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ON RELAXATION OF STATE CONSTRAINED OPTIMAL CONTROL PROBLEM FOR A PDE-ODE MODEL OF SUPPLY CHAINS

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Abstract. We discuss the optimal control problem (OCP) stated as the minimization of the queues and the difference between the effective outflow and a desired one for the continuous model of supply chains, consisting of a PDE for the density of processed parts and an ODE for the queue buffer occupancy. The main goal is to consider this problem with pointwise control and state constraints. Using the so-called Henig delation, we propose the relaxation approach to characterize the solvability and regularity of the original problem by analyzing the corresponding relaxed OCP.

1. Introduction.

The aim of this article is to analyze an optimal control problem for a coupled system of partial and ordinary differential equations describing the dynamic of supply chains. Here we focus on the model introduced in [15] by Goettlich, Herty and Klar (briefly GHK model), where supply chains are concatenations of suppliers. The latter is composed of a processor for assembling and construction and a buffer for unprocessed parts, called queue. The evolution of parts inside the processor is given by a conservation law for the density of parts, $\rho(t,x)$. The dynamics of each queue is given by an ODE for the queue buffer occupancy $q(t)$. Since an important matter for applications is to reduce the dead times, to avoid bottlenecks and improve productivity, it leads to the reasonable questions: can we control the processing rates, or the processing velocities, or the input flow in such way to minimize queues and to achieve an expected outflow?

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It is worth to note that the mathematical simulation of supply chain is characterized by (at least) two different approaches: discrete event simulations, considering trajectories of individual parts, and continuous models, based either on ordinary (see [5], [18]) or on partial differential equations. The first continuous model for supply chains, based on partial differential equations, was introduced by D. Armbruster et al. (see [1]). The authors, taking the limit on the number of parts and suppliers, obtained a conservation law for the part density, with flux given by the minimum between the physical flow and the maximal processing capacity. In the meantime, it is not easy matter to define solutions to the model of [1], also because of delta waves, thus other fluid-dynamic models for supply chains have been introduced in [3], [8], [9] and [15]. The works [3], [8], [9] and [10] deal with a mixed continuum-discrete model consisting of a system of two conservation laws, one for the density part and one for the processing rate, and Riemann solvers at fixed nodes. A comprehensive description of such models can be found in the recent monograph [6].

In [16] two optimal control problems for supply chains on networks were considered. First the problem of determining optimal velocities for each individual processing unit is addressed for a supply chain consisting of three processors. Then, given a supply network with a vertex of dispersing type, the distribution rate has been controlled in such way to minimize queues.

A variational method is proposed for the modeling, computation, and optimization of a class of continuous supply chain networks in [17]. The optimization problems are formulated as mixed integer programs, requiring much less computational effort than the one based on the PDE-ODE system. In a test network for both cases of buffers with finite or infinite capacity the total throughput of the network within the time horizon has been maximized fixing the inflow and using as control variables the time-varying flow allocation. Moreover the problem of maximizing the throughput of the network while keeping the queuing at a minimum level has been faced controlling not only the allocation rates, but also the inflow profile.

In [11] and [12], the authors consider the input flow as a control in order to minimize queues and provide the best approximation of a desired supply chain outflow. As a result, the optimization is realized defining a cost functional J in the form of two parts. The first is simply the time integral of queue buffers occupancies, while the second is the quadratic distance of the outflow from the desired one. Having used the concept of generalized tangent vectors to a piecewise control representing shifts of discontinuities, the authors prove the existence of an optimal control and show that this solution can be defined on the whole supply chain for bounded variation inflows.

A nonlinear and non local PDE modeling a factory whose cycle time depends nonlinearly on the work in progress is considered in [23]. One of the few ways to influence the output of such a factory is by adjusting the start rate in a time dependent manner. Two prototypical control problems for this case are investigated: i) demand tracking where the start rate that generates an output rate which optimally tracks a given time dependent demand rate is determined and ii) backlog tracking which optimally tracks the cumulative demand. The method is based on the formal adjoint method for constrained optimization, incorporating the hyperbolic PDE as a constraint of a nonlinear optimization problem.

A supply network where the flow of parts can be controlled at the vertices of the network is considered in [20]. Discrete adjoint equations which are subsequently validated by the continuous adjoint calculus are derived. The continuous optimality system is obtained and it was shown that the mixed-integer formulation is also a valid discretization of the discretized continuous optimality system, i.e., both approaches discretize-then-optimize and optimize-then-discretize lead to the same continuous optimal control if the discretization width tends to zero.

The characteristic feature of OCP we deal with in this article is the fact that the density of parts for each processor is restricted by nonlinear pointwise constraints in L^2-spaces. In fact, the ordering cone of positive elements in L^2-spaces is typically nonsolid, i.e. it has an empty topological interior. Following Lagrange multiplier rule, which gives a necessary optimality condition for local solutions to state constrained OCPs, the constraint qualifications such as the Slater condition or the Robinson condition should be applied in this case. However, these conditions cannot be verified for cones such as $L^2_+(\Omega)$ due to int $\left(L^2_+(\Omega)\right) = \emptyset$. Therefore, our main intention in this article is to propose a suitable relaxation of the pointwise state constraints in the form of some inequality conditions involving a so-called Henig approximation $\left(L^2_+(\Omega)\right)_\varepsilon(B)$ of the ordering cone of positive elements $L^2_+(\Omega)$. Here, B is a fixed closed base of $L^2_+(\Omega)$. Due to fact that $L^2_+(\Omega) \subset \left(L^2_+(\Omega)\right)_\varepsilon(B)$ for all $\varepsilon > 0$, we can replace the cone $L^2_+(\Omega)$ by its approximation $\left(L^2_+(\Omega)\right)_\varepsilon(B)$. As a result, it leads to some relaxation of the inequality constraints of the considered problem, and, hence, to an approximation of the set of admissible pairs to OCP. The main issue is to show that admissibility and solvability of a given class of OCPs can be characterized by solving the corresponding Henig relaxed problems in the limit $\varepsilon \to 0+$.

As for the class of admissible controls, we consider it as inflows with uniform bounded variation. Such choice is motivated both by theoretical needs of having good properties of solutions and practical one for the presence of possible costs for adjusting the supply chain inflow. As was shown in [11], the supply chain dynamics is well defined for every control with bounded variation.

The outline of the paper is the following. In Section 2 we report some preliminaries and notation we need in the sequel. In Sections 3 we discuss the GHK model of supply chains in the spirit of [15] and describe the main assumptions on the initial data and control functions. The precise statement of the corresponding optimal control problem for supply chains is formulated in Section 4. The aim of Section 5 is to give a precise definition of the set of admissible solutions. Since the supply chain model is a coupled control system of partial and ordinary differential equations with initial and boundary conditions, the solutions of such system may develop discontinuities (after a finite time). In view of this we give the most important properties of entropy solutions to such systems and specify the sets of admissible states for the densities of parts at the boundary points. In Section 6 we provide results concerning solvability of the original problem with control and state constraints. We show that this problem admits at least one solution if and only if this problem is regular. Section 7 contains the main result of this article, where we show that the pointwise state constraints can be replaced by the weakened conditions coming from Henig relaxation of ordering cones. As a result, we give the precise definition of relaxed optimization problems and show that the solvability and regularity of the original OCP can be characterized by associated relaxed problems. In particular, we prove that the optimal solution to the original problem can be attained in the limit by the optimal solution of the relaxed problem.

