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FINITE MECHANICAL PROXIES FOR A CLASS OF REDUCIBLE CONTINUUM SYSTEMS

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Abstract. We present the exact finite reduction of a class of nonlinearly perturbed wave equations –typically, a non-linear elastic string– based on the Amann–Conley–Zehnder paradigm. By solving an inverse eigenvalue problem, we establish an equivalence between the spectral finite description derived from A–C–Z and a discrete mechanical model, a well definite finite spring–mass system. By doing so, we decrypt the abstract information encoded in the finite reduction and obtain a physically sound proxy for the continuous problem.

1. Introduction.

Since the dawn of analytical mechanics, the behavior of continuum materials has been modeled by observing a large collections of small particles, in the limit when the number of elements approaches infinity. The archetypal example, the equations for the large vibrations of the one dimensional elastic string in the plane, was first derived by Euler in 1744 by considering a chain of N beads connected by springs of length $L/(N-1)$ and by letting $N \to \infty$ keeping both total mass and length L constant.[1] Discrete systems converging to continua occur also for other physical problems. The Sine-Gordon equation is obtained as a thermodynamic limit of a finite discrete system, the Frenkel–Kontorova model, as the number of discrete components goes to infinity [27]. Zabusky and Kruskal [46] studied the continuum limit of the Fermi-Pasta-Ulam model and obtained the Korteweg-de Vries equation, previously derived [30] for describing weakly nonlinear shallow water waves.[2]

Also the applications to continuous problems in scientific calculus are based on an analogous limit of a system of finite elements towards an infinite system. The mesh size is chosen according to the desired approximation accuracy, or more often to the available computational resources.

Of course, there are plenty of successful physical and engineering applications, however, one cannot conclude that every solution of a continuum physical problem can be approximated consistently with the solutions of a suitable discretized problem. Indeed, as observed in [6], even though one can deduce the one dimensional wave equation as the limit of a sequence of discrete ODEs, each one consisting in a

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[1]See [6, Chapter 2.3, page 25].

[2]We point out also recent applications of continuum limits to other physical domains [18, 19, 35].

chain of beads connected with springs, where the total mass and the length are the same, it does not follow that the solutions of the latter converge to solutions of the former in some physically reasonable sense.

Originally, in a view firstly proposed by Von Neumann [44], it seemed that the solutions for the positions of the beads in the ODEs, together with their time derivatives and suitable difference quotients would converge respectively to the position, velocity and strain fields for the PDE. In fact, this convergence is valid where classical smooth solutions for the PDE exist.

On the other hand, where velocity and strain suffer jump discontinuities, acceleration waves or shocks, the solutions of the discrete problem develop high–frequency oscillations that persist in the limit as the number of particles N tend to infinity. Consequently the limiting stress results incorrect.[3]

In this paper we actually do not come across this difficulty, in a sense, it is overcome in an inverse direction: we will consider a class of nonlinear wave equations for which we will define a totally equivalent discrete nonlinear system composed by beads and springs. We will avoid possible problems with discontinuities in the PDE solutions because we will not rely on thermodynamical limits. Actually, because we consider the continuous problem (the PDE) as a starting point, and then we attain an exactly equivalent ODE system, we will not encounter approximation errors, as it will be described below in details.

The approach and the results proposed here rely on the Amann–Conley–Zehnder reduction (ACZ in what follows), a global Lyapunov–Schmidt technique [1, 2, 3, 4, 20, 21] which transforms infinite dimensional variational principles into equivalent finite dimensional functionals. This method has been employed in conjunction to topological techniques, e.g., Conley index, Morse theory, Lusternik–Schnirelmann category and degree theory, for proving results of existence and multiplicity of solutions for nonlinear differential equations, in particular for semilinear Dirichlet problems, Hamiltonian systems and nonlinear wave equations [8, 9, 10, 13, 15, 16, 22, 23, 24, 31, 33, 36, 39, 40, 42].

It has been pointed out [25], that the (finite number of) parameters involved in the above exact reduction scheme can be regarded as a sort of *collective variables*, which represent a rather known and strengthened way to describe, in molecular dynamics and in other allied fields, complex phenomena occurring on distinct space-time scales inside systems with infinite degrees of freedom. The different scales correspond to the various ways we may watch to the system, for example by observing either local variables which depend only on a few degrees of freedom, or else collective variables which describe the global behavior as a whole. This fruitful point of view has been developed since some years and it is utilized in many branches; we recall, e.g., some references in statistical mechanics: [11, 34, 45].

The nonlinear elastic string, when processed with ACZ, produces a nonlinear discrete system composed by (i) a finite set of uncoupled harmonic oscillators, plus (ii) an overall nonlinear coupling term. The finite variables μ_k can be clearly interpreted as collective variables, but are of spectral nature and completely abstract. Nevertheless, there exists a privileged coordinate transformation which restores a strong physical meaning to the reduced system. Making use of well established results about inverse eigenvalue problems[4], the linear core of the reduced system, i.e.,

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[3] Thorough discussion on this issue can be found in [29].

[4] See [7, 26, 28, 37] and the references therein.

the set of uncoupled linear oscillators, is transformed in an elastic chain of beads and springs, having precisely the same spectral representation.

This theoretical excursion is summarized in Figure 1. By doing so we think to provide a further motivation for the ACZ reduction coming from the microphysics of the matter.

