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GROUP PINNING CONSENSUS UNDER FIXED AND RANDOMLY SWITCHING TOPOLOGIES WITH ACYCLIC PARTITION

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Abstract. This paper addresses group consensus problems in generic linear multi-agent systems with directed information flow over (i) fixed topology and (ii) randomly switching topology governed by a continuous-time homogeneous Markov process. We propose two types of pinning control protocols to ensure group consensus regardless of the magnitude of the coupling strengths among the agents. In the case of randomly switching topology, we show that the group consensus behavior is unrelated to the magnitude of the couplings among agents if the union of the topologies corresponding to the positive recurrent states of the Markov process possesses an acyclic partition. Sufficient conditions for achieving group consensus are presented in terms of simple graphic conditions, which are easy to be checked compared to conventional algebraic criteria. Simulation examples are also presented to validate the effectiveness of the theoretical results.

1. **Introduction.** In recent years, the study of distributed coordination in multi-agent systems has attracted increasing attention from researchers in diverse fields of engineering, physics, biology, and mathematics. An important problem in cooperative control of multiple agents is to design appropriate protocols such that the states of a group of agents converge to a consistent value with information exchanges between each other. Such a problem is usually called the consensus problem, in which design of consensus algorithms, convergence analysis and consensus speed are well-researched topics [16]. The consistent state in consensus problems could represent the position and velocity in multivehicle formation control, anticipated processing rate in distributed task management, or common phase pattern in distributed oscillator networks, etc. Consensus problems have a long history in control theory starting with the pioneering works [4, 34]. Seminal theoretical frameworks for solving consensus problems were introduced by Olfati-Saber and Murray [17] and Jadbabaie, Lin, and Morse [11]. Since then various consensus problems, such as average consensus [26], asynchronous consensus [35], finite-time consensus [24], and stochastic consensus [33], have been extensively studied. For details, we refer readers to survey paper [16] and the references therein.

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Note that the above works concern the complete consensus, where all the agents in a network share a common value. However, in the course of accomplishing some complicated tasks or reaching some goals, a group of dynamic agents may evolve into several subgroups in response to unanticipated situations or changes. This might give rise to agreements that are different with the changes of environments, situations, cooperative tasks or even time [27, 28, 38]. Group consensus was first introduced by Yu and Wang [38] to represent such consensus where the states of of all agents in the same subgroup achieve the same consistent state while the states of agents in different subgroups may not coincide. Group average consensus problems for continuous-time single-integrator agents under fixed and undirected topology were explored in [38]. By using the Lyapunov direct method and double-tree-form transformations, the same authors later extended the results to the case containing switching topologies and communication delays in [39]. A common assumption in these above works that the sum of adjacent weights to every agent in one group coming from all agents in another group is equal to zero (called the *in-degree balance condition*) was further relaxed in [32] by transforming the system into a reduced-order system. This technique turns out to be viable even in the presence of random noises and time delays [27]. Algebraic criterions for group consensus in agents with discrete-time single-integrator dynamics were established in [10] and were later adapted to the situation accommodating stochastic inputs [25]. Group consensus for second-order multi-agent systems and linear time-invariant systems under fixed communication topology was addressed in [9] and [28], respectively. Non-linear agent dynamics was considered in [31] via pinning control.

We mention that another line of research in parallel with (but precedes temporally) group consensus is the group and cluster synchronization, which have been mainly studied in the physics literature. The difference between cluster synchronization and group synchronization is that the former analyzes the case that the uncoupled systems are all identical, while the latter considers a situation where the systems in each group are characterized by different local dynamics [7]. Group and cluster synchronization can be viewed as a generalization of the group consensus problem to encompass nonlinear dynamics. We refer to [1, 3, 7, 29] for more details and history in this exciting field.

In all the previously mentioned publications on group consensus, certain algebraic criteria were proposed to ensure the consensus. These conditions, as pointed out in [19], are often very difficult to be checked. For example, linear matrix inequality conditions (without discussing the feasibility) were introduced in [38, 39], and other conditions involving eigenvalues of interaction topologies were proposed in [9, 27, 31, 32]. Recently, Qin and Yu [19] investigated the group consensus problems for generic linear time-invariant systems from a novel perspective; under the in-degree balance condition, they showed that group consensus can be achieved regardless of the magnitude of the coupling strengths among the agents when the underlying communication graph has an acyclic partition. This result is both theoretically interesting because it only requires simple graphic conditions to guarantee the group consensus, and practically appealing because it accommodates the realistic situation where coupling strengths among agents are not allowed to be large. The consensus speed was also estimated in [19].

In this paper, continuing with previous works, we tackle the group consensus problems in continuous-time linear time-invariant systems via pinning control. Firstly, we investigate the group consensus over fixed directed topology by looking into two different types of pinning control protocols. The first protocol was used in [19], under which we will show that the group consensus can be achieved with any speed when the agent dynamics is controllable. Likewise, we propose a second protocol which guarantees group consensus regardless of the magnitude of the coupling strengths among the agents if the communication topology admits an acyclic partition. Under this protocol, the in-degree balance condition can be largely relaxed to only demand that the sum of adjacent weights to every node in one group coming from all nodes in another group is a constant. Thanks to this relaxation, we are able to discuss how to choose appropriate pinning controllers for reaching group consensus, which offers further insights on the influence of network topology on group consensus behaviors. In general, selecting pinning nodes is a key problem in pinning control of complex networks [5, 37], since it is costly and literally impossible to control every node in a network.

Next, we extend our theoretical framework to address group consensus for a network of dynamic agents whose communication topology is modeled by a randomly switching graph. Specifically, the switching is driven by a continuous-time homogenous Markov process. Each communication (possibly directed) graph corresponds to a state of the Markov process. Such stochastic consensus is desirable since systems in real world are often operating under random uncertain environments. For both protocols considered here, we show that group consensus can be achieved in mean square and almost sure senses regardless of the magnitude of the coupling strengths among the agents if the union of topologies corresponding to the positive recurrent states of the Markov process has an acyclic partition. When the agent dynamics is controllable, the speed to group consensus can be achieved arbitrarily fast. We mention that there have been some works concerning consensus problems over Markovian switching networks, see e.g. [13, 14, 15, 23, 36], where nevertheless only complete consensus is addressed. In all these works, without using pinning nodes (or virtual leaders), the ultimate consensus value is either not specified or the average of initial states of all agents.

The rest of the paper is organized as follows. In Section 2, some preliminaries, lemmas and the problem formulation are introduced. Section 3 deals with group consensus under fixed topology, and Section 4 addresses the case for Markovian switching topologies. Simulation examples are presented in Sections 3 and 4 respectively to facilitate discussions and illustrate our theoretical results. Conclusions are finally drawn in Section 5.

Notation. The following notations will be used throughout the paper. $\mathbb{R}$ ($\mathbb{C}$) denotes the set of real (complex) numbers. Let 1_E be the indicator function of an event E. Let $\mathbf{1}_n$ be the n-dimensional column vector with all entries equal to one. I_n is the n-dimensional identity matrix. We often drop the subscript n when the dimension is compatible with the context. We say $A > B$ ($A \geq B$) if $A - B$ is positive definite (semi-definite), where A and B are symmetric matrices of same dimensions. $\lambda_{\min}(A)$ denotes the smallest eigenvalue of symmetric matrix A. A^T (A^H) is the transpose (conjugate transpose) of matrix A. $\mathrm{diag}(A_1, \cdots, A_r)$ is the block diagonal matrix with the i-th main diagonal block being square matrix A_i. Let $\|x\|$ signify the Euclidean norm of a vector x. $A \otimes B$ refers to the Kronecker product of two matrices A and B [21]. The notation $*$ represents the matrix elements to be determined.

2. Preliminaries and problem formulation.

2.1. Communication graphs.
Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order N, in which $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ is the set of nodes (or agents), $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges, and $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is the associated weighted adjacency matrix. A directed edge from node v_i to node v_j is represented as an ordered pair (v_i, v_j), indicating that agent v_j can obtain information from agent v_i. Here, v_i and v_j are called parent node and child node, respectively. The entry $a_{ij} \neq 0$ if $(v_j, v_i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Moreover, we assume $a_{ii} = 0$ for all i. The set of neighbors of the agent v_i in $\mathcal{G}$ is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. The Laplacian matrix $\mathcal{L} = (l_{ij})$ of the weighted directed graph $\mathcal{G}$ is defined as $l_{ij} = -a_{ij}$, $i \neq j$, and $l_{ii} = \sum_{j=1,j\neq i}^{N} a_{ij}$. Clearly, $\mathcal{L}\mathbf{1}_N = 0$. $\sum_{j=1}^{N} a_{ij}$ is called the in-degree of agent v_i.

A directed path from agent v_{i_1} to agent v_{i_k} consists of a sequence of nodes $v_{i_1}, v_{i_2}, \cdots, v_{i_k}$, such that $(v_{i_{j-1}}, v_{i_j}) \in \mathcal{E}$ for $j = 2, \cdots, k$. If, in addition, $v_{i_1} = v_{i_k}$, then it is called a directed cycle. $\mathcal{G}$ is said to be a *directed acyclic graph* (DAG) if it contains no directed cycles. The following result is a useful property of DAG, which simplifies its matrix representation [2, 19].

Lemma 2.1. *Each DAG can be relabeled in such a way that the index of the parent node is smaller than the index of the child node for each edge. Therefore, each DAG can be relabeled such that its adjacency (and Laplacian) matrix is lower triangular.*

We say $\mathcal{G}$ contains a spanning tree if there exists an agent (referred to as root) such that every other agent can be connected via a directed path originating from the root. It follows from [20, Lemma 3.3] that $\mathcal{G}$ has a spanning tree if and only if $\mathcal{L}$ has a zero eigenvalue with algebraic multiplicity one and all the other nonzero eigenvalues are with positive real parts. For an integer s, the union of s graphs $\mathcal{G}^{(1)} = (\mathcal{V}, \mathcal{E}^{(1)}, \mathcal{A}^{(1)}), \cdots, \mathcal{G}^{(s)} = (\mathcal{V}, \mathcal{E}^{(s)}, \mathcal{A}^{(s)})$ is defined as $\cup_{k=1}^{s}\mathcal{G}^{(k)} = (\mathcal{V}, \cup_{k=1}^{s}\mathcal{E}^{(k)}, \sum_{k=1}^{s}\mathcal{A}^{(k)})$. Clearly, if the union is a DAG, so is each $\mathcal{G}^{(k)}$.

