NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2015.10.1
©American Institute of Mathematical Sciences
Volume 10, Number 1, March 2015 pp. 1–16

TRANSFERABILITY OF COLLECTIVE TRANSPORTATION LINE NETWORKS FROM A TOPOLOGICAL AND PASSENGER DEMAND PERSPECTIVE

Eva Barrena

Interuniversity Research Center on Network Enterprise
Logistics and Transportation (CIRRELT)
HEC Montréal, 3000 chemin de la Côte-Sainte-Catherine
Montréal, H3T 2A7, Canada

Alicia De-Los-Santos

Department of Applied Mathematics II
Higher Technical School of Engineering. University of Seville
Camino de los Descubrimientos s/n
Seville, 41092, Spain

Gilbert Laporte

Interuniversity Research Center on Network Enterprise
Logistics and Transportation (CIRRELT)
HEC Montréal, 3000 chemin de la Côte-Sainte-Catherine
Montréal, H3T 2A7, Canada

Juan A. Mesa

Department of Applied Mathematics II
Higher Technical School of Engineering. University of Seville
Camino de los Descubrimientos s/n
Seville, 41092, Spain

Abstract. We analyze the transferability of collective transportation line networks (CTLN) with the help of hypergraphs, their linearization, and connectivity measures from Complex Network Theory. In contrast to other existing works in the literature, where transferability is analyzed at a topological level, we are also concerned with passenger system level, introducing data on the travel patterns. This will allow us to have a more complete view of the functioning of the transfer system of a CTLN.

1. **Introduction.** One of the fields in which the Science of Complex Systems has been applied is that of Technological Networks. In particular, transportation network and, especially, urban transit networks are complex systems for which several features as efficiency, vulnerability and robustness, have been studied from the viewpoint of the Complex Network Science. Two properties of complex networks are of special relevance. These are the scale-free pattern and the small-world effect. Whereas the concept of scale-free network was introduced by Barabasi and Albert [1], that of the small-world network was first coined by Watts and Strogratz [16],

2010 *Mathematics Subject Classification.* Primary: 05C82, 05C65, 05C40; Secondary: 90B10, 68M10, 90B06.

Key words and phrases. Collective transportation line networks, transferability measures, passengers demand, complex networks, hypergraphs.

in analogy with the small-world phenomenon observed in social networks and popularly known as the six degrees of separation [13].

A small-world network is characterized by a high average local clustering coefficient and a low average shortest distance between nodes. The clustering coefficient, which measures the average cliquisness of the nodes, and characteristic path length were introduced by Watts and Strogratz [16]. These two measures are not well defined in some cases and only apply to the topological setting. For these reasons Latora and Marchiori [11] defined two substitutive measures, the local and global efficiency, and applied them to metric networks, that is, those in which the edges have an associated weight. In particular, the local and global efficiency concepts have been applied to the Boston subway [12]. Indices to evaluate the robustness of a railway network against interruptions in the normal functioning of its links (both accidental interruptions and intentional attacks) have been introduced in [7]. Other relevant papers regarding issues related with the complexity of metro networks from the viewpoint of the topological setting are [6], [8], [14] and [15].

Collective transportation networks can be decomposed into three layers: the infrastructure network, the line network, which uses the infrastructure one as support, and the passengers system, which uses the lines for traveling. For purposes of efficiency of the collective transportation line networks (CTLN), an edge for which the line traversing it only carries one passenger should not be weighted the same as one that transports one thousand passengers. Recently, Barrena et al. [3] have introduced several measures on the line network by means of hypergraph theory [4, 5]. Hypergraphs are the natural extension of graphs and allow us to describe and apply different concepts which cannot be used by graphs. The new transferability measures defined on hypergraphs and their corresponding linear graphs allow assessing how easy or how hard it is to transfer from one line to another [3].

In this paper we are concerned with the passenger system level, where data on the travel patterns are introduced. From this perspective, the transferability measures are better to evaluate the difficulty of transferring between lines. We make the following hypotheses:

- The CTLN is connected.
- Passengers use their shortest paths.
- There is no maximal capacity on stations (stops), nor on lines or edges.
- There is no other means of transportation competing with that of the CTLN, therefore demand is fixed.
- The number of passengers wishing to use the CLTN is greater than or equal to one for each pair of different nodes.
- All transfers are considered similar.

The remainder of the paper is organized as follows. In Section 2 we formally describe different representations of a CTLN topology, as well as the travel patterns to be used in the following sections in order to adapt the topological transferability measures to a passenger system level. In Section 3 we adapt the usual topological connectivity indicators to a CTLN and its demand patterns in order to have transferability indicators at a topological and passenger system level. We analyze the properties of the new passenger-oriented transferability indicators. In Section 4, we apply our indicators to four typical CTLN configurations and compare the pure topological with the passenger-oriented indicators. Finally, in Section 5, we present the conclusions of our work.

2. **Previous definitions and demand patterns.** In this section we will present the notation and definitions needed to define our measures. We assume the existence of a CTLN represented by a set of lines $\{L_1, \ldots, L_\ell\}$, each of them characterized by its set of nodes and itinerary. More precisely, a line L_p is a chain graph $L_p = (S_p, E_p)$, where $S_p = \{s_1^i, \ldots, s_{k_p}^p\}$ is the node (station, stop) set and $E_p = \{\{s_1^i, s_2^i\}, \ldots, \{s_{k_p-1}^p, s_{k_p}^p\}\}$ is the edge (inter-station space) set describing the itinerary. Thus, a collective transportation line network G can be described by the union of chain graphs $G = (\cup S_p, \cup E_p), p = 1, \ldots, \ell$.

2.1. **Topology of the CTLN. Previous definitions.** As in [2], depending on the level of given information about the network, the appropriated structure graph for representing a $CTLN$ can be different. We will consider that a CTLN G can be represented by means of hypergraphs, linear graphs, and multigraphs as follows.

- Hypergraph

  Let $V(\mathbb{H}) = \{s_1, \ldots, s_k\}$ be the set containing all the stations of G, $S_1, \ldots,$ S_ℓ the station (stop) sets of lines $L_1, \ldots, L_\ell$, and $E(\mathbb{H}) = \{S_1, \ldots, S_\ell\}$. Then, $\mathbb{H} = (V(\mathbb{H}), E(\mathbb{H}))$ is the hypergraph associated to G. Note that, as opposed to standard graphs, the elements in $E(\mathbb{H})$ are not necessarily pairs of elements of $V(\mathbb{H})$, but sets of elements. From now on, we call this hypergraph $\mathbb{H}$ the *transit hypergraph*.