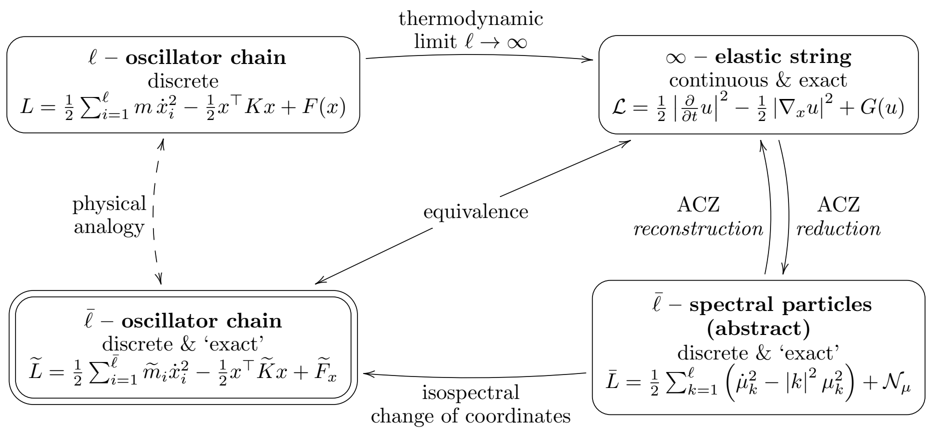


FIGURE 1. A suitable change of variables gives a physically sound interpretation to the ACZ reduction for a class of nonlinear wave equations.

2. **Preliminaires: Exact reductions in stationary field theory.** For the sake of clarity, we preliminarily present the ACZ exact reduction technique in the simpler case of a static semilinear elliptic problem, inspired by elastostatics. We want to find a function $u(x) \in H := H_0^1(\Omega)$, where $x \in \Omega \subseteq \mathbb{R}^n$ is a Stokes domain (i.e., with a piecewise smooth boundary) satisfying the following nonlinearly perturbed elliptic equation:

$$\begin{cases} -\Delta u = F(u), & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

As nonlinear source term F we consider a Nemitski operator

$$F : H \longrightarrow H, \qquad u \longmapsto F(u), \tag{2}$$

$$F(u)(x) := f \circ u(x), \tag{3}$$

associated to a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ with constant C:

$$\|F(u_1) - F(u_0)\| \leqslant C\,\|u_1 - u_0\|\,.$$

By means of the Amann–Conley–Zehnder reduction, a completely equivalent *algebraic* equation can be obtained (9). First of all, a spectral decomposition of $H_0^1(\Omega)$ w.r.t. $-\Delta$ is applied, and the solutions of the Dirichlet problem (1) can be represented in the countably infinite dimensional space $\ell^2(\Omega)$,

$$H \ni u = u_1\hat{u}_1 + \cdots + u_j\hat{u}_j + u_{j+1}\hat{u}_{j+1} + \ldots,$$

where the $\hat{u}_j$ are the eigenvectors of the Laplace operator in Ω:

$$-\Delta\hat{u}_j = \lambda_j\hat{u}_j, \qquad 0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \qquad \forall j = 1, 2, \ldots.$$

The spectral decomposition allows to write the inverse operator of the Laplacian:

$$g : H \longrightarrow H, \qquad \sum u_j \hat{u}_j \longmapsto \sum \frac{u_j}{\lambda_j} \hat{u}_j,$$
$$-\Delta \circ g(u) = g \circ (-\Delta)(u) = u.$$

The equation (1) can be rewritten setting $u = g(v)$,

$$v = F(g(v)), \qquad v \in H, \qquad v = \sum_{j=1}^{\infty} v_j \hat{u}_j,$$

and a splitting into a finite dimensional *core* and an infinite dimensional *tail* can be performed for every cutoff $\ell \in \mathbb{N}$, applying the projection operators derived from the spectral decomposition:

$$v = v_1 \hat{u}_1 + v_2 \hat{u}_2 + \cdots = \tag{4}$$
$$= \mathbb{P}_\ell v + \mathbb{Q}_\ell v = \mu + \eta = \tag{5}$$
$$:= \underbrace{\mu_1 \hat{u}_1 + \cdots + \mu_\ell \hat{u}_\ell}_{\mu} + \underbrace{\eta_{\ell+1} \hat{u}_{\ell+1} + \dots}_{\eta} \tag{6}$$

$$\begin{cases} \mu = \mathbb{P}_\ell F(g(\mu + \eta)), & \text{finite core} \\ \eta = \mathbb{Q}_\ell F(g(\mu + \eta)), & \text{infinite tail} \end{cases} \tag{7}$$

The key point now is to show that for sufficiently large values of ℓ the infinite part of the equation is always uniquely solved for every fixed finite part μ. This is proved if we can show that the operator

$$\eta \longmapsto \mathbb{Q}_\ell F(g(\mu + \eta))$$

is contractive for ℓ sufficiently large. Indeed,

$$\begin{aligned} \|\mathbb{Q}_\ell F(g(\mu + \eta_1)) - \mathbb{Q}_\ell F(g(\mu + \eta_2))\| \leqslant \|F(g(\mu + \eta_1)) - F(g(\mu + \eta_2))\| \leqslant \\ \leqslant C \|g(\mu + \eta_1) - g(\mu + \eta_2)\| \leqslant \frac{C}{\lambda_{\ell+1}} \|\eta_1 - \eta_2\|. \end{aligned} \tag{8}$$

We have $\frac{C}{\lambda_{\ell+1}} < 1$ for ℓ suitably large because $\lambda_\ell \to \infty$.

Thus we can denote by $\widetilde{\eta}(\mu)$ the unique fixed point of the previous contraction, and we can (at least formally) substitute in the finite equation, which definitively does represent the very equation of our problem, the determination of $\mu = (\mu_1, \dots, \mu_\ell)$:

$$\mu = \mathbb{P}_\ell F(g(\mu + \widetilde{\eta}(\mu))), \qquad (\mu_1, \dots, \mu_\ell) \in \mathbb{R}^\ell. \tag{9}$$

2.1. **Application of the reduction to the variational principle in the static case.** By applying the Volterra–Vainberg Theorem [5, 41, 43] one can associate to the original Dirichlet problem (1) a variational principle, an Euler–Lagrange functional:

$$J[u(\cdot)] := \int_{t=0}^{t=1} \langle -\Delta(tu) - F(tu), u \rangle \, dt = \int_\Omega \Big[\frac{1}{2} |\nabla u|^2 - \int_0^u f(s) ds \Big] dx \tag{10}$$

which critical points $dJ[u] = 0$ are the (weak) solutions of (1).

