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A REVISIT TO THE CONSENSUS FOR LINEARIZED VICSEK MODEL UNDER JOINT ROOTED LEADERSHIP VIA A SPECIAL MATRIX

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Abstract. We address the exponential consensus problem for the linearized Vicsek model which was introduced by Jadbabaie *et al.* in [10] under a joint rooted leadership via the (*sp*) matrices. This model deals with self-propelled particles moving in the plane with the same speed but different headings interacting with neighboring agents by a linear relaxation rule. When the time-varying switching topology of the neighbor graph satisfies some weak connectivity condition, namely, "*joint connectivity condition*" in the spatial-temporal domain, it is well known that the consensus for the linearized Vicsek model can be achieved asymptotically. In this paper, we extend the theory of (*sp*) matrices and apply it to revisit this asymptotic consensus problem and give an explicit estimate on the maximum Lyapunov exponent, when the underlying network topology satisfies the joint rooted leadership which is directed and non-symmetric.

1. **Introduction.** Consensus problem on dynamic networks is an active topic in many different disciplines such as computer sciences, communications and control theory due to its engineering applications in the formation control of robots, unmanned aerial vehicles, underwater gliders and sensor networks etc. In [15], Vicsek *et al.* proposed a simple discrete-time multi-agent model consisting of self-propelled particles moving in the plane with the same speed but different headings. The agents interact with neighboring agents within a finite range to update their heading angles through a nonlinear coupling rule. By numerical simulations, they demonstrated that the finite range interaction rule can lead to consensus for their headings, despite the absence of centralized coordination and the fact that the neighbor graph changes with time. An analytic study on the original Vicsek model was given in [13]. In 2003, Jadbabaie *et al.* [10] studied a variant version of the Vicsek model, where the coupling between the test particle and its neighboring particles is given in a linearized form. They gave the first mathematical proof for the emergence of asymptotic consensus for a large class of switching signals fulfilling some *infinity*

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conditions on the connectivity of the undirected neighbor graph time series. After its publication, it has attracted lots of attention from various communities, in particular, it brought attention to the authors of [5] in which the Cucker-Smale flocking model was proposed and its flocking estimates were studied in terms of initial configurations and spatial decay rate of communication weights.

The approach in [10] is closely related to undirected neighbor graph and switching systems. In [7], Gao *et al.* pointed out that the proof of the consensus result for leader-following case in [10] is questionable and gave an amendment with an extension to directed graphs. Other considerations of this model appear in [9]. In this paper, we revisit the consensus problem for the linearized Vicsek model by improving the theory of (sp) matrices (where the adjective (sp) was coined from "**s***im***p***le*") and its switching stability theory [17]. Moreover, we also give an explicit estimate for the Lyapunov exponent for the linearized Vicsek model under "joint rooted leadership" across time intervals.

Recently, the idea of (sp) matrices has been applied to the discrete-time Cucker-Smale consensus model to analyze its flocking behavior [11, 12] and was further extended to the study of the model in continuous-time [8] in a large coupling limit. However, for the general continuous-time Cucker-Smale model under rooted leadership, the asymptotic consensus has not been verified analytically, although the numerical simulations suggest asymptotic consensus.

The paper after this introduction is organized as follows. In Section 2, we present the linearized Vicsek model under joint rooted leadership and some related results about (sp) matrices. In Section 3, we give our main result and its proof. Finally, Section 4 is devoted to the summary of our result.

2. The consensus model and (sp) matrices.

In this section, we recall the linearized Vicsek model and its graph theoretical interpretation, and present some new results on the theory of (sp) matrices for a later use.

2.1. The linearized Vicsek model for consensus.

In this paper we will apply the theory of (sp) matrices to the linearized Vicsek model with a leader-follower topology. In this case, the system consists of N autonomous follower agents, labeled by $1, 2, \ldots, N$, plus a leader agent, labeled by 0. The leader moves independently with a fixed heading, say θ_0, whereas the headings of followers are updated such that they approach to the local average headings in their neighbor set. Here j is a neighbor of i means that the agent i is influenced by the agent j, i.e., j sends some information to i. In this situation, the heading angles $\theta_i,\ i = 0, 1, \ldots, N$ are governed by the following difference system:

$$\begin{aligned}
&\theta_0(t+1) = \theta_0(t), \quad t \in \mathbb{N},\\
&\theta_i(t+1) = \theta_i(t) + u_i(t), \quad i = 1, 2, \ldots, N,\\
&u_i(t) := \frac{1}{1 + n_i(t) + b_i(t)} \left[b_i(t)(\theta_0(t) - \theta_i(t)) + \sum_{0 \neq j \in \mathcal{N}_i(t)} (\theta_j(t) - \theta_i(t)) \right], \qquad (1)\\
&b_i(t) := \begin{cases} 1, & 0 \in \mathcal{N}_i(t),\\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

where $\mathcal{N}_i(t)$ is the agent i's neighbor set and $n_i(t)$ is its cardinality.

Note that the heading of the leader agent is fixed, i.e., $\theta_0(t) = \theta_0(0) =: \theta_0$ and the feedback control term $u_i(t)$ is defined to push the i-th agent's heading angles to move toward the local heading average of its neighbor set. Thus, to study the consensus, we can consider the fluctuation $\hat{\theta}_i := \theta_i - \theta_0$ instead of the original state θ_i. As we shall see it in the sequel, the consensus problem is linked to (sp) matrices.

We next review a graph theoretic interpretation of the Vicsek model (1) from the view point of a directed graph (or digraph) without self-loops as in [10]. The graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ associated with the system (1) is defined as follows:

$$\bar{\mathcal{V}} := \{0, 1, \dots, N\}, \qquad \bar{\mathcal{E}} := \left\{(j,i) \in \bar{\mathcal{V}} \times \bar{\mathcal{V}} :\ i \neq j,\ \ j \text{ is a neighbor of } i\right\}.$$

The graph $\bar{\mathcal{G}}$ can be regarded as the information flow chart of the network structure, so we write

$$j \to i \qquad \Longleftrightarrow \qquad (j,i) \in \bar{\mathcal{E}}.$$

A directed path from j to i means a sequence of distinct arcs of the form $j \to k_1 \to k_2 \to \cdots \to k_n \to i$. In the sequel we use the symbol $\mathcal{I}$ to denote a set indexing all such digraphs $\bar{\mathcal{G}}_p = (\bar{\mathcal{V}}_p, \bar{\mathcal{E}}_p)$. For $p \in \mathcal{I}$, we use $\mathcal{G}_p = (\mathcal{V}_p, \mathcal{E}_p)$ to denote the proper subgraph of $\bar{\mathcal{G}}_p$ by deleting vertex 0 and all edges issued from the vertex 0. Next we introduce three classes of matrices associated to the digraphs $\mathcal{G}_p$ and $\bar{\mathcal{G}}_p$:

(a) A_p: the adjacency matrix of $\mathcal{G}_p$, i.e.,

$$(A_p)_{ij} = \begin{cases} 1, & j \neq i,\ (j,i) \in \mathcal{E}_p, \\ 0, & j \neq i,\ (j,i) \notin \mathcal{E}_p, \\ 0, & j = i. \end{cases}$$

(b) D_p: the diagonal matrix of valences of $\mathcal{G}_p$, i.e., $(D_p)_{ii} = \sum_{j=1}^{N} (A_p)_{ij}$.