2.2. Group pinning control over fixed topology.
Consider a multi-agent system containing N agents with interaction graph represented by a fixed directed graph $\mathcal{G}$. For an integer r, $\{\mathcal{V}_1, \cdots, \mathcal{V}_r\}$ is said to be a partition of the node set $\mathcal{V}$ if $\mathcal{V}_\ell \neq \emptyset$, $\cup_{\ell=1}^{r}\mathcal{V}_\ell = \mathcal{V}$, and $\mathcal{V}_\ell \cap \mathcal{V}_{\ell'} = \emptyset$ for $\ell \neq \ell'$. Without loss of generality, we assume $\mathcal{V}_1 = \{v_1, \cdots, v_{N_1}\}$, $\mathcal{V}_2 = \{v_{N_1+1}, \cdots, v_{N_1+N_2}\}$, $\cdots$, $\mathcal{V}_r = \{v_{\sum_{i=1}^{r-1} N_i+1}, \cdots, v_N\}$ and $\sum_{\ell=1}^{r} N_\ell = N$. The dynamics of agent v_i takes the following form

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \cdots, N, \quad t \geq 0, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the state and control input of agent v_i; $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices. It is apparent that (1) reduces to the single-integrator dynamics by setting $n = m$, $A = 0$, and $B = I_n$. To exclude any trivial case, we assume that A is not Hurwitz (If A is Hurwitz, the group consensus can be achieved by taking zero gain in view of Definitions 2.2 and 2.3 below).

Let $y_1(t), \cdots, y_r(t)$ be the solutions of the homogeneous system $\dot{y}(t) = Ay(t)$ such that $\lim_{t\to\infty} \|y_\ell(t) - y_{\ell'}(t)\| > 0$ for $\ell \neq \ell'$. Note that it is feasible since A is not Hurwitz. We remark that a solution of an isolated node, i.e., $\dot{y}(t) = Ay(t)$, is called a *synchronized state*. A possible approach to study stability of the synchronized solution is through a master stability function (see [1, 18]), which provides necessary and sufficient conditions for local stability of this solution.

The following group pinning consensus protocol

$$u_i(t) = K\Big(\sum_{v_j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) + d_i(y_{\bar{i}}(t) - x_i(t))\Big) \tag{2}$$

is adopted in [19]. Here, $K \in \mathbb{R}^{m\times n}$ is the common consensus gain matrix to be designed, and $\bar{i}$ is the subscript of the subset to which the agent v_i belongs, i.e., if $v_i \in \mathcal{V}_\ell$ then $\bar{i} = \ell$. Moreover, $d_i > 0$ if agent v_i is pinned, and $d_i = 0$ otherwise; $a_{ij} \geq 0$ if $\bar{i} = \bar{j}$, and $a_{ij} \in \mathbb{R}$ otherwise.

Motivated by the protocols used in [9, 27, 32, 39], we also address in this work another group pinning consensus protocol as follows.

$$u_i(t) = K\Big(\sum_{v_j\in\mathcal{N}_{\bar{i}i}} a_{ij}(x_j(t) - x_i(t)) + \sum_{v_j\in\mathcal{N}_i\setminus\mathcal{N}_{\bar{i}i}} a_{ij}x_j(t) + d_i(y_{\bar{i}}(t) - x_i(t))\Big), \tag{3}$$

where K, a_{ij}, and d_i are defined as above; $\mathcal{N}_{\bar{i}i} = \{v_j \in \mathcal{V}_{\bar{i}} : (v_j, v_i) \in \mathcal{E}\}$ means the set of neighbors of v_i in $\mathcal{V}_{\bar{i}}$. By definition, $\cup_{\ell=1}^r \mathcal{N}_{\ell i} = \mathcal{N}_i$.

Definition 2.2. The multi-agent system (1) under the control law (2) or (3) is said to achieve group pinning consensus if there exists a consensus gain K such that for any $x_i(0) \in \mathbb{R}^n$,

$$\lim_{t\to\infty} \|x_i(t) - y_{\bar{i}}(t)\| = 0, \quad \text{for } i = 1, \cdots, N.$$

In addition, if there exist positive numbers κ, C, and t_0 such that $\|x_i(t) - y_{\bar{i}}(t)\| \leq Ce^{-\kappa t}$ for all i and $t > t_0$, we say the consensus is achieved exponentially fast with a *speed* κ.

$\mathcal{G}$ is said to possess an *acyclic partition* $\{\mathcal{V}_1, \cdots, \mathcal{V}_r\}$ [19] if the contracted graph—obtained by replacing each subset $\mathcal{V}_\ell$ with a single node and adding an edge $(\mathcal{V}_\ell, \mathcal{V}_{\ell'})$ if there exist $u \in \mathcal{V}_\ell$ and $v \in \mathcal{V}_{\ell'}$ such that $(u, v) \in \mathcal{E}$—is a DAG. Acyclic partition turns out to be a critical topological condition for reaching group consensus irrespective of how weak or strong the couplings among the agents are. We make the following assumption.

Assumption 1. *The communication topology $\mathcal{G}$ has an acyclic partition $\{\mathcal{V}_1, \cdots, \mathcal{V}_r\}$.*

Under Assumption 1, thanks to Lemma 2.1 we can relabel the indices of all the nodes in $\mathcal{G}$ such that its adjacency matrix is in a block lower triangular form

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{A}_{r1} & \cdots & \mathcal{A}_{rr} \end{bmatrix}, \tag{4}$$

where $\mathcal{A}_{\ell\ell}$ specifies the information exchange within subgroup $\mathcal{V}_\ell$, and $\mathcal{A}_{\ell\ell'}$ specifies the information exchange from subgroup $\mathcal{V}_{\ell'}$ to $\mathcal{V}_\ell$. Similar notation can be made for the Laplacian matrix $\mathcal{L}$. For $\ell = 1, \cdots, r$, the induced subnetwork of $\mathcal{G}$ on $\mathcal{V}_\ell$ is denoted by $\mathcal{G}_\ell$. In what follows, we assume the above relabeling and hence the adjacency and Laplacian matrices of $\mathcal{G}$ take the form in (4). We mention that the special block matrix form (4), resulting from Assumption 1, remarkably enables us to apply the Algebraic Riccati Equation tool and simplifies the proof in the convergence analysis later.

Assumption 2. *For $\ell > \ell'$, each row sum of $\mathcal{A}_{\ell\ell'}$ is equal to $\alpha_{\ell\ell'}$, where $\alpha_{\ell\ell'}$ are constant.*

Note that it is not necessary that all $\alpha_{\ell\ell'}$ are equal to zero, which implies that the commonly used in-degree balance condition (see e.g. [9, 19, 31, 39]) is considerably relaxed. When all $\alpha_{\ell\ell'}$ are equal to zero, it is easy to check (c.f. (4)) that $\mathcal{L}_{\ell\ell}$ becomes the Laplacian matrix of $\mathcal{G}_\ell$ for $\ell = 1, \cdots, r$.

2.3. **Group pinning control over random topologies.** When the interaction topology among agents is described by a randomly switching graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$, the group pinning consensus protocols (2) and (3) can be modified, respectively, as follows:

$$u_i(t) = K\Big(\sum_{v_j\in\mathcal{N}_i(t)} a_{ij}(t)(x_j(t) - x_i(t)) + d_i(t)(y_{\bar{i}}(t) - x_i(t))\Big) \tag{5}$$

and

$$\begin{aligned} u_i(t) =& K\Big(\sum_{v_j\in\mathcal{N}_{\bar{i}i}(t)} a_{ij}(t)(x_j(t) - x_i(t)) \\ &+ \sum_{v_j\in\mathcal{N}_i(t)\setminus\mathcal{N}_{\bar{i}i}(t)} a_{ij}(t)x_j(t) + d_i(t)(y_{\bar{i}}(t) - x_i(t))\Big). \end{aligned} \tag{6}$$

The randomly switching graph $\mathcal{G}(t)$ is governed by a time-homogeneous Markov process $\theta(t)$, taking value in a finite set $\mathcal{S} = \{1, 2, \cdots, s\}$. More specifically, $\mathcal{G}(t) \in \{\mathcal{G}^{(1)}, \cdots, \mathcal{G}^{(s)}\}$, where $\mathcal{G}^{(k)} = (\mathcal{V}, \mathcal{E}^{(k)}, \mathcal{A}^{(k)})$; $\mathcal{G}(t) = \mathcal{G}^{(k)}$ if and only if $\theta(t) = k$ for $k \in \mathcal{S}$. In addition, assume that $d_i(t) = d_i^{(k)}$ $(i = 1, \cdots, N)$ if and only if $\theta(t) = k$ for $k \in \mathcal{S}$. For $\ell = 1, \cdots, r$, the induced subnetwork of $\mathcal{G}(t)$ on $\mathcal{V}_\ell$ is denoted by $\mathcal{G}_\ell(t)$. Similar notations will be made for $\mathcal{G}_\ell^{(k)}$ for $k \in \mathcal{S}$. The protocols (5) and (6) indicate that the communication topology (together with its pinned nodes) is switching among a set of s systems—each of which can be viewed as a static one introduced in the previous section—as time goes on.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ represent the underlying probability space for the Markov process $\theta(t)$. Its generator $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{s\times s}$ is formally delineated by

$$\mathbb{P}(\theta(t+h) = j|\theta(t) = i) = \begin{cases} \gamma_{ij}h + o(h), & \text{if } i \neq j, \\ 1 + \gamma_{ii}h + o(h), & \text{if } i = j, \end{cases}$$

where $o(h)$ denotes an infinitesimal of higher order than h, i.e., $\lim_{h\to 0^+} o(h)/h = 0$. Here γ_{ij} is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. For ease of presentation, we assume that $\theta(t)$ is ergodic (see Remark 7 below for the case of non-ergodic processes). Hence, each state of the process is reachable from any other state, and there exists a unique invariant distribution $\pi = (\pi_1, \cdots, \pi_s)^T$ such that $\pi_k > 0$ for each $k \in \mathcal{S}$ [22].