The reduction applies as well to the variational principle, simply substituting u by $g(\mu + \widetilde{\eta}(\mu))$, obtaining a finite dimensional functional:

$$I : \mathbb{R}^\ell \longrightarrow \mathbb{R}, \qquad I(\mu) := J[g(\mu + \widetilde{\eta}(\mu))]. \tag{11}$$

Indeed one can show this straightforwardly obtained functional is the energy functional of the finite dimensional equation (9), in the sense that the respective solution sets coincide:

$$dI(\mu) = 0 \quad \Longleftrightarrow \quad \mu = \mathbb{P}_\ell F(g(\mu + \widetilde{\eta}(\mu))). \tag{12}$$

As already observed in the introduction, this construction lends itself naturally to topological methods, which can be exploited in a very fruitful way in finite dimensional spaces for obtaining existence and multiplicity of solutions results. More details on the ACZ reduction for the static nonlinear elasticity can be found in [14, 15, 17].

3. **A perturbed wave equation.** Consider the problem of finding T-periodic solutions for the 1-dimensional nonlinear wave equation:

$$\begin{cases} \Box u = F(u), & t \in \mathbb{R}, 0 \leqslant x \leqslant \pi, \\ u(t,x) = u(t+T,x), \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{13}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ is the D'Alembert wave operator with unitary phase velocity. We assume hereafter $T = 2\pi$, but we point out that the results presented in what follows can be obtained also if the period is a rational multiple of 2π.

We consider the functional framework in [38], indeed let

$$C_0^\infty = \Big\{ \varphi(t,x) \in C^\infty(\mathbb{R} \times [0,\pi], \mathbb{R}) \Big| \; \varphi(t+2\pi, x) = \varphi(t,x), \\ \varphi(t,0) = \varphi(t,\pi) = 0 \Big\} \tag{14}$$

and let H_0 be the completion of C_0^∞ with respect to the space-time L^2-norm:

$$\|\varphi\|^2 = \langle \varphi, \varphi \rangle = \int_0^{2\pi} \int_0^\pi |\varphi(t,x)|^2 \, dx dt. \tag{15}$$

For the period 2π, or any rational multiple of 2π, the D'Alembertian has a nontrivial kernel N. Indeed, any function of the form

$$u(t,x) := p(t+x) - p(t-x), \tag{16}$$

where p is 2π periodic (and $\int_0^{2\pi} |p(a)|^2 \, da < +\infty$) is a solution of $\Box u = 0$. The kernel N can be characterized also in terms of Fourier series,

$$N = \left\{ \varphi \in H_0 \Big| \; \varphi(t,x) = \sum_{k=-\infty}^{+\infty} a_k \sin(kx) e^{ikt} \quad \text{with} \quad \sum_{k=-\infty}^{+\infty} |a_k|^2 < +\infty \right\}, \tag{17}$$

while its orthogonal complement $N^\perp$ can be written as

$$N^\perp = \left\{ \varphi \in H_0 \Big| \; \varphi(t,x) = \sum_{\substack{k=1 \\ k \neq |n|}}^{\infty} \sum_{n=-\infty}^{+\infty} a_{k,n} \sin(kx) e^{int}, \quad \sum_{k,n} |a_{k,n}|^2 < +\infty \right\}. \tag{18}$$

In place of the original problem (13) we consider the problem restricted on $N^\perp$,

$$\Box u = F(u), \qquad u \in N^\perp, \tag{19}$$

where F is a nonlinear operator

$$F : N^\perp \to N^\perp, \qquad \text{with global Lipschitz constant } C. \tag{20}$$

The restricted problem appears in [12, 22, 38] among others. Adopting this format the kernel of the D'Alembertian becomes trivial and the existence of a bounded inverse $\Box^{-1}$ is assured. In this way it is possible for instance to relax monotonicity conditions on F and obtain existence results as well [22].[5]

Indeed, consider the linear problem in $N^\perp$,

$$\Box w = f, \qquad f(t,x) = \sum_{\substack{k=1 \\ k\neq|n|}}^{\infty} \sum_{n=-\infty}^{+\infty} f_{k,n} \sin(kx) e^{int}. \tag{21}$$

It is straightforward to check that

$$w = \Box^{-1} f = \sum_{k\neq|n|} w_{k,n} \sin(kx) e^{int} = \sum_{k\neq|n|} \frac{f_{k,n}}{k^2-n^2} \sin(kx) e^{int}, \tag{22}$$

is the unique solution in $N^\perp$. The original nonlinear problem can therefore be rewritten in the form of a fixed point problem from $N^\perp$ into itself:

$$u \longmapsto \Box^{-1} F(u). \tag{23}$$

For fixed $\ell \in N$, let us consider the following orthogonal projections involving the space coordinate x,

$$\mathbb{P}_\ell \varphi = \sum_{\substack{k\leqslant\ell, \\ k\neq|n|}} \varphi_{k,n} \sin(kx) e^{int} \qquad \mathbb{Q}_\ell \varphi = \sum_{\substack{k>\ell, \\ k\neq|n|}} \varphi_{k,n} \sin(kx) e^{int}, \tag{24}$$

$N^\perp = \mathbb{P}_\ell N^\perp \oplus \mathbb{Q}_\ell N^\perp$, associated to a splitting of the problem (19):

$$\begin{cases} \mu = \mathbb{P}_\ell \Box^{-1} F(\mu+\eta), \\ \eta = \mathbb{Q}_\ell \Box^{-1} F(\mu+\eta). \end{cases} \tag{25}$$

Proposition 1. *If F is globally Lipschitz, the following map*

$$\mathbb{Q}_\ell N^\perp \longrightarrow \mathbb{Q}_\ell N^\perp \tag{26}$$

$$\eta \longmapsto \mathbb{Q}_\ell \Box^{-1} F(\mu+\eta), \tag{27}$$

is a contraction for ℓ sufficiently large and for every fixed $\mu \in \mathbb{P}_\ell N^\perp$.