2. **Notation and preliminaries.** Let $N \in \mathbb{N}$ be a fixed positive integer and let $\{I_j := (a_j, b_j) \subset \mathbb{R}\}_{j=1}^N$ be a given collection of bounded open intervals such that $b_{j-1} = a_j$ for all $j = 2, \dots, N$. For each $j = 1, \dots, N$ and for a given $T > 0$, we set $\overline{I} = [a_j, b_j]$, $\Omega_T = (0,T) \times \mathbb{R}$, and $\Omega_j = (0,T) \times I_j$. Let $L^p_{loc}(\Omega_T)$, with $1 \le p \le \infty$, be the locally convex space of all measurable functions $g : \Omega_T \to \mathbb{R}$ such that $g|_{(0,T)\times K} \in L^p((0,T) \times K)$ for all compact sets $K \subset \mathbb{R}$. Hereinafter, $L^p_+(\Omega_j)$ denotes the natural ordering cone of positive elements in $L^p(\Omega_j)$, i.e.

$$L^p_+(\Omega_j) = \left\{ f \in L^p(\Omega_j) \;\middle|\; f \ge 0 \text{ almost everywhere in } \Omega_j \right\}.$$

Let I be a bounded open interval in $\mathbb{R}$. By $BV(I)$ we denote the space of all functions in $L^1(I)$ with bounded variation. Under the norm $\|f\|_{BV(I)} = \|f\|_{L^1(I)} + TV(f)$, $BV(I)$ is a Banach space, where

$$TV(f) := \int_I |Df| = \sup\left\{ \int_I f\,\varphi'\,dx \;:\; \varphi \in C^1_0(I),\ |\varphi(x)| \le 1 \text{ for } x \in I \right\}.$$

It is well-known the following compactness result for BV-spaces (Helly's selection theorem, see [2, 14]):

Theorem 2.1. *The uniformly bounded sets in BV-norm are relatively compact in $L^1(I)$, that is, if $\{f_k\}_{k=1}^\infty \subset BV(I)$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{BV(I)} < +\infty$, then there exists a subsequence of $\{f_k\}_{k=1}^\infty$ strongly converging in $L^1(I)$ to some $f \in BV(I)$ such that $Df_k \overset{*}{\rightharpoonup} Df$ weakly-$*$ in the space of Radon measures $\mathcal{M}(I)$, i.e.*

$$\lim_{k\to\infty} \int_I \varphi\, Df_k = \int_I \varphi\, Df, \quad \forall\, \varphi \in C_0(I).$$

For an arbitrary function $g \in L^\infty(0,T; BV(I_j))$ we say that $g(t,a_j)$ and $g(t,b_j)$ are its traces at the points $x = a_j$ and $x = b_j$ respectively, if the following relations

$$g(t,a_j) = -\lim_{\delta\to 0+} \int_{a_j}^{a_j+\delta} g(t,x)\chi'_\delta(x-a_j)\,dx, \tag{1}$$

$$g(t,b_j) = \lim_{\delta\to 0+} \int_{b_j-\delta}^{b_j} g(t,x)\chi'_\delta(x-b_j)\,dx \tag{2}$$

hold true for almost all $t \in (0,T)$ and for any test function $\chi_\delta \in C^2_0(-\delta,\delta)$ such that

$$0 \le \chi_\delta(x) \le 1, \quad \chi_\delta(0) = 1, \quad |\chi'_\delta(x)| \le C\delta^{-1}, \quad \forall\, x \in (-\delta,\delta)$$

with a constant $C > 0$ independent of δ (see [2, 25]).

3. **GHK model of supply chains.** In this section we focus on the model introduced in [15] by Göttlich, Herty and Klar, where supply chains are concatenations of suppliers which process parts. Further, each supplier is composed of a processor for assembling and construction and a buffer for unprocessed parts, called queue. Formally we adopt the following definition.

Definition 3.1. We say that a graph $G = (\mathcal{V}, \mathcal{J})$ with a finite set of arcs $\mathcal{J} = \{J_1, J_2, \dots, J_N\}$ and vertices $\mathcal{V} = \{q_2, q_3, \dots, q_N\}$ is the formal representation of a supply chain if each arc $J_j \in \mathcal{J}$ can be associated with a processor and vertices q_j represent the corresponding queues in front of the corresponding processor J_j, except the first. Moreover, each arc J_j is parametrized by a bounded closed interval $\overline{I}_j = [a_j, b_j]$ with $b_{j-1} = a_j$ for all $j = 2, ..., N$.

Thus, we consider the type of supply chains when each vertex is connected to exactly one incoming supplier and one outgoing supplier.

Let $\mu = [\mu_1, \mu_2, \dots, \mu_N]^t \in \mathbb{R}^N_+$ be a given vector, where $\mu_j > 0$ stands for the maximal processing rate of the corresponding processor. Let $\zeta^{max}(\cdot) = [\zeta_1^{max}(\cdot), \zeta_2^{max}(\cdot), \dots, \zeta_N^{max}(\cdot)]^t \in \prod_{j=1}^N L^2_+(\Omega_j)$ be a given vector-valued function. We also assume that each of processors is characterized by a length $L_j = b_j - a_j > 0$ and processing time $T_j > 0$. The quantity $V_j = L_j/T_j$ is thus the processing velocity. As a result, having assumed that the dynamics of parts inside the processor J_j is given by a conservation law for the density of parts, $\rho_j(t,x)$, with flux given by the minimum between the physical flow and the maximal processing capacity, we arrive at the following state constrained initial-boundary value problem (see [11, 15])

$$\partial_t \rho_j(t,x) + \partial_x \min\{\mu_j, V_j \rho_j(t,x)\} = 0 \quad \text{in } \ \Omega_j := (0,T)\times(a_j,b_j), \tag{3}$$

$$\rho_j(0,x) = \rho_{j,0}(x) \quad \text{on } \ I_j = (a_j,b_j), \tag{4}$$

$$\rho_j(t,a_j) = \frac{f_j^{inc}(t)}{V_j} \quad \text{for a.a. } \ t\in(0,T), \tag{5}$$

$$0 \le \rho_j(t,x) \quad \text{and} \quad \mathcal{B}_j(\rho_j)(t,x) \le \zeta_j^{max}(t,x) \quad \text{a.e. in } \ \Omega_j. \tag{6}$$

Here, $\mathcal{B}_j : L^\infty(0,T;BV(I_j)) \to L^2(\Omega_j)$ is a bounded compact operator (i.e. image of any bounded set in $L^\infty(0,T;BV(I_j))$ under $\mathcal{B}_j$ is relatively compact in $L^2(\Omega_j)$), $\rho_j(t,x)$ is the unknown function, representing the density of parts, while the initial datum $\rho_{j,0} \in L^\infty_+(I_j)$ and the inflow $f_j^{inc} \in L^\infty_+(0,T)$ are given functions. It is worth to emphasize that we consider the inflow for the first arc f_1^{inc} as a control function $f_1^{inc} = u(t)$ subjected to the constraints

$$u \in U_\partial = \left\{ v \in BV(0,T) \;\middle|\; \begin{array}{c} 0 \le u(t) \le \mu_1 \quad \text{for a.a. } t\in(0,T), \\ TV(u) \le \widehat{C}, \end{array} \right\} \tag{7}$$

where $\widehat{C} > 0$ is a given constant.

As follows from relations 3–6, the GHK model for the j-th processor has the following interpretation: the supplier has to process as many parts as possible. If the outgoing buffer is empty, then it processes all incoming parts but at most μ_j, otherwise it can always process at rate μ_j.

As for the dynamics of the queues, we assume that each queue buffer occupancy can be described as a time-dependent function $t \to q_j(t)$ for $j = 2, \dots, N$, satisfying the following Cauchy problem for ordinary differential equation

$$\frac{d\, q_j(t)}{dt} = f_{j-1}(\rho_{j-1}(t,b_{j-1})) - f_j^{inc}(t) \quad \text{in } \ (0,T), \tag{8}$$

$$q_j(0) = q_{j,0}. \tag{9}$$

Here, $q_0 = [q_{2,0}, q_{3,0}, \dots, q_{N,0}]^t \in \mathbb{R}^N_+$ is a given positive vector, and functions $f_j^{inc} \in L^\infty_+(0,T)$ and $f_{j-1} \in L^\infty_+(\mathbb{R}_+)$ are defined for $j = 2, \dots, N$ as follows

$$f_j^{inc}(t) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(t,b_{j-1})), \mu_j\} & \text{if } q_j(t) = 0, \\ \mu_j & \text{if } q_j(t) > 0, \end{cases} \tag{10}$$

$$f_{j-1}(\rho) = \min\{\mu_{j-1}, V_{j-1}\rho\}, \quad \forall\, \rho \ge 0. \tag{11}$$

Finally, the supply chain model is a coupled control system of partial and ordinary differential equations with initial and boundary conditions given by

$$\begin{cases} \partial_t \rho_j(t,x) + \partial_x \min\{\mu_j, V_j\rho_j(t,x)\} = 0 & \text{in } \ \Omega_j, & j=1,...,N, \\ \frac{d\,q_j(t)}{dt} = f_{j-1}(\rho_{j-1}(t,b_{j-1})) - f_j^{inc}(t) & \text{in } \ (0,T), & j=2,...,N, \\ \rho_j(0,x) = \rho_{j,0}(x) & \text{on } \ I_j, & j=1,...,N, \\ \rho_j(t,a_j) = \frac{f_j^{inc}(t)}{V_j} & \text{for a.a. } \ t\in(0,T), & j=1,...,N, \\ q_j(0) = q_{j,0} & & j=2,...,N, \\ f_1^{inc}(t) = u(t) & & \end{cases} \tag{12}$$

where $f_j^{inc}(t)$ and $f_{j-1}(\rho)$ are given by 10–11 for $j=2,...,N$.