(c) B_p: $N \times N$ diagonal matrix such that

$$(B_p)_{ii} := \begin{cases} 1, & (0,i) \in \bar{\mathcal{E}}_p, \\ 0, & (0,i) \notin \bar{\mathcal{E}}_p. \end{cases}$$

The adjacency and valence matrices of the full digraph $\bar{\mathcal{G}}_p$ can be defined similarly. Using these notations, system (1) can be written as a compact form:

$$\theta(t+1) = \left(I + D_{\sigma(t)} + B_{\sigma(t)}\right)^{-1} \left((I + A_{\sigma(t)})\theta(t) + B_{\sigma(t)} \mathbf{1} \theta_0\right), \tag{2}$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_N)^\top$, $\mathbf{1} = (1, 1, \dots, 1)^\top \in \mathbb{R}^N$ and $\sigma : \mathbb{N} \to \mathcal{I}$ is a switching signal. Consider the fluctuation vector

$$\hat{\theta}(t) := \theta(t) - \theta_0 \mathbf{1}.$$

Then the fluctuation $\hat{\theta}$ satisfies

$$\hat{\theta}(t+1) = F_{\sigma(t)} \hat{\theta}(t), \tag{3}$$

which gives a compact form for the fluctuation, where F_p is given by

$$F_p := \left(I + D_p + B_p\right)^{-1} (I + A_p), \qquad p \in \mathcal{I}. \tag{4}$$

Lemma 2.1. *The matrix F_p in (4) has the following properties:*

(1) F_p is a nonnegative matrix, and $(F_p)_{ij} > 0$ if and only if $i = j$ or $(j,i) \in \mathcal{E}_p$;

(2) $\sum_{j=1}^{N} (F_p)_{ij} \leq 1$, and $\sum_{j=1}^{N} (F_p)_{ij} < 1$ if and only if $(0,i) \in \bar{\mathcal{E}}_p$.

Proof. (1) Since the matrices D_p and B_p are diagonal matrices with nonnegative elements, the matrix $\left(I + D_p + B_p\right)^{-1}$ is a diagonal matrix with positive diagonals. This means that the matrices F_p and $I + A_p$ have the same sign pattern, i.e.,

$(F_p)_{ij} > 0\ (= 0) \iff (I + A_p)_{ij} > 0\ (= 0)$. On the other hand, the matrix $I + A_p$ is a nonnegative matrix with positive diagonals and for $i \neq j$,

$$(I + A_p)_{ij} > 0 \iff (A_p)_{ij} > 0 \iff (j, i) \in \mathcal{E}_p.$$

Therefore, we conclude the property (I).

(2) First of all, we notice that $\big(I + D_p + B_p\big)^{-1}$ is a diagonal matrix with positive diagonals given by

$$\big((I + D_p + B_p)^{-1}\big)_{ii} = \frac{1}{1 + (B_p)_{ii} + \sum_{j=1}^N (A_p)_{ij}}.$$

Therefore, we have

$$\begin{aligned}\sum_{j=1}^N (F_p)_{ij} &= \big((I + D_p + B_p)^{-1}\big)_{ii} \left(\sum_{j=1}^N \big(I + A_p\big)_{ij}\right)\\ &= \frac{1 + \sum_{j=1}^N (A_p)_{ij}}{1 + (B_p)_{ii} + \sum_{j=1}^N (A_p)_{ij}}.\end{aligned}$$

If $(0, i) \in \bar{\mathcal{E}}_p$, then we have

$$(B_p)_{ii} = 1 \quad \text{and} \quad \sum_{j=1}^N (F_p)_{ij} < 1.$$

If $(0, i) \notin \bar{\mathcal{E}}_p$, then we have

$$(B_p)_{ii} = 0 \quad \text{and} \quad \sum_{j=1}^N (F_p)_{ij} = 1.$$

□

Remark 1. (1) Because the updating rule (1) or (2) is given by the averaging algorithm, we have the following observations:

(i) For any $i = j$ or $(j, i) \in \mathcal{E}_p$;, we have $(F_p)_{ij} \geq \frac{1}{N+1}$.

(ii) For any i with $(0, i) \in \bar{\mathcal{E}}_p$, we have $\sum_{j=1}^N (F_p)_{ij} \leq \frac{N}{N+1}$.

(2) Lemma 2.1 implies that the updating rule of the fluctuation is given by sub-stochastic matrices, i.e., nonnegative matrices with row sum less than or equal to 1. We refer to [17] and references therein for brief introduction to this kind of matrices. In literature [3, 4], the consensus problem was studied via the theory of stochastic matrices since the updating rule of the (original) state variables, f or example, the variables $\theta_0, \theta_1, \dots, \theta_N$ in system (1), can be described by stochastic matrices, i.e., nonnegative matrices with rows sum to 1. In this work, we will focus on the fluctuation and employ the (sp) matrices, a special class of sub-stochastic matrices, to investigate the consensus problem for system (1) with a leader-follower structure.

For an undirected graph with bidirectional information exchange [10], one says that the follower agents $1, 2, \dots, N$ are linked to the leader 0 across an interval $[t, \tau)$ $(t, \tau \in \mathbb{N}, t < \tau)$ if the collection of undirected graphs $\big\{\bar{\mathcal{G}}_{\sigma(t)}, \bar{\mathcal{G}}_{\sigma(t+1)}, \dots, \bar{\mathcal{G}}_{\sigma(\tau-1)}\big\}$ encountered along the time interval is jointly connected, i.e., the union graph $\bar{\mathcal{G}}_{\sigma(t)} \bigcup \bar{\mathcal{G}}_{\sigma(t+1)} \bigcup \cdots \bigcup \bar{\mathcal{G}}_{\sigma(\tau-1)}$ is connected. In the framework of digraphs, the concept of "*joint rooted leadership*" is given as follows.

Definition 2.2 (Joint rooted leadership)**.** The system (1) is under *joint rooted leadership* across the time interval $[t, \tau)$ if the agent 0 does not have incoming path from others, whereas each agent in $\{1, 2, \dots, N\}$ has a directed path from 0 in the union graph of $\bar{\mathcal{G}}_{\sigma(t)}, \bar{\mathcal{G}}_{\sigma(t+1)}, \dots, \bar{\mathcal{G}}_{\sigma(\tau-1)}$.

Note that the union graph topology is exactly under rooted leadership in the sense of [8, 12]. As shown in [12], the rooted leadership structure is closely related to an (*sp*) matrix which will be presented in the next subsection (see Definition 2.3).

2.2. **Properties of (*sp*) matrices.** In this subsection, we will present some theory of (*sp*) matrices independently of Vicsek model. Actually, the idea of (*sp*) matrices has been applied to the Cucker-Smale consensus model with leader-follower interaction topologies [8, 11, 12]. Thus, this theory is not only limited to the consensus problem of Vicsek model. For the readers' convenience, we next give some backgrounds of (*sp*) matrices and related results.