Proof. Let $\varphi^{(\iota)} = F(\mu+\eta_\iota)$, $\iota = 1,2$,

$$\varphi^{(\iota)}(t,x) = \sum_{\substack{k\in\mathbb{N}^+,\ n\in\mathbb{Z} \\ k\neq|n|}} \varphi^{(\iota)}_{k,n} \sin(kx) e^{int}. \tag{28}$$

Then the contractivity of (27) holds for $2\ell+1 > C$, indeed:

$$\left\| \mathbb{Q}_\ell \Box^{-1} F(\mu+\eta_1) - \mathbb{Q}_\ell \Box^{-1} F(\mu+\eta_2) \right\| \;=\; \left\| \sum_{\substack{k>\ell \\ k\neq|n|}} \frac{\varphi^{(1)}_{k,n} - \varphi^{(2)}_{k,n}}{k^2-n^2} \sin(kx)\ e^{int} \right\| \leqslant$$

[5] Invertibility of $\Box$ can be obtained also for irrational periods of *constant type*, i.e., satisfying suitable diophantine conditions [23]. However, in such a case the map (27) will not be contractive in our setting and alternative reductions should be searched in other directions.

$$\overset{(\star)}{\leqslant} \frac{1}{\min\limits_{\substack{k>\ell\\k\neq|n|}} |k^2-n^2|} \left\| \sum_{\substack{k>\ell\\k\neq|n|}} \left(\varphi^{(1)}_{k,n} - \varphi^{(2)}_{k,n}\right) \sin(kx)\, e^{\mathrm{i}nt} \right\| \leqslant$$

$$\leqslant \frac{\|F(\mu+\eta_1) - F(\mu+\eta_2)\|}{2\ell+1} \leqslant \frac{C}{2\ell+1} \|\eta_1 - \eta_2\| , \quad (29)$$

where $(\star)$ holds because

$$\left\| \sum_{\substack{k>\ell\\k\neq|n|}} \frac{\varphi^{(1)}_{k,n} - \varphi^{(2)}_{k,n}}{|k^2-n^2|^2} \sin(kx)\, e^{\mathrm{i}nt} \right\|^2 = \sum_{\substack{k>\ell\\k\neq|n|}} \frac{\left|\varphi^{(1)}_{k,n} - \varphi^{(2)}_{k,n}\right|^2}{|k^2-n^2|^2} \leqslant$$

$$\leqslant \frac{1}{\min|k^2-n^2|^2} \sum_{\substack{k>\ell\\k\neq|n|}} \left|\varphi^{(1)}_{k,n} - \varphi^{(2)}_{k,n}\right|^2 =$$

$$= \frac{1}{\min|k^2-n^2|^2} \left\| \sum_{\substack{k>\ell\\k\neq|n|}} (\varphi^{(1)}_{k,n} - \varphi^{(2)}_{k,n}) \sin(kx)\, e^{\mathrm{i}nt} \right\|^2 . \quad (30)$$

□

Therefore we conclude the existence of a unique fixed point $\eta \in \mathbb{Q}_\ell N^\perp$ for each given $\mu \in \mathbb{P}_\ell N^\perp$, if ℓ is sufficiently large. Denoting as above by $\widetilde{\eta}(\mu)$ such a fixed point, and substituting in (25), we gain the finite reduction of the problem on the space variable x, obtaining the *bifurcation equation*:

$$\mu = \mathbb{P}_\ell \Box^{-1} F(\mu + \widetilde{\eta}(\mu)). \quad (31)$$

The temporal modes are still infinite, while the spatial modes, associated to the variable x are now in finite number.

4. **Reduced variational formulation for the wave equation.** We will now exhibit the variational principle associated to the wave equation. We denote by G a primitive of F, $G(s) = \int_0^s F(\tau)d\tau$, and we write the Euler–Lagrange functional:

$$J[u] := \frac{1}{2} \langle \Box u, u\rangle - \int_0^{2\pi} \int_0^{\pi} G(u) dx dt =$$

$$= - \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{2} \left| \frac{\partial}{\partial t} u \right|^2 - \frac{1}{2} \left| \frac{\partial}{\partial x} u \right|^2 + G(u) \right] dx dt. \quad (32)$$

Consider also the following pair of maps:

$$\mathbb{P}_\ell N^\perp \ni \mu \longmapsto \widetilde{u}(\mu) := \mu + \widetilde{\eta}(\mu) \in N^\perp, \quad (33)$$

$$N^\perp \ni u \longmapsto \overline{\mu}(u) := \mathbb{P}_\ell u \in \mathbb{P}_\ell N^\perp. \quad (34)$$

By combining the E-L functional with the fixed point map $\mu \mapsto \widetilde{\eta}(\mu)$ (27), we obtain a variational principle depending on a finite number of spatial modes:

$$I[\mu] = J[\widetilde{u}(\mu)] = J[\mu + \widetilde{\eta}(\mu)]. \quad (35)$$

The reduced variational principle is equivalent to the original one, in the same way that the reduced problem (31) is equivalent to the original problem (19).

Theorem 4.1. *There exists a one to one correspondence between the critical points of J and the critical points of I, i.e., the reduced variational principle (35) is exactly equivalent to the original variational principle (32).*

Proof. Such a correspondence is given through the maps (33) and (34). Let $N^\perp \ni h = h_1 + h_2 \in \mathbb{P}_\ell N^\perp \oplus \mathbb{Q}_\ell N^\perp$, and $h, h_1, h_2 \in C^\infty$. First of all, we prove that (35) is the variational principle associated to the equation (31). We have,

$$\begin{aligned}
d_\mu I[\mu] \cdot h_1 = d_u J[\widetilde{u}(\mu)] \cdot \frac{\partial \widetilde{u}}{\partial \mu}(\mu) \cdot h_1 &= \left\langle \Box \widetilde{u}(\mu) - F(\widetilde{u}(\mu)), \frac{\partial \widetilde{u}}{\partial \mu}(\mu) \cdot h_1 \right\rangle \\
&= \left\langle \Box \widetilde{u}(\mu) - F(\widetilde{u}(\mu)), h_1 + \frac{\partial \widetilde{\eta}}{\partial \mu}(\mu) \cdot h_1 \right\rangle \\
= \langle \mathbb{P}_\ell(\Box \widetilde{u}(\mu) - F(\widetilde{u}(\mu))), h_1 \rangle &+ \left\langle \mathbb{Q}_\ell(\Box \widetilde{u}(\mu) - F(\widetilde{u}(\mu))), \frac{\partial \widetilde{\eta}}{\partial \mu}(\mu) \cdot h_1 \right\rangle \\
&= \left\langle \Box \left(\mu - \mathbb{P}_\ell \Box^{-1} F(\mu + \widetilde{\eta}(\mu)) \right), h_1 \right\rangle + \\
&+ \left\langle \Box \left(\widetilde{\eta}(\mu) - \mathbb{Q}_\ell \Box^{-1} F(\mu + \widetilde{\eta}(\mu)) \right), \frac{\partial \widetilde{\eta}}{\partial \mu}(\mu) \cdot h_1 \right\rangle = \\
&= \left\langle \left(\mu - \mathbb{P}_\ell \Box^{-1} F(\mu + \widetilde{\eta}(\mu)) \right), \Box h_1 \right\rangle + \\
&+ \left\langle \underbrace{\widetilde{\eta}(\mu) - \mathbb{Q}_\ell \Box^{-1} F(\mu + \widetilde{\eta}(\mu))}_{\equiv 0, \quad \text{by def of fixed point of (27)}}, \Box \left(\frac{\partial \widetilde{\eta}}{\partial \mu}(\mu) \cdot h_1 \right) \right\rangle, \qquad (36)
\end{aligned}$$

therefore,

$$d_\mu I[\mu] \cdot h_1 = 0, \ \forall h_1, \qquad \Longleftrightarrow \qquad \left\langle \mu - \mathbb{P}_\ell \Box^{-1} F(\mu + \widetilde{\eta}(\mu)), h_1 \right\rangle = 0, \ \forall h_1. \qquad (37)$$

In other words, the bifurcation equation (31) is variational. Now we prove the equivalence between the two variational principles. We have that,

$$\begin{aligned}
d_u J[u] \cdot h = \langle \Box u - F(u), h \rangle &= \left\langle u - \Box^{-1} F(u), \Box h \right\rangle = \\
&= \left\langle \mu - \mathbb{P}_\ell \Box^{-1} F(\mu + \eta), \Box h_1 \right\rangle + \left\langle \eta - \mathbb{Q}_\ell \Box^{-1} F(\mu + \eta), \Box h_2 \right\rangle, \qquad (38)
\end{aligned}$$

thus, if $d_u J[u] \cdot h = 0$ for all h, we have that $\left\langle \eta - \mathbb{Q}_\ell \Box^{-1} F(\mu + \eta), h_2 \right\rangle = 0$ for all h_2, i.e., η is the unique fixed point of the map (27), $\eta = \widetilde{\eta}(\mu)$, and also $\left\langle \mu - \mathbb{P}_\ell \Box^{-1} F(\mu + \widetilde{\eta}(\mu)), h_1 \right\rangle = 0$, which implies $d_\mu I[\overline{\mu}(u)] \cdot h_1$, for all h_1. Conversely, it is clear that $d_\mu I[\mu] \cdot h_1 = 0$ implies that $d_u J[\widetilde{u}(\mu)] \cdot h = d_u J[\mu + \widetilde{\eta}(\mu)] \cdot h = 0$ by (37) and (38). □

5. From continua to discrete.

5.1. **Physical interpretation.** The aim of this section consists in rewriting the reduced E-L functional (35) as the Hamilton variational principle corresponding to a certain discrete system, i.e.,

$$I[\mu] = \frac{\pi}{2} \int_0^{2\pi} \widehat{\mathcal{L}}_\mu(t) dt. \qquad (39)$$

Let us write more explicitly $I[\mu]$:

$$\begin{aligned}
I[\mu] = J[\mu + \widetilde{\eta}(\mu)] = \\
= \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{2} \left| \frac{\partial \mu}{\partial t} \right|^2 - \frac{1}{2} \left| \frac{\partial \mu}{\partial x} \right|^2 + G(\mu + \widetilde{\eta}(\mu)) \right] dx dt + \qquad (40)
\end{aligned}$$

$$+\int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{2}\left|\frac{\partial\widetilde{\eta}(\mu)}{\partial t}\right|^2-\frac{1}{2}\left|\frac{\partial\widetilde{\eta}(\mu)}{\partial x}\right|^2\right]dxdt,$$

where the mixed terms in the expansion of the squared terms have vanished after integration, being μ and $\widetilde{\eta}(\mu)$ orthogonal, as well as their derivatives.

From the weak formulation of the fixed point problem $(25)_2$:

$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial}{\partial t}\widetilde{\eta}(\mu)\frac{\partial}{\partial t}h-\frac{\partial}{\partial x}\widetilde{\eta}(\mu)\cdot\frac{\partial}{\partial x}h+\mathbb{Q}_\ell F(\mu+\widetilde{\eta}(\mu))\cdot h\right]dxdt=0,\quad \forall h\in\mathbb{Q}_\ell H. \tag{41}$$

Note that we can write F in place of $\mathbb{Q}_\ell F$, being h orthogonal to $\mathbb{P}_\ell F$. If we substitute $\widetilde{\eta}(\mu)$ for h, we get:

$$\int_0^{2\pi}\int_0^{\pi}\left[\left|\frac{\partial}{\partial t}\widetilde{\eta}(\mu)\right|^2-\left|\frac{\partial}{\partial x}\widetilde{\eta}(\mu)\right|^2\right]dxdt=-\int_0^{2\pi}\int_0^{\pi}F(\mu+\widetilde{\eta}(\mu))\cdot\widetilde{\eta}(\mu)dxdt. \tag{42}$$