Definition 2.3. The multi-agent system (1) under the control protocol (5) or (6) is said to achieve group pinning consensus if there exists a consensus gain K such that for any $x_i(0) \in \mathbb{R}^n$ and initial distribution of $\theta(0)$,

$$\lim_{t\to\infty} \mathbb{E}(\|x_i(t) - y_{\bar{i}}(t)\|^2) = 0, \quad \text{for } i = 1, \cdots, N,$$

where the expectation $\mathbb{E}$ is taken under the measure $\mathbb{P}$. In addition, if there exist positive numbers κ, C, and t_0 such that $\mathbb{E}(\|x_i(t) - y_{\bar{i}}(t)\|^2) \leq Ce^{-\kappa t}$ for all i and $t > t_0$, we say the consensus is achieved exponentially fast with a *speed* κ.

The group pinning consensus in Definition 2.3 is defined in the sense of mean square convergence. This further implies that the consensus can also be achieved in the almost sure sense (see [8, 14]). Denote by $\mathcal{G}^{\mathcal{S}}$ the union of the topologies in the set $\mathcal{S}$, i.e., $\mathcal{G}^{\mathcal{S}} = \cup_{k=1}^{s} \mathcal{G}^{(k)}$. For $\ell = 1, \cdots, r$, the induced subnetwork of $\mathcal{G}^{\mathcal{S}}$ on $\mathcal{V}_\ell$ is denoted by $\mathcal{G}^{\mathcal{S}}_\ell$. We make the following assumption.

Assumption 3. *The union communication topology $\mathcal{G}^{\mathcal{S}}$ has an acyclic partition $\{\mathcal{V}_1, \cdots, \mathcal{V}_r\}$.*

Under Assumption 3, we can similarly relabel the indices of all the nodes in $\mathcal{G}^{\mathcal{S}}$ such that its adjacency matrix is in a block lower triangular form

$$\mathcal{A}^{\mathcal{S}} = \begin{bmatrix} \mathcal{A}^{\mathcal{S}}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{A}^{\mathcal{S}}_{r1} & \cdots & \mathcal{A}^{\mathcal{S}}_{rr} \end{bmatrix}, \tag{7}$$

where $\mathcal{A}^{\mathcal{S}}_{\ell\ell}$ specifies the information exchange within subgroup $\mathcal{V}_\ell$, and $\mathcal{A}^{\mathcal{S}}_{\ell\ell'}$ specifies the information exchange from subgroup $\mathcal{V}_{\ell'}$ to $\mathcal{V}_\ell$. Since Assumption 3 implies that each $\mathcal{G}^{(k)}$ $(k \in \mathcal{S})$ has the same acyclic partition as $\mathcal{G}^{\mathcal{S}}$, similar notations can be made for $\mathcal{A}^{(k)}$, $\mathcal{A}(t)$, and their Laplacian counterparts $\mathcal{L}^{\mathcal{S}}$, $\mathcal{L}^{(k)}$, and $\mathcal{L}(t)$, for $k \in \mathcal{S}$, $t \geq 0$. In what follows, we assume the above relabeling and thus the related matrices mentioned above all take the same form as in (7). The following assumption is similar to Assumption 2.

Assumption 4. *For $\ell > \ell'$ and $k \in \mathcal{S}$, each row sum of $\mathcal{A}^{(k)}_{\ell\ell'}$ is equal to $\alpha^{(k)}_{\ell\ell'}$, where $\alpha^{(k)}_{\ell\ell'}$ are constant.*

To conclude this section, we collect a couple of lemmas which will be used in the convergence analysis.

Lemma 2.4. *(Schur complement [21]) Let X, Y, Z be given matrices such that $Z > 0$. Then*

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$$

if and only if $X - YZ^{-1}Y^T > 0$.

Lemma 2.5. *([6]) Suppose that $f(t)$ is $\mathcal{F}$-measurable and that $\mathbb{E}\left(f(t)1_{\{\theta(t)=k\}}\right)$ exists. Then for $k \in \mathcal{S}$,*

$$\mathbb{E}\left(f(t)d(1_{\{\theta(t)=k\}})\right) = \sum_{j=1}^{s} \gamma_{jk} \mathbb{E}\left(f(t)1_{\{\theta(t)=j\}}\right) dt + o(dt).$$

3. **Group pinning consensus analysis: Fixed topology.** In this section, under Assumptions 1 and 2, we focus on group pinning consensus of system (1) with control laws (2) and (3) over fixed network topology $\mathcal{G}$. In Section 3.1, we first address a special case of protocol (3) with $r = 2$, i.e., the network $\mathcal{G}$ consists of two subnetworks, and then generalize the result to general r. In Section 3.2, we complement the consensus result obtained in [19] with protocol (2) by applying an analogous argument in Section 3.1. We discuss the choice of pinning nodes, group partition, and compare the two protocols in Section 3.3.

3.1. **Convergence results for system (1) under protocol (3).** Without loss of generality, we first assume that the network $\mathcal{G}$ consists of $N_1 + N_2 = N$ agents such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, i.e., the case of $r = 2$.

Let $\mathcal{D}_1 = \mathrm{diag}(d_1, \cdots, d_{N_1})$ and $\mathcal{D}_2 = \mathrm{diag}(d_{N_1+1}, \cdots, d_N)$. Define an $(N+2)$-dimensional matrix

$$\bar{\mathcal{H}} = \begin{bmatrix} \bar{\mathcal{H}}_1 & 0 \\ \bar{\mathcal{H}}_{21} & \bar{\mathcal{H}}_2 \end{bmatrix},$$

where

$$\bar{\mathcal{H}}_{21} = \begin{bmatrix} -\mathcal{A}_{21} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(N_2+1)\times(N_1+1)},$$

$$\bar{\mathcal{H}}_\ell = \begin{bmatrix} \mathcal{L}_\ell + \mathcal{D}_\ell & -\mathcal{D}_\ell \mathbf{1}_{N_\ell} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(N_\ell+1)\times(N_\ell+1)},$$

and $\mathcal{L}_\ell$ is the Laplacian matrix of $\mathcal{G}_\ell$ for $\ell = 1, 2$.

Set $\bar{x}(t) = (x_1^T(t), \cdots, x_{N_1}^T(t), y_1^T(t), x_{N_1+1}^T(t), \cdots, x_N^T(t), y_2^T(t))^T \in \mathbb{R}^{n(N+2)}$. As a consequence of Assumption 1, the system (1) under protocol (3) can be recast in the following compact form

$$\dot{\bar{x}}(t) = (I_{N+2} \otimes A - \bar{\mathcal{H}} \otimes BK)\bar{x}(t), \tag{8}$$

where we have used the fact that $y_1(t)$ and $y_2(t)$ are solutions of the system $\dot{y}(t) = Ay(t)$. Let $\delta(t) = \big(x_1^T(t) - x_2^T(t), \cdots, x_1^T(t) - x_{N_1}^T(t), x_1^T(t) - y_1^T(t), x_{N_1+1}^T(t) - x_{N_1+2}^T(t), \cdots, x_{N_1+1}^T(t) - x_N^T(t), x_{N_1+1}^T(t) - y_2^T(t)\big)^T \in \mathbb{R}^{nN}$. We have the following result regarding the evolution of error dynamics.

Proposition 1. *Under Assumptions 1 and 2, we have for* $t \geq 0$,

$$\dot{\delta}(t) = \big(I_N \otimes A - \mathcal{Z} \otimes BK\big)\delta(t), \tag{9}$$

where $\mathcal{Z} = \left[\begin{array}{c:c} \mathcal{Z}_1 & 0 \\ \hdashline \mathcal{Z}_{21} & \mathcal{Z}_2 \end{array}\right] = F \left[\begin{array}{c:c} U_1^H \bar{\mathcal{H}}_1 U_1 & 0 \\ \hdashline U_2^H \bar{\mathcal{H}}_{21} U_1 & U_2^H \bar{\mathcal{H}}_2 U_2 \end{array}\right] F^{-1}$, $F = \mathrm{diag}(R_1U_1, R_2U_2)$, $R_1 = [\mathbf{1}_{N_1} \;\; -I_{N_1}]$, $R_2 = [\mathbf{1}_{N_2} \;\; -I_{N_2}]$, *and* $U_1 \in \mathbb{C}^{(N_1+1)\times N_1}$ *and* $U_2 \in \mathbb{C}^{(N_2+1)\times N_2}$ *are arbitrary matrices such that* $\Phi_1 := \big[\mathbf{1}_{N_1+1}/\sqrt{N_1+1} \quad U_1\big]$ *and* $\Phi_2 := \big[\mathbf{1}_{N_2+1}/\sqrt{N_2+1} \quad U_2\big]$ *are two unitary matrices.*

Proof. Set $R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$. It follows from (8) that

$$\dot{\delta}(t) = (R \otimes I_n)\dot{\bar{x}}(t) = (R \otimes A - R\bar{\mathcal{H}} \otimes BK)\bar{x}(t). \tag{10}$$