In 2007, Xue *et al.* introduced a subclass of nonnegative matrices, so called (*sp*) matrices, by describing the distribution of non-zero elements. For the original definition of an (*sp*) matrix, please refer to [16, Definition 2.1]. The motivation to introduce (*sp*) matrices is to characterize a class of asymptotically stable discrete-time linear systems given by a sub-stochastic matrix, i.e.,

$$A \in \mathcal{S} := \left\{ (a_{ij}) \in \mathbb{R}^{N\times N} : \; a_{ij} \geq 0, \; i, j = 1, 2, \dots, N, \quad \sum_{k=1}^{N} a_{ik} \leq 1 \right\}.$$

In [17], an alternative definition for the (*sp*) matrix was given in the terminology of graphs, which appears rather simple. It is known that any nonnegative square matrix of size N, say $A = (a_{ij})$, can be associated with some weighted digraph (see [6]) with vertices $1, 2, \dots, N$. The graph is defined so that (j, i) is a directed arc from j to i if and only if $a_{ij} > 0$. Then the definition of (*sp*) matrix in the terminology of graphs is as follows.

Definition 2.3. [17] Let $A = (a_{ij}) \in \mathcal{S}$, and $(\mathcal{V}, \mathcal{E})$ be its associated digraph.

(1) The vertex i is called non-saturated (saturated) if and only if the i-th row sum of A is strictly less than 1 (exactly 1).
(2) The matrix A is an (*sp*) matrix if and only if each saturated vertex has a directed path from a non-saturated vertex.

Remark 2. (1) For $A = (a_{ij}) \in \mathcal{S}$, we use $I_1(A)$ and $I_2(A)$ to denote the non-saturated and saturated vertex sets, respectively:

$$I_1(A) := \left\{ i : \sum_{j=1}^{N} a_{ij} < 1 \right\}, \quad I_2(A) := \left\{ i : \sum_{j=1}^{N} a_{ij} = 1 \right\}.$$

We denote by $\mathcal{SP}$ the set of all (*sp*) matrices in $\mathcal{S}$. If all row sums of A are strictly less than 1, i.e., $I_2(A) = \emptyset$, then A is called a trivial (*sp*) matrix, denoted by $A \in \mathcal{SP}_0$. Obviously, $A \in \mathcal{SP}_0$ is equivalent to

$$\|A\|_\infty := \max_{1\leq i\leq N} \sum_{j=1}^{N} |a_{ij}| < 1$$

as long as A is a nonnegative matrix. Throughout this paper we use $\|\cdot\|_\infty$ to denote the infinity norm of vectors or matrices.

(2) Note that the above definition appears different from the previous one in [17]: each saturated vertex has a directed path "from" a non-saturated vertex, whereas in [17] has a path "to" a non-saturated vertex. This is because, in this paper, we interpret the neighbor graph as an information flow chart and we want to make Definition 2.3 to be consistent with the information flow meaning of a neighbor graph.
(3) By Lemma 2.1 (2), we see that the non-saturated vertices for F_p are exactly the agents directly led by the root agent 0.

The reason for introducing the concept of (*sp*) matrices is apparent from the following proposition.

Proposition 1. [16] *Suppose that the matrix A is in the set $\mathcal{S}$. Then the linear discrete-time system*

$$x(t+1) = Ax(t), \quad x(t) \in \mathbb{R}^N, \ t \in \mathbb{N} \tag{5}$$

is asymptotically stable if and only if A is an (sp) matrix.

Given a finite set of $N \times N$ matrices $\mathcal{M} = \{A_1, A_2, \ldots, A_m\}$, the associated switched system of (5) is

$$x(t+1) = A_{\sigma(t)}x(t), \quad x(t) \in \mathbb{R}^N, \ t \in \mathbb{N}, \tag{6}$$

where $\sigma : \mathbb{N} \to \{1, 2, \ldots, m\}$ is a switching signal and the time-invariant system (5) with matrix A_i is called a subsystem. We next present a subset of $\mathcal{S}$ which was introduced in [17]:

$$\mathcal{P} := \left\{ (a_{ij}) \in \mathcal{S} : \ a_{ii} \neq 0 \text{ for } i \text{ with } \sum_{j=1}^{N} a_{ij} = 1 \right\}.$$

To make a switching system (6) uniformly asymptotically stable, it requires that all subsystems should be asymptotically stable. Here we say "uniformly" with respect to the arbitrarity of the switching signal. We now state a result about the stability of the system (6) arising from a given finite set of (*sp*) matrices.

Proposition 2. [17] *If $A_1, A_2, \ldots, A_m \in \mathcal{P} \bigcap \mathcal{SP}$, then the associated switched linear system (6) is uniformly exponentially stable, i.e., there exist positive constants $\bar{C}$ and $\bar{\mu}$ such that*

$$\|x(t)\| < \bar{C}e^{-\bar{\mu}t}, \quad \forall \ t \in \mathbb{N}, \ \forall \ \sigma.$$

Remark 3. In [17, Theorem 4.1], the uniform asymptotic stability is concluded. Furthermore, as mentioned in [17, Remark 4.1], the proof of Proposition 2 gives an estimate below 1 for the joint spectral radius $\rho(\mathcal{M})$ [1]:

$$\rho(\mathcal{M}) := \limsup_{t\to\infty} \max \left\{ \|A_{i_k}A_{i_{k-1}} \cdots A_{i_1}\|^{\frac{1}{k}} : A_{i_j} \in \mathcal{M} \right\},$$

where $\|\cdot\|$ is some norm of matrices. It is known that the logarithm of the joint spectral radius of a set of matrices coincides with the Lyapunov exponent (asymptotic growth rate) of the associated linear switched system (see [1]). Thus, $\rho(\mathcal{M}) < 1$ implies the uniform exponential stability of system (6).

We next present two lemmas on (*sp*) matrices which will be crucially used in our approach for consensus analysis for the linearized Vicsek model. For an ordered set of two $N \times N$ matrices A_1, A_2, we set

$$I_e(A_2, A_1) := \{i \in I_2(A_2) : \ \exists \ l \in I_1(A_1) \text{ such that } (A_2)_{il} > 0\}. \tag{7}$$

In other words, $i \in I_e(A_2, A_1)$ means that i is a saturated vertex for A_2 and has some information inflow in the digraph of A_2 from a non-saturated vertex for A_1. The set $I_e(\cdot, \cdot)$ describes the expanding of non-saturated vertices under the matrix multiplication. More precisely, we have the following results.