This equality can be used in the second summand of the reduced functional (40), obtaining

$$I[\mu]=\int_0^{2\pi}\int_0^{\pi}\Big[\frac{1}{2}\left|\frac{\partial\mu}{\partial t}\right|^2-\frac{1}{2}\left|\frac{\partial\mu}{\partial x}\right|^2+ +G(\mu+\widetilde{\eta}(\mu))-\frac{1}{2}F(\mu+\widetilde{\eta}(\mu))\widetilde{\eta}(\mu)\Big]dxdt. \tag{43}$$

Writing the Fourier representation of μ (24)

$$\mu(t,x)=\sum_{k=1}^{\ell}\mu_k(t)\sin(kx) \tag{44}$$

and integrating on the space variable x the first two terms, we obtain

$$I[\mu]=\frac{\pi}{2}\left\{\int_0^{2\pi}\sum_{1\leqslant k\leqslant\ell}\left(\frac{1}{2}\left|\dot{\mu}_k(t)\right|^2-\frac{|k|^2}{2}\left|\mu_k(t)\right|^2\right)dt+ +\int_0^{2\pi}\underbrace{\left[\frac{2}{\pi}\int_0^{\pi}G(\mu+\widetilde{\eta}(\mu))-\frac{1}{2}F(\mu+\widetilde{\eta}(\mu))\widetilde{\eta}(\mu)dx\right]}_{=:\,\mathcal{N}_\mu(t)}dt\right\}. \tag{45}$$

As a result, the 'Lagrangian function' corresponding to the variational principle (35) can be written as:

$$\widehat{\mathcal{L}}_\mu(t):=\mathcal{L}(\mu,\dot{\mu})+\mathcal{N}_\mu(t)=\underbrace{\sum_{1\leqslant k\leqslant\ell}\left(\frac{1}{2}\left|\dot{\mu}_k\right|^2-\frac{|k|^2}{2}\left|\mu_k\right|^2\right)}_{\substack{\text{system of }\ell\\ \text{harmonic oscillators}}}+\underbrace{\mathcal{N}_\mu(t).}_{\text{nonlinear coupling}} \tag{46}$$

Thus, the finite dimensional reduced functional can be interpreted as a classical Action with Lagrangian corresponding to a spectral lattice consisting in ℓ harmonic oscillators with displacement μ_k, unit mass, stiffness $|k|^2$, and a global nonlinear coupling term $\mathcal{N}_\mu$ computable from (45).

5.2. **Reconstruction of mass–spring systems from eigenvalue sequences.** Let us consider a one dimensional elastic string of length L. With minor changes (13) becomes:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u = F(u), \\ u(t,x) = u(t+2\pi, x), \qquad \forall x \in [0,L],\ t \in \mathbb{R} \\ u(t,0) = u(t,L) = 0, \qquad \forall t \in \mathbb{R}. \end{cases} \tag{47}$$

The eigensystem of $-\frac{\partial^2}{\partial x^2}$ is given by

$$0 < \lambda_1 < \lambda_2 < \dots \quad < \lambda_k = \left(\frac{\pi k}{L}\right)^2 < \quad \dots \tag{48}$$

$$\widehat{u}_k(x) = \sqrt{\frac{L}{2}}\frac{1}{\pi k}\sin\left(\frac{\pi k}{L}x\right). \tag{49}$$

So the solutions u in $N^\perp$ can be written as a linear combination with time dependent coefficients of these spatial eigenfunctions

$$u(t,x) = \sum_{k=1}^{\infty} a_k(t)\widehat{u}_k(x), \qquad a_k(t) = \sum_{|n|\neq k} a_{k,n}e^{\mathrm{i}nt} \tag{50}$$

$$u(t,x) = \sum_{k=1}^{\ell} a_k(t)\widehat{u}_k(x) + \sum_{k=\ell+1}^{\infty} a_k(t)\widehat{u}_k(x) = \sum_{k=1}^{\ell}\mu_k(t) + \sum_{k=\ell+1}^{\infty}\eta_k(t) \tag{51}$$

The above ACZ construction leads to the following finite dimensional Lagrangian (renormalized with respect to (46)):

$$\widehat{\mathcal{L}}_\mu = \sum_{k=0}^{\ell}\left(\frac{1}{2}|\dot{\mu}_k|^2 - \frac{1}{2}\left(\frac{\pi k}{L}\right)^2|\mu_k|^2\right) + \mathcal{N}_\mu. \tag{52}$$

This exact finite spectral formulation does correspond to the description of a large number of –more or less realistic– finite dynamical systems. The direct reconnaissance of (52) shows a system of ℓ unit mass harmonic oscillators, whose positions are described by the spectral variables μ_j, with elastic constants $\lambda_k = \left(\frac{\pi k}{L}\right)^2$; the term $\mathcal{N}_\mu$ represents the nonlinear correction.

We recall that the one dimensional wave equation, i.e., equation (47) with $F \equiv 0$, originates as the thermodynamic limit of a finite chain of ℓ small beads connected with springs, sending ℓ to infinity while keeping mass density and elastic tension constant [32]. On the other hand, the quadratic part in (52) does not correspond to a chain of springs and beads, but to a set of uncoupled harmonic oscillators (see Figure 2(b) and (c)).

Question. *Is it possible to change the variables in (52), obtaining a genuine linear spring-mass system (plus higher order terms) but maintaining the core of the spectral sequence (48), i.e.,* $\lambda_k = \left(\frac{\pi k}{L}\right)^2$, $k = 1, \dots, \ell$*?*

If so, the ACZ reduction would give back a finite mechanical system with a relevant physical meaning, because of the analogy with the elements of the thermodynamic limit sequence.

To this purpose, we consider the equation of motion of a chain of ℓ masses $m_1, \dots, m_\ell$ connected with springs k_i and with fixed extrema. We could consider also a possible nonlinear contribution F_u, where $u = (u_1, \ldots, u_\ell)$ is the vector of the displacements of the masses from their rest position.

$$M\ddot{u} + Ku = \mathsf{F}_u, \tag{53}$$

$$M = \begin{bmatrix} m_1 & 0 & \ldots & \\ 0 & m_2 & 0 & \\ \vdots & & \ddots & 0 \\ & & 0 & m_\ell \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & \ldots & \\ -k_2 & k_2 + k_3 & -k_3 & \\ \vdots & & \ddots & -k_\ell \\ & & -k_\ell & k_\ell + k_{\ell+1} \end{bmatrix}$$

We are interested in the quadratic part of the Lagrangian of this system:

$$L(u, \dot{u}) = \frac{1}{2}\dot{u}^\top M\dot{u} - \frac{1}{2}u^\top Ku, \tag{54}$$

and we want to find a linear coordinate transformation $u = \Upsilon v$ such that (54) becomes the quadratic part of (52), maintaining the same proper oscillation modes.