4. **Statement of optimal control problem.** In this subsection we introduce an optimal control problem for the model 12 with control and state constraints 6–7. Let $\beta \in L^1_+(0,T)$ and $\psi, \alpha_2, \alpha_3, \dots, \alpha_N \in L^\infty_+(0,T)$ be given positive weight functions. We define the cost functional as follows

$$J(u,\rho,q) = \sum_{j=2}^{N} \int_0^T \alpha_j(t) q_j(t)\,dt + \int_0^T \beta(t) \Big| V_N\rho_N(t,b_N) - \psi(t) \Big|\,dt, \tag{13}$$

where the term $V_N\rho_N(\cdot,b_N)$ represents the outflow of the supply chain. The main reason to consider the cost functional in the form 13 is to minimize the queues $q(t) = [q_2(t), q_3(t), \dots, q_N(t)]^t$ in front of processors and the distance between the effective outflow $V_N\rho_N(t,b_N)$ and the pre-assigned one $\psi(t)$, using the supply chain input $u \in U_\partial$ as a control. We consider the distance in the second term of 13 instead of the quadratic distance of the outflow from the pre-assigned function ψ because the cost functional should be well defined for each admissible $u \in U_\partial$, $q_j \in W^{1,1}(0,T)$, and $\rho_N \in \big(C^{0,1}([0,T];L^1(I_j)) \cap L^\infty(0,T;BV(I_j))\big)$, where the given choice of functional spaces is a consequence of the existence result for the coupled control system 12 (see [6, 11]).

Thus, our main goal is to analyze the following minimization problem: for given $\widehat{C} > 0$,

$$\begin{aligned} q_0 &= [q_{2,0}, q_{3,0}, \dots, q_{N,0}]^t \in \mathbb{R}^{N-1}_+, \\ \mu &= [\mu_1, \mu_2, \dots, \mu_N]^t \in \mathbb{R}^N_+, \\ &\big\{\mathcal{B}_j : L^\infty(0,T;BV(I_j)) \to L^2(\Omega_j)\big\}_{j=1}^N \text{ are bounded and compact,} \\ V &= [V_1, V_2, \dots, V_N]^t \in \mathbb{R}^N_+, \\ \rho_0(\cdot) &= [\rho_{1,0}(\cdot), \rho_{2,0}(\cdot), \dots, \rho_{N,0}(\cdot)]^t \in \prod_{j=1}^N L^\infty_+(I_j), \quad \text{and} \\ \zeta^{max}(\cdot) &= [\zeta_1^{max}(\cdot), \zeta_2^{max}(\cdot), \dots, \zeta_N^{max}(\cdot)]^t \in \prod_{j=1}^N L^2_+(\Omega_j), \end{aligned} \tag{14}$$

$$\text{minimize } \ J(u,\rho,q) \ \text{ subject to constraints 6–7 and 12.} \tag{15}$$

To start we need to make sure that minimization problem 15 is regular, i.e. the dynamics of supply chain 12 with control and state constraints 6–7 is well defined for every admissible control $u \in U_\partial$. It is well known that even for arbitrary smooth functions $u(t)$ and $\rho_{j,0}(x)$, the initial-boundary value problem 3–6 is not well-posed. Indeed it can admit non-unique solutions and moreover, these solutions may develop discontinuous after a finite time (see [4, 6, 24]), which makes it necessary to consider the concept of weak solutions. In the meanwhile, weak solutions of 3–6 are, in general, not unique and, in order to select the "physically" relevant solution, some additional conditions and special formulation of Dirichlet boundary conditions must be imposed.

On the other hand, because of the state constraints 6 the regularity of the OCP 15 is an open question even for the simplest situation. So, the first question to be answered for this problem is about admissibility: does there exist at least one triple (u, ρ, q) such that $u \in U_\partial$, $0 \le \rho$, $\mathcal{B}(\rho) \le \zeta^{max}$, and (ρ, q) would be a physically relevant solution to 12? In fact, one needs the set of admissible solutions to be sufficiently rich in some sense, otherwise the OCP 15 becomes trivial. However, from a mathematical point of view, to deal directly with all constraints above presented is typically very difficult and, except for some special cases, this question is largely open [7, 21, 22]. Nevertheless, in many applications it is an important task to find an admissible (or at least an approximately admissible, in a sense to be made precise) solution when both control and state constraints for the OCP are given. Thus, if the set of admissible solutions is rather "thin", it is reasonable to weaken the requirements on admissible solutions to the original OCP. In particular, it would be reasonable to assume that the optimality property for the solutions $(u, y(u))$ holds not strictly but rather with some (possibly high) accuracy. In this sense, an extremal problem may have an approximate or suboptimal solution even if it is not solvable.

5. **On admissible solutions to coupled control system (12).** Following [4, 13], we use the entropy-admissibility conditions, coming from physical considerations.

Definition 5.1. A couple of functions $(\eta_j, \nu_j) : \mathbb{R}_+ \to \mathbb{R}^2$ for a given $j = 1, \dots, N$ is an entropy-flux for the equation 3, if $\eta_j : \mathbb{R}_+ \to \mathbb{R}$ is convex and they are related by the equality

$$D\eta_j(v) \cdot Df_j(\rho) = D\nu_j(\rho) \qquad \text{for a.a.} \quad \rho \in \mathbb{R}_+, \tag{16}$$

where $f_j(\rho) := \min\{\mu_j, V_j\rho\}$.

We note that such definition makes sense because any convex function η defined on an open set is locally Lipschitz, and therefore $D\eta_j$ and Df_j are defined almost everywhere. As an example of the entropy-flux pair, it can be considered the following one

$$\eta_j(\rho) := |\rho - k|, \quad \nu_j(\rho) := (f_j(\rho) - f_j(k))\,\mathrm{sign}\,(\rho - k), \quad \forall\, k \in \mathbb{R}.$$

Further, by analogy with Dubois and Lefloch [13], we specify the sets of admissible states at the boundary points $\{x = a_j\}_{j=1}^N$.

Definition 5.2. For a given $j = 1, \dots, N$ and given boundary value $\frac{f_j^{inc}(t)}{V_j}$, the set $\mathcal{E}_j\left(\frac{f_j^{inc}(t)}{V_j}\right)$ of admissible values of the function $\rho_j(t, x)$ at the boundary point $x = a_j$ is defined as all the states $v(t)$ in $\mathbb{R}_+$ such that

$$\nu_j\big(v(t)\big) - \nu_j\Big(\frac{f_j^{inc}(t)}{V_j}\Big) - D\eta_j\Big(\frac{f_j^{inc}(t)}{V_j}\Big)\Big(f_j\big(v(t)\big) - f_j\Big(\frac{f_j^{inc}(t)}{V_j}\Big)\Big) \le 0 \tag{17}$$

for each pair (η_j, ν_j) of entropy-flux.

As a result, the boundary conditions for the problem 3–6 can be reformulated as follows

$$\rho_j\left(t,a_j\right) \in \mathcal{E}_j\Big(\frac{f_j^{inc}(t)}{V_j}\Big), \qquad \text{for a.a. } \; t\in(0,T) \; \text{ and all } \; j=1,...,N. \tag{18}$$

Note that the function $\frac{f_j^{inc}(t)}{V_j}$ itself belongs to $\mathcal{E}_j(\frac{f_j^{inc}(t)}{V_j})$. But generally the set $\mathcal{E}_j(\frac{f_j^{inc}(t)}{V_j})$ is not reduced to this point, and, therefore, conditions 18 should be considered as extensions of the usual Dirichlet boundary conditions 6.

In view of this, we adopt the following definition of entropy solutions to the initial-boundary value problem 3–6.

Definition 5.3. For given $j=1,\dots,N$, $\rho_{j,0}\in L^\infty_+(I_j)$, and $f_j^{inc}\in L^\infty_+(0,T)$, function $\rho_j(t,x)$ is called an entropy solution of 3–6 if for each entropy-flux pair (η_j,ν_j) the entropy inequality

$$D_t\eta_j(\rho_j)+D_x\nu_j(\rho_j)\le 0 \quad \text{in} \quad \mathcal{D}'(\Omega_j) \tag{19}$$

holds true (in the sense of distributions), the boundary condition 6 is valid in the sense of inclusion 18, and fulfilment of the initial condition 5 is assumed as follows

$$\lim_{t\to 0+}\int_a^b |\rho_j(t,x)-\rho_{j,0}(x)|\,dx=0, \quad \forall\,(a,b)\in I_j. \tag{20}$$

As for the dynamics of queues, it is easy to see that the right-hand side of equation 8 is a discontinuous function with respect to q_j. Hence, having rewritten 8 in the form of the corresponding differential inclusion, it is plausible to define a weak solution to 8–9 in the interpretation of Filippov (for the details we refer to [6, 11]).

The existence and uniqueness of entropy solution for the coupled control system 12 under assumptions given above was established in [6] by the front tracking method (see also [11, 19]). The following result collects the most important properties of entropy solutions to the system 12.

Theorem 5.4 ([6, 11])**.** *Let*

$$\widehat{C}>0, \quad q_0\in\mathbb{R}^{N-1}_+, \quad \mu,V\in\mathbb{R}^N_+, \quad \text{and} \;\; \rho_0\in\prod_{j=1}^N L^\infty_+(I_j)\cap BV(I_j)$$

be given data. Then for every admissible control $u\in U_\partial$ there exists a unique pair of vector-valued function

$$\rho(t,x)=[\rho_1(t,x),\rho_2(t,x),\dots,\rho_N(t,x)]^t, \quad q(t)=[q_2(t),q_3(t),\dots,q_N(t)]^t$$

such that

1. $(\rho,q)\in\prod_{j=1}^N\left(C^{0,1}([0,T];L^1(I_j))\cap L^\infty(0,T;BV(I_j))\right)\times\prod_{j=2}^N W^{1,1}(0,T)$;
2. *for each $j=1,\dots,N$, $\rho_j(t,x)$ is the entropy solution to the corresponding initial-boundary value problem 3–6 in the sense of Definition 5.3;*
3. *for each $j=2,\dots,N$, $q_j(t,x)$ is a Filippov weak solution to the Cauchy problem 8–9 such that the relations*
$$\dot q_j(t)=f_{j-1}\left(\rho_{j-1}\left(t,b_{j-1}\right)\right)-f_j^{inc}(t), \quad j=2,\dots,N,$$
are valid for almost all $t\in(0,T)$;
4. *$q_j(t)\ge 0$ for all $t\in[0,T]$ and $j=2,\dots,N$;*
5. *$\rho_j(t,x)\ge 0$ for almost all $(t,x)\in\Omega_j=(0,T)\times I_j$ and $j=1,\dots,N$;*

6. there exists a constant $C = C(T) > 0$ such that

$$\sum_{j=1}^{N} \operatorname*{ess\,sup}_{t\in(0,T)} \|\rho_j(t,\cdot)\|_{BV(I_j)} \le C\Big(\sum_{j=1}^{N} \|\rho_{j,0}\|_{BV(I_j)} + \|u\|_{BV(0,T)} + \|q_0\|_{\mathbb{R}^{N-1}}\Big); \quad (21)$$

$$\sum_{j=2}^{N} \|q_j\|_{W^{1,1}(0,T)} \le C\Big(\sum_{j=1}^{N} \|\rho_{j,0}\|_{BV(I_j)} + \|u\|_{BV(0,T)} + \|q_0\|_{\mathbb{R}^{N-1}}\Big). \quad (22)$$

Moreover, if $u, \widetilde{u} \in U_\partial$ are arbitrary controls and (ρ, q) and $(\widetilde{\rho}, \widetilde{q})$ are the corresponding solutions to 12, then the following estimate holds true

$$\sum_{j=1}^{N} \max_{t\in[0,T]} \|\rho_j(t,\cdot) - \widetilde{\rho}_j(t,\cdot)\|_{L^1(I_j)} + \sum_{j=2}^{N} \|q_j - \widetilde{q}_j\|_{C([0,T])} \le \frac{1}{V_1}\|u - \widetilde{u}\|_{L^1(0,T)}. \quad (23)$$

Remark 1. As immediately follows from Theorem 5.4 and Theorem 10.2.1 in [2], for every $j = 1, \dots, N$ there exist linear continuous trace operators

$$\gamma_{a_j} : L^\infty(0,T; BV(I_j)) \to L^\infty(0,T) \quad \text{and} \quad \gamma_{b_j} : L^\infty(0,T; BV(I_j)) \to L^\infty(0,T)$$

such that

$$\gamma_{a_j}(\rho) = \rho\big|_{x=a_j} \quad \text{and} \quad \gamma_{b_j}(\rho) = \rho\big|_{x=b_j} \quad \text{for all } \rho \in L^\infty(0,T; BV(I_j) \cap C(\overline{I}_j)).$$

Moreover, in this case the following generalized Green's formulae

$$\int_0^T \alpha(t)\gamma_{b_j}(\rho(t,\cdot))\,dt = \int_0^T \Big[\int_{I_j} \varphi D\rho + \int_{a_j}^{b_j} \varphi'(x)\rho(t,x)\,dx\Big]\,dt, \quad (24)$$

$$\int_0^T \alpha(t)\gamma_{a_j}(\rho(t,\cdot))\,dt = \int_0^T \Big[\int_{I_j} \psi D\rho + \int_{a_j}^{b_j} \psi'(x)\rho(t,x)\,dx\Big]\,dt \quad (25)$$

hold true for every $\rho \in L^\infty(0,T; BV(I_j))$, $\alpha \in L^1(0,T)$, and $\varphi, \psi \in C^1(\overline{I}_j)$ such that

$$\varphi(a_j) = 0, \quad \varphi(b_j) = 1, \quad \psi(a_j) = -1, \quad \psi(b_j) = 0.$$

Following the definition of the trace operators γ_{a_j} and γ_{b_j}, it is easy to see that the equalities

$$\gamma_{a_j}(\rho_j(t,x)) = \rho_j(t,a_j), \qquad \gamma_{b_j}(\rho_j(t,x)) = \rho_j(t,b_j), \quad \forall\, j = 1, \dots, N,$$

where functions $\rho_j(t,a_j)$ and $\rho_j(t,b_j)$ are defined by 1–2, hold true for any entropy solution $\rho(t,x) = [\rho_1(t,x), \rho_2(t,x), \dots, \rho_N(t,x)]^t$ of 12.

Taking into account the fact that conditions $\rho_j \in L^2_+(\Omega_j)$ and $\zeta_j^{max} - \mathcal{B}_j(\rho_j) \in L^2_+(\Omega_j)$ are equivalent to the state constraints $0 \le \rho_j$ and $\mathcal{B}_j(\rho_j) \le \zeta_j^{max}$ a.e. in Ω_j, we specify the following set

$$\Upsilon_\partial = \left\{ \rho = [\rho_1, \dots, \rho_N]^t \;\middle|\; \begin{array}{c} \rho_j \in L^\infty(0,T; BV(I_j)), \\ \rho_j \in L^2_+(\Omega_j), \quad \zeta_j^{max} - \mathcal{B}_j(\rho_j) \in L^2_+(\Omega_j), \\ j = 1, 2, \dots, N. \end{array} \right\} \quad (26)$$

Definition 5.5. We say that a triplet

$$(u, \rho, q) \in BV(0,T) \times \prod_{j=1}^{N} C\left([0,T]; L^1(I_j)\right) \times \prod_{j=2}^{N} W^{1,1}(0,T)$$

is an admissible solution to the OCP 15 if $u \in \mathcal{U}_\partial$, $\rho \in \Upsilon_\partial$, $J(u,\rho,q) < +\infty$, and pair $(\rho, q) = (\rho(u), q(u))$ is an entropy solution to 12 in the sense of Definition 5.3.

Let Ξ be the set of all admissible triplets to the problem 15. We say that a triplet (u^0, ρ^0, q^0) is optimal for the problem 15 if

$$(u^0, \rho^0, q^0) \in \Xi \quad \text{and} \quad J(u^0, \rho^0, q^0) = \inf_{(u,\rho,q)\in\Xi} J(u, \rho, q).$$

6. **Existence of optimal solutions.** Since Theorem 5.4 does not ensure the existence of admissible controls $u \in U_\partial$ such that the corresponding entropy solutions (ρ, q) of 12 would satisfy the state constraints $\zeta_j^{max} - \mathcal{B}_j(\rho_j) \in L^2_+(\Omega_j)$ for all $j = 1, \dots, N$, the following assumption is crucial for our further analysis:

(A1): The OCP 15 is regular in the following sense: for a given vector-valued function $\zeta^{max}(\cdot) = [\zeta_1^{max}(\cdot), \zeta_2^{max}(\cdot), \dots, \zeta_N^{max}(\cdot)]^t \in \prod_{j=1}^N L^2_+(\Omega_j)$ and a given collection of bounded compact operators $\left\{\mathcal{B}_j : L^\infty(0,T; BV(I_j)) \to L^2(\Omega_j)\right\}_{j=1}^N$ there exists at least one triplet (u, ρ, q) such that $(u, \rho, q) \in \Xi$.

Then the sufficient conditions for the existence of an optimal solution to the problem 15 can be stated as follows.