Lemma 2.4. *Let $A_1, A_2 \in \mathcal{P} \bigcap \mathcal{S}P$. Then*

$$I_1(A_2A_1) = I_1(A_2) \bigcup I_1(A_1) \bigcup I_e(A_2, A_1). \tag{8}$$

Proof. (i) We first show that

$$I_1(A_2) \bigcup I_1(A_1) \bigcup I_e(A_2, A_1) \subset I_1(A_2A_1). \tag{9}$$

Step 1. We claim that

$$I_1(A_2) \bigcup I_1(A_1) \subseteq I_1(A_2A_1). \tag{10}$$

For $i \in I_1(A_2)$, we have

$$\begin{aligned} \sum_{j=1}^{N} (A_2A_1)_{ij} &= \sum_{j=1}^{N} \sum_{k=1}^{N} (A_2)_{ik}(A_1)_{kj} \\ &= \sum_{k=1}^{N} (A_2)_{ik} \sum_{j=1}^{N} (A_1)_{kj} \\ &< \sum_{k=1}^{N} (A_2)_{ik} < 1. \end{aligned} \tag{11}$$

For $i \in I_1(A_1) \backslash I_1(A_2)$, since $A_2 \in \mathcal{P}$, we have

$$(A_2)_{ii} > 0, \quad \sum_{j=1}^{N} (A_1)_{ij} < 1.$$

This yields

$$\begin{aligned} \sum_{j=1}^{N} (A_2A_1)_{ij} &= \sum_{j,k=1}^{N} (A_2)_{ik}(A_1)_{kj} = \sum_{k=1}^{N} (A_2)_{ik} \left(\sum_{j=1}^{N} (A_1)_{kj} \right) \\ &= \sum_{k=1, k\neq i}^{N} (A_2)_{ik} \left(\sum_{j=1}^{N} (A_1)_{kj} \right) + (A_2)_{ii} \left(\sum_{j=1}^{N} (A_1)_{ij} \right) \\ &\leq \sum_{k=1, k\neq i}^{N} (A_2)_{ik} + (A_2)_{ii} \left(\sum_{j=1}^{N} (A_1)_{ij} \right) \\ &< \sum_{k=1, k\neq i}^{N} (A_2)_{ik} + (A_2)_{ii} \leq 1. \end{aligned} \tag{12}$$

We now combine (11) and (12) to get (10).

Step 2. We claim that

$$I_e(A_2, A_1) \subset I_1(A_2A_1). \tag{13}$$

Let $i \in I_e(A_2, A_1)$, that is, $i \in I_2(A_2)$ and there exists $l \in I_1(A_1)$ such that $(A_2)_{il} > 0$. Then we have

$$\begin{aligned}\sum_{j=1}^N (A_2A_1)_{ij} &= \sum_{k=1}^N (A_2)_{ik}\big(\sum_{j=1}^N (A_1)_{kj}\big)\\ &= \sum_{k=1,k\neq l}^N (A_2)_{ik}\big(\sum_{j=1}^N (A_1)_{kj}\big) + (A_2)_{il}\big(\sum_{j=1}^N (A_1)_{lj}\big)\\ &\leq \sum_{k=1,k\neq l}^N (A_2)_{ik} + (A_2)_{il}\big(\sum_{j=1}^N (A_1)_{lj}\big)\\ &< \sum_{k=1,k\neq l}^N (A_2)_{ik} + (A_2)_{il} = 1.\end{aligned}$$

This yields $i \in I_1(A_2A_1)$, and (13) follows. Then we combine (10) and (13) to find (9).

(ii) We next show that

$$I_1(A_2A_1) \subset I_1(A_2) \bigcup I_1(A_1) \bigcup I_e(A_2, A_1). \tag{14}$$

Suppose that $i \notin I_1(A_2) \bigcup I_1(A_1) \bigcup I_e(A_2, A_1)$. Then we have

$$\sum_{j=1}^N (A_2)_{ij} = 1, \quad \text{and} \quad (A_2)_{ik} = 0 \ \text{ for any } k \in I_1(A_1).$$

Therefore, we have

$$\begin{aligned}\sum_{j=1}^N (A_2A_1)_{ij} &= \sum_{k=1}^N (A_2)_{ik}\big(\sum_{j=1}^N (A_1)_{kj}\big)\\ &= \sum_{k\in I_1(A_1)}^N (A_2)_{ik}\big(\sum_{j=1}^N (A_1)_{kj}\big) + \sum_{k\in I_2(A_1)}^N (A_2)_{ik}\big(\sum_{j=1}^N (A_1)_{kj}\big)\\ &= \sum_{k\in I_2(A_1)}^N (A_2)_{ik}\big(\sum_{j=1}^N (A_1)_{kj}\big)\\ &= \sum_{k\in I_2(A_1)}^N (A_2)_{ik} = 1,\end{aligned}$$

which implies $i \notin I_1(A_2A_1)$. Thus we have (14). Finally, (9) and (14) implies the desired result (8). ☐

Lemma 2.5. *Let $A_1, A_2 \in \mathcal{P} \bigcap \mathcal{SP}$, $A_1 \notin \mathcal{SP}_0$. Then we have*

$$I_1(A_1) \subsetneq I_1(A_2A_1).$$

Proof. It follows from Lemma 2.4 that

$$I_1(A_1) \subset I_1(A_2A_1),$$

thus to derive the desired estimate, we need to show

$$I_1(A_2A_1) \setminus I_1(A_1) \neq \emptyset.$$

We consider two cases according to the relationship between $I_1(A_1)$ and $I_1(A_2)$.

• **Case 1.** If $I_1(A_2)$ has some vertex $i_0 \notin I_1(A_1)$, then again by Lemma 2.4, we have $i_0 \in I_1(A_2A_1)$ and thus $i_0 \in I_1(A_2A_1) \setminus I_1(A_1)$.
• **Case 2.** If $I_1(A_2) \subset I_1(A_1)$, we claim that $I_e(A_2, A_1) \neq \emptyset$. Suppose not, i.e., $I_e(A_2, A_1) = \emptyset$, then by the definition of $I_e(A_2, A_1)$ in (7), any vertex $i \in I_2(A_2) \neq \emptyset$ cannot be reached by vertices in $I_1(A_1)$ in the digraph of A_2. Note that $I_1(A_2) \subset I_1(A_1)$, thus any vertex $i \in I_2(A_2)$ cannot be reached by vertices in $I_1(A_2)$ in the digraph of A_2, which contradicts to the fact that $A_2 \in \mathcal{SP}$. Therefore, we have $I_e(A_2, A_1) \neq \emptyset$. Again by Lemma 2.4 we find that $I_e(A_2, A_1) \subset I_1(A_2A_1)$ and thus,

$$\emptyset \neq I_e(A_2, A_1) \subset I_1(A_2A_1) \setminus I_1(A_1).$$

We combine Case 1 and Case 2 to get the desired result. □

Proposition 3. *Let $A_i \in \mathcal{P} \bigcap \mathcal{SP},\ i = 1, 2, \cdots, N$ be $N \times N$ matrices. Then*

$$A_N \cdots A_2A_1 \in \mathcal{SP}_0, \quad \textit{i.e., is a trivial } (sp) \textit{ matrix.}$$

Proof. Recall that a trivial (*sp*)-matrix is the (*sp*)-matrix that all row sums are strictly less than 1. For the proof, we divide it into two cases.
• **Case 1.** Suppose that the matrix A_1 is in $\mathcal{SP}_0$, i.e., $\|A_1\|_\infty < 1$. In this case, we use $\|A_i\|_\infty \leq 1,\ i = 2, \cdots, N$ to get

$$\|A_N \cdots A_2A_1\|_\infty \leq \|A_N\|_\infty \cdots \|A_2\|_\infty \|A_1\|_\infty < 1.$$