If we were to take a system with equal masses and equal springs, as in the sequence of the thermodynamic limit, we would obtain the following eigenvalues:

$$\widetilde{\lambda}_{k,\ell+1} = \frac{4(\ell+1)^2}{L^2}\sin^2\left(\frac{\pi}{2(\ell+1)}k\right), \quad \underset{\ell\to\infty}{\longrightarrow} \quad \left(\frac{\pi k}{L}\right)^2, \tag{55}$$

which, as shown in figure 3, are close to the desired eigenvalues only in the first part of the sequence. Of course the k-th eigenvalue of the ℓ-th chain, $\widetilde{\lambda}_{k,\ell+1}$ will converge to λ_k as ℓ tends to infinity, because $\frac{\pi}{2(\ell+1)}k \ll 1$ and the sinus in (55) can be substituted with its argument. Nevertheless, the largest eigenvalues will maintain a definite distance from the corresponding eigenvalues of the continuum. More precisely,

$$\frac{\widetilde{\lambda}_{\ell,\ell}}{\lambda_\ell} \cong \frac{4(\ell+1)^2}{L^2}\Big/\frac{\pi^2\ell^2}{L^2} \to \frac{4}{\pi^2}(\cong 0.4052847). \tag{56}$$

Therefore, for matching exactly the whole sequence of eigenvalues we must consider a system with variable masses and springs. However, the reconstructed system can be symmetric, as it will be specified below.

Let us consider the diagonal matrix $D^{-1} := \mathrm{diag}(\frac{1}{\sqrt{m_1}}, \ldots, \frac{1}{\sqrt{m_\ell}})$. Clearly we have that $M = D^2$, therefore, by setting

$$u = D^{-1}y, \tag{57}$$

we have that (54) becomes

$$\widetilde{L}(y, \dot{y}) = L(u, \dot{u}) = \frac{1}{2}\sum_{k=1}^{\ell}\dot{y}_k^2 - \frac{1}{2}y^\top D^{-1}KD^{-1}y. \tag{58}$$

If moreover we consider the orthogonal matrix U of the normalized eigenvectors of $D^{-1}KD^{-1}$, by setting

$$y = Uq, \tag{59}$$

the Lagrangian function (54) is represented by

$$\widetilde{\widetilde{L}}(q, \dot{q}) = L(u, \dot{u}) = \frac{1}{2}\sum_{k=1}^{\ell}\dot{q}_k^2 - \frac{1}{2}q^\top\widetilde{\Lambda}q, \tag{60}$$

where the diagonal matrix $\tilde{\Lambda} := \mathrm{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_\ell)$ contains the eigenvalues of D^{-1} KD^{-1}.

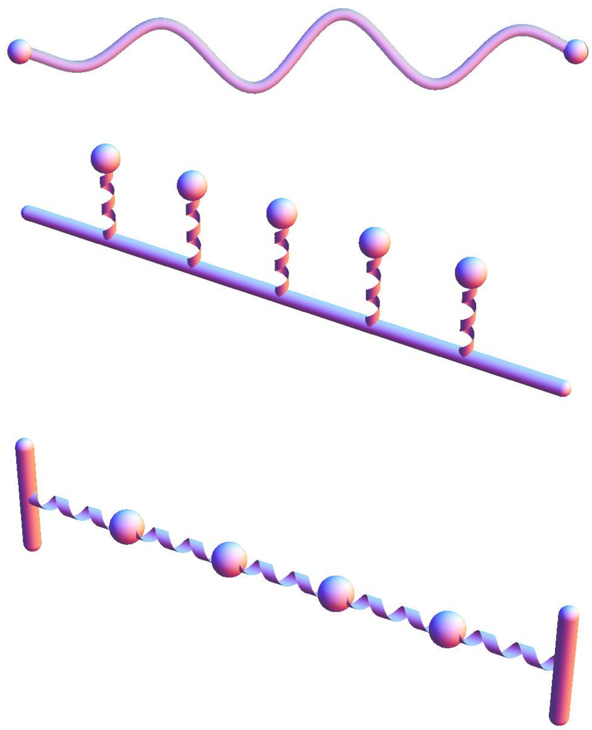
(a)
(b)
(c)


FIGURE 2. Equivalence among continuous and discrete systems: (a) elastic string, (b) uncoupled oscillators and (c) elastic chain.

Therefore, the answer to our question is positive if it is possible to choose masses $m_1, \ldots, m_\ell$ and springs $k_1, \ldots, k_{\ell+1}$ such that the proper frequencies $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_\ell$ coincide with the sequence $\lambda_1, \ldots, \lambda_\ell$ in (48).

This problem is an *inverse eigenvalue problem*, and has a unique solution in the form of a *persymmetric* system, i.e., a chain of masses and springs symmetric with respect to the mid point [6].