Theorem 6.1. *Let*

$$\widehat{C} > 0, \quad q_0 \in \mathbb{R}^{N-1}_+, \quad \mu, V \in \mathbb{R}^N_+, \quad \rho_0 \in \prod_{j=1}^N L^\infty_+(I_j) \cap BV(I_j), \tag{27}$$

$$\zeta^{max}(\cdot) = [\zeta_1^{max}(\cdot), \zeta_2^{max}(\cdot), \dots, \zeta_N^{max}(\cdot)]^t \in \prod_{j=1}^N L^2_+(\Omega_j), \tag{28}$$

$$\beta, \alpha_2, \alpha_3, \dots, \alpha_N \in L^1_+(0,T), \quad \text{and} \quad \psi \in L^\infty_+(0,T) \tag{29}$$

be an arbitrary given initial data. Let $\left\{\mathcal{B}_j : L^\infty(0,T; BV(I_j)) \to L^2(\Omega_j)\right\}_{j=1}^N$ be an arbitrary family of bounded compact operators. Then the OCP 15 admits an optimal solution $(u^0, y^0) \in \Xi$ if and only if Hypothesis (A1) is satisfied.

Proof. Hypothesis (A1) and Theorem 5.4 (see items 4 and 5) imply that $\Xi \neq \emptyset$ and the cost functional $J(u, \rho, q)$ is bounded below on the set Ξ, namely, $J(u, \rho, q) \geq 0$ for all admissible triplets $(u, \rho, q) \in \Xi$.

Let $\left\{(u^k, \rho^k, q^k)\right\}_{k\in\mathbb{N}} \subset \Xi$ be a minimizing sequence for the original problem, i.e.

$$\lim_{k\to\infty} J(u^k, \rho^k, q^k) = \inf_{(u,\rho,q)\in\Xi} J(u, \rho, q) \geq 0.$$

It is well known that the set of all step functions on $(0, T)$ is dense in $L^1(0, T)$. Hence, without loss of generality, we may assume that $\left\{u^k\right\}_{k\in\mathbb{N}}$ is a sequence of piecewise constant controls. Since the sequence $\left\{u^k\right\}_{k\in\mathbb{N}} \subset U_\partial$ is uniformly bounded in BV-norm, it follows from a priory estimates 21–22 and Theorem 2.1 that there exists a triplet

$$(u^*, \rho^*, q^*) \in U_\partial \times \prod_{j=1}^N L^\infty(0,T; BV(I_j)) \times \prod_{j=2}^N W^{1,1}(0,T)$$

such that, within a subsequence, we have (see [2, 6])

$$u^k \to u^* \quad \text{in} \;\; L^1(0,T), \tag{30}$$

$$Du^k \overset{*}{\rightharpoonup} Du^* \quad \text{in} \;\; \mathcal{M}(0,T), \tag{31}$$

$$q_j^k \rightharpoonup q_j^* \quad \text{in} \;\; W^{1,1}(0,T), \quad \forall\, j = 2, \dots, N, \tag{32}$$

$$q_j^k \to q_j^* \quad \text{in} \quad L^1(0,T), \quad \forall\, j = 2, \ldots, N, \tag{33}$$

$$\rho_j^k(t,x) \to \rho_j(t,x) \quad \text{a.e. in} \quad \Omega_j, \quad \forall\, j = 1, \ldots, N, \tag{34}$$

$$\lim_{k\to\infty} \int_0^T \phi(t) \int_{I_j} |\rho_j^k(t,x) - \rho_j^*(t,x)|\, dx\, dt = 0,\ \forall\, \phi \in L^1(0,T),\ \forall\, j = \overline{1,N}, \tag{35}$$

where 35 is a direct consequence of the BV-compactness criterion and Banach-Alaoglu Theorem.

Let us show that $(u^*, \rho^*, q^*) \in \Xi$. Let $\{(\eta_j, \nu_j)\}_{j=1}^N$ be an arbitrary collection of entropy-flux pairs related to the system 12. Since $(u^k, \rho^k, q^k) \in \Xi$, it follows that for every $k \in \mathbb{N}$ the following relations hold true

$$\int_0^T \int_{I_j} \left[\eta_j(\rho_j^k(t,x))\phi_t + \nu_j(\rho_j^k(t,x))\phi_x\right] dx\, dt \geq 0,\ \forall\, \phi \in C_0^\infty(\Omega_j),\ j = \overline{1,N}, \tag{36}$$

$$\lim_{t\to 0+} \int_a^b |\rho_j^k(t,x) - \rho_{j,0}(x)|\, dx = 0, \quad \forall\, (a,b) \in I_j,\ j = \overline{1,N}, \tag{37}$$

$$\begin{gathered} \nu_j\big(\rho_j^k\,(t,a_j)\,\big) - \nu_j\left(g_j^k(t)\right) \\ -D\eta_j\left(g_j^k(t)\right)\left(f_j\big(\rho_j^k\,(t,a_j)\,\big) - f_j\left(g_j^k(t)\right)\right) \leq 0 \text{ a.e. in } (0,T),\ j = \overline{1,N}, \end{gathered} \tag{38}$$

$$0 \leq \rho_j^k(t,x) \quad \text{a.e. in} \quad \Omega_j,\ j = \overline{1,N}, \tag{39}$$

$$\mathcal{B}_j\left(\rho_j^k\right)(t,x) \leq \zeta_j^{max}(t,x) \quad \text{a.e. in} \quad \Omega_j,\ j = \overline{1,N}, \tag{40}$$

$$q_j^k(0) = q_{j,0},\ j = \overline{2,N}, \tag{41}$$

$$\begin{gathered} \int_0^T \left[-\dot{q}_j^k(t) + f_{j-1}\left(\rho_{j-1}^k\,(t,b_{j-1})\right) - V_j g_j^k(t)\right]\phi(t)\, dt = 0, \\ \forall\, \phi \in C_0^\infty(\mathbb{R}),\ j = \overline{2,N}, \end{gathered} \tag{42}$$

where

$$g_j^k(t) = \frac{1}{V_j}\begin{cases} \min\left\{f_{j-1}\left(\rho_{j-1}^k\,(t,b_{j-1})\right), \mu_j\right\} & \text{if } q_j^k\,(t) = 0, \\ \mu_j & \text{if } q_j^k\,(t) > 0. \end{cases} \tag{43}$$

These relations are the direct consequence of Definition 5.3 and the fact that functions $q_j^k \in W^{1,1}(0,T)$ are the solutions of the Cauchy problems 8-9 in the sense of Caratheodory (see item 3 of Theorem 5.4).

Thus, in order to prove the inclusion $(u^*, \rho^*, q^*) \in \Xi$ it is enough to pass to the limit in 36–42 as $k \to \infty$ and show that the limit triplet (u^*, ρ^*, q^*) satisfies the same relations. Since the operators $\mathcal{B}_j : L^\infty(0,T; BV(I_j)) \to L^2(\Omega_j)$ are compact and the sequence $\left\{\rho_j^k\right\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T; BV(I_j))$, it follows from 35 that

$$\mathcal{B}_j\left(\rho_j^k\right) \to \mathcal{B}_j\left(\rho_j^*\right) \quad \text{strongly in} \quad L^2(\Omega_j).$$

So, we may assume that, up to a subsequence, we have the pointwise convergence of $\mathcal{B}_j\left(\rho_j^k\right)(t,x)$ to $\mathcal{B}_j\left(\rho_j^*\right)(t,x)$ almost everywhere in Ω_j for every $j = 1, \ldots, N$. Hence, taking this observation and properties 34–35 into account, we can pass to the limit in relations 36, 37, 39, and 40. As a result, we obtain

$$\begin{gathered} \int_0^T \int_{I_j} \left[\eta_j(\rho_j^*(t,x))\phi_t + \nu_j(\rho_j^*(t,x))\phi_x\right] dx\, dt \geq 0, \\ \forall\, \phi \in C_0^\infty(\Omega_j),\ j = \overline{1,N}, \end{gathered} \tag{44}$$

$$\lim_{t\to 0+}\int_a^b |\rho_j^*(t,x)-\rho_{j,0}(x)|\,dx = 0, \quad \forall\,(a,b)\in I_j,\ j=\overline{1,N}, \tag{45}$$

$$0\le \rho_j^*(t,x) \quad \text{a.e. in } \ \Omega_j,\ j=\overline{1,N}, \tag{46}$$

$$\mathcal{B}_j\left(\rho_j^*\right)(t,x)\le \zeta_j^{max}(t,x) \quad \text{a.e. in } \ \Omega_j,\ j=\overline{1,N}. \tag{47}$$