  On this structure, the distance $d_{\mathbb{H}}(s_i, s_j)$ on the elements of $V(\mathbb{H})$ is the length of the shortest ordinary (s_i, s_j)-chain. So, all nodes belonging to the same hyperedge are one unit of distance apart. More precisely, $d_{\mathbb{H}}(s_i, s_j)$ is the minimum number of different lines one needs in order to travel from station s_i to station s_j. For the sake of readability we will identify a station by its index whenever this creates no confusion.
- Linear graph

  Let $\mathcal{L}(\mathbb{H}) = (V(\mathcal{L}(\mathbb{H})), E(\mathcal{L}(\mathbb{H})))$ be the linear graph associated to hypergraph $\mathbb{H}$. Its node set $V(\mathcal{L}(\mathbb{H})) = \{L_1, \ldots, L_\ell\}$, represents the network lines (hyperedges of $\mathbb{H}$) and its edge set $E(\mathcal{L}(\mathbb{H}))$ is the set of transfer edges connecting lines with intersections between them. These transfer edges are denoted by e_{pq}. Observe that each hyperedge in $\mathbb{H}$ corresponds to a node in $\mathcal{L}(\mathbb{H})$, and two nodes in $\mathcal{L}(\mathbb{H})$ are linked if and only if the corresponding hyperedges in $\mathbb{H}$ have a non-empty intersection. For the sake of readability, we will identify a line by its index whenever this creates no confusion.

  In this graph, the concept of distance is the usual topological distance in graphs. Specifically, the distance $d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)$ from node L_p to L_q is the minimum number of edges of a shortest path between L_p and L_q. From the point of view of transfers, this distance indicates the number of transfers one needs to make when traveling from one line to different lines in the CTLN.
- Linear multigraph

  Note that the linear graph $\mathcal{L}(\mathbb{H})$ is assumed to be a simple or strict graph, where multiple edges between nodes are not allowed. Recall that an edge e_{pq} in this graph means that the lines L_p and L_q have a non-empty intersection, that is, these lines have at least one common station in $\mathbb{H}$. However, it does not indicate the number of common stations between the lines, which is interesting in order to measure how easy it is to transfer between lines. For some purposes, it can become helpful to allow the linear graph to have multiple edges, thus becoming the multilinear graph $\mathcal{L}^M(\mathbb{H})$, i.e., a graph in which multiple edges are permitted. The latter case, the number of edges connecting two lines in $\mathcal{L}(\mathbb{H})$ will be equal to the number of transfer stations between them in $\mathbb{H}$. In the rest of the paper $\mathcal{L}^M(\mathbb{H})$ will be referred to as the *linear multigraph* of G.

As in the linear graph, the distance defined on the multigraph is the topological distance.

2.2. **Passenger system level. Demand patterns.** As mentioned, the goal of this work is to analyze the development of a CTLN with respect to the number of transfers from a passenger demand perspective. To this end, we consider the passenger demand between stations, as well as between the lines of the network by means of origin-destination matrices. Let k and ℓ be the number of stations and the number of lines forming the CTLN, respectively.

Let $OD \in \mathcal{M}_{k\times k}$ be the origin-destination demand matrix, whose elements $OD(i,j)$, $i,j \in \{1,\dots,k\}$ represent the number of passengers traveling from station s_i to station s_j and the diagonal elements are equal zero since there is no demand within a station. We assume that $OD(i,j) \geq 1$, for all $i \neq j, i,j \in \{1,\dots,k\}$. Regarding travels between lines, let $LOD \in \mathcal{M}_{\ell\times\ell}$ be the corresponding origin-destination line-demand matrix where its elements $LOD(p,q)$, $p,q \in \{1,\dots,\ell\}$ represent the number of passengers traveling from line L_p to line L_q, and its diagonal elements $LOD(p,p)$ represent the number of passengers traveling within line L_p. Starting from OD, LOD is computed as follows:

$$LOD(p,q) = \sum_{i\in L_p}\sum_{j\in L_q} OD(i,j), \qquad p,q \in \{1,\dots,\ell\}.$$

Note that the matrices OD and LOD are not necessarily symmetric. The total demand in the CTLN can be obtained by means of the elements of OD or LOD. Let N be the total demand expressed as the sum of all demands $OD(i,j)$, $i,j \in \{1,\dots,k\}$ and let N^L be defined as the sum of all demands $LOD(p,q)$, $p \neq q$. Note that the total demand N can also be defined by means of linear graphs, i. e., $N = N^L + \sum_p LOD(p,p)$.

3. **Passenger system level. Passenger-oriented transferability measures.** Interesting topological measures to evaluate the connectivity of a collective transportation line network are the *characteristic path length*, the *clustering coefficient*, and the *local and global efficiency*. In this section we show how to adapt the topological measures to a passenger system level.

3.1. **Characteristic path passenger-oriented length.** In a previous work [3], the characteristic path length was defined on the transit hypergraph, its associated linear graph and linear multi-graph. From a topological point of view, this measure indicates the separation, in terms of number of transfers, between pairs of nodes in a CTLN. However, this measure does not take into account the mobility patterns and should be extended. The following sections introduce the characteristic path passenger-oriented length on the transit hypergraph and its linearizations.

3.1.1. *Characteristic path passenger-oriented length on the transit hypergraph.* Over the hypergraph level of information, the characteristic path length will provide an average measure of how the topological configuration of the CTLN affects the passengers who transfer between the stations of a CTLN. This measure will depend on the number of transfers that passengers must make in order to reach their destination station.

Definition 3.1. We define the *characteristic path passenger-oriented length* of the transit hypergraph $\mathbb{H}$, with $|V(\mathbb{H})| > 1$, as the average passenger-oriented distance in $\mathbb{H}$, i.e.,

$$\mathfrak{L}_d(\mathbb{H}) = \sum_{s_i \neq s_j} d_{\mathbb{H}}(s_i, s_j) \frac{OD(i,j)}{N}.$$

So, the higher the number of passengers who travel between stations s_i and s_j, the higher the weight we assign to the hypergraph distance between these two stations. This way, the solutions of the problem of designing a line network minimizing the characteristic path length will tend to have better connections between lines with more passengers traveling between them.

Our intention is to analyze the topology of the CTLN from a passengers perspective. The pairs of stations with more passengers demand between them have more influence on the resulting *characteristic path passenger-oriented length*, and the pairs of stations with no passengers demand between them, have no influence. The same is true for the linear graph $\mathcal{L}(\mathbb{H})$.

The following example shows the effect on considering the travel patterns in a transit hypergraph associated to a CTLN.

Example 3.1. Consider a simple case in which two possible line configurations (see Figure 2) are described for the same infrastructure network (see Figure 1). The infrastructure network is formed by six stations and five edges as follows.

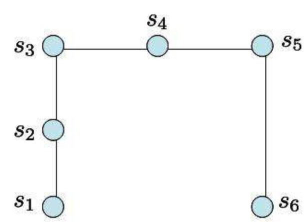


FIGURE 1. Infrastructure network.

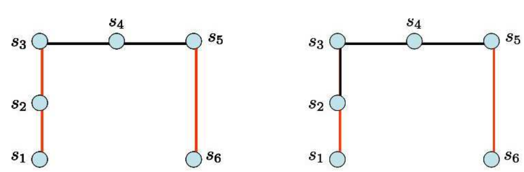


FIGURE 2. Two CTLN G and $\bar{G}$ associated to the infrastructure network of Figure 1.

The first CTLN G is defined by three lines:

- $L_1 = (S_1, E_1) = (\{s_1, s_2, s_3\}, \{\{s_1, s_2\}, \{s_2, s_3\}\})$
- $L_2 = (S_2, E_2) = (\{s_3, s_4, s_5\}, \{\{s_3, s_4\}, \{s_4, s_5\}\})$
- $L_3 = (S_3, E_3) = (\{s_5, s_6\}, \{\{s_5, s_6\}\})$.

The second CTLN $\bar{G}$ is defined by the lines:

- $\bar{L}_1 = (\bar{S}_1, \bar{E}_1) = (\{s_1, s_2\}, \{\{s_1, s_2\}\})$

- $\bar{L}_2 = (\bar{S}_2, \bar{E}_2) = (\{s_2, s_3, s_4, s_5\}, \{\{s_2, s_3\}, \{s_3, s_4\}, \{s_4, s_5\}\})$
- $\bar{L}_3 = L_3$.

Figure 3 illustrates the hypergraphs representations $\mathbb{H}$ and $\bar{\mathbb{H}}$ of the CTLNs G and $\bar{G}$, respectively.

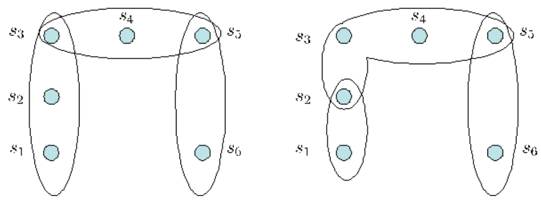


FIGURE 3. The two hypergraph $\mathbb{H}$ and $\bar{\mathbb{H}}$ associated to G and $\bar{G}$, respectively.

The characteristic path lengths associated to $\mathbb{H}$ and $\bar{\mathbb{H}}$ are $\mathfrak{L}(\mathbb{H}) = 50/30 > 44/30 = \mathfrak{L}(\bar{\mathbb{H}})$.

In order to compute the characteristic path passenger-oriented length and observe the differences with respect to the usual characteristic path length, we assume the following demand pattern as $OD(1,3) = 6$, $OD(i,j) = 1, \forall i \neq j, (i,j) \neq (1,3)$. So, $\mathfrak{L}_d(\mathbb{H}) = 60/40 < \mathfrak{L}_d(\bar{\mathbb{H}}) = 64/40$. It can be observed that when this demand pattern is introduced, the order is different. This result is due to the maximum demand pair $(1,3)$, which is better connected in $\mathbb{H}$ than in $\bar{\mathbb{H}}$. If we consider a constant demand for all origin-destination pairs, such as for example, $OD(i,j) = \alpha, \forall i \neq j, \alpha > 0$, then $\mathfrak{L}_d(\mathbb{H}) = 50/30 > 44/30 = \mathfrak{L}_d(\bar{\mathbb{H}})$.

The next lemma proves that the characteristic path passenger-oriented length above defined is a natural extension of $\mathfrak{L}(\mathbb{H})$ defined in [3].

Lemma 3.2. *$\mathfrak{L}_d(\mathbb{H})$ is an extension of $\mathfrak{L}(\mathbb{H})$, which yields the same result if the number of passengers between each pair of stations s_i, s_j is the same, that is, all the elements of matrix OD, except its diagonal elements, are the same.*

Proof. Trivially, if we consider a constant number $\alpha > 0$ of passengers for each OD pair of different stations (i.e. $OD(i,j) = \alpha$, $i \neq j$), then the total number of passengers N is equal to the number of pairs of stations multiplied by α, that is $N = \alpha(|V(\mathbb{H})|(|V(\mathbb{H})| - 1))$. The following expression is thus obtained:

$$\mathfrak{L}_d(\mathbb{H}) = \sum_{i \neq j} d_{\mathbb{H}}(s_i, s_j) \frac{\alpha}{N} = \sum_{i \neq j} d_{\mathbb{H}}(s_i, s_j) \frac{\alpha}{\alpha(|V(\mathbb{H})|(|V(\mathbb{H})| - 1))}$$
$$= \frac{1}{|V(\mathbb{H})|(|V(\mathbb{H})| - 1)} \sum_{i \neq j} d_{\mathbb{H}}(s_i, s_j)).$$

Since the distance $d_{\mathbb{H}}$ is symmetric and $d_{\mathbb{H}}(s_i, s_i) = 0$, then

$$\frac{1}{|V(\mathbb{H})|(|V(\mathbb{H})| - 1)} \sum_{i \neq j} d_{\mathbb{H}}(s_i, s_j) = \frac{2}{|V(\mathbb{H})|(|V(\mathbb{H})| - 1)} \sum_{i < j} d_{\mathbb{H}}(s_i, s_j) = \mathfrak{L}(\mathbb{H}).$$

□

The following proposition states that the characteristic path length of the transit hypergraph, which gives a measure of the transferability of a CTLN, satisfies the same two basic properties of this type of measures: it lies within a predefined range and satisfies a monotonicity property.

Proposition 1. *Consider a CTLN G, and let $\mathbb{H}$ be its associated transit hypergraph. Let OD be a non-empty demand matrix between stations. The passenger-oriented characteristic path length $\mathfrak{L}_d(\mathbb{H})$ of the transit hypergraph satisfies the following two properties:*

1. $1 \leq \mathfrak{L}_d(\mathbb{H}) \leq \frac{OD_{max}}{N}\frac{\ell(\ell+1)(\ell+2)}{3}$, *where* $OD_{max} = \max_{i\neq j} OD(i,j)$.
2. *Let G' be a CTLN obtained when adding one new line joining two lines of G without additional stations, and let $\mathbb{H}'$ be the associated hypergraph. Then* $\mathfrak{L}_d(\mathbb{H}) \geq \mathfrak{L}_d(\mathbb{H}')$.

Proof. 1. If $s_i, s_j \in V(\mathbb{H})$, then $1 \leq d_{\mathbb{H}}(s_i, s_j)$ holds. The best situation regarding $d_{\mathbb{H}}(s_i, s_j)$ is when there is only one line, in which case $d_{\mathbb{H}}(s_i, s_j) = 1, \forall s_i, s_j \in V(\mathbb{H}), i \neq j$. Then, in this case,

$$\mathfrak{L}_d(\mathbb{H}) = \sum_{i\neq j} d_{\mathbb{H}}(s_i, s_j)\frac{OD(i,j)}{N} = 1.$$

The worst situation regarding $d_{\mathbb{H}}(s_i, s_j)$ is when there are ℓ lines, each of them consisting of two stations forming a chain graph. So, the lines are $L_1, ..., L_\ell$, and the sets of nodes are $V(L_k) = \{s_k, s_{k+1}\}$, for $k = 1, ..., \ell$. It is then fulfilled that $d_{\mathbb{H}}(s_k, s_{k+k'}) = k'$. Therefore

$$\begin{aligned}
\mathfrak{L}_d(\mathbb{H}) =& \sum_{k_1\neq k_2} d_{\mathbb{H}}(s_{k_1}, s_{k_2})\frac{OD(k_1,k_2)}{N} \\
=& 2\sum_{k_1<k_2} d_{\mathbb{H}}(s_{k_1}, s_{k_2})\frac{OD(k_1,k_2)}{N} \\
\leq& \frac{OD_{max}}{N}\sum_{k_1=1}^{\ell}\sum_{k_2=k_1+1}^{\ell+1}(k_2-k_1) \\
=& \frac{OD_{max}}{N}2\sum_{k_1=1}^{\ell}\Big[\frac{(\ell+1-k_1)(\ell+2-k_1)}{2}\Big] \\
=& \frac{OD_{max}}{N}\sum_{k_1=1}^{\ell}[(\ell+1)(\ell+2) - k_1(\ell+1) - k_1(\ell+2) + k_1^2] \\
=& \frac{OD_{max}}{N}\Big[\frac{\ell(\ell+1)(\ell+2)}{3}\Big].
\end{aligned}$$

2. Adding a new line joining two lines without additional stations results in $d_{\mathbb{H}}(s_i, s_j) \geq d_{\mathbb{H}'}(s_i, s_j)$. Since the node set $V(\mathbb{H}')$ does not change, the result follows.

□

3.1.2. *Characteristic path passenger-oriented length on the linear graph.* The following definition is the natural extension of the characteristic path length defined in [3], but considering the passenger demand. The characteristic path length in $\mathcal{L}(\mathbb{H})$ provides information on how the number of transfers (resulting from the topology of the network) affects the passengers. It gives an average measure of how easy it is for passengers to transfer between the lines of a CTLN.

Definition 3.3. We define the *characteristic path passenger-oriented length* of the linear graph $\mathcal{L}(\mathbb{H})$ with $|V(\mathcal{L}(\mathbb{H}))| > 1$ as the average passenger-oriented distance in $\mathcal{L}(\mathbb{H})$, i.e.,

$$\mathfrak{L}_d(\mathcal{L}(\mathbb{H})) = \sum_{p\neq q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)\frac{LOD(p,q)}{N^L},$$

where $LOD(p,q)/N^L$ is the proportion of passengers transferring from line L_p to line L_q over all passengers who transfer.

So, the higher the number of passengers traveling between lines L_p and L_q, the higher the weight we assign to the distance between these two lines. This way, when the characteristic path passenger-oriented length is minimized among the feasible linear graphs, lines with more passengers traveling between them will have better connections. Another possibility is to consider in the denominator, instead of the number N^L of passengers who transfer, the total number of passenger N in the network.

The next lemma proves that the characteristic path passenger-oriented length above defined is a natural extension of $\mathfrak{L}(\mathcal{L}(\mathbb{H}))$ defined in [3].

Lemma 3.4. *$\mathfrak{L}_d(\mathcal{L}(\mathbb{H}))$ is an extension of $\mathfrak{L}(\mathcal{L}(\mathbb{H}))$, which yields the same result if the number of passengers between each pair of lines L_p, L_q, is the same, that is, all the elements of matrix LOD, except its diagonal elements, are the same.*

Proof. Trivially, if we consider a constant number $\alpha > 0$ of passengers for each LOD pair (i.e. $LOD(p,q) = \alpha$), then the number of passengers who tranfer is equal to the number of ordered pairs of lines multiplied by α, that is, $N^L = \alpha(|V(\mathcal{L}(\mathbb{H}))|(|V(\mathcal{L}(\mathbb{H}))|-1))$. The following expression is thus obtained:

$$\begin{aligned}\mathfrak{L}_d(\mathcal{L}(\mathbb{H})) =& \sum_{p\neq q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)\frac{\alpha}{N^L} = \sum_{p\neq q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)\frac{\alpha}{\alpha(|V(\mathcal{L}(\mathbb{H}))|(|V(\mathcal{L}(\mathbb{H}))|-1))}\\ =& \frac{1}{|V(\mathcal{L}(\mathbb{H}))|(|V(\mathcal{L}(\mathbb{H}))|-1)}\sum_{p\neq q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q).\end{aligned}$$

Since the distance $d_{\mathcal{L}(\mathbb{H})}$ is symmetric and $d_{\mathcal{L}(\mathbb{H})}(L_p, L_p) = 0$, then

$$\begin{aligned}&\frac{1}{|V(\mathcal{L}(\mathbb{H}))|(|V(\mathcal{L}(\mathbb{H}))|-1)}\sum_{p\neq q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)\\ &= \frac{2}{|V(\mathcal{L}(\mathbb{H}))|(|V(\mathcal{L}(\mathbb{H}))|-1)}\sum_{p< q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q) = \mathfrak{L}(\mathcal{L}(\mathbb{H})).\end{aligned}$$

□

The next proposition proves two properties which allow us to use the passenger-oriented characteristic path length as a transferability measure for collective transportation line networks. These two properties are: staying within a predefined range of variation and monotonicity.

Proposition 2. *Consider a CTLN G, and let $\mathcal{L}(\mathbb{H})$ be its associated linear graph. The passenger-oriented characteristic path length $\mathfrak{L}_d(\mathcal{L}(\mathbb{H}))$ of the linear graph satisfies the following properties:*

1. $1 \leq \mathfrak{L}_d(\mathcal{L}(\mathbb{H})) \leq \ell - 1$.
2. *$\mathfrak{L}_d(\mathcal{L}(\mathbb{H}))$ is monotone decreasing in the sense that, if G' is obtained when adding a new link to G connecting two lines without additional stations (with at least one passenger), we have that $\mathfrak{L}_d(\mathcal{L}(\mathbb{H})') \leq \mathfrak{L}_d(\mathcal{L}(\mathbb{H}))$, where $\mathcal{L}(\mathbb{H})'$ is the linear graph of G'.*

Proof. 1.

$$\mathfrak{L}_d(\mathcal{L}(\mathbb{H})) = \sum_{p\neq q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)\frac{LOD(p,q)}{N^L} \geq \sum_{p\neq q}\frac{LOD(p,q)}{N^L} = 1.$$

This bound is attained when $d_{\mathcal{L}(\mathbb{H})}(L_p, L_q) = 1, \forall p \neq q$, i.e. each two lines intersect at a transfer station/stop and, thus, the linear graph is complete.

$$\ell - 1 = \sum_{p\neq q}(\ell-1)\frac{LOD(p,q)}{N^L} \geq \sum_{p\neq q} d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)\frac{LOD(p,q)}{N^L} = \mathfrak{L}_d(\mathcal{L}(\mathbb{H})).$$

2. When adding a line that connects two lines that were previously connected in G, $d_{\mathcal{L}(\mathbb{H})}$ does not vary and neither does $\mathfrak{L}_d(\mathcal{L}(\mathbb{H}))$. When adding a line that connects two lines that were not directly connected, say L_p and L_q, now $d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)$ decreases to 1, and $d_{\mathcal{L}(\mathbb{H})}$ either remains constant or decreases for any other pair of lines. Therefore $\mathfrak{L}_d(\mathcal{L}(\mathbb{H})') < \mathfrak{L}_d(\mathcal{L}(\mathbb{H}))$.

□

Similar definitions and properties hold for the linear multigraph $\mathcal{L}^M(\mathbb{H})$.

3.2. **Clustering passenger-oriented coefficient.** Transitivity is an important concept in social network analysis. A well-known indicator to measure the local degree of clustering, known as the clustering coefficient, is defined in [16]. In [3], it is adapted to the different topological representations of a CTLN. The clustering coefficient takes into account the number of transfers one needs to make to travel between neighbors of a node when this is deleted. In the following sections we will present the clustering coefficient at the different topological representations of CTLN, but also including travel demand patterns.

3.2.1. *Clustering passenger-oriented coefficient on linear graphs $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}^M(\mathbb{H})$.* The clustering coefficient in $\mathcal{L}(\mathbb{H})$ takes into account the number of transfers one needs to make to travel between neighbors of a line when this is deleted, as well as the number of passengers travelling between lines. The number of stations is therefore not considered, nor are the trips between pairs of stations of the same line. These aspects will be taken into consideration in Section 3.2.2.

Definition 3.5. Let G be a CTLN and let $\mathcal{L}(\mathbb{H})$ be its associated linear graph. We consider the *passenger-oriented clustering coefficient* C_d on the linear graph $\mathcal{L}(\mathbb{H})$ as an extension considering demand of the clustering coefficient presented in [16]. Therefore, for each node $p \in V(\mathcal{L}(\mathbb{H}))$, the subgraph $\mathcal{L}_p(\mathbb{H})$ formed by all first neighbors of p is considered. In this subgraph, node p and all edges incidents to p are eliminated. If node p has k_p neighbors, then $\mathcal{L}_p(\mathbb{H})$ will have k_p nodes and at most $k_p(k_p-1)/2$ edges. C_p is the fraction of these edges that actually exist and C_d is the average of C_p, calculated over all nodes:

$$C_d(\mathcal{L}(\mathbb{H})) = \frac{1}{\sum_{p\in V(\mathcal{L}(\mathbb{H}))} N_p}\sum_{p\in V(\mathcal{L}(\mathbb{H}))} C_p(\mathcal{L}_p(\mathbb{H})),$$

where

$$C_p(\mathcal{L}_p(\mathbb{H})) = \frac{\text{number of edges in } \mathcal{L}_p(\mathbb{H})}{k_p(k_p-1)/2} N_p,$$

where N_p is the total number of passengers taking line L_p. Note that $C_d(\mathcal{L}(\mathbb{H})) \in [0,1]$.

We consider that if $|\mathcal{L}_p(\mathbb{H})| = 1$, then $C_p(\mathcal{L}_p(\mathbb{H})) = 0$.

The next lemma proves that the passenger-oriented clustering coefficient just defined, is a natural extension of $C(\mathcal{L}(\mathbb{H}))$ defined in [3].

Lemma 3.6. *$C_d(\mathcal{L}(\mathbb{H}))$ is an extension of $C(\mathcal{L}(\mathbb{H}))$, which yields the same result if the number of passengers N_p traversing each line is the same, that is, if there exists a constant $\alpha > 0$ such that $N_p = \alpha$.*

Proof. Trivially, if there exists a constant $\alpha > 0$ such that $N_p = \alpha$, the following expression is obtained:

$$\begin{aligned} C_d(\mathcal{L}(\mathbb{H})) =& \frac{1}{\sum_{p \in V(\mathcal{L}(\mathbb{H}))} \alpha} \sum_{p \in V(\mathcal{L}(\mathbb{H}))} \frac{\text{number of edges in } \mathcal{L}_p(\mathbb{H})}{k_p(k_p-1)/2} \alpha \\ =& \frac{1}{|V(\mathcal{L}(\mathbb{H}))|} \sum_{p \in V(\mathcal{L}(\mathbb{H}))} \frac{\text{number of edges in } \mathcal{L}_p(\mathbb{H})}{k_p(k_p-1)/2} = C(\mathcal{L}(\mathbb{H})). \end{aligned}$$

□

The passenger-oriented clustering coefficient in $\mathcal{L}(\mathbb{H})$ measures the degree of transferability between its nodes from a passenger perspective, but does not take into account the number of transfer nodes in $\mathbb{H}$. In order to consider these, not only if passengers can transfer between two lines, but also the number of possibilities to transfer, we propose the following definition of the passenger-oriented clustering coefficient in $\mathcal{L}^M(\mathbb{H})$.

Definition 3.7. Let $\mathcal{L}_p^M(\mathbb{H})$ be the neighbor multigraph associated to $p \in V(\mathcal{L}^M(\mathbb{H}))$ and U^{max} a threshold that represents the maximum number of transfer nodes that can exist between two lines. If node p has k_p neighbors, then $\mathcal{L}_p^M(\mathbb{H})$ will have at most $U^{max}(k_p(k_p-1))/2$ multi-edges. $C_p^{MA}(\mathcal{L}_p^M(\mathbb{H}))$ is the fraction of these edges that actually exist and the *passenger-oriented clustering coefficient* $C_d^{MA}(\mathcal{L}^M(\mathbb{H}))$ on the linear multigraph $\mathcal{L}^M(\mathbb{H})$ is the average of $C_p^{MA}(\mathcal{L}_p^M(\mathbb{H}))$, calculated over all nodes:

$$C_d^{MA}(\mathcal{L}^M(\mathbb{H})) = \frac{1}{\sum_{p \in V(\mathcal{L}^M(\mathbb{H}))} N_p} \sum_{p \in V(\mathcal{L}^M(\mathbb{H}))} C_p^{MA}(\mathcal{L}_p^M(\mathbb{H})),$$

where

$$C_p^{MA}(\mathcal{L}_p^M(\mathbb{H})) = \frac{\text{number of edges in } \mathcal{L}_p^M(\mathbb{H})}{U^{max} k_p(k_p-1)/2} N_p.$$

Note that $C_d^{MA}(\mathcal{L}^M(\mathbb{H})) \in [0,1]$.

Multiplying C_p by N_p we weight the nodes (lines) in such a way that the lines with more passengers weight more. In this way, we give more relevance to the busier lines. Moreover, if these indices are to be minimized, then these more busy lines will be preferred to have better connections in order not to weight that much in the objective function.

Trivially, Lemma 3.6 also applies to the passenger-oriented clustering coefficient on the linear multigraph.

3.2.2. *Passenger-oriented clustering coefficient on transit hypergraph* $\mathbb{H}$. The clustering coefficient in $\mathbb{H}$ will take into account the passengers traveling between stations. In order to define the local clustering coefficient on hypergraphs, we refer to the primal graph of a hypergraph and will make calculations on it by using the terminology of hypergraphs.

Definition 3.8. Let $G_{\mathbb{H}}$ be the primal graph (see [4]) of the transit hypergraph $\mathbb{H}$. We will consider the passenger-oriented clustering coefficient of $G_{\mathbb{H}}$ defined as an extension of the clustering coefficient introduced in [16] and denoted by $C_d(G_{\mathbb{H}})$. Let $G_{\mathbb{H}_i}$ be the subgraph formed by all first neighbors of i, then the passenger-oriented clustering coefficient of $G_{\mathbb{H}}$ is defined as follows:

$$C_d(G_{\mathbb{H}}) = \frac{1}{\sum_{i\in V(G_{\mathbb{H}})} n_i} \sum_{i\in V(G_{\mathbb{H}})} C_d(G_{\mathbb{H}_i}),$$

where

$$C_d(G_{\mathbb{H}_i}) = \frac{\text{number of edges in } G_{\mathbb{H}_i}}{k_i(k_i-1)/2} n_i,$$

k_i being the number of nodes of $G_{\mathbb{H}_i}$ and n_i the number of passengers traversing station i. Note that $C_d(G_{\mathbb{H}}) \in [0,1]$.

Note that if a station has no passenger traversing it, then $n_i = 0$ and therefore this station is not being considered in the calculation of the *passenger-oriented clustering coefficient*. Our intention is to analyze the topology of the CTLN from a passenger perspective. Thus the stations with more passengers have more influence on the resulting coefficient, and the stations with no passengers have no influence. The same applies to the linear graph $\mathcal{L}(\mathbb{H})$: if a line has no passengers, it is not considered in the computation of the *passenger-oriented clustering coefficient* C_d on $\mathcal{L}(\mathbb{H})$, and the lines with more passengers have more influence on this coefficient.

The next lemma proves that the passenger-oriented clustering coefficient on the primal graph $G_{\mathbb{H}}$ is a natural extension of $C(G_{\mathbb{H}})$ defined in [3].

Lemma 3.9. *$C_d(G_{\mathbb{H}})$ is an extension of $C(G_{\mathbb{H}})$, which yields the same result if the number of passengers traversing each station s_i is the same, that is, if there exists a constant $\alpha > 0$ such that $n_i = \alpha$.*

Proof. Trivially, if there exists a constant $\alpha > 0$ such that $n_i = \alpha$, the following expression is obtained:

$$\begin{aligned} C_d(G_{\mathbb{H}}) =& \frac{1}{\sum_{i\in V(G_{\mathbb{H}})} \alpha} \sum_{i\in V(G_{\mathbb{H}})} \frac{\text{number of edges in } G_{\mathbb{H}_i}}{k_i(k_i-1)/2} \alpha \\ =& \frac{1}{|V(G_{\mathbb{H}})|} \sum_{i\in V(G_{\mathbb{H}})} \frac{\text{number of edges in } G_{\mathbb{H}_i}}{k_i(k_i-1)/2} = C(G_{\mathbb{H}}). \end{aligned}$$

□

3.3. **Passenger-oriented local and global efficiency.** Global and local efficiency [11] in a CTLN environment [3] measure how efficiently one can move from one node to another from a global and a local point of view, respectively. In this section, we introduce this measures considering not only the topology of the CTLN, but also the passenger demand patterns.

3.3.1. *Passenger-oriented local and global efficiency on the linear graph $\mathcal{L}(\mathbb{H})$.* The passenger-oriented local and global efficiency on the linear graph will measure how efficiently the passenger demand can move from one line to another from a global and a local point of view, respectively.

Definition 3.10. We define the *passenger oriented global efficiency indicator* of the linear graph $\mathcal{L}(\mathbb{H})$ as the average of the inverse of the passenger-oriented distances in $\mathcal{L}(\mathbb{H})$, that is,

$$E_d^{glob}(\mathcal{L}(\mathbb{H})) = \sum_{p \neq q} \frac{LOD(p,q)}{N^L d_{\mathcal{L}(\mathbb{H})}(L_p, L_q)},$$

where $LOD(p,q)/N^L$ is the proportion of passengers transferring from line L_p to line L_q over all passengers who transfer.

The next proposition provides two properties which allow us to use the passenger-oriented global efficiency as a transferability measure for collective transportation line networks. These two properties are: staying within a predefined range of variation and monotonicity.

Proposition 3. *Consider a CTLN G, and let $\mathcal{L}(\mathbb{H})$ be its associated linear graph. We have that:*

1. $\frac{1}{\ell-1} \leq E_d^{glob}(\mathcal{L}(\mathbb{H})) \leq 1$*. The closer to 1, the more interconnected the lines are.* $E_d^{glob}(\mathcal{L}(\mathbb{H})) = 1$ *means that for all pairs of lines of G there is a transfer station that directly connects them.*
2. $E_d^{glob}(\mathcal{L}(\mathbb{H}))$ *is monotone increasing in the sense that, if G' is obtained when adding a new line without additional stations that connects two lines in G, we have that* $E_d^{glob}(\mathcal{L}(\mathbb{H})') \geq E_d^{glob}(\mathcal{L}(\mathbb{H}))$*, where $\mathcal{L}(\mathbb{H})'$ is the linear graph of G'. Moreover,* $E_d^{glob}(\mathcal{L}(\mathbb{H})') > E_d^{glob}(\mathcal{L}(\mathbb{H}))$ *if and only if the new link connects two lines that were not directly connected in G.*

Proof. This proof is analogous to that of Proposition 2. □

Definition 3.11. We define the *passenger-oriented local efficiency indicator* of the linear graph $\mathcal{L}(\mathbb{H})$ as the average passenger oriented global efficiency of the subgraph $\mathcal{L}_p(\mathbb{H}) = (V_p, E_p)$, formed by all first neighbors of L_p in $\mathcal{L}(\mathbb{H})$, where $V_p = V(\mathcal{L}_p(\mathbb{H}))$ and $E_p = E(\mathcal{L}_p(\mathbb{H}))$. Mathematically,

$$E_d^{loc}(\mathcal{L}(\mathbb{H})) = \frac{1}{\sum_{p \in V(\mathcal{L}(\mathbb{H}))} N_p} \sum_{p \in V(\mathcal{L}(\mathbb{H}))} E_d^{glob}(\mathcal{L}_p(\mathbb{H})) N_p,$$

where N_p is the number of passengers traveling within line L_p.

3.3.2. *Passenger-oriented local and global efficiency on the hypergraph $\mathbb{H}$.* The *passenger-oriented local and global efficiency on the hypergraph $\mathbb{H}$* measure how efficiently passenger demand can move from one station to another from a global and a local point of view, respectively. This is defined as the average of the inverse of the distances $d_{\mathbb{H}}(s_i, s_j)$ (Section 2.1), similar to what is done in Definitions 3.10 and 3.11.

Note that for constant demand patterns, the passenger oriented global and local efficiency indicators for CTLN are the same as the usual global and local efficiency indicators for CTLN studied in [3]. This can be proved analogously to what is done for the passenger-oriented characteristic path length in Lemmas 3.4 and 3.2.

4. **Computational experiments.** Most CTLNs have been classified according to simple topological patterns ([10]) and the number of lines in the network, such as single lines, two lines with one or two crosses, circular lines, stars, grids, cartwheels and triangles. In order to show the applicability of our measures, we have tested several basic network configurations (see Figure 4), composed of three lines each.

Specifically, we have considered four topological CTLN configurations: a cartwheel configuration with 29 nodes and 32 edges, a parallel configuration with 29 nodes and 28 edges, a star configuration with 29 nodes and 28 edges and a triangle configuration with 29 nodes and 29 edges.

In order to introduce demand patterns, we have generated an OD demand matrix for each of these configurations, following the trip distribution described in [9]. To this end, we have considered each network as a graph, where each node is represented as point by a coordinate in the plane, and each edge represents a direct connection between a pair of nodes. As in [9], each configuration describes a CTLN for a circular city with two disjoint areas: a central business district B and a residential area A. For each configuration, we have defined A as the area corresponding to a circle of radius 2 and centre a (see Figure 4), and B as the area lying outside of A.

In Table 1 we report the values of the transferability measures introduced in Section 3 obtained for each configuration and its associated OD matrix. More precisely, the rows represent the different configurations, and the columns the transferability measures for the pure topological case [3], as well as for the passenger-oriented case (Section 3).

It can be observed that with the exception of the parallel configuration, all $\mathcal{L}(\mathbb{H})$ measures are equal to one, both for the topological and the passenger-based cases. This is due to the topology of the linear graph, which is a complete graph for the cartwheel, star and triangle configurations, thus yielding the best possible results for the measures at this level. However, differences can be observed for the clustering at the multi-linear graph, the cartwheel configuration being the one achieving the best results for both passenger oriented and topological cases. This is due to the number of transfer possibilities between any two lines. The cartwheel is the only configuration with more than one transfer station between two lines. These transfer stations correspond to multiple edges in corresponding the multi-linear graph thus yielding better results. The multi-linear graph allows us to know not only if there are possibilities to transfer between two lines, but also the number of transfers.

Regarding the clustering coefficient in $\mathbb{H}$, the best results are achieved by the star configuration when no passenger patterns are considered since the existing transfer node is very well connected. However, when we calculate the passenger-oriented clustering coefficient, the parallel configuration yields better results. This is due to the shape of the OD matrices, which have more passengers between pairs of stations in the best connected area for the parallel case.

Since the demand is mostly concentrated in a small central area, the passenger-oriented characteristic path length and the efficiency in $\mathbb{H}$ is smaller than the pure topological characteristic path length for all configurations. The pair of stations that are the closest are given more weight, thus yielding better results for characteristic path length and efficiency when passenger patterns are taken into consideration. In spite of the fact that $\mathfrak{L}_d$ and E_d^{glob} are better than $\mathfrak{L}$ and E^{glob}, respectively, for all configurations, there are configurations that improve more than others depending on the demand distribution associated with the network topology. For example, the cartwheel configuration is the one yielding the best results for $\mathfrak{L}$ and E^{glob}, whereas when demand patterns are introduced, the triangular configuration is the best one with respect to the measures $\mathfrak{L}_d$ and E_d^{glob}.

Table 2 shows the average values of Table 1. It can be seen that on average, all measures except the clustering coefficient in $\mathbb{H}$, improve for the passenger-oriented case. As explained before, this is due to the trip distribution [9], which concentrates in a small central area, so that the closest OD pairs have more passengers than the most separated ones.

Network	Levels	$\mathfrak{L}$	C	E^{glob}	$\mathfrak{L}_d$	C_d	E_d^{glob}
Cartwheel	Hypergraph	1.571	0.918	0.714	1.524	0.820	0.737
	Linear	1	1	1	1	1	1
	Multi-linear		0.555			0.548	
Parallel	Hypergraph	1.886	0.964	0.638	1.666	0.837	0.702
	Linear	1.333	0.666	0.833	1.187	0.839	0.906
	Multi-linear		0.222			0.181	
Star	Hypergraph	1.642	0.976	0.678	1.522	0.829	0.738
	Linear	1	1	1	1	1	1
	Multi-linear		0.333			0.333	
Triangle	Hypergraph	1.62	0.938	0.689	1.479	0.825	0.76
	Linear	1	1	1	1	1	1
	Multi-linear		0.333			0.333	

TABLE 1. Results of the different configurations.

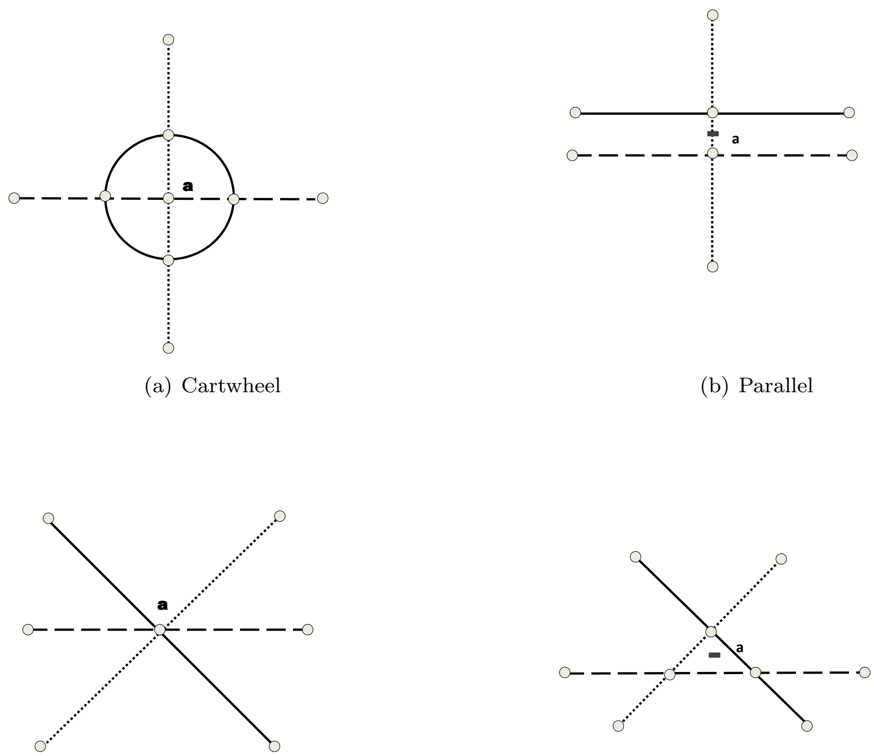
Network	Levels	$\mathfrak{L}$	C	E^{glob}	$\mathfrak{L}_d$	C_d	E_d^{glob}
Average	Hypergraph	1.679	0.949	0.679	1.547	0.827	0.734
	Linear	1.083	0.916	0.958	1.046	0.959	0.976
	Multi-linear		0.361			0.505	

TABLE 2. Average statistic for Table 1.

Sensitivity analysis with respect to passenger demand for a fixed CTLN. We analyze the behavior of our measures with respect to small changes in the OD matrix. For this sensitivity analysis, we focus on the cartwheel configuration. Let $\overline{OD}$ be the OD matrix obtained from [9] for the cartwheel configuration. Starting from $\overline{OD}$, we have randomly generated 120 different OD matrices by considering small changes in its elements. To this end, we consider a parameter τ representing the maximum change in the number of passengers allowed for each OD pair. We consider four different values of τ and for each of them we randomly create 30 different OD matrices $\overline{OD}_\tau^r$, $r = 1, \dots, 30$. The relative error of each $\overline{OD}_\tau^r$ with respect to $\overline{OD}$ is computed as $Error(\overline{OD}_\tau^r) = \|\overline{OD} - \overline{OD}_\tau^r\|/\|\overline{OD}\|$, and the average relative error $Error(\overline{OD}_\tau)$ for each τ is shown in Table 3. We compute the transferability measures defined in previous sections for each $\overline{OD}_\tau^r$.

In Table 3 the first column shows the values of τ; the second presents the average relative error $Error(\overline{OD}_\tau)$; the third column is the average relative error of the total demand; the fourth, fifth and sixth column report the average relative error of $\mathfrak{L}_d$, C_d and E_d^{glob}, respectively. The last three columns present the average results of $\mathfrak{L}_d$, C_d and E_d^{glob}, respectively.

It can be observed from Table 3 that for small changes in the OD matrices, small average relative error are obtained for the passenger-oriented characteristic path length, clustering coefficient and global efficiency. Our measures are stable



(a) Cartwheel (b) Parallel

(c) Star (d) Triangle

FIGURE 4. Four simple network configurations.

τ	$Error(\overline{OD}_\tau)$	$Error(demand)$	$Error(\mathfrak{L}_d)$	$Error(C_d)$	$Error(E_d^{glob})$	$aver(\mathfrak{L}_d)$	$aver(C_d)$	$aver(E_d^{glob})$
1	0.01834	0.0035	0.0019	0.0002	0.0016	1.5248	0.8205	0.0027
2	0.0335	0.0034	0.0037	0.0004	0.0034	1.5253	0.8205	0.7373
3	0.0474	0.0050	0.0058	0.0006	0.0048	1.5257	0.8205	0.7372
4	0.0607	0.0103	0.0118	0.0006	0.0092	1.5276	0.8207	0.7362

TABLE 3. Average demand sensitivity results for cartwheel configuration.

against small changes in OD matrices. These results show that they are satisfactory regarding stability against small changes in the OD matrix.

5. **Conclusions.** We have presented an extension of topological transferability coefficients for CTLN, by also considering passenger demand, thus yielding a more complete analysis of the transferability of these networks. We have analytically demonstrated that when the demand between each OD pair is constant, the passenger demand has no influence on the resulting coefficients, and the results are the same as those obtained by considering only the topology of the networks. We have also seen that for the non-constant demand case, the influence of passenger demand is reflected in the coefficients, yielding better results when demand is concentrated

between well connected areas, and worse results when it is higher between poorly connected areas. Our computational experiments support the analytical results and have allowed us to analyze the behavior of the coefficients for four typical CTLN configurations.

Acknowledgments. This research work was partially supported by Ministerio de Economía y Competitividad (Spain)/FEDER under grant MTM2012-37048, by Junta de Andalucía (Spain)/FEDER under excellence projects P09-TEP-5022 and P10-FQM-5849, and by the Canadian Natural Sciences and Engineering Research Council under grant 39682-10. This support is gratefully acknowledged. Thanks are due to the referees for their valuable comments.

REFERENCES

[1] A. Barabasi and R. Albert, Emergence of scaling in random networks, *Science*, **286** (1999), 509–512.

[2] E. Barrena, A. De-Los-Santos, G. Laporte and J. A. Mesa, Passenger flow connectivity in collective transportation line networks, *International Journal of Complex Systems in Science*, **3** (2013), 1–10.

[3] E. Barrena, A. De-Los-Santos, J. A. Mesa and F. Perea, Analyzing connectivity in collective transportation line networks by means of hypergraphs, *European Physical Journal. Special Topics*, **215** (2013), 93–108.

[4] C. Berge, *Graphes et Hypergraphes*, Elsevier Science, Paris, 1973.

[5] C. Berge, *Hypergraphs: Combinatorics of Finite Sets*, North-Holland Mathematical Library, North-Holland, Amsterdam, 1989. Available from: http://books.google.es/books?id=jEyfse-EKf8C.

[6] R. Criado, B. Hernández-Bermejo and M. Romance, Efficiency, vulnerability and cost: An overview with applications to subway networks worldwide, *International Journal of Bifurcation and Chaos*, **17** (2007), 2289–2301.

[7] A. De-Los-Santos, G. Laporte, J. Mesa and F. Perea, Evaluating passenger robustness in a rail transit network, *Transportation Research Part C: Emerging Technologies*, **20** (2012), 34–46.

[8] S. Derrible and C. Kennedy, The complexity and robustness of metro networks, *Physica A: Statistical Mechanics and its Applications*, **389** (2010), 3678–3691.

[9] G. Laporte, J. Mesa and F. Ortega, Assessing the efficiency of rapid transit configurations, *TOP*, **5** (1997), 95–104.

[10] G. Laporte, J. Mesa and F. Ortega, Optimization methods for the planning of rapid transit systems, *European Journal of Operational Research*, **122** (2000), 1–10.

[11] V. Latora and M. Marchiori, Efficient behavior of small-world networks, *Physical Review Letters*, **87** (2001), 198701-1–198701-4.

[12] V. Latora and M. Marchiori, Is the Boston subway a small-world network?, *Physica A*, **314** (2002), 109–113.

[13] S. Milgram, The small world problem, *Psychology Today*, **1** (1967), 60–67.

[14] C. Roth, S. Kang, M. Batty and M. Barthelemy, A long-time limit for world subway networks, *Journal of The Royal Society Interface*, **9** (2012), 2540–2550.

[15] K. Seaton and L. Hackett, Stations, trains and small-world networks, *Physica A: Statistical Mechanics and its Applications*, **339** (2004), 635–644.

[16] D. Watts and S. Strogatz, Collective dynamics of small-world networks, *Nature*, **393** (1998), 440–442.

Received July 2014; revised December 2014.

E-mail address: eva.barrena@cirrelt.ca
E-mail address: aliciasantos@us.es
E-mail address: gilbert.laporte@cirrelt.ca
E-mail address: jmesa@us.es