Theorem 5.1. *For any sequence of non negative distinct numbers* $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_\ell$, *there exists a unique persymmetric mass–spring system (53-54),* $m_1, \ldots, m_\ell$, $k_1, \ldots, k_{\ell+1}$, *with specified total mass* $m = \sum_{k=1}^{\ell} m_k$, *having the sequence* $\lambda_1, \ldots, \lambda_\ell$ *as the set of proper oscillation modes.*

Proof. The lines of the proof follows [28, Chapters 3 and 4]. A *Jacobi matrix* is a positive semi-definite symmetric tridiagonal matrix with (strictly) negative codiagonal entries. If the elastic constants $k_1, \ldots, k_{\ell+1}$ are all positive, the stiffness matrix K in (54) is a Jacobi matrix. Moreover, if also the masses $m_1, \ldots, m_\ell$ are all positive, the matrix $D^{-1}KD^{-1}$ is still a Jacobi matrix. The proof is completed by invoking

[6]Thorough discussion on the problem of finding special springs–masses systems reproducing specified eigenvalues can be found in [28, 37]

THEOREM ([28, Theorem 4.3.2]) *There is a* unique*, up to multiplicative constants, persymmetric Jacobi matrix* J *for any assigned spectrum* $\sigma(J) = (\lambda_j)_{j=1}^{\ell}$*, satisfying* $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_\ell$.

As described in details in [28, Section 4.4-4.6] it is possible to reconstruct uniquely a spring-mass system with specified total mass from such a persymmetric Jacobi matrix J. □

For instance, if we consider the first five eigenvalues of an elastic string and reconstruct an equivalent persymmetric elastic chain with five beads and total mass 1, we obtain the mechanical system:

$$\textbf{masses:} \qquad \left\{\tfrac{5}{21}, \tfrac{5}{28}, \tfrac{1}{6}, \tfrac{5}{28}, \tfrac{5}{21}\right\}, \tag{61}$$

$$\textbf{springs:} \qquad \tfrac{\pi^2}{L^2}\left\{\tfrac{25}{42}, \tfrac{15}{14}, \tfrac{5}{4}, \tfrac{5}{4}, \tfrac{15}{14}, \tfrac{25}{42}\right\}. \tag{62}$$

The corresponding normalized eigenvectors of the discrete *simulacrum* are shown in figure 4(b), along with the first five eigenvectors of the continuous system for comparison (c). The eigenvalues of a chain with equal masses and springs are compared with the eigenvalues of the string in Figure 3, for a system with 5 masses and 25 masses. Being the largest eigenvalue of the finite chain less than half the corresponding eigenvalue of the string (56), the reconstructed system will not tend to a homogeneous system but a definite difference among masses and springs is maintained as the system size grows.

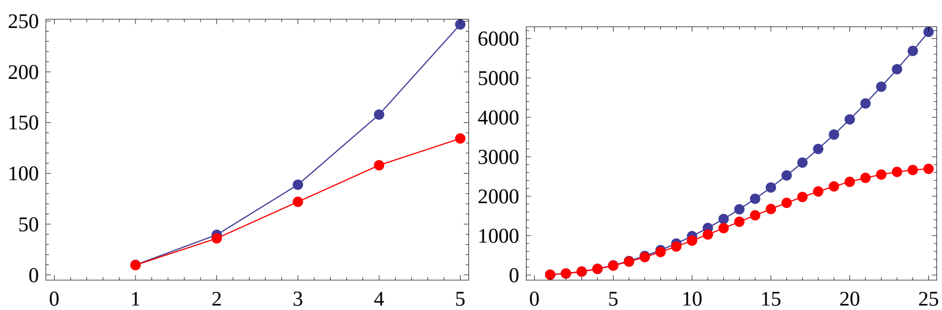


FIGURE 3. Right panel: first five eigenvalues of the elastic string (blue) and eigenvalues of the elastic chain with five equal masses and springs (red). Left panel: first 25 eigenvalues of the elastic string (blue) and eigenvalues of the elastic chain with 25 equal masses and springs (red).

6. **Conclusions.** The above exploration around the meaning of the exact finite reduction of the PDE setting of a static or dynamic continuous system, here sketched by the problems (1) and (13) respectively, has led us to the following interpretation: such exact reduced system is precisely a discrete 'simulacrum' of the original system, reminiscent of all its main features, and it can be perfectly interpreted as a genuine physical system, strictly analogous to the systems of the hierarchy employed in the thermodynamic limit, unless that the masses and springs are not all equal, but persymmetric, i.e., symmetric with respect to the middle point of the system.

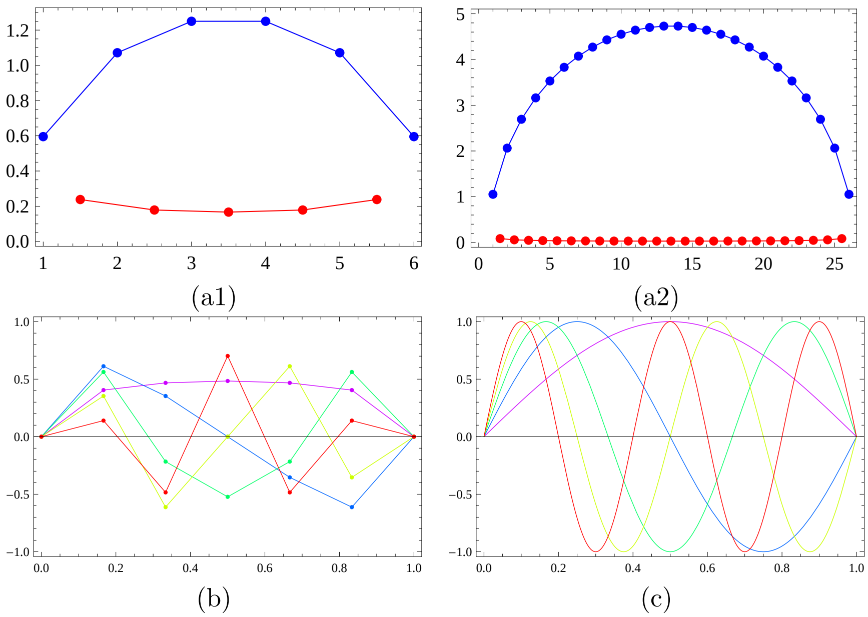


Figure 4. Five-beads elastic chain with the eigenvalues equal to the first five eigenvalues of the elastic string. Panel (a1-2): springs (blue) and masses (red) of the reconstructed system (61-62) with five elements (a1) and 25 elements (a2). Panel (b): normalized eigenvectors for the chain with 5 elements. Panel (c): first five eigenvectors for the elastic string.

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