Let $\bar{R}_1 = [R_1^T \; \mathbf{1}_{N_1+1}]^T$ and $\bar{R}_2 = [R_2^T \; \mathbf{1}_{N_2+1}]^T$. Then $\bar{R}_1$ and $\bar{R}_2$ are invertible. For $\ell = 1, 2$, $R_\ell \mathbf{1}_{N_\ell+1} = 0$ and $\bar{\mathcal{H}}_\ell \mathbf{1}_{N_\ell+1} = 0$. Assumption 2 implies that $\mathcal{A}_{21}\mathbf{1}_{N_1} = \alpha_{21}\mathbf{1}_{N_2}$. Hence, a straightforward computation yields

$$R\bar{\mathcal{H}} = R\bar{\mathcal{H}}\left(\bar{R}^T(\bar{R}\bar{R}^T)^{-1}\bar{R}\right) = R\bar{\mathcal{H}}R^T(RR^T)^{-1}R. \tag{11}$$

Combining (11) with (10), we derive

$$\begin{aligned} \dot{\delta}(t) =& (I_N \otimes A)(R \otimes I_n)\bar{x}(t) - \left(R\bar{\mathcal{H}}R^T(RR^T)^{-1} \otimes BK\right)(R \otimes I_n)\bar{x}(t) \\ =& \left(I_N \otimes A - R\bar{\mathcal{H}}R^T(RR^T)^{-1} \otimes BK\right)\delta(t). \end{aligned} \tag{12}$$

Since $\Phi := \mathrm{diag}(\Phi_1, \Phi_2)$ is unitary, it is direct to check that

$$\begin{aligned} R\bar{\mathcal{H}}R^T(RR^T)^{-1} \otimes BK =&(R\Phi)(\Phi^H\bar{\mathcal{H}}\Phi)(R\Phi)^H \left((R\Phi)(R\Phi)^H\right)^{-1} \\ =&F \begin{bmatrix} U_1^H & 0 \\ 0 & U_2^H \end{bmatrix} \bar{\mathcal{H}} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} F^{-1}, \end{aligned}$$

where $F = \mathrm{diag}(R_1U_1, R_2U_2)$. This concludes the proof in view of (12). $\square$

For $\ell = 1, 2$, since $\bar{\mathcal{H}}_\ell \mathbf{1} = 0$, by the Schur decomposition theorem [21] we can choose U_ℓ in Proposition 1 such that $\bar{\mathcal{H}}_\ell$ admits an upper triangulation, i.e.,

$$\Phi_\ell^T \bar{\mathcal{H}}_\ell \Phi_\ell = \begin{bmatrix} 0 & * \\ 0 & U_\ell^T \bar{\mathcal{H}}_\ell U_\ell \end{bmatrix} \tag{13}$$

with

$$U_\ell^T \bar{\mathcal{H}}_\ell U_\ell = \begin{bmatrix} \lambda_2(\bar{\mathcal{H}}_\ell) & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{N_\ell+1}(\bar{\mathcal{H}}_\ell) \end{bmatrix},$$

where $\{\lambda_1(\bar{\mathcal{H}}_\ell) = 0, \lambda_2(\bar{\mathcal{H}}_\ell), \cdots, \lambda_{N_\ell+1}(\bar{\mathcal{H}}_\ell)\}$ form the spectrum of $\bar{\mathcal{H}}_\ell$.

For every ℓ, let $\bar{\mathcal{G}}_\ell$ denote the "extended" version of subnetwork $\mathcal{G}_\ell$ such that a virtual agent y_ℓ and a directed edge from y_ℓ to the node in $\mathcal{G}_\ell$ that is pinned are added.

Theorem 3.1. *Suppose that (A, B) is stabilizable. Under Assumptions 1 and 2, if the agents in subgroups are pinned such that each $\bar{\mathcal{G}}_\ell$, $\ell = 1, \cdots, r$, has a spanning tree, then the multi-agent system (1) under protocol (3) can achieve group pinning consensus exponentially fast.*

Moreover, if (A, B) is controllable, the group pinning consensus can be achieved with any speed.

Proof. Set $r = 2$. If we label the virtual leader y_ℓ as the last node in each subgroup, the Laplacian matrix of $\bar{\mathcal{G}}_\ell$ is exactly $\bar{\mathcal{H}}_\ell$ for $\ell = 1, 2$. In the light of the comments in Section 2.1, the condition that $\bar{\mathcal{G}}_\ell$ contains a spanning tree indicates that all eigenvalues of $\mathcal{L}_\ell + \mathcal{D}_\ell$ are with positive real parts. By (13) and Proposition 1, this means all the eigenvalues of $\mathcal{Z}_\ell$, $\ell = 1, 2$, are also with positive real parts. The Lyapunov algebraic equation then implies that there exists a $\Xi_\ell > 0$ such that $\Xi_\ell \mathcal{Z}_\ell + \mathcal{Z}_\ell^T \Xi_\ell > 0$, $\ell = 1, 2$. Thanks to Lemma 2.4, we can choose some $\varepsilon_1, \varepsilon_2 > 0$ such that $\Delta\Xi\mathcal{Z} + \mathcal{Z}^T\Delta\Xi > 0$, where $\Delta = \mathrm{diag}(\varepsilon_1 I_{N_1}, \varepsilon_2 I_{N_2})$ and $\Xi = \mathrm{diag}(\Xi_1, \Xi_2)$.

Define $Q = (\sqrt{\Delta\Xi})^{-1}(\Delta\Xi\mathcal{Z} + \mathcal{Z}^T\Delta\Xi)(\sqrt{\Delta\Xi})^{-1}$. We have $\lambda_{\min}(Q) > 0$. Since (A, B) is stabilizable, by the algebraic Riccati inequality, there exists a $P > 0$ such that

$$PA + A^TP - \lambda_{\min}(Q)PBB^TP < 0.$$

Moreover, we can find a $\beta > 0$ such that

$$PA + A^TP - \lambda_{\min}(Q)PBB^TP + \beta P < 0. \tag{14}$$

Define the following Lyapunov function

$$V(t) = \delta^T(t)(\Delta\Xi \otimes P)\delta(t),$$

and take the gain matrix $K = B^T P$. We obtain

$$\begin{aligned}\dot{V}(t) =&\delta^T(t)(\sqrt{\Delta\Xi}\otimes I_n)\left(I_N\otimes(A^TP+PA)-Q\otimes PBB^TP\right)(\sqrt{\Delta\Xi}\otimes I_n)\delta(t)\\ \le&\delta^T(t)(\sqrt{\Delta\Xi}\otimes I_n)\left(I_N\otimes(A^TP+PA-\lambda_{\min}(Q)PBB^TP)\right)(\sqrt{\Delta\Xi}\otimes I_n)\delta(t)\\ \le&-\beta V(t),\end{aligned}$$

where we have used (9) and (14). By the comparison principle, we obtain $V(t) \le V(0)e^{-\beta t}$, which implies that the multi-agent system (1) under protocol (3) can achieve group pinning consensus exponentially fast with a speed β.

Moreover, if (A, B) is controllable, there always exists a $P > 0$ solving (14) for any positive β [12]. Therefore, the group pinning consensus can be achieved with any speed. □

Remark 1. For the case of general r, Theorem 3.1 can be proved in the similar way as above. We leave the details to the reader.

Remark 2. A similar (but non-linear) pinning protocol is also used in [31] to ensure group consensus. Their paper differs from the current work mainly in the following two points: (a) Assumption 2 is also proposed in [31], but all $\alpha_{\ell\ell'}$ are required to be zero. (b) Some complicated algebraic conditions are introduced on interaction topology as well as pinning control gain d_i in [31] to guarantee group consensus, while our paper focuses on seeking group consensus on graphs with acyclic partition regardless of the magnitude of the coupling strength and d_i. The methodology used are totally different.

Remark 3. In the practical application, it is desirable to make the number of controllers as small as possible. Theorem 3.1 reveals that only r controllers may drive the multi-agent system (1) under protocol (3) to group consensus with r subgroups. Favorably, this number does not rely on the coupling strength. In complete consensus problems, the question of determining controller number is discussed in e.g. [5, 37], where small number of controllers typically requires large coupling strength.

3.2. **Convergence results for system (1) under protocol (2).** The same technique developed in the proof of Theorem 3.1 can be used to show that multi-agent system (1) with protocol (2) will achieve pinning group consensus exponentially fast with any speed when (A, B) is controllable, which complements/modifies the convergence result in [19]. We reformulate the result as follows and omit the proof.

Theorem 3.2. *Suppose that (A, B) is stabilizable. Under Assumptions 1 and 2 with all $\alpha_{\ell\ell'} \equiv 0$, if the agents in subgroups are pinned such that each $\bar{\mathcal{G}}_\ell$, $\ell = 1, \cdots, r$, has a spanning tree, then the multi-agent system (1) under protocol (2) can achieve group pinning consensus exponentially fast* [19, Theorem 1].

Moreover, if (A, B) is controllable, the group pinning consensus can be achieved with any speed.

Comparing Theorem 3.1 with Theorem 3.2, we see that the protocol (3) surpasses protocol (2) in the sense that no in-degree balance condition is needed. We will see in Section 3.3 that this really makes a difference in some situations.

A word is in order now about the necessity. The acyclic partition condition, i.e., Assumption 1, unfortunately is not a necessary condition for group pinning consensus. (A counterexample is constructed in [19, Fig. 6], where the communication graph consisting of 5 nodes is partitioned into two subgroups with bidirectional edges running between them.) That said, we will soon see in Section 3.3 that the acyclic partition is not too limited.

3.3. **Discussion on partition scheme.** In this section, we discuss on how to partition the network and choose the controllers appropriately so that the desired group consensus could be achieved under our pinning protocols.

In view of the results developed in the previous sections, two natural questions are in order. First, could the acyclic partition condition (i.e., Assumption 1) be too restrictive? A remarkable result of Stanley [30] asserts that the number of labeled DAGs on r nodes grows approximately as $1.7 \cdot 2^{\binom{r}{2}} r!(2/3)^r$, which ensures a sufficient pool of candidates for any finite r. Indeed, the first few exact numbers for $r = 1, 2, 3, 4, 5, \cdots$ are $1, 3, 25, 543, 29281, \cdots$. One also notices that if the contracted graph is a tree, then it must be a DAG.