Therefore, we have

$$A_N \cdots A_2A_1 \in \mathcal{SP}_0.$$

• **Case 2.** Suppose that the matrix A_1 is not in $\mathcal{SP}_0$, i.e.,

$$I_2(A_1) \neq \emptyset, \quad \text{i.e.,} \quad I_1(A_1) \subsetneq \{1, \ldots, N\}.$$

Consider the set $I_1(A_kA_{k-1} \cdots A_1)$. If there exists $k_0 \in \{2, \ldots, N-1\}$ such that

$$I_1(A_{k_0}A_{k_0-1} \cdots A_1) = \{1, \ldots, N\}, \quad \text{i.e.,}\ A_{k_0}A_{k_0-1} \cdots A_1 \in \mathcal{SP}_0.$$

Then by the same arguments in Case 1, we have

$$A_kA_{k-1} \cdots A_1 \in \mathcal{SP}_0, \quad \text{for } k_0 \leq k \leq N.$$

If all $A_kA_{k-1} \cdots A_1,\ k = 2, \ldots, N$ satisfy

$$I_1(A_kA_{k-1} \cdots A_1) \subsetneq \{1, \ldots, N\},$$

then by Lemma 2.5, the finite sequence $I_1(A_kA_{k-1} \cdots A_1)$ is strictly increasing such that

$$I_1(A_1) \subsetneq I_1(A_2A_1) \subsetneq \cdots \subsetneq I_1(A_NA_{N-1} \cdots A_1) \subsetneq \{1, \ldots, N\}.$$

This yields

$$1 \leq |I_1(A_1)| < |I_1(A_2A_1)| < \cdots < |I_1(A_NA_{N-1} \cdots A_1)| < N, \tag{15}$$

where $|G|$ is the cardinality of the set G. One the other hand, since $|I_1(A_kA_{k-1} \cdots A_1)| - |I_1(A_{k-1}A_{k-2} \cdots A_1)| \geq 1$, we have

$$|I_1(A_NA_{N-1} \cdots A_1)| \geq N.$$

This contradicts to (15). □

3. **Asymptotic exponential consensus.** In this section, we apply the relevant theory of (*sp*) matrices to estimate the asymptotic exponential consensus. Before that we consider a simple motivating example consisting of four agents (one leader, three followers).

3.1. **A motivating example.** Consider a four-agent model governed by (1). The neighbor graph has node set $\bar{\mathcal{V}} = \{0,1,2,3\}$ and the arc set in each time is given by the following possibilities:

$$\bar{\mathcal{E}}_1 = \{(0,1)\},\ \bar{\mathcal{E}}_2 = \{(1,2)\},\ \bar{\mathcal{E}}_3 = \{(2,3)\}.$$

We assume that across each time interval $[3k, 3k+3)$, $k \in \mathbb{N}$, of length 3, the graphs $\bar{\mathcal{G}}_{\sigma(t)}$'s exactly consist of $(\bar{\mathcal{V}}, \bar{\mathcal{E}}_1)$, $(\bar{\mathcal{V}}, \bar{\mathcal{E}}_2)$, $(\bar{\mathcal{V}}, \bar{\mathcal{E}}_3)$, whereas the ordering of the three graphs across time interval $[3k, 3k+3)$ changes with k. Obviously, the union of the three graphs is given by $0 \to 1 \to 2 \to 3$, which means that the topology is under jointly rooted leadership at 0 across each time interval.

We set $\hat{\theta}_i(t) = \theta_i(t) - \theta_0$, and $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)^\top$, we find that

$$\hat{\theta}(t+1) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{\theta}(t), \qquad \text{if } \sigma(t) = 1, \text{ i.e., } (\bar{\mathcal{V}}, \bar{\mathcal{E}}_1) \text{ is active,}$$

$$\hat{\theta}(t+1) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{\theta}(t), \qquad \text{if } \sigma(t) = 2, \text{ i.e., } (\bar{\mathcal{V}}, \bar{\mathcal{E}}_2) \text{ is active,}$$

$$\hat{\theta}(t+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \hat{\theta}(t), \qquad \text{if } \sigma(t) = 3, \text{ i.e., } (\bar{\mathcal{V}}, \bar{\mathcal{E}}_3) \text{ is active.}$$

We denote the three matrices above by F_1, F_2, and F_3 respectively. By assumption, across each time interval $[3k, 3k+3)$, the updating rule of the fluctuation $\hat{\theta}$ is given by some product of the three matrices, i.e., $F_{\tau(1)}F_{\tau(2)}F_{\tau(3)}$ where τ is a permutation of $\{1,2,3\}$. we now calculate all possible $F_{\tau(1)}F_{\tau(2)}F_{\tau(3)}$ to get 6 outputs. We can easily verify that all the 6 outputs are (*sp*) matrices. Then we choose three of them (note that multiplicity is allowed here) and multiply in any ordering to get 6^3 outputs. Taking the infinity norms for the 6^3 outputs, we find that the maximum of the infinity norms is $\frac{7}{8}$. Note that under the assumption on the switching signal, all possible products of the updating matrices across time interval $[3k, 3k+9), k \in \mathbb{N}$ must be one of the 6^3 outputs, so we find that the updating matrix from $t = 3k$ to $t = 3k+8$ has an infinity norm dominated by $\frac{7}{8}$. Then we easily see that for the above class of switching signals,

$$\|\hat{\theta}(t)\|_\infty \le \Big(\frac{7}{8}\Big)^{[\frac{t}{9}]} \|\hat{\theta}(0)\|_\infty \le \Big(\frac{7}{8}\Big)^{\frac{t}{9}-1} \|\hat{\theta}(0)\|_\infty = \frac{8}{7}\|\hat{\theta}(0)\|_\infty e^{-\frac{1}{9}(\ln 8 - \ln 7)t}, \quad t \in \mathbb{N},$$

where $\|\cdot\|_\infty$ denotes the infinity norm of vectors and $[a]$ denotes the integer part of a positive number a. This gives an exponential decay estimate for the consensus.

3.2. **Results on the consensus model.** In this section we apply the theory of (*sp*) matrices to the consensus problem of model (1) or (3) in compact form.