In order to guarantee the fulfilment of boundary conditions for the limit functions ρ_j^*, we apply the following trick coming from Green's formula for BV-functions

$$\begin{aligned}
&\lim_{k\to\infty}\min\left\{f_{j-1}\left(\rho_{j-1}^k\left(t,b_{j-1}\right)\right),\mu_j\right\}\\
&\overset{\text{by } 24}{=} \min\left\{f_{j-1}\Big(\lim_{k\to\infty}\Big(\int_{I_{j-1}}\varphi D\rho_{j-1}^k+\int_{a_{j-1}}^{b_{j-1}}\varphi'(x)\rho_{j-1}^k(t,x)\,dx\Big)\Big),\mu_j\right\}\\
&\overset{\text{by } 35}{=} \min\left\{f_{j-1}\Big(\Big(\int_{I_{j-1}}\varphi D\rho_{j-1}^*+\int_{a_{j-1}}^{b_{j-1}}\varphi'(x)\rho_{j-1}^*(t,x)\,dx\Big)\Big),\mu_j\right\}\\
&\overset{\text{by } 24}{=} \min\left\{f_{j-1}\Big(\rho_{j-1}^*\Big(t,b_{j-1}\Big)\Big),\mu_j\right\} \quad \text{for a.a. } \ t\in(0,T),
\end{aligned} \tag{48}$$

where $\varphi\in C^1(\overline{I}_j)$ is an arbitrary function with $\varphi(a_{j-1})=0$ and $\varphi(b_{j-1})=1$.

In the meantime, in view of the continuous embedding $W^{1,1}(0,T)\hookrightarrow C([0,T])$, the convergence properties 32–33 imply that $q^*\in C([0,T])$. Moreover, since $\left\{u^k\right\}_{k\in\mathbb{N}}$ is a sequence of piecewise constant controls, the Front-Tracking procedure ensures the uniform convergence $q_j^k\to q_j^*$ on $[0,T]$. Hence, in view of 48, we immediately arrive at the following equality

$$g_j^*(t):=\lim_{k\to\infty}g_j^k(t)=\frac{1}{V_j}\begin{cases}\min\left\{f_{j-1}\left(\rho_{j-1}^*\left(t,b_{j-1}\right)\right),\mu_j\right\} & \text{if } q_j^*\left(t\right)=0,\\ \mu_j & \text{if } q_j^*\left(t\right)>0,\end{cases} \tag{49}$$

which should be interpreted in the sense of almost everywhere in $(0,T)$. Taking this fact into account, we can pass to the limit in relations 41–42 as $k\to\infty$. As a result, for each $j=2,\dots,N$, we get

$$q_j^*(0)=q_{j,0}, \tag{50}$$

$$\int_0^T\left[-\dot q_j^*(t)+f_{j-1}\left(\rho_{j-1}^*\left(t,b_{j-1}\right)\right)-V_jg_j^*(t)\right]\phi(t)\,dt=0,\ \forall\,\phi\in C_0^\infty(\mathbb{R}). \tag{51}$$

It remains to pass to the limit in the entropy inequality 38 which is a weak formulation of the boundary conditions. To this end, we can apply the same trick as we did it in 48. As a result, by analogy with 48, it can be shown that

$$\lim_{k\to\infty}f_j\big(\rho_j^k\left(t,a_j\right)\big)=f_j\big(\rho_j^*\left(t,a_j\right)\big),$$

and, therefore, the limit passage in 38 leads us to the relation

$$\begin{aligned}
&\nu_j\big(\rho_j^*\left(t,a_j\right)\big)-\nu_j\left(g_j^*(t)\right)\\
&\quad -D\eta_j\left(g_j^*(t)\right)\left(f_j\big(\rho_j^*\left(t,a_j\right)\big)-f_j\left(g_j^*(t)\right)\right)\le 0 \text{ a.e. in } \ (0,T),\ j=\overline{1,N}.
\end{aligned} \tag{52}$$

Thus, summing up the relations 44–52 with condition $u^*\in U_\partial$, we finally conclude: the pair (ρ^*,q^*) is an entropy solution to the system 12 for $u=u^*$ and, hence, by Theorem 5.4,

$$(u^*,\rho^*,q^*)\in U_\partial\times\prod_{j=1}^N\Big(C^{0,1}([0,T];L^1(I_j))\cap L^\infty(0,T;BV(I_j))\Big)\times\prod_{j=2}^N W^{1,1}(0,T).$$

Thus, the limit triple (u^*,ρ^*,q^*) is an admissible solution to the OCP 15.

To conclude the proof, it remains to show the lower semicontinuity property for the cost functional, i.e.

$$\liminf_{k\to\infty} J(u^k,\rho^k,q^k) \geq J(u^*,\rho^*,q^*).$$

To this end, we make use of the following chain of transformations. Let $\varphi \in C^1(\overline{I}_N)$ be an arbitrary test function on $\overline{I}_N = [a_N, b_N]$ such that $\varphi(a_N) = 0$ and $\varphi(b_N) = 1$. Then, using generalized Green's formula 24 and convergence property 35, we get

$$\begin{aligned}
&\inf_{(u,\rho,q)\in\Xi} J(u,\rho,q) = \lim_{k\to\infty} J(u^k,\rho^k,q^k)\\
&\quad= \lim_{k\to\infty}\sum_{j=2}^{N}\int_0^T \alpha_j(t)q_j^k(t)\,dt + \lim_{k\to\infty}\int_0^T \beta(t)\Big|V_N\rho_N^k(t,b_N)-\psi(t)\Big|\,dt\\
&\quad\overset{\text{by } 33}{=} \sum_{j=2}^{N}\int_0^T \alpha_j(t)q_j^*(t)\,dt + \lim_{k\to\infty}\int_0^T \beta(t)\Big|V_N\rho_N^k(t,b_N)-\psi(t)\Big|\,dt\\
&\overset{\text{by } 24}{=} \lim_{k\to\infty}\int_0^T \Big|\beta(t)V_N\int_{I_N}\varphi D\rho_N^k + V_N\beta(t)\int_{a_N}^{b_N}\varphi'(x)\rho_N^k(t,x)\,dx - \beta(t)\psi(t)\Big|\,dt\\
&\qquad\qquad + \sum_{j=2}^{N}\int_0^T \alpha_j(t)q_j^*(t)\,dt \overset{\text{by } 34\text{–}35}{=} \sum_{j=2}^{N}\int_0^T \alpha_j(t)q_j^*(t)\,dt\\
&\quad + \int_0^T \Big|\beta(t)V_N\int_{I_N}\varphi D\rho_N^* + V_N\beta(t)\int_{a_N}^{b_N}\varphi'(x)\rho_N^*(t,x)\,dx - \beta(t)\psi(t)\Big|\,dt\\
&\overset{\text{by } 24}{=} \sum_{j=2}^{N}\int_0^T \alpha_j(t)q_j^*(t)\,dt + \int_0^T \beta(t)\Big|V_N\rho_N^*(t,b_N)-\psi(t)\Big|\,dt = J(u^*,\rho^*,q^*).
\end{aligned} \tag{53}$$

Thus, (u^*,ρ^*,q^*) is an optimal solution for the problem 15. □

7. **Henig relaxation of state-constrainted OCP 15.** As follows from Theorem 6.1, the existence of optimal solutions to the problem 15 can be obtained by using compactness arguments and the regularity assumption (A1). However, because of the state constraints $\rho \in \Upsilon_\partial$ the regularity of the OCP 15 (see (A1)) is an open question even for the simplest situation. As was mentioned above, the pointwise inequality constraints

$$\mathcal{B}_j(\rho_j)(t,x) \leq \zeta_j^{max}(t,x) \quad \text{a.e. in } \ \Omega_j, \quad \forall\, j = 1,\dots,N$$

can be equivalently rewritten as $\zeta_j^{max} - \mathcal{B}_j(\rho_j) \in L^2_+(\Omega_j)$ for all $j = 1,\dots,N$, where $L^2_+(\Omega_j)$ stands for the natural ordering cone of positive elements in $L^2(\Omega_j)$. From practical point of view it means that we cannot apply to the OCP 15 any constraint qualifications like the Slater condition or the Robinson condition because each of those approaches is essentially based on non-emptiness of the interiors of the ordering cones $L^2_+(\Omega_j)$. However, in our case we have $\operatorname{int}(L^2_+(\Omega_j)) = \emptyset$ for all $j = 1,\dots,N$. Therefore, the main goal of this section is to provide a regularization of the pointwise state constraints by replacing the ordering cones $\Lambda^j := L^2_+(\Omega_j)$ by their solid Henig approximations $(\Lambda^j)_\varepsilon(D_j)$ (see [27]) and show that admissibility and solvability of OCP 15 can be characterized by solving the corresponding Henig relaxed problems in the limit $\varepsilon \to 0$. Here,

$$D_j := \Big\{\xi \in L^2_+(\Omega_j) \Big| \int_0^T \int_{a_j}^{b_j} \xi(t,x)\,dx\,dt = 1\Big\} \text{ is a closed base of } \Lambda^j, \tag{54}$$

for all $j = 1, \dots, N$.

For given $j = 1, \dots, N$, base D_j, and $\varepsilon > 0$, we adopt the following concept.

Definition 7.1 ([27]). We say that $(\Lambda^j)_\varepsilon(D_j)$ is the Henig dilating cone if for every $j = 1, \dots, N$ it is defined as follows

$$(\Lambda^j)_\varepsilon(D_j) := \mathrm{cl}\,(\mathrm{cone}\,(D_j + B_\varepsilon(0))) := \mathrm{cl}\,(\{\lambda\xi \ :\ \lambda \geq 0, \xi \in D_j + B_\varepsilon(0)\}),$$

where $\frac{1}{\varepsilon}B_\varepsilon(0) := \big\{\xi \in L^2(\Omega_j) \ :\ \|\xi\|_{L^2(\Omega_j)} \leq 1\big\}$ is the closed unit ball in $L^2(\Omega_j)$.

As follows from this definition, $\mathrm{int}\,(\Lambda^j)_\varepsilon(D_j) \neq \emptyset$ for every $\varepsilon > 0$, i.e. Henig dilating cones are proper solid. Moreover, we have the following properties of such cones (see [26, 27]).

Proposition 1. *For a fixed $j = 1, \dots, N$, let D_j be the set defined by 54. Then choosing $\varepsilon \in (0, \delta)$, where $\delta := \inf\big\{\|\xi\|_{L^2(\Omega_j)} \ :\ \xi \in D_j\big\}$, the following statements hold true:*

(i) $(\Lambda^j)_\varepsilon(D_j)$ *is a pointed cone, i.e.* $(\Lambda^j)_\varepsilon(D_j) \cap \big(-(\Lambda^j)_\varepsilon(D_j)\big) = \{0\}$;
(ii) $(\Lambda^j)_\varepsilon(D_j) \subset (\Lambda^j)_{\varepsilon+\gamma}(D_j)$ *for all* $\gamma > 0$;
(iii) $(\Lambda^j)_\varepsilon(D_j) = \mathrm{cone}\,\big(\mathrm{cl}\, L^2_+(\Omega_j)\big)$;
(iv) $L^2_+(\Omega_j) = \bigcap_{0<\varepsilon<\delta}(\Lambda^j)_\varepsilon(D_j)$;
(v) $L^2_+(\Omega_j) \subset \{0\} \cup \mathrm{int}\,\big((\Lambda^j)_\varepsilon(D_j)\big)$;
(vi) The sequence of Henig dilating cones $\big\{(\Lambda^j)_\varepsilon(D_j)\big\}_{\varepsilon>0}$ *converges to the cone of positive elements* $L^2_+(\Omega_j)$ *in Kuratowski sense with respect to the norm topology of* $L^2(\Omega_j)$ *as* $\varepsilon \to 0$;
(vii) The implication

$$\xi \in (\Lambda^j)_\varepsilon(D_j) \quad \Longrightarrow \quad \frac{\varepsilon}{\kappa+\varepsilon}\|\xi\|_{L^2(\Omega_j)} + \xi \notin L^2_-(\Omega_j), \tag{55}$$
$$i.\ e.\ \ \xi \not<_{\Lambda^j} -\frac{\varepsilon}{\kappa+\varepsilon}\|\xi\|_{L^2(\Omega_j)}$$

holds true with $\kappa = \sup\big\{\|\xi\|_{L^2(\Omega_j)} \ :\ \xi \in D_j\big\}$.

Taking these results into account, we associate with OCP 15 the following Henig relaxed problem

$$\inf_{(u,\rho,q)\in\Xi_\varepsilon} J(u,\rho,q), \quad \forall\, \varepsilon \in (0,\delta) \tag{56}$$

where

$$\delta = \min_{j=1,\dots,N}\Big[\inf\big\{\|\xi\|_{L^2(\Omega_j)} \ :\ \xi \in D_j\big\}\Big], \tag{57}$$

and we define the set of admissible solutions

$$\Xi_\varepsilon \subset BV(0,T) \times \prod_{j=1}^N C\left([0,T]; L^1(I_j)\right) \times \prod_{j=2}^N W^{1,1}(0,T)$$

as follows: $(u,\rho,q) \in \Xi_\varepsilon$ if and only if $u \in \mathcal{U}_\partial$, $J(u,\rho,q) < +\infty$, the pair $(\rho,q) = (\rho(u), q(u))$ is an entropy solution to 12 in the sense of Definition 5.3, and

$$\rho \in \Upsilon^\varepsilon_\partial = \left\{ \rho = [\rho_1, \dots, \rho_N]^t \ \middle|\ \begin{array}{l} \zeta_j^{max} - \mathcal{B}_j(\rho_j) \in (\Lambda^j)_\varepsilon(D_j), \\ \rho_j \in L^2_+(\Omega_j),\ j = 1, 2, \dots, N. \end{array} \right\} \tag{58}$$

Since, by property (v) (see Proposition 1), the inclusion $\Xi \subseteq \Xi_\varepsilon$ holds true for all $\varepsilon > 0$, it is reasonable to call the OCP 56 a relaxation of OCP 15. Moreover, as obviously follows from Proposition 1 (see property (vi)), the convergence $\Xi_\varepsilon \stackrel{\varepsilon\to 0}{\to} \Xi$ in Kuratowski sense holds true with respect to the topology on $BV(0,T)\times \prod_{j=1}^N L^\infty(0,T;BV(I_j))\times\prod_{j=2}^N W^{1,1}(0,T)$ given by relations 30–35.

We are now in a position to show that using the relaxation approach we can reduce the main suppositions of Theorem 6.1. In particular, we characterize Hypothesis (A1) by the regularity properties of the corresponding Henig relaxed problems.

Theorem 7.2. *Let $\{\varepsilon_k\}_{k\in\mathbb{N}}\subset(0,\delta)$ be a monotonically decreasing sequence converging to 0 as $k\to\infty$. Then for given initial data 27–29 and given family of bounded compact operators $\left\{\mathcal{B}_j : L^\infty(0,T;BV(I_j))\to L^2(\Omega_j)\right\}_{j=1}^N$ the Henig relaxed problems 56 are regular for all $\varepsilon=\varepsilon_k$, $k\in\mathbb{N}$ if and only if Hypothesis (A1) is satisfied. Moreover, for any sequence $\left\{(u^k,\rho^k,q^k)\right\}_{k\in\mathbb{N}}$ satisfying condition $(u^k,\rho^k,q^k)\in\Xi_{\varepsilon_k}$ for all $k\in\mathbb{N}$, there is a subsequence $\left\{(u^{k_i},\rho^{k_i},q^{k_i})\right\}_{i\in\mathbb{N}}$ of $\left\{(u^k,\rho^k,q^k)\right\}_{k\in\mathbb{N}}$ such that*

$$(u^{k_i},\rho^{k_i},q^{k_i}) \stackrel{i\to\infty}{\longrightarrow} (u,\rho,q) \quad \text{in the sense of 30–35, and} \quad (u,\rho,q)\in\Xi.$$

Proof. Since the implication $\big(\Xi\neq\emptyset\big)\Longrightarrow\big(\Xi_\varepsilon\neq\emptyset \text{ for all } \varepsilon>0\big)$ is obvious by Proposition 1 (see property (vi)), we concentrate on the proof of the inverse statement — regularity of the Henig relaxed problems $\inf_{(u,\rho,q)\in\Xi_{\varepsilon_k}} J(u,\rho,q)$ for all $k\in\mathbb{N}$ implies the existence of at least one triple (u,ρ,q) such that $(u,\rho,q)\in\Xi$.

Let $\left\{(u^k,\rho^k,q^k)\right\}_{k\in\mathbb{N}}$ be an arbitrary sequence with property: $(u^k,\rho^k,q^k)\in\Xi_{\varepsilon_k}$ for all $k\in\mathbb{N}$. Since the set U_∂ and a priory estimates 21–23 do not depend on parameter ε_k, it follows by compactness arguments (see the proof of Theorem 6.1) that there exist a subsequence of $\left\{(u^k,\rho^k,q^k)\right\}_{k\in\mathbb{N}}$ (still denoted by the same index) and a triplet

$$(u^*,\rho^*,q^*)\in U_\partial\times\prod_{j=1}^N L^\infty(0,T;BV(I_j))\times\prod_{j=2}^N W^{1,1}(0,T)$$

such that $(u^k,\rho^k,q^k)\stackrel{k\to\infty}{\longrightarrow}(u^*,\rho^*,q^*)$ in the sense of 30–35. Moreover, in view of the compactness properties of operators $\mathcal{B}_j : L^\infty(0,T;BV(I_j))\to L^2(\Omega_j)$, we may suppose that

$$\xi^k := \zeta_j^{max}-\mathcal{B}_j(\rho_j^k)\to\zeta_j^{max}-\mathcal{B}_j(\rho_j^*) \quad \text{strongly in } \ L^2(\Omega_j),\ \forall\, j=1,\dots,N. \tag{59}$$

Closely following the proof of Theorem 6.1, it can be shown that the limit triplet (u^*,ρ^*,q^*) is such that $u^*\in\mathcal{U}_\partial$, $J(u^*,\rho^*,q^*)<+\infty$, and the pair $(\rho^*,q^*)=(\rho^*(u^*),q^*(u^*))$ is an entropy solution to 12 in the sense of Definition 5.3. It remains to prove that $\rho^*\in\Upsilon_\partial$, i.e. we have to establish the inclusions

$$\zeta_j^{max}-\mathcal{B}_j(\rho_j^*)\in L^2_+(\Omega_j), \quad \forall\, j=1,\dots,N. \tag{60}$$

By contraposition, let us assume that $\xi^*:=\zeta_j^{max}-\mathcal{B}_j(\rho_j^*)\in L^2(\Omega_j)\setminus L^2_+(\Omega_j)$ for some $j\in\{1,\dots,N\}$ (here, we apply the arguments of R. Schiel from [26]). Since the cone $L^2_+(\Omega_j)$ is closed, it follows that there is a neighborhood $\mathcal{N}(\xi^*)$ of ξ^* in $L^2(\Omega_j)$ such that $\mathcal{N}(\xi^*)\cap L^2_+(\Omega_j)=\emptyset$. Using the fact that

$$L^2_+(\Omega_j)\subset(\Lambda^j)_{\varepsilon_k}(D_j)\subseteq(\Lambda^j)_{\varepsilon_l}(D_j), \quad \forall\, k\geq l$$

and using the property (vi) of Proposition 1, it is easy to conclude from definition of the Kuratowski limit the existence of an index $k_0 \in \mathbb{N}$ such that

$$\mathcal{N}(\xi^*) \cap (\Lambda^j)_{\varepsilon_k}(D_j) = \emptyset, \quad \forall\, k \geq k_0. \tag{61}$$

However, in view of the strong convergence property 59, there is an index $k_1 \in \mathbb{N}$ satisfying

$$\xi^k \in \mathcal{N}(\xi^*), \quad \forall\, k \geq k_1. \tag{62}$$

Combining 61 and 62, we finally obtain

$$\xi^k = \zeta_j^{max} - \mathcal{B}_j(\rho_j^k) \in L^2(\Omega_j) \setminus (\Lambda^j)_{\varepsilon_k}(D_j), \quad \forall\, k \geq \max\{k_0, k_1\}.$$

This, however, is a contradiction to

$$\rho_j^k \in \Upsilon_\partial^{\varepsilon_k}, \quad \forall\, k \in \mathbb{N}.$$

Thus, $\zeta_j^{max} - \mathcal{B}_j(\rho_j^*) \in L_+^2(\Omega_j)$, i.e. $\rho^* \in \Upsilon_\partial$, and hence, the triplet (u^*, ρ^*, q^*) is admissible for the OCP 15. □

Next result is crucial in this article. We show that optimal solutions for the original OCPs 15 can be attained by solving the corresponding Henig relaxed problems 56.

Theorem 7.3. *Assume that the initial data for OCP 15 are given as in Theorem 6.1. Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \delta)$ be a monotonically decreasing sequence such that $\varepsilon_k \to 0$ as $k \to \infty$, where $\delta > 0$ is defined by 57. Let $\left\{(u^{k,0}, \rho^{k,0}, q^{k,0}) \in \Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ be a sequence of optimal solutions to the Henig relaxed problems 56. Then there is a subsequence $\left\{(u^{k_i,0}, \rho^{k_i,0}, q^{k_i,0})\right\}_{i\in\mathbb{N}}$ of $\left\{(u^{k,0}, \rho^{k,0}, q^{k,0})\right\}_{k\in\mathbb{N}}$ and a triple (u^0, ρ^0, q^0) such that*

$$(u^{k_i,0}, \rho^{k_i,0}, q^{k_i,0}) \overset{i\to\infty}{\longrightarrow} (u^0, \rho^0, q^0) \quad \text{in the sense of 30–35}, \tag{63}$$

$$(u^0, \rho^0, q^0) \in \Xi, \quad \text{and} \quad J(u^0, \rho^0, q^0) = \inf_{(u,\rho,q)\in\Xi} J(u, \rho, q). \tag{64}$$

Proof. Since the compactness property 63 and the inclusion $(u^0, \rho^0, q^0) \in \Xi$ are a direct consequence of Theorem 7.2, it remains to show that the limit triplet (u^0, ρ^0, q^0) is a solution to the OCP 15. Indeed, the condition $(u^0, \rho^0, q^0) \in \Xi$ implies regularity of the original OCP 15. Hence, by Theorem 6.1, this problem has a nonempty set of solutions. Let (u^*, ρ^*, q^*) be one of them. Then the following inequality is obvious

$$J(u^*, \rho^*, q^*) \leq J(u^0, \rho^0, q^0), \quad \forall\, i \in \mathbb{N}. \tag{65}$$

On the other hand, by Proposition 1 (see property (iv)), we have $(u^*, \rho^*, q^*) \in \Xi_{\varepsilon_{k_i}}$ for every $i \in \mathbb{N}$. Since $\left\{(u^{k_i,0}, \rho^{k_i,0}, q^{k_i,0})\right\}_{i\in\mathbb{N}}$ are the solutions to the corresponding relaxed problems 56, it follows that

$$\inf_{(u,\rho,q)\in\Xi_{\varepsilon_{k_i}}} J(u, \rho, q) = J(u^{k_i,0}, \rho^{k_i,0}, q^{k_i,0}) \leq J(u^*, \rho^*, q^*), \quad \forall\, i \in \mathbb{N}. \tag{66}$$

As a result, taking into account the relations 65 and 66, and the continuity property 53 for the cost functional J with respect to the convergence 30–35, we finally get

$$\begin{aligned}\inf_{(u,\rho,q)\in\Xi} J(u,\rho,q) = J(u^*,\rho^*,q^*) &\overset{\text{by } 66}{\geq} \limsup_{i\to\infty} J(u^{k_i,0},\rho^{k_i,0},q^{k_i,0}) \\ &\geq \liminf_{i\to\infty} J(u^{k_i,0},\rho^{k_i,0},q^{k_i,0}) \overset{\text{by } 53 \text{ and } 63}{=} J(u^0,\rho^0,q^0) \\ &\overset{\text{by } 65}{\geq} J(u^*,\rho^*,q^*).\end{aligned}$$

Thus,

$$\inf_{(u,\rho,q)\in\Xi} J(u,\rho,q) = \lim_{i\to\infty} J(u^{k_i,0},\rho^{k_i,0},q^{k_i,0}) = J(u^0,\rho^0,q^0),$$

and we arrive at the desired property 64_2. The proof is complete. □

8. Final remarks.

As was mentioned in the previous section, the main benefit of the relaxed optimal control problems 56 comes from the fact that the Henig dilating cones $(\Lambda^j)_\varepsilon(D_j)$ have a nonempty topological interior. Hence, it gives a possibility to apply the Slater condition or the Robinson condition in order to characterize the optimal solutions for state constrained OCP. On the other hand, this approach provides nice convergence properties for the solutions of relaxed problems 56. However, as follows from Theorems 7.2 and 7.3, the most restrictive assumption deals with the regularity of the relaxed problems 56 for all $\varepsilon \in (0,\delta)$. So, if we reject the Hypothesis (A1), it becomes unclear, in general, whether the relaxed sets of admissible solutions Ξ_ε are nonempty for all $\varepsilon \searrow 0$. In this case it makes sense to provide further relaxation for each of Henig problems 56. In particular, using the methods of vector-valued optimization theory, it can be shown that each of Henig problems 56 admits the existence of the so-called weakened approximate solution which can be interpreted as the generalized solution to some vector optimization problem of special form (for the details we refer to [7, 22]). In the meantime, the question of numerical treatment of Henig relaxation for the PDE-ODE supply problems, however, remains open in general.

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