Second, what if the desired partition of the communication graph $\mathcal{G}$ does not admit an acyclic partition?

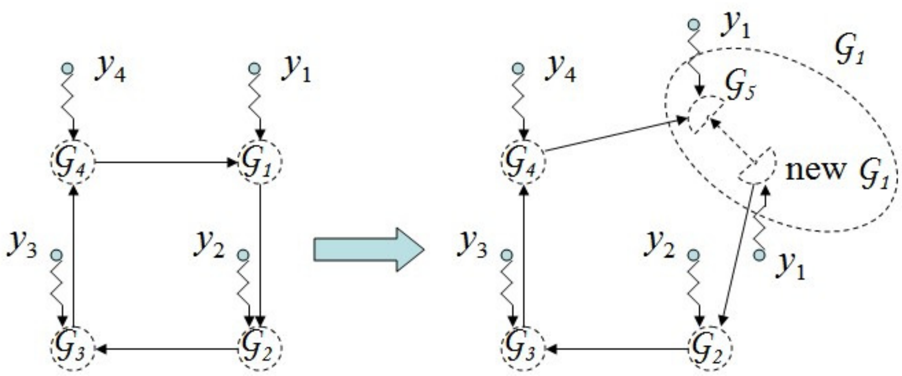


FIGURE 1. An example where the original partition of $\mathcal{G}$ results in a directed cycle in the contracted graph (see the left panel). The subnetwork $\mathcal{G}_1$ is decomposed further into two subnetworks to “break” the cycle (see the right panel, where the dashed arrow indicates possible edges from new $\mathcal{G}_1$ to $\mathcal{G}_5$). Notice that no edges or nodes are actually added/deleted—the network structure remains the same except an additional node is pinned.

A generic scenario example is depicted in Fig. 1, where the contracted graph of the whole network $\mathcal{G}$ contains a directed cycle. (Recall that each node in the contracted graph represents a set of agents.) From our results above, the desired pinning group consensus for all kinds of coupling strength cannot be guaranteed under protocol (2) or (3). However, there are possible solutions at the expense of extra controllers as indicated in Fig. 1—splitting a subnetwork into two new ones so that either they are not linked to each other or they are linked but in a way that inhibits directed cycles; exercising the same value of y_ℓ on each new subnetwork. Note that the protocol (2) imposes a stronger condition on Assumption 2, which limits largely its applicability in many situations compared to protocol (3).

To further illustrate the idea mentioned above, a concrete example is worked out as follows. Consider a group of $N = 6$ agents with two subgroups $\mathcal{G}_1 =$

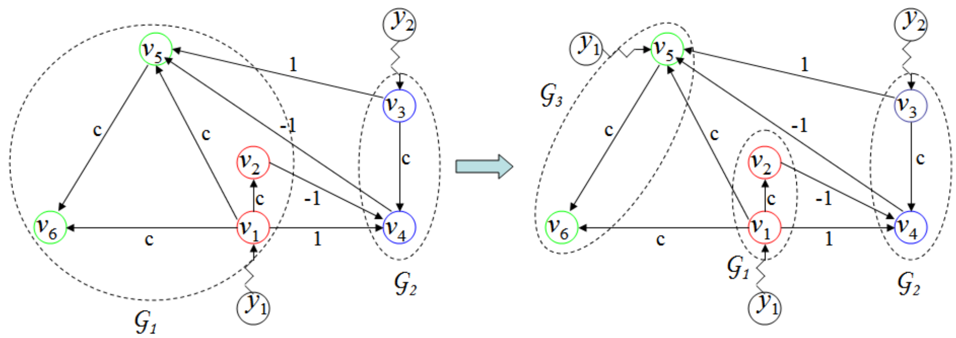


FIGURE 2. Partitioning a network to meet Assumptions 1 and 2, while maintain desired group consensus.

$\{v_1, v_2, v_5, v_6\}$ and $\mathcal{G}_2 = \{v_3, v_4\}$ (see Fig. 2 left panel). Clearly, the in-degree balance condition holds true but there is a directed cycle in the contracted graph. Theorems 3.1 and 3.2 cannot be applied directly. To overcome this problem, we make a further partition and get three subgroups $\mathcal{G}_1 = \{v_1, v_2\}$, $\mathcal{G}_2 = \{v_3, v_4\}$, $\mathcal{G}_3 = \{v_5, v_6\}$ (see Fig. 2 right panel). This partition satisfies Assumptions 1 and 2, but not the in-degree balance condition (since the edge weight c is non-zero.)

Take $n = 3$, $m = 1$, and let the agent dynamics (1) be specified as

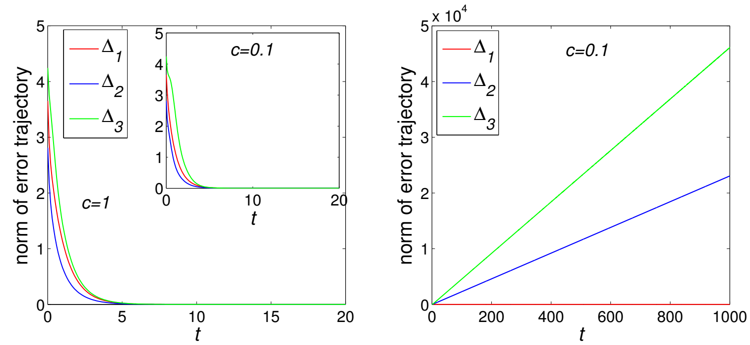
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The pair (A, B) is stabilizable. We choose $\mathcal{D}_\ell = \mathrm{diag}(1, 0)$, $\ell = 1, 2, 3$ as the pinning control gain, and $\Delta\Xi = \mathrm{diag}(1.2, 1.2, 0.75, 0.5, 0.2, 0.2)$. It is easy to solve the Riccati equation (14) that $P = \begin{bmatrix} 4.1624 & 3.0015 & -0.1445 \\ 3.0015 & 7.7823 & 1.0241 \\ -0.1445 & 1.0241 & 2.0920 \end{bmatrix}$ with $\lambda_{\min}(Q) = 0.4272$ and $\beta = 0.2$. Thus, the gain matrix $K = B^T P = (3.0015, 7.7823, 1.0241)$. We take the initial states $x_i(0)$, $i = 1, \cdots, 6$ randomly in $[-2, 2]^3$, and the initial values $y_1(0)$ and $y_2(0)$ arbitrarily just to satisfy $\lim_{t\to\infty} \|y_1(t) - y_2(t)\| \neq 0$.

Define the quantities $\Delta_1(t) = \|x_1(t) - y_1(t)\| + \|x_2(t) - y_1(t)\|$, $\Delta_2(t) = \|x_3(t) - y_2(t)\| + \|x_4(t) - y_2(t)\|$, and $\Delta_3(t) = \|x_5(t) - y_1(t)\| + \|x_6(t) - y_1(t)\|$, which measure the norm of error trajectories within subnetworks. Fig. 3(a) shows that the desired group consensus is achieved for $c = 1$ and $c = 0.1$ under protocol (3). This agrees with Theorem 3.1. Since the in-degree balance condition fails, Theorem 3.2 does not apply. By taking a small c, for example $c = 0.1$, as shown in Fig. 3(b), the group consensus cannot be achieved under protocol (2). Of course, this shows the limitation of protocol (2), but by no means implies the necessity of the conditions in Theorem 3.2.

4. Group pinning consensus analysis: Randomly switching topologies.

In this section, we reveal the effect of Markovian switching topology $\mathcal{G}(t)$ on the group pinning consensus problem under Assumptions 3 and 4. The stationary distribution π of the Markov process $\theta(t)$ is shown to be vital in the design of common consensus gain. The system (1) with control laws (5) and (6) is tackled in Sections 4.1 and



(a) Δ_ℓ under protocol (3) (b) Δ_ℓ under protocol (2)

FIGURE 3. Norm of error trajectories $\Delta_\ell(t)$ for multi-agent system (1) with fixed topology. (a): protocol (3), $c = 1$ in the main panel, and $c = 0.1$ in the inset; (b): protocol (2) and $c = 0.1$.

4.2, respectively. In Section 4.3, simulation results are presented to illustrate the effectiveness of the theoretical results.

4.1. **Convergence results for system (1) under protocol (5).** For $i = 1, \cdots, N$, let $\delta_i(t) = x_i(t) - y_{\bar{i}}(t)$. If Assumption 4 holds with all $\alpha_{\ell\ell'}^{(k)} \equiv 0$, we obtain for any $t \geq 0$,

$$\begin{aligned}\sum_{j=1}^{N} a_{ij}(t)(x_j(t) - x_i(t)) &= -\sum_{\ell=1}^{r}\sum_{v_j\in\mathcal{V}_\ell} l_{ij}(t)(x_j(t) - y_\ell(t) + y_\ell(t)) \\ &= -\sum_{j=1}^{N} l_{ij}(t)\delta_j(t) - \sum_{\ell=1}^{r}\Big(\sum_{v_j\in\mathcal{V}_\ell} l_{ij}(t)\Big)y_\ell(t) \\ &= \sum_{j=1}^{N} a_{ij}(t)(\delta_j(t) - \delta_i(t)),\end{aligned}$$

where $\mathcal{L}(t) = (l_{ij}(t))$ is the Laplacian matrix of $\mathcal{G}(t)$ assuming the lower triangular form as in (7). Recall that $\dot{y}_\ell(t) = Ay_\ell(t)$ for $\ell = 1, \cdots, r$. We are led to the conclusion that system (1) with protocol (5) can be rewritten as

$$\dot{\delta}_i(t) = A\delta_i(t) + BK\Big(\sum_{j=1}^{N} a_{ij}(t)(\delta_j(t) - \delta_i(t)) - d_i(t)\delta_i(t)\Big)$$

for $i = 1, \cdots, N$.

