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DYNAMICS AND KINETIC LIMIT FOR A SYSTEM OF NOISELESS d-DIMENSIONAL VICSEK-TYPE PARTICLES

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Abstract. We analyze the continuous time evolution of a d-dimensional system of N self propelled particles with a kinematic constraint on the velