} \nabla w_1^\varepsilon(x) \int_0^t g_{21}(s)dW_2(s)\nabla_x\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P}=0$$
follows from the condition (2.9) and the definiton of w_1^ε.

Now, the same computation used to show (5.15) yields:
$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T\int_{D^\varepsilon}\int_0^t g_{21}(s)dW_2(s)\phi^\varepsilon(\omega,t,x)dxdtd\mathbb{P}=$$
$$|Y^*|\int_\Omega\int_0^T\int_{D}\int_0^t g_{21}(s)dW_2(s)\phi(\omega,t,x)dxdtd\mathbb{P}.$$

To show (5.20), let $(e_{i2}(x))_{i=1}^\infty$ and $(\lambda_{i2})_{i=1}^\infty$ previously defined in (2.9). Denote by $h_i(s)=g_{22}(s)e_{i2}\in L^2(\partial O)$, for each $i\in\mathbb{Z}_+$ and $s\in[0,T]$. We infer from (2.9) that
$$\sup_{s\in[0,T]}\sum_{i=1}^\infty \lambda_{i2}||h_i(s)||^2_{L^2(\partial O)}<+\infty. \tag{5.21}$$

We will define $w_i(s)$ similarly as w_1 to be the unique element in $H^1(Y^*)$ that solves:
$$\begin{cases} -\Delta w_i(s) &= -\dfrac{1}{|Y^*|}\displaystyle\int_{\partial O} h_i(s)d\sigma \quad \text{in } Y^*,\\ \dfrac{\partial w_i(s)}{\partial n} &= h_i(s) \quad \text{on } \partial O,\\ \displaystyle\int_{Y^*} w_i(s) &= 0,\\ w_i(s) & -Y-periodic, \end{cases} \tag{5.22}$$
and $w_i^\varepsilon(s)=\varepsilon^2 w_i(s)\left(\dfrac{\cdot}{\varepsilon}\right)$ that will solve
$$\begin{cases} -\Delta w_i^\varepsilon(s) &= -\dfrac{1}{|Y^*|}\displaystyle\int_{\partial O} h_i(s)d\sigma \quad \text{in } D^\varepsilon,\\ \dfrac{\partial w_i^\varepsilon(s)}{\partial n} &= \varepsilon\mathcal{R}^\varepsilon h_i(s) \quad \text{on } \partial O^\varepsilon. \end{cases} \tag{5.23}$$

There exists a constant C independent of i and s such that $||w_i(s)||_{H^1(Y^*)^n}\le C||h_i(s)||_{L^2(\partial O)^n}$ and using (5.21) we also have:
$$\sup_{s\in[0,T]}\sum_{i=1}^\infty \lambda_{i2}||w_i(s)||^2_{H^1(Y^*)^n}<\infty. \tag{5.24}$$

Using the decomposition of ϕ^ε and previously used computations, we have to show that:
$$\int_\Omega\left|\int_{\partial O^\varepsilon}\int_0^t \varepsilon\mathcal{R}^\varepsilon g_{22}(s)dW_2(s)\phi_2(x')d\sigma(x')-\int_{\partial O}\int_0^t g_{22}(s)dW_2(s)d\sigma\int_D\phi_2(x)dx\right|^2 d\mathbb{P}$$
converges to 0 uniformly for $t\in[0,T]$. We perform the following calculations:
$$\int_\Omega\left|\int_{\partial O^\varepsilon}\int_0^t \varepsilon\mathcal{R}^\varepsilon g_{22}(s)dW_2(s)\phi_2(x')d\sigma(x')-\int_{\partial O}\int_0^t g_{22}(s)dW_2(s)d\sigma\int_D\phi_2(x)dx\right|^2 d\mathbb{P}=$$

$$\int_\Omega \sum_{i=1}^\infty \lambda_{i2} \left| \int_{\partial O^\varepsilon} \int_0^t \varepsilon \mathcal{R}^\varepsilon g_{22}(s) e_{i2} d\beta_i(s) \phi_2(x') d\sigma(x') - \int_{\partial O} \int_0^t g_{22}(s) e_{i2} d\beta_i(s) d\sigma(x') \int_D \phi_2(x) dx \right|^2 d\mathbb{P} =$$
$$\int_\Omega \sum_{i=1}^\infty \lambda_{i2} \left| \int_0^t \int_{\partial O^\varepsilon} \varepsilon h_i^\varepsilon(s) \phi_2(x') d\sigma(x') d\beta_i(s) - \int_0^t \int_{\partial O} h_i(s) d\sigma \int_D \phi_2(x) dx d\beta_i(s) \right|^2 d\mathbb{P} =$$
$$\sum_{i=1}^\infty \lambda_{i2} \int_0^t ds \left| \int_{\partial O^\varepsilon} \varepsilon h_i^\varepsilon(s) \phi_2(x') d\sigma(x') - \int_{\partial O} h_i(s) d\sigma \int_D \phi_2(x) dx \right|^2 =$$
$$\sum_{i=1}^\infty \lambda_{i2} \int_0^t ds \left| \int_{\partial O^\varepsilon} \frac{\partial w_i^\varepsilon(s)}{\partial n} \phi_2(x') d\sigma(x') - \int_{\partial O} h_i(s) d\sigma \int_D \phi_2(x) dx \right|^2 =$$
$$\sum_{i=1}^\infty \lambda_{i2} \int_0^t ds \left| \int_{D^\varepsilon} \left(\Delta w_i^\varepsilon(s) \phi_2(x) + \nabla w_i^\varepsilon(s) \nabla \phi_2(x)\right) dx - \int_{\partial O} h_i(s) d\sigma \int_D \phi_2(x) dx \right|^2 =$$
$$\sum_{i=1}^\infty \lambda_{i2} \int_0^t ds \left| \int_D \int_{\partial O} h_i(s) d\sigma \left(\frac{1}{|Y^*|} \mathbb{1}_{D^\varepsilon}(x) - 1 \right) \phi_2(x) dx + \int_{D^\varepsilon} \nabla w_i^\varepsilon(s) \nabla \phi_2(x) dx \right|^2 \leq$$
$$C \sum_{i=1}^\infty \lambda_{i2} \int_0^t ds \left| \int_D \int_{\partial O} h_i(s) d\sigma \left(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|\right) \phi_2(x) dx \right|^2 + C \sum_{i=1}^\infty \lambda_{i2} \int_0^t ds \left| \int_{D^\varepsilon} \nabla w_i^\varepsilon(s) dx \right|^2.$$

The second term of the sum will be bounded by

$$C\varepsilon^2 \sum_{i=1}^\infty \lambda_{i2} \int_0^T ds \left| \int_{Y^*} \nabla w_i(s) dx \right|^2.$$

Using (5.24) it converges to 0 uniformly for $t \in [0, T]$.

We decompose the first term similarly to (5.15), into

$$\sum_{i=1}^N \lambda_{i2} \int_0^t ds \left| \int_D \int_{\partial O} h_i(s) d\sigma \left(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|\right) \phi_2(x) dx \right|^2 +$$
$$\sum_{i=N+1}^\infty \lambda_{i2} \int_0^t ds \left| \int_D \int_{\partial O} h_i(s) d\sigma \left(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|\right) \phi_2(x) dx \right|^2.$$

The first sum goes to 0 for any fixed N because of the weak convergence to 0 in $L^2(D)$ of $(\mathbb{1}_{D^\varepsilon}(x) - |Y^*|)\phi_2(x)$. The second sum goes to 0 when $N \to \infty$ because of (5.21).

Using the limits obtained in the variational formulation we get

$$|Y^*| \int_\Omega \int_0^T \int_D u(\omega, t, x) \phi(\omega, t, x) dx dt d\mathbb{P} - |Y^*| \int_\Omega \int_0^T \int_D u_0(\omega, x) \phi(\omega, t, x) dx dt d\mathbb{P} +$$
$$\int_\Omega \int_0^T \int_D \int_{Y^*} \int_0^t \nu \left(\nabla_x u(\omega, s, x) + \nabla_y u_1(\omega, s, x, y)\right) \left(\nabla_x \phi(\omega, t, x) + \nabla_y \psi(\omega, t, x, y)\right) ds dy dx dt d\mathbb{P}$$
$$- \int_\Omega \int_0^T \int_D \int_{Y^*} P(\omega, t, x, y) \left(\operatorname{div}_x \phi(\omega, x) + \operatorname{div}_y \psi(\omega, t, x, y)\right) dy dx dt d\mathbb{P} +$$
$$|\partial O| \int_\Omega \int_0^T \int_D \int_0^t b u(\omega, s, x) \phi(\omega, t, x) ds dx dt d\mathbb{P} =$$
$$|Y^*| \int_\Omega \int_0^T \int_D \int_0^t f(s, x) \phi(\omega, t, x) ds dx dt d\mathbb{P} +$$
$$|Y^*| \int_\Omega \int_0^T \int_D \int_0^t g_1(s) dW_1(s) \phi(\omega, t.x) dx dt d\mathbb{P} +$$
$$|\partial O| \int_\Omega \int_0^T \int_D \int_0^t g_{21}(s) dW_2(s) \phi(\omega, t, x) dx dt d\mathbb{P} +$$
$$\int_\Omega \int_0^T \int_{\partial O} \int_0^t g_{22}(s) dW_2(s) d\sigma \int_D \phi(\omega, t, x) dx dt d\mathbb{P}. \tag{5.25}$$

□

6. **Homogenized equation.** This section contains the main result of this paper. We will prove that the two-scale limit (u, P) that satisfies the variational formulation (5.25) will be solution of a stochastic partial differential equation coupled with a cell problem. Moreover, we will write the SPDE associated with the weak limit $u^*(\omega, t, x) = \int_Y u(\omega, t, x, y)dy$.

6.1. **The cell problem.** We separate the terms in (5.25) according to ϕ and ψ and consider their decompositions introduced in section 5, which means that U_1 and P satisfy the following equation for every $t \in [0, T]$, almost every $x \in D$ and $\omega \in \Omega$:

$$\int_{Y^*} \nu\left(\nabla_x U(\omega, t, x) + \nabla_y U_1(\omega, t, x, y)\right) \nabla_y \psi_4(y) - P(\omega, t, x, y) \operatorname{div}_y \psi_4(y) dy = 0, \tag{6.1}$$

for every $\psi_4 \in C^\infty_\#(Y)^n$.

We have also, as a consequence of (5.5) that

$$\operatorname{div}_x U(\omega, t, x) + \operatorname{div}_y U_1(\omega, t, x, y) = 0, \tag{6.2}$$

being the two scale limit of the sequence $\widetilde{\operatorname{div}_x U}^{\varepsilon}(\omega, t, x)$.

Equations (6.1)–(6.2) lead us to define for every $\Lambda \in \mathbb{R}^{n\times n}$, $w_\Lambda \in H^1_\#(Y^*)^n$ and $q_\Lambda \in L^2_\#(Y^*)$, as the solution of the following problem in Y^*:

$$\left\{ \begin{array}{rll} \displaystyle\int_{Y^*} \nu\nabla_y\left(\Lambda y + w_\Lambda(y)\right) \nabla_y \phi(y) - q_\Lambda(y) \operatorname{div}_y \phi(y) dy & = 0 & \text{for all } \phi \in H^1_\#(Y)^n \\ \operatorname{div}_y\left(\Lambda y + w_\Lambda(y)\right) & = 0 & \text{in } Y^* \end{array} \right. \tag{6.3}$$

which is equivalent to the system:

$$\left\{ \begin{array}{rll} -\nu\Delta w_\Lambda(y) + \nabla q_\Lambda(y) & = 0 & \text{in } Y^* \\ \operatorname{div}_y\left(\Lambda y + w_\Lambda(y)\right) & = 0 & \text{in } Y^* \\ \nu\dfrac{\partial\left(\Lambda y + w_\Lambda(y)\right)}{\partial n} - q_\Lambda(y) n & = 0 & \text{on } \partial O \\ w_\Lambda,\ q_\Lambda & Y - periodic & \\ \displaystyle\int_{Y^*} w_\Lambda(y)dy & = 0 & \end{array} \right. \tag{6.4}$$

Define $\mathcal{C}$ the linear form on $\mathbb{R}^{n\times n}$, by:

$$\mathcal{C}\Lambda = \int_{Y^*} \left(\nu\left(\Lambda + \nabla_y w_\Lambda(y)\right) - q_\Lambda(y) I\right) dy, \tag{6.5}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

Let $\mathbf{e}_{ij} \in \mathbb{R}^{n\times n}$ be defined by $(\mathbf{e}_{ij})_{kh} = \delta_{ik}\delta_{jh}$, for every $1 \le k, h \le n$, and let w_{ij} and q_{ij} be the solutions of the cell problem (6.4) corresponding to $\Lambda = \mathbf{e}_{ij}$. Then, by linearity:

$$w_\Lambda(y) = \sum_{i,j=1}^n w_{ij}(y)\Lambda_{ij} \quad \text{and} \quad q_\Lambda(y) = \sum_{i,j=1}^n q_{ij}(y)\Lambda_{ij}, \tag{6.6}$$

and the linear form $\mathcal{C}$ can be given componentwise:

$$(\mathcal{C}\mathbf{e}_{ij})_{kh} = \int_{Y^*} \left(\nu\delta_{ik}\delta_{jh} + \nu\frac{\partial w^k_{ij}}{\partial y_h}(y) - q_{ij}\delta_{kh}\right) dy. \tag{6.7}$$

6.2. **The homogenized system.** According to (6.6) the solutions U_1 and P of (6.1)–(6.2) are given by

$$U_1(\omega,t,x,y)=\sum_{i,j=1}^{n} w_{ij}(y)\frac{\partial (U)_i}{\partial x_j}(\omega,t,x), \tag{6.8}$$

$$P(\omega,t,x,y)=\sum_{i,j=1}^{n} q_{ij}(y)\frac{\partial (U)_i}{\partial x_j}(\omega,t,x), \tag{6.9}$$

and the equation (5.25) becomes

$$\begin{aligned}
&|Y^*|\int_D u(\omega,t,x)\phi_3(x)dx-|Y^*|\int_D u_0(\omega,x)\phi_3(x)dx+\int_D \left(\mathcal{C}\nabla U(\omega,t,x)\right)\nabla\phi_3(x)dx-\\
&|\partial O|\int_D\int_0^t bu^\varepsilon(\omega,s,x)\phi_3(x)dsdx=|Y^*|\int_D\int_0^t f(s,x)\phi_3(x)dsdx+\\
&|Y^*|\int_D\int_0^t g_1(s)dW_1(s)\phi_3(x)dx+|\partial O|\int_D\int_0^t g_{21}(s)dW_2(s)\phi_3(x)dx+\\
&\int_{\partial O}\int_0^t g_{22}(s)dW_2(s)d\sigma\int_D\phi_3(x)dx.
\end{aligned} \tag{6.10}$$

for every $\phi_3\in C_0^\infty(D)^n$, for every $t\in[0,T]$ and a.s. $\omega\in\Omega$, which implies that u is the solution of the following stochastic partial differential equation:

$$\begin{cases}
du(t) &= \left[\operatorname{div}_x\left(\mathcal{C}\nabla_x u(t)\right)+\dfrac{|\partial O|}{|Y^*|}bu(t)+f(t)\right]dt\\
&\quad +g_1(t)dW_1(t)+\dfrac{1}{|Y^*|}\left(|\partial O|g_{21}(t)+\displaystyle\int_{\partial O}g_{22}(t)d\sigma\right)dW_2(t) \quad \text{in } D,\\
u(0) &= u_0 \quad \text{in } D.
\end{cases} \tag{6.11}$$

If we denote by u^* the weak limit in $L^2(\Omega\times[0,T]\times D)^n$ of the sequence $\widetilde{u}^\varepsilon$, according to (5.2), then it will solve a similar equation:

$$\begin{cases}
du^*(t) &= \left[\operatorname{div}_x\left(\mathcal{C}\nabla_x u^*(t)\right)+|\partial O|bu^*(t)+|Y^*|f(t)\right]dt\\
&\quad +|Y^*|g_1(t)dW_1(t)+\left(|\partial O|g_{21}(t)+\displaystyle\int_{\partial O}g_{22}(t)d\sigma\right)dW_2(t) \quad \text{in } D,\\
u^*(0) &= |Y^*|u_0 \quad \text{in } D.
\end{cases} \tag{6.12}$$

Let us define now the operator $A^*: D(A^*)\subset L^2(D)^n\mapsto L^2(D)^n$, by $A^*u=-\operatorname{div}(\mathcal{C}\nabla u)$ for every $u\in D(A^*)$, where

$$D(A^*)=H_0^1(D)^n\cap H^2(D)^n,$$

and $\mathcal{C}$ is introduced in (6.5).

Lemma 6.1. *(Properties of the operator A^*) The linear operator A^* is positive and self-adjoint in $L^2(D)^n$.*

Proof. All we have to show is that the operator $\mathcal{C}$ defined in (6.5) is symmetric and positive definite. Let Λ_1 and Λ_2 be two matrices from $\mathbb{R}^{n\times n}$ and let us compute $\mathcal{C}\Lambda_1\cdot\Lambda_2$:

$$\mathcal{C}\Lambda_1\cdot\Lambda_2=\int_{Y^*}\left(\nu\left(\Lambda_1+\nabla_y w_{\Lambda_1}(y)\right)-q_{\Lambda_1}(y)I\right)\cdot\Lambda_2 dy.$$

Using w_{Λ_2} as a test function in (6.4), for $\Lambda = \Lambda_1$ we get that:

$$\int_{Y^*} (\nu (\Lambda_1 + \nabla_y w_{\Lambda_1}(y)) - q_{\Lambda_1}(y)I) \cdot \nabla w_{\Lambda_2} dy = 0,$$

and given that $\operatorname{div}_y q_{\Lambda_1}(y) = \operatorname{div}_y q_{\Lambda_2}(y) = 0$ in Y^*,

$$\int_{Y^*} (\nu \nabla_y (\Lambda_1 y + w_{\Lambda_1}(y))) \cdot q_{\Lambda_2}(y) dy = \int_{Y^*} (\nu \nabla_y (\Lambda_2 y + w_{\Lambda_2}(y))) \cdot q_{\Lambda_1}(y) dy = 0.$$

Using these relations, elementary calculations will give us that

$$\mathcal{C}\Lambda_1 \cdot \Lambda_2 = \int_{Y^*} \nu (\Lambda_1 + \nabla_y w_{\Lambda_1}(y)) \cdot (\Lambda_2 + \nabla_y w_{\Lambda_2}(y)) \, dy, \tag{6.13}$$

which implies that $\mathcal{C}$ is symmetric and positive definite. □

As a consequence, we have the following existence result:

Theorem 6.2. *(Well posedness of the homogenized equation) Let $S^*(t)_{t\geq 0}$ be the semigroup generated by the operator A^* and assume that assumption (2.9) is satisfied. Then, the equation (6.12) admits a unique mild solution u^* such that*

$$u^* \in L^2(\Omega; C([0,T], L^2(D)^n) \cap L^2(0,T; H_0^1(D)^n))$$

given by

$$\begin{aligned} u^*(t) =& |Y^*| S^*(t) u_0^* + |\partial O| \int_0^t S^*(t-s) b u^*(s) ds + |Y^*| \int_0^t S^*(t-s) f(s) ds \\ &+ |Y^*| \int_0^t S^*(t-s) g_1(s) dW_1(s) + |\partial O| \int_0^t S^*(t-s) g_{21}(s) dW_2(s) \\ &+ \int_0^t S^*(t-s) \int_{\partial O} g_{22}(s) d\sigma dW_2(s), \quad t \in [0,T]. \end{aligned} \tag{6.14}$$

Remark 6.3.
1. The solution of (6.12) is not divergence free, so the homogenized equation does not contain a pressure term. However, the effect of the pressure $P^\varepsilon(t)$ appears implicitely through the matrix $\mathcal{C}$, hence the cell problem.
2. The overall effect of the boundary condition is seen in the homogenized equation through the Brinkman term $|\partial O| b u^*(t)$ and through an extra stochastic forcing acting in the volume, $\left(|\partial O| g_{21}(t) + \int_{\partial O} g_{22}(t) d\sigma \right) \dot{W}_2(t)$.
3. The convergence of the sequence u^ε to the limit u^* is strong in the probabilistic sense, but weak in the deterministic sense. In particular, the sequence $\widetilde{u}^\varepsilon$ of the extensions by 0 of u^ε inside O^ε converges to u^* weakly in $L^2(\Omega \times [0,T] \times D)^n$.

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E-mail address: bessaih@uwyo.edu
E-mail address: efendiev@math.tamu.edu
E-mail address: florinmaris@gmail.com