Lemma 3.1. *Consider the consensus system* (1) *or a compact form* (3)*. Suppose that this system is under joint rooted leadership across the time interval* $[t_0, t_0+T)$*. Then, the product of iterative matrices in* (3) *across this time interval is an* (*sp*) *matrix:*

$$F_{\sigma(t_0+T-1)} \cdots F_{\sigma(t_0+1)} F_{\sigma(t_0)} \in \mathcal{P} \bigcap \mathcal{SP}.$$

Proof. By the definition of F_p in (4) and Lemma 2.1, we see that

$$F_p \in \mathcal{S} \quad \text{and} \quad (F_p)_{ii} > 0, \quad i = 1, 2, \ldots, N,\ \forall p \in \mathcal{I},$$

which means that any product satisfies

$$F_{\sigma(t+s)} \cdots F_{\sigma(t+1)} F_{\sigma(t)} \in \mathcal{S} \bigcap \mathcal{P}, \quad t, s \in \mathbb{N}. \tag{16}$$

Next we verify the graph condition in Definition 2.3 for the product

$$F_{\sigma(t_0+T-1)} \cdots F_{\sigma(t_0+1)} F_{\sigma(t_0)}.$$

We set

$$I_* := \left\{ i_* \in \{1, 2, \ldots, N\} : \exists\, t_*(i_*) \in [t_0, t_0 + T) \text{ such that } (0, i_*) \in \bar{\mathcal{E}}_{\sigma(t_*(i_*))} \right\}.$$

By the joint rooted leadership, I_* is not empty. Moreover, from the property (2) in Lemma 2.1, we know that any $i_* \in I_*$ is a non-saturated vertex for $F_{\sigma(t_*(i_*))}$. We now present two properties for the graph associated with the product $F_{\sigma(t_0+T-1)} \cdots F_{\sigma(t_0+1)} F_{\sigma(t_0)}$.

(1) Every $i_* \in I_*$ is a non-saturated vertex for the product. This is simply a consequence of Lemma 2.4.

(2) If $(j, i) \in \bar{\mathcal{E}}_{\sigma(t_0)} \bigcup \cdots \bigcup \bar{\mathcal{E}}_{\sigma(t_0+T-1)}, i, j \neq 0$, then (j, i) is also a directed arc in the graph associated to $F_{\sigma(t_0+T-1)} \cdots F_{\sigma(t_0+1)} F_{\sigma(t_0)}$. To show this, we suppose $(j, i) \in \bar{\mathcal{E}}_{\sigma(t_0)} \bigcup \cdots \bigcup \bar{\mathcal{E}}_{\sigma(t_0+T-1)}$, then there exists a time $t \in [t_0, t_0 + T)$ such that $(j, i) \in \mathcal{E}_{\sigma(t)}$. Note that $(F_{\sigma(t)})_{ii} > 0$ for all $t \geq 0$ and $i = 1, 2, \ldots, N$. Then the following implication gives the desired result:

$$(F_p)_{ij} > 0 \implies \begin{cases} (F_p F_q)_{ij} \geq (F_p)_{ij} (F_q)_{jj} > 0 \\ (F_q F_p)_{ij} \geq (F_q)_{ii} (F_p)_{ij} > 0. \end{cases}$$

Now we observe that the first assertion means that any agent i with $(0, i) \in \bar{\mathcal{E}}_{\sigma(t)}$ for some time $t \in [t_0, t_0 + T)$ must be a non-saturated vertex in the digraph associated to $F_{\sigma(t_0+T-1)} \cdots F_{\sigma(t_0+1)} F_{\sigma(t_0)}$. The second one means that any arc $(j, i), i, j \neq 0$ in $\bar{\mathcal{E}}_{\sigma(t_0)} \bigcup \cdots \bigcup \bar{\mathcal{E}}_{\sigma(t_0+T-1)}$ is still an arc in the digraph associated to $F_{\sigma(t_0+T-1)} \cdots F_{\sigma(t_0+1)} F_{\sigma(t_0)}$. Combining with the definition of joint rooted leadership, we find

$$F_{\sigma(t_0+T-1)} \cdots F_{\sigma(t_0+1)} F_{\sigma(t_0)} \in \mathcal{P} \bigcap \mathcal{SP}.$$

□

Proposition 4. *Suppose that the state θ_0 for the leader agent is fixed and σ is a switching signal for which there exists an infinite sequence of contiguous, nonempty, bounded, time-intervals $[t_i, t_{i+1})$, $i \geq 0$, starting at $t_0 = 0$, with the property that the system is under joint rooted leadership across each such interval. Then the solution of system (1) or (2) exponentially converges to the consensus state, i.e., there exist positive constants C and λ such that*

$$|\theta_i(t) - \theta_0| < Ce^{-\lambda t}, \ \forall\, t \in \mathbb{N}, \ i = 1, 2, \ldots, N.$$

Proof. From Lemma 3.1 we see that

$$F_{\sigma(t_{i+1}-1)} \cdots F_{\sigma(t_i+1)} F_{\sigma(t_i)} \in \mathcal{P} \bigcap \mathcal{SP}, \quad \forall\, i. \tag{17}$$

The iterative product of the matrices $F_{\sigma(t)}$, $t \in \mathbb{N}$, can be viewed as the product of a sequence of (sp) matrices, each of which is a product as in (17). Then, by Proposition 2 we can conclude the desired result, since there is only a finite number of possible neighbor digraphs for the products in (17). □

Proposition 4 qualitatively gives the exponential convergence of $\hat{\theta}(t)$. Next, we refine the estimates on (sp) matrices to get an explicit estimate for consensus given by a simple formula. To do this, we first give the following lemma.

Lemma 3.2. *The matrices $F_{\sigma(t-1)}\cdots F_{\sigma(1)}F_{\sigma(0)}$ satisfy the following property:*

(1) If

$$(F_{\sigma(t-1)}\cdots F_{\sigma(1)}F_{\sigma(0)})_{ij} > 0,$$

then $(F_{\sigma(t-1)}\cdots F_{\sigma(1)}F_{\sigma(0)})_{ij} \geq (\frac{1}{N+1})^t$;

(2) If

$$i \in I_1(F_{\sigma(t-1)}\cdots F_{\sigma(1)}F_{\sigma(0)}),$$

then $\displaystyle\sum_{j=1}^{N}(F_{\sigma(t-1)}\cdots F_{\sigma(1)}F_{\sigma(0)})_{ij} \leq 1 - \Big(\frac{1}{N+1}\Big)^t.$

Proof. (1) This statement simply follows from the fact that every non-zero element of $F_{\sigma(i)}$ must be not less than $\frac{1}{N+1}$ (see Lemma 2.1 and Remark 1 (1)).
(2) We prove the second assertion by induction on t.
• **Initialization step** $(t = 1)$**.** By Lemma 2.1 (2) and Remark 1 (1), we have

$$\sum_{j=1}^{N}(F_{\sigma(0)})_{ij} \leq 1 - \frac{1}{N+1} \qquad \text{for any } i \in I_1(F_{\sigma(0)}).$$

In the sequel, for notational simplicity, we denote $F^{t_0} := F_{\sigma(t_0-1)}\cdots F_{\sigma(1)}F_{\sigma(0)}$.
• **Induction step.** Assume that the desired estimate holds for $t = t_0$, i.e., for any $i \in I_1(F_{\sigma(t_0-1)}\cdots F_{\sigma(1)}F_{\sigma(0)})$,

$$\sum_{j=1}^{N}(F^{t_0})_{ij} \leq 1 - \Big(\frac{1}{N+1}\Big)^{t_0}. \tag{18}$$

We now consider (t_0+1)-product $F_{\sigma(t_0)}F^{t_0}$.
⋄ **Case 1** $(F^{t_0} \in \mathcal{SP}_0)$**.** By induction hypothesis (18) and $\|F_{\sigma(t_0)}\|_\infty \leq 1$, we have

$$\|F_{\sigma(t_0)}F^{t_0}\|_\infty \leq \|F_{\sigma(t_0)}\|_\infty\|F^{t_0}\|_\infty \leq 1 - \Big(\frac{1}{N+1}\Big)^{t_0} < 1 - \Big(\frac{1}{N+1}\Big)^{t_0+1}.$$