Set $\delta(t) = (\delta_1^T(t), \cdots, \delta_N^T(t))^T \in \mathbb{R}^{nN}$, and $\mathcal{D}(t) = \mathrm{diag}(d_1(t), \cdots, d_N(t))$. Then we have the following compact form of the error dynamics

$$\dot{\delta}(t) = ((I_N \otimes A) - (\mathcal{L}(t) + \mathcal{D}(t)) \otimes BK)\delta(t). \tag{15}$$

Recall that, for $\ell = 1, \cdots, r$, $\mathcal{G}_\ell^{\mathcal{S}}$ is the induced subnetwork of the union interaction topology $\mathcal{G}^{\mathcal{S}}$ on $\mathcal{V}_\ell$. Similarly as in the previous sections, let $\bar{\mathcal{G}}_\ell^{\mathcal{S}}$ denote the "extended" version of subnetwork $\mathcal{G}_\ell^{\mathcal{S}}$ such that a virtual agent y_ℓ and a directed edge from y_ℓ to the node in $\mathcal{G}_\ell^{\mathcal{S}}$ that is pinned at some time t are added. Note that

$\bar{\mathcal{G}}_\ell^{\mathcal{S}}$ is not unique (more precisely, has at most $\min\{s, N_\ell\}$ versions) by definition since the pinned nodes in $\mathcal{G}_\ell^{(k)}$ can be different for different $k \in \mathcal{S}$ (c.f. Fig. 4).

Theorem 4.1. *Suppose that (A, B) is stabilizable. Under Assumptions 3 and 4 with all $\alpha_{\ell\ell'}^{(k)} \equiv 0$ $(\ell > \ell')$, if the agents in subgroups are pinned such that each $\bar{\mathcal{G}}_\ell^{\mathcal{S}}$, $\ell = 1, \cdots, r$, has a spanning tree, then the multi-agent system (1) under protocol (5) can achieve group pinning consensus exponentially fast.*

Moreover, if (A, B) is controllable, and $K = \sigma B^T P$, where the coefficient $\sigma \geq (\min_{k\in\mathcal{S}} \pi_k)^{-1}$ and $P > 0$ is given in (16) below, the group pinning consensus can be achieved with any speed.

Before proceeding to the proof, we remark that the spanning tree condition in Theorem 4.1 is imposed on only one version of $\bar{\mathcal{G}}_\ell^{\mathcal{S}}$ for each ℓ. To see why this may suffice, we recall that the Markov process $\theta(t)$ is ergodic. Hence, given $k \in \mathcal{S}$, $\mathcal{G}_\ell^{(k)}$ will be visited infinite times, which is reminiscent of the well-known "frequently-connected" condition proposed for achieving consensus in deterministic switching networks [20].

Proof. Under Assumption 3, the Laplacian matrix of the union topology $\mathcal{G}^{\mathcal{S}}$, denoted by $\mathcal{L}^{\mathcal{S}}$, has the lower triangular form as in (7). Therefore,

$$\bar{\mathcal{L}}^{\mathcal{S}} := \mathcal{L}^{\mathcal{S}} + \mathcal{D}^{\mathcal{S}} = \begin{bmatrix} \mathcal{L}_{11}^{\mathcal{S}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{r1}^{\mathcal{S}} & \cdots & \mathcal{L}_{rr}^{\mathcal{S}} \end{bmatrix} + \mathrm{diag}(\mathcal{D}_1^{\mathcal{S}}, \cdots, \mathcal{D}_r^{\mathcal{S}}),$$

where $\mathcal{D}^{\mathcal{S}} = \mathrm{diag}(\mathcal{D}_1^{\mathcal{S}}, \cdots, \mathcal{D}_r^{\mathcal{S}}) = \sum_{k=1}^{s} \mathcal{D}^{(k)}$, $\mathcal{D}^{(k)} = \mathrm{diag}(d_1^{(k)}, \cdots, d_N^{(k)})$.

Since Assumption 4 holds with all $\alpha_{\ell\ell'}^{(k)} \equiv 0$ $(\ell > \ell')$, $\mathcal{L}_{\ell\ell}^{\mathcal{S}}$ is the Laplacian matrix of graph $\mathcal{G}_\ell^{\mathcal{S}}$ for $\ell = 1, \cdots, r$. The Laplacian matrix of the "extend" graph $\bar{\mathcal{G}}_\ell^{\mathcal{S}}$ is thus given by $\begin{bmatrix} \mathcal{L}_{\ell\ell}^{\mathcal{S}} + \mathcal{D}_\ell^{\mathcal{S}} & -\mathcal{D}_\ell^{\mathcal{S}} \mathbf{1}_{N_\ell} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(N_\ell+1)\times(N_\ell+1)}$. As in the proof of Theorem 3.1, the spanning tree conditions imply that all the eigenvalues of $\mathcal{L}_{\ell\ell}^{\mathcal{S}} + \mathcal{D}_\ell^{\mathcal{S}}$ are with positive real parts. Thanks to the Lyapunov algebraic equation, there exists a $\Xi_\ell > 0$ such that $\Xi_\ell(\mathcal{L}_{\ell\ell}^{\mathcal{S}} + \mathcal{D}_\ell^{\mathcal{S}}) + (\mathcal{L}_{\ell\ell}^{\mathcal{S}} + \mathcal{D}_\ell^{\mathcal{S}})^T \Xi_\ell > 0$, $\ell = 1, \cdots, r$. It follows from Lemma 2.4 that we can select some $\varepsilon_\ell > 0$ such that $\Delta\Xi\bar{\mathcal{L}}^{\mathcal{S}} + (\bar{\mathcal{L}}^{\mathcal{S}})^T \Delta\Xi > 0$, where $\Delta = \mathrm{diag}(\varepsilon_1 I_{N_1}, \cdots, \varepsilon_r I_{N_r})$ and $\Xi = \mathrm{diag}(\Xi_1, \cdots, \Xi_r)$.

Define $Q = (\sqrt{\Delta\Xi})^{-1}\left(\Delta\Xi\bar{\mathcal{L}}^{\mathcal{S}} + (\bar{\mathcal{L}}^{\mathcal{S}})^T \Delta\Xi\right)(\sqrt{\Delta\Xi})^{-1}$. We have $\lambda_{\min}(Q) > 0$. Since (A, B) is stabilizable, by the algebraic Riccati inequality, there exists a $P > 0$ such that

$$PA + A^T P - \lambda_{\min}(Q) PBB^T P < 0.$$

Furthermore, we can select a $\beta > 0$ such that

$$PA + A^T P - \lambda_{\min}(Q) PBB^T P + \beta P < 0. \tag{16}$$

Define the Lyapunov functions by

$$V(t) = \mathbb{E}\left(\delta^T(t)(\Delta\Xi \otimes P)\delta(t)\right)$$

and

$$V_k(t) = \mathbb{E}\left(\delta^T(t)(\Delta\Xi \otimes P)\delta(t) 1_{\{\theta(t)=k\}}\right)$$

for $k \in \mathcal{S}$. By Lemma 2.5 and (15), we obtain

$$\begin{aligned} dV_k(t) =&\mathbb{E}\Big(d\delta^T(t)(\Delta\Xi \otimes P)\delta(t)1_{\{\theta(t)=k\}} + \delta^T(t)(\Delta\Xi \otimes P)d\delta(t)1_{\{\theta(t)=k\}}\Big) \\ &+ \sum_{j=1}^{s} \gamma_{jk} V_j(t)dt + o(dt) \\ =&\mathbb{E}\Big(\delta^T(t)\Big(\Delta\Xi \otimes (PA + A^T P) - \big((\mathcal{L}(t) + \mathcal{D}(t))^T \Delta\Xi \otimes K^T B^T P \\ &+ \Delta\Xi(\mathcal{L}(t) + \mathcal{D}(t)) \otimes PBK\big)\Big)\delta(t)1_{\{\theta(t)=k\}}\Big) \\ &+ \sum_{j=1}^{s} \gamma_{jk} V_j(t)dt + o(dt). \end{aligned} \tag{17}$$

Since the Markov process $\theta(t)$ is ergodic, without loss of generality, we assume that it starts from the invariant distribution π. Note that $V(t) = \sum_{k=1}^{s} V_k(t)$, $\sum_{k=1}^{s} \gamma_{jk} = 0$, and $\bar{\mathcal{L}}^{\mathcal{S}} = \sum_{k=1}^{s}(\mathcal{L}^{(k)} + \mathcal{D}^{(k)})$. By taking $K = \sigma B^T P$ with $\sigma \geq (\min_{k\in\mathcal{S}} \pi_k)^{-1} \geq \pi_k^{-1}$ for any $k \in \mathcal{S}$, we derive from (17) that

$$\begin{aligned} \dot{V}(t) \leq&\mathbb{E}\Big(\delta^T(t)\Big(\Delta\Xi \otimes (PA + A^T P) - \big((\bar{\mathcal{L}}^{\mathcal{S}})^T \Delta\Xi + \Delta\Xi\bar{\mathcal{L}}^{\mathcal{S}}\big) \otimes PBB^T P\Big)\delta(t)\Big) \\ =&\mathbb{E}\Big(\delta^T(t)(\sqrt{\Delta\Xi} \otimes I_n)\big(I_N \otimes (PA + A^T P) - Q \otimes PBB^T P\big)(\sqrt{\Delta\Xi} \otimes I_n)\delta(t)\Big). \end{aligned}$$

Employing (16) we further have

$$\dot{V}(t) \leq -\beta\mathbb{E}\Big(\delta^T(t)(\sqrt{\Delta\Xi} \otimes I_n)\big(I_N \otimes P)(\sqrt{\Delta\Xi} \otimes I_n)\delta(t)\Big) = -\beta V(t).$$

By the comparison principle, we obtain $V(t) \leq V(0)e^{-\beta t}$, which implies that the multi-agent system (1) under protocol (5) can achieve group pinning consensus exponentially fast with a speed β.

In addition, when (A, B) is controllable, there always exists a $P > 0$ solving (16) for any positive β similarly as in the proof of Theorem 3.1. Hence, the group pinning consensus can be achieved with any speed. □

Remark 4. From the perspective of the design of gain matrix K, the choice of positive definite matrix P in (16) depends on both the agent dynamics and the network topologies. This differs from the design in [36, Theorem 1] for complete consensus, where the selected matrix is independent of graphs.

Remark 5. It is worth noting that the same group consensus protocol over deterministic switching topology was considered in [19]. Given a partition $\{\mathcal{V}_1, \cdots, \mathcal{V}_r\}$. It is assumed that the contracted graphs of the communication topologies are time-invariant and acyclic (see Assumption 4 therein). Indeed, it suffices to only require that the contracted graph of the union topology over time is acyclic. The same proof presented there is still valid in such circumstances.

Remark 6. Note that we do not require balance of the underlying graphs (a graph is called *balanced* if the out-degree is equal to the in-degree for every node [17]). This provides additional flexibility on many applications as compared to the continuous-time complete consensus over Markovian switching graphs, where the balance condition is commonly assumed; see e.g. [14, 23, 36]. The balance condition is also imposed in [19] to ensure group consensus over deterministic switching topology.

Remark 7. When the Markov process $\theta(t)$ in question is not ergodic, the state space $\mathcal{S} = \{1, 2, \cdots, s\}$ can be decomposed uniquely into the form $\mathcal{S} = \{\mathcal{J} \cup \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_q\}$, where each $\mathcal{S}_j$ $(j = 1, \cdots, q)$ is a closed set of positive recurrent states and $\mathcal{J}$ is a set of transient states [22]. The assumption in Theorem 4.1 needs to be modified into that the agents in subgroups are pinned such that each $\bar{\mathcal{G}}_\ell^{\mathcal{S}_j}$, $\ell = 1, \cdots, r$, and $j = 1, \cdots, q$, has a spanning tree. Similar treatment has been proposed in [15, 36] for complete consensus problems.

4.2. **Convergence results for system (1) under protocol (6).** As in Section 3.1 we here only focus on the case of $r = 2$ for ease of presentation. The general case can be proved in the same way but the details are left to the readers.

Recall that $\delta(t) = \big(x_1^T(t) - x_2^T(t), \cdots, x_1^T(t) - x_{N_1}^T(t), x_1^T(t) - y_1^T(t), x_{N_1+1}^T(t) - x_{N_1+2}^T(t), \cdots, x_{N_1+1}^T(t) - x_N^T(t), x_{N_1+1}^T(t) - y_2^T(t)\big)^T \in \mathbb{R}^{nN}$. $\bar{\mathcal{G}}_\ell^{\mathcal{S}}$ denotes the "extended" version of subnetwork $\mathcal{G}_\ell^{\mathcal{S}}$ such that a virtual agent y_ℓ and a directed edge from y_ℓ to the node in $\mathcal{G}_\ell^{\mathcal{S}}$ that is pinned at some (any) time t are added.

Theorem 4.2. *Suppose that (A, B) is stabilizable. Under Assumptions 3 and 4, if the agents in subgroups are pinned such that each $\bar{\mathcal{G}}_\ell^{\mathcal{S}}$, $\ell = 1, \cdots, r$, has a spanning tree, then the multi-agent system (1) under protocol (6) can achieve group pinning consensus exponentially fast.*

Moreover, if (A, B) is controllable, and $K = \sigma B^T P$, where the coefficient $\sigma \geq (\min_{k \in \mathcal{S}} \pi_k)^{-1}$ and $P > 0$ is given in (19) below, the group pinning consensus can be achieved with any speed.

Proof. Set $r = 2$. In view of Proposition 1, we obtain for $t \geq 0$,

$$\dot{\delta}(t) = \big(I_N \otimes A - \mathcal{Z}(t) \otimes BK\big)\delta(t), \tag{18}$$

where $\mathcal{Z}(t) = \left[\begin{array}{c:c} \mathcal{Z}_1(t) & 0 \\ \hdashline \mathcal{Z}_{21}(t) & \mathcal{Z}_2(t) \end{array}\right] = F \left[\begin{array}{c:c} U_1^H \bar{\mathcal{H}}_1(t) U_1 & 0 \\ \hdashline U_2^H \bar{\mathcal{H}}_{21}(t) U_1 & U_2^H \bar{\mathcal{H}}_2(t) U_2 \end{array}\right] F^{-1}$, $\bar{\mathcal{H}}_{21}(t) = \begin{bmatrix} -\mathcal{A}_{21}(t) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(N_2+1)\times(N_1+1)}$, $\bar{\mathcal{H}}_\ell(t) = \begin{bmatrix} \mathcal{L}_\ell(t) + \mathcal{D}_\ell(t) & -\mathcal{D}_\ell(t)\mathbf{1}_{N_\ell} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(N_\ell+1)\times(N_\ell+1)}$, $\mathcal{D}_1(t) = \mathrm{diag}(d_1(t), \cdots, d_{N_1}(t))$, $\mathcal{D}_2(t) = \mathrm{diag}(d_{N_1+1}(t), \cdots, d_N(t))$, and $\mathcal{L}_\ell(t)$ is the Laplacian matrix of $\mathcal{G}_\ell(t)$, $\ell = 1, 2$; U_ℓ is determined by the Schur decomposition of $\bar{\mathcal{H}}_\ell^{\mathcal{S}}$ (i.e., replacing $\bar{\mathcal{H}}_\ell$ with $\bar{\mathcal{H}}_\ell^{\mathcal{S}}$ in (13)) for $\ell = 1, 2$; F is defined as in Proposition 1. Here, $\bar{\mathcal{H}}_\ell^{\mathcal{S}} = \begin{bmatrix} \mathcal{L}_\ell^{\mathcal{S}} + \mathcal{D}_\ell^{\mathcal{S}} & -\mathcal{D}_\ell^{\mathcal{S}}\mathbf{1}_{N_\ell} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(N_\ell+1)\times(N_\ell+1)}$, where $\mathcal{L}_\ell^{\mathcal{S}}$ is the Laplacian matrix of $\mathcal{G}_\ell^{\mathcal{S}}$ and $\mathcal{D}_\ell^{\mathcal{S}}$ is defined as in the proof of Theorem 4.1, for $\ell = 1, 2$. Similar notations will be made for $\mathcal{Z}^{\mathcal{S}}$, $\mathcal{Z}_\ell^{\mathcal{S}}$, $\mathcal{Z}^{(k)}$, $\mathcal{Z}_\ell^{(k)}$, $\bar{\mathcal{H}}_{21}^{(k)}$, $\bar{\mathcal{H}}_\ell^{(k)}$, etc. for $\ell = 1, 2$, $k \in \mathcal{S}$.

Following the similar reasoning as in Theorem 3.1, we define a positive definite matrix

$$Q = (\sqrt{\Delta\Xi})^{-1}(\Delta\Xi\mathcal{Z}^{\mathcal{S}} + (\mathcal{Z}^{\mathcal{S}})^T\Delta\Xi)(\sqrt{\Delta\Xi})^{-1}$$

with $\lambda_{\min}(Q) > 0$. Here, $\Delta = \mathrm{diag}(\varepsilon_1 I_{N_1}, \varepsilon_2 I_{N_2})$ and $\Xi = \mathrm{diag}(\Xi_1, \Xi_2)$ are taken such that $\Xi_\ell \mathcal{Z}_i^{\mathcal{S}} + (\mathcal{Z}_\ell^{\mathcal{S}})^T \Xi_\ell > 0$ $(\ell = 1, 2)$, and $\Delta\Xi\mathcal{Z}^{\mathcal{S}} + (\mathcal{Z}^{\mathcal{S}})^T\Delta\Xi > 0$.

Since (A, B) is stabilizable, again by the algebraic Riccati inequality, there exist a $P > 0$ such that

$$PA + A^T P - \lambda_{\min}(Q) PBB^T P < 0,$$

and a $\beta > 0$ such that

$$PA + A^T P - \lambda_{\min}(Q) PBB^T P + \beta P < 0. \tag{19}$$

Similarly, define the Lyapunov candidates by

$$V(t) = \mathbb{E}\left(\delta^T(t)(\Delta\Xi \otimes P)\delta(t)\right)$$

and

$$V_k(t) = \mathbb{E}\left(\delta^T(t)(\Delta\Xi \otimes P)\delta(t)1_{\{\theta(t)=k\}}\right)$$

for $k \in \mathcal{S}$. By Lemma 2.5 and (18), we obtain

$$\begin{aligned} dV_k(t) =& \mathbb{E}\Big(d\delta^T(t)(\Delta\Xi \otimes P)\delta(t)1_{\{\theta(t)=k\}} + \delta^T(t)(\Delta\Xi \otimes P)d\delta(t)1_{\{\theta(t)=k\}}\Big) \\ &+ \sum_{j=1}^{s} \gamma_{jk} V_j(t)dt + o(dt) \\ =& \mathbb{E}\Big(\delta^T(t)\Big(\Delta\Xi \otimes (PA + A^T P) - \big(\mathcal{Z}^T(t)\Delta\Xi \otimes K^T B^T P \\ &+ \Delta\Xi\mathcal{Z}(t) \otimes PBK\big)\Big)\delta(t)1_{\{\theta(t)=k\}}\Big) + \sum_{j=1}^{s} \gamma_{jk} V_j(t)dt + o(dt). \end{aligned} \tag{20}$$

Without loss of generality, we assume that the Markov process $\theta(t)$ starts from the invariant distribution π. Note that $V(t) = \sum_{k=1}^{s} V_k(t)$, $\sum_{k=1}^{s} \gamma_{jk} = 0$, and

$$\bar{\mathcal{H}}^{\mathcal{S}} = \begin{bmatrix} \bar{\mathcal{H}}_1^{\mathcal{S}} & 0 \\ \bar{\mathcal{H}}_{21}^{\mathcal{S}} & \bar{\mathcal{H}}_2^{\mathcal{S}} \end{bmatrix} = \sum_{k=1}^{s} \begin{bmatrix} \bar{\mathcal{H}}_1^{(k)} & 0 \\ \bar{\mathcal{H}}_{21}^{(k)} & \bar{\mathcal{H}}_2^{(k)} \end{bmatrix} = \sum_{k=1}^{s} \bar{\mathcal{H}}^{(k)}.$$

Hence, $\mathcal{Z}^{\mathcal{S}} = \sum_{k=1}^{s} \mathcal{Z}^{(k)}$ as F, U_1, and U_2 are time-invariant. Taking $K = \sigma B^T P$ with $\sigma \geq (\min_{k\in\mathcal{S}} \pi_k)^{-1} \geq \pi_k^{-1}$ for any $k \in \mathcal{S}$, we derive from (20) that

$$\begin{aligned} \dot{V}(t) \leq& \mathbb{E}\Big(\delta^T(t)\Big(\Delta\Xi \otimes (PA + A^T P) - \big((\mathcal{Z}^{\mathcal{S}})^T\Delta\Xi + \Delta\Xi\mathcal{Z}^{\mathcal{S}}\big) \otimes PBB^T P\Big)\delta(t)\Big) \\ =& \mathbb{E}\Big(\delta^T(t)(\sqrt{\Delta\Xi} \otimes I_n)\big(I_N \otimes (PA + A^T P) - Q \otimes PBB^T P\big)(\sqrt{\Delta\Xi} \otimes I_n)\delta(t)\Big). \end{aligned}$$

Accordingly, we obtain from (19) that

$$\dot{V}(t) \leq -\beta\mathbb{E}\Big(\delta^T(t)(\sqrt{\Delta\Xi} \otimes I_n)\big(I_N \otimes P\big)(\sqrt{\Delta\Xi} \otimes I_n)\delta(t)\Big) = -\beta V(t).$$

The rest of the proof follows as in Theorem 4.1. Therefore, we conclude the proof as desired. □

The remarks after Theorem 4.1 can be applied here analogously.

4.3. **Simulations.** Consider the multi-agent system (1) with $N = 6$ agents divided into two subgroups $\mathcal{V}_1 = \{v_1, v_2, v_3\}$ and $\mathcal{V}_2 = \{v_4, v_5, v_6\}$. The communication topology among agents will randomly switch between $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ (see Fig. 4 first row) following a time-homogeneous Markovian process $\theta(t)$ with generator $\Gamma = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$ and state space $\mathcal{S} = \{1, 2\}$. The initial distribution of $\theta(t)$ is given by its invariant distribution $\pi = (2/3, 1/3)^T$.

Take $n = 2$, $m = 1$, and let the agent dynamics (1) be specified as

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

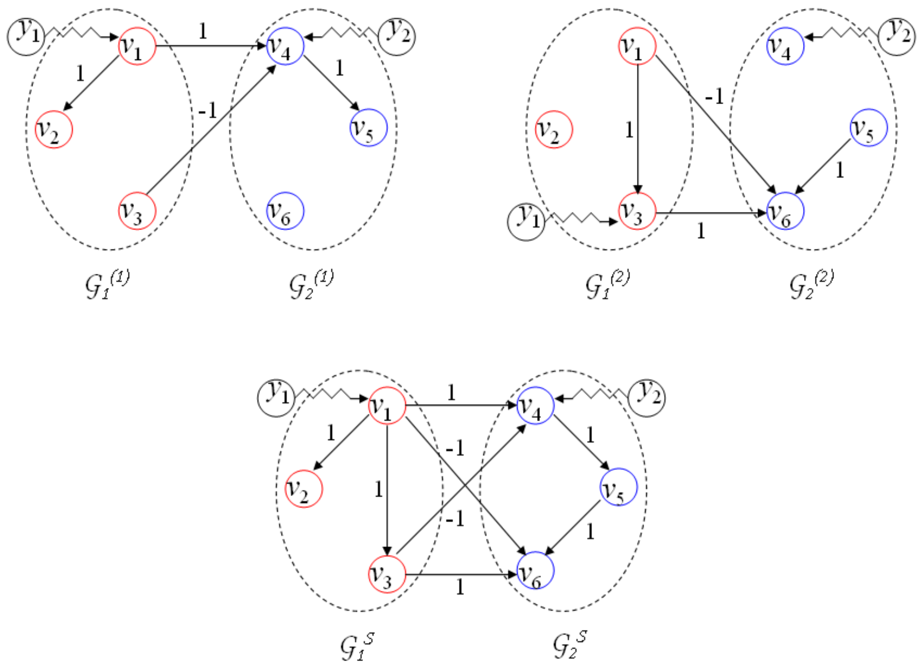


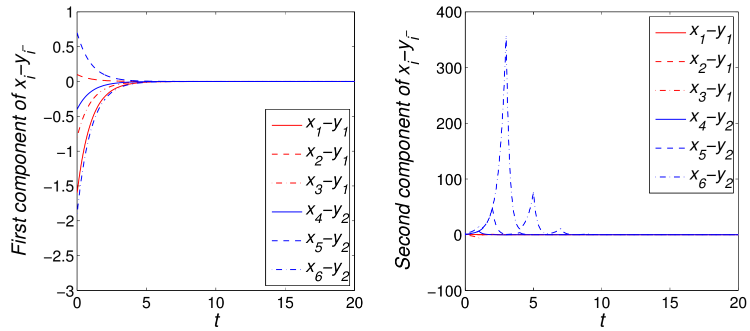
FIGURE 4. Network topologies: $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, and their union $\mathcal{G}^{\mathcal{S}} = \mathcal{G}^{(1)} \cup \mathcal{G}^{(2)}$. Note that we have chosen one version of $\bar{\mathcal{G}}_1^{\mathcal{S}}$ out of two possible ones.

The pair (A, B) is stabilizable. With $\bar{\mathcal{G}}_\ell^{\mathcal{S}}$ $(\ell = 1, 2)$ displayed in Fig. 4 last row, it is straightforward to check that all assumptions in Theorems 4.1 and 4.2 are satisfied. We take the initial states $x_i(0)$, $i = 1, \cdots, 6$ randomly in $[-2, 2]^2$, and the initial values $y_1(0)$ and $y_2(0)$ arbitrarily just to satisfy $\lim_{t\to\infty} \|y_1(t) - y_2(t)\| \neq 0$.

Firstly, we consider the system (1) with protocol (5). Take $\mathcal{D}^{(1)} = \mathrm{diag}(1, 0, 0, 1, 0, 0)$, $\mathcal{D}^{(2)} = \mathrm{diag}(0, 0, 1, 1, 0, 0)$ and $\Delta\Xi = \mathrm{diag}(1, 0.1, 0.5, 0.3, 0.2, 0.1)$. It is easy to solve the Riccati equation (16) that $P = \begin{bmatrix} 1.0396 & 1.5701 \\ 1.5701 & 5.0228 \end{bmatrix}$ with $\lambda_{\min}(Q) = 0.8360$ and $\beta = 0.1$. The gain matrix is solved as $K = (4.7103, 15.0684)$. Fig. 5 shows a sample path of the consensus seeking process, which agrees with Theorem 4.1.

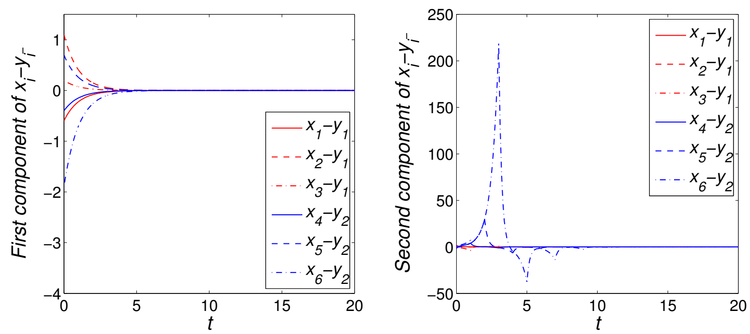
Next, we consider the system (1) with protocol (6). Take $\mathcal{D}^{(1)}$, $\mathcal{D}^{(2)}$ as above, and $\Delta\Xi = \mathrm{diag}(1, 1, 0.8, 0.3, 0.1, 0.1)$. Solving the Riccati equation (19) with $\lambda_{\min}(Q) = 0.5721$ and $\beta = 0.1$ gives $P = \begin{bmatrix} 1.2852 & 2.3049 \\ 2.3049 & 7.2334 \end{bmatrix}$. The gain matrix is solved as $K = (6.9147, 21.7002)$. Fig. 6 shows a sample path of the consensus seeking process, which is consistent with Theorem 4.2.

5. **Conclusion.** In this paper, the group consensus problem of continuous-time linear multi-agent systems has been studied. Two different types of pinning control protocols are proposed to ensure group consensus regardless of the magnitude of the coupling strengths among the agents. Sufficient conditions guaranteeing the group consensus under directed fixed interaction topology and randomly switching



(a) First component of $x_i - y_{\bar{i}}$ (b) Second component of $x_i - y_{\bar{i}}$

FIGURE 5. Error trajectories for multi-agent system (1) under randomly switching topology using protocol (5).



(a) First component of $x_i - y_{\bar{i}}$ (b) Second component of $x_i - y_{\bar{i}}$

FIGURE 6. Error trajectories for multi-agent system (1) under randomly switching topology using protocol (6). The same sample path of $\theta(t)$ is used as in Fig. 5.

topology are derived in terms of simple graphic conditions. For one of the protocols studied here, the commonly used in-degree balance condition can be largely relaxed. In the case of randomly switching topology, where the underlying networks are governed by a continuous-time Markov process, it is shown that the group consensus behavior is unrelated to the magnitude of the couplings among agents if the union of the topologies corresponding to the positive recurrent states of the Markov process has an acyclic partition. Numerical simulations are presented to illustrate the effectiveness of our theoretical results and facilitate discussion on the choice of pinning schemes.

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