⋄ **Case 2** $(F^{t_0} \notin \mathcal{SP}_0)$**.** In this case, Lemma 2.4 yields that the set $I_1(F_{\sigma(t_0)}F^{t_0})$ consists of three disjoint parts, that is,

$$I_1(F_{\sigma(t_0)}F^{t_0}) = I_1(F_{\sigma(t_0)})\bigcup\big(I_1(F^{t_0})\setminus I_1(F_{\sigma(t_0)})\big)\bigcup(I_e(F_{\sigma(t_0)},F^{t_0})\setminus I_1(F_{\sigma(t_0)})).$$

We consider there subcases according to this partition.
▹ **Subcase 2.1.** If $i \in I_1(F_{\sigma(t_0)})$, then

$$\begin{aligned}\sum_{j=1}^{N}(F_{\sigma(t_0)}F^{t_0})_{ij} &= \sum_{k=1}^{N}(F_{\sigma(t_0)})_{ik}\sum_{j=1}^{N}(F^{t_0})_{kj}\\ &\leq \sum_{k=1}^{N}(F_{\sigma(t_0)})_{ik} \leq 1 - \frac{1}{N+1}\\ &< 1 - \Big(\frac{1}{N+1}\Big)^{t_0+1}.\end{aligned}$$

▷ **Subcase 2.2.** If $i \in I_1(F^{t_0}) \setminus I_1(F_{\sigma(t_0)})$, we have

$$\begin{aligned}
1-\sum_{j=1}^{N}(F_{\sigma(t_0)}F^{t_0})_{ij} &= 1-\sum_{k=1}^{N}(F_{\sigma(t_0)})_{ik}\big(\sum_{j=1}^{N}(F^{t_0})_{kj}\big)\\
&= 1-\sum_{k=1,k\neq i}^{N}(F_{\sigma(t_0)})_{ik}\big(\sum_{j=1}^{N}(F^{t_0})_{kj}\big)\\
&\quad -(F_{\sigma(t_0)})_{ii}\big(\sum_{j=1}^{N}(F^{t_0})_{ij}\big)\\
&\geq 1-\sum_{k=1,k\neq i}^{N}(F_{\sigma(t_0)})_{ik}-(F_{\sigma(t_0)})_{ii}\big(\sum_{j=1}^{N}(F^{t_0})_{ij}\big)\\
&\geq (F_{\sigma(t_0)})_{ii}-(F_{\sigma(t_0)})_{ii}\Big[1-\Big(\frac{1}{N+1}\Big)^{t_0}\Big]\\
&= (F_{\sigma(t_0)})_{ii}\Big(\frac{1}{N+1}\Big)^{t_0} \geq \Big(\frac{1}{N+1}\Big)^{t_0+1},
\end{aligned}$$

where we used the facts:

$$\begin{aligned}
&\sum_{j=1}^{N}(F^{t_0})_{kj} < 1, \quad \text{and}\\
&\sum_{k=1}^{N}(F_{\sigma(t_0)})_{ik} = \sum_{k=1,k\neq i}^{N}(F_{\sigma(t_0)})_{ik}+(F_{\sigma(t_0)})_{ii} \leq 1.
\end{aligned}$$

▷ **Subcase 2.3.** If $i \in I_e(F_{\sigma(t_0)}, F^{t_0}) \setminus I_1(F_{\sigma(t_0)})$, we first recall that

$$I_e(F_{\sigma(t_0)}, F^{t_0}) := \big\{ i \in I_2(F_{\sigma(t_0)}) : \exists\, l \in I_1(F^{t_0}) \text{ such that } (F_{\sigma(t_0)})_{il} > 0 \big\}.$$

By the similar arguments as in Subcase 2.2, we have

$$\begin{aligned}
1-\sum_{j=1}^{N}(F_{\sigma(t_0)}F^{t_0})_{ij} &= 1-\sum_{k=1}^{N}(F_{\sigma(t_0)})_{ik}\big(\sum_{j=1}^{N}(F^{t_0})_{kj}\big)\\
&= 1-\sum_{k=1,k\neq l}^{N}(F_{\sigma(t_0)})_{ik}\big(\sum_{j=1}^{N}(F^{t_0})_{kj}\big)\\
&\quad -(F_{\sigma(t_0)})_{il}\big(\sum_{j=1}^{N}(F^{t_0})_{lj}\big)\\
&\geq (F_{\sigma(t_0)})_{il}-(F_{\sigma(t_0)})_{il}\Big[1-\Big(\frac{1}{N+1}\Big)^{t_0}\Big]\\
&= (F_{\sigma(t_0)})_{il}\Big(\frac{1}{N+1}\Big)^{t_0} \geq (\frac{1}{N+1})^{t_0+1},
\end{aligned}$$

which implies $\sum_{j=1}^{N}(F_{\sigma(t_0)}F^{t_0})_{ij} \leq 1-(\frac{1}{N+1})^{t_0+1}$. We now combine subcases 2.1-2.3 to see that the estimate is true for $t = t_0+1$. Therefore, by induction we conclude the estimate in (2). □

We are now ready to present the consensus estimate for the linearized Vicsek model.

Theorem 3.3. *Consider the consensus system* (1) *or a compact form* (3). *Suppose that the conditions in Proposition 4 hold, and we denote the uniform bound of the time interval* $[t_i, t_{i+1})$ *by* $T \geq 1$. *Then we have:*

(1) The product of iterative matrices for the fluctuation across each time interval is an (sp) *matrix:*

$$F_{\sigma(t_{i+1}-1)} \cdots F_{\sigma(t_i+1)} F_{\sigma(t_i)} \in \mathcal{P} \bigcap \mathcal{SP}.$$

(2) The fluctuation decays at least exponentially fast:

$$\|\hat{\theta}(t)\|_\infty \leq \|\hat{\theta}(0)\|_\infty \left(1 + \frac{1}{(N+1)^{NT} - 1}\right) e^{-\frac{\ln(N+1)^{NT} - \ln\left((N+1)^{NT}-1\right)}{NT} t}, \quad t \in \mathbb{N}.$$

Proof. In Lemma 3.1, we already found that along any time interval $[t_i, t_{i+1} - 1)$,

$$F_{\sigma(t_{i+1}-1)} \cdots F_{\sigma(t_i+1)} F_{\sigma(t_i)} \in \mathcal{SP}.$$

Thus, we only need to refine the qualitative estimate in Proposition 4 to verify the inequality in Theorem 3.3 (2). Lemma 3 claims that an N-product of (sp) matrices becomes trivial; thus,

$$F_{\sigma(t_{i+N}-1)} \cdots F_{\sigma(t_{i+1}-1)} \cdots F_{\sigma(t_i+1)} F_{\sigma(t_i)} \in \mathcal{SP}_0.$$

By Lemma 3.2, we derive that the infinity norm of

$$F_{\sigma(t_{i+N}-1)} \cdots F_{\sigma(t_{i+1}-1)} \cdots F_{\sigma(t_i+1)} F_{\sigma(t_i)}$$

is dominated by the deterministic constant $1 - (\frac{1}{N+1})^{NT}$. Thus, we have the following estimate:

$$\begin{aligned}
\|\hat{\theta}(t)\|_\infty &\leq \|\hat{\theta}(0)\|_\infty \Big(1 - (\frac{1}{N+1})^{NT}\Big)^{[\frac{t}{NT}]} \\
&\leq \|\hat{\theta}(0)\|_\infty \Big(1 - (\frac{1}{N+1})^{NT}\Big)^{\frac{t}{NT}-1} \\
&= \|\hat{\theta}(0)\|_\infty \left(1 + \frac{1}{(N+1)^{NT} - 1}\right) e^{-\frac{\ln(N+1)^{NT} - \ln\left((N+1)^{NT}-1\right)}{NT} t}, \quad t \in \mathbb{N}.
\end{aligned}$$

Here $[\frac{t}{NT}]$ denotes the integer part of $\frac{t}{NT}$. □

3.3. **Remarks.** In the proof of Theorem 3.3, we used the fact that any NT-product of $F_{\sigma(t)}$ is a trivial (sp) matrix, which is implied by Lemma 3.1 and Proposition 3. We note that NT is indeed necessary to build a trivial (sp) matrix. To show this, we consider the configurations in Subsection 3.1 for which $N = 3$ and $T = 3$. For F_1, F_2 and F_3 in this case, Lemma 3.1 claims that any product $F_{\tau(1)} F_{\tau(2)} F_{\tau(3)}$ is an (sp) matrix where τ is a permutation of $\{1, 2, 3\}$. Then, by Proposition 3 we see that any 9-product of F_p's built from the joint rooted leadership, for example, $\underbrace{F_3F_1F_2}\,\underbrace{F_1F_2F_3}\,\underbrace{F_1F_2F_3}$, must be a trivial ($sp$) matrix. However, the 8-product

$$F_1F_2\,\underbrace{F_1F_2F_3}\,\underbrace{F_1F_2F_3} = \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ \frac{3}{8} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

is not a trivial (sp) matrix. This shows that the number NT is not improvable. This example also shows that the product of $(N-1)$ (sp) matrices may build a non-trivial (sp) matrix; in other words, the result in Proposition 3 is not improvable.

Theorem 3.1 gives an upper bound for the maximum Lyapunov exponent (denoted by λ) of the linearized Vicsek model:

$$\lambda \le -\frac{\ln(N+1)^{NT} - \ln\big((N+1)^{NT}-1\big)}{NT} < 0.$$

We set the bound for the decay exponent by $f(N,T)$:

$$\begin{aligned} f(N,T) &:= -\frac{\ln(N+1)^{NT} - \ln\big((N+1)^{NT}-1\big)}{NT} \\ &= \frac{1}{NT}\ln\Big[\frac{(N+1)^{NT}-1}{(N+1)^{NT}}\Big] < 0. \end{aligned}$$

Consider the function defined on the continuous-time domain, i.e., $t \in \mathbb{R}^+$:

$$f_N(t) := -\ln(N+1) + \frac{\ln\big((N+1)^{Nt}-1\big)}{Nt} < 0.$$

Note that for all $t > 0$,

$$\begin{aligned} f_N'(t) &= \frac{\frac{(N+1)^{Nt}}{(N+1)^{Nt}-1} Nt\ln(N+1) - \ln\big((N+1)^{Nt}-1\big)}{Nt^2} \\ &> \frac{\ln(N+1)^{Nt} - \ln\big((N+1)^{Nt}-1\big)}{Nt^2} > 0. \end{aligned}$$

This means $f(N,T)$ is an monotonically increasing function in T, thus our upper bound gives faster exponential decay when the time length for the joint connectivity is smaller. This is consistent with the intuition. On the other hand, we observe that

$$\lim_{T\to\infty} f(N,T) = 0, \quad \lim_{N\to\infty} f(N,T) = 0.$$

This means that as $T \to \infty$ or $N \to \infty$, our upper bound does not provide the decay estimate, which suggests plausible occurrence of glassy behavior in the convergence toward the consensus in the asymptotic limits.

Variant consensus problems have been largely studied in the past decades, see for example [2, 3, 4, 7, 9, 14] and references therein. Many of them have been carried out by studying the product of stochastic matrices [2, 3, 4, 9, 14]. In [3, 9], systematic studies on the graph and consensus problem were carried out. We acknowledge that our consensus result is not new since the assumption of joint rooted leadership can be covered by the interaction topologies in related literature. For example, in [9, Theorem 9.2], the consensus is established with an explicit decay estimate if there exists a sequence of contiguous intervals of bounded length over each of which at least one vertex is connected to all others. Here, they say "j is connected to i over an interval $[t_1, t_2]$" if "there exists a sequence of vertices $j = v_{t_1}, v_{t_1+1}, \dots, v_{t_2} = i$ such that $(v_t, v_{t+1}) \in E_t$ holds for any $t \in [t_1, t_2]$" (see [9, pp. 149]). Then, in [9, Corollary 9.1] the consensus is concluded if the union graph over the time interval $[t, t+T]$ contains a spanning tree for any t; this assumption indeed covers the joint rooted leadership. But, in [9, Corollary 9.1] there is no direct bound given for B to fulfill the conditions in [9, Theorem 9.2] by which a decay rate is derived. The usage of (*sp*) matrices in this work gives the direct bound NT for B in [9, Theorem 9.2] and $\alpha = \frac{1}{N+1}$. Hence, the results in Theorem 3.1 are the same as [9, Theorem 9.2] provided one knows an upper bound for B in [9, Theorem 9.2]. In [3, Proposition 12] the consensus is established if the neighbor graphs satisfy "repeatedly jointly rooted"; this also covers the joint rooted leadership in this paper. An exponential decay estimate is concluded in [3, Proposition 12] but it is given by the maximum of all possible products of those matrices built from the repeatedly jointly rooted interactions. The approach of (*sp*) matrices in this work gives an explicit and direct estimate on the decay rate.

The main purpose of this paper is to extend the theory of (*sp*) matrices and use it to revisit the linearized Vicsek model with a joint rooted leadership. Due to the leader-follower structure, we study the consensus problem by considering the zero-convergence of fluctuation and thus the discrete-time dynamics can be modeled by sub-stochastic matrices. Then we can employ the approach of (*sp*) matrices. This approach has been applied in Cucker-Smale flocking [11, 12] and we hope to see it in a wider context.

4. **Conclusion.** In this paper, we revisited the consensus problem for the linearized Vicsek model. We improved the theory of (*sp*) matrices to cover the joint rooted leadership topology, and applied the improved theory to the consensus problem for the linearized Vicsek model. Our consensus estimate provides an explicit upper bound for the Lyapunov exponent of the discrete-time linearized Vicsek model under the "joint connectivity" across time intervals. Our results essentially employ the theory of (*sp*) matrices and its switching stability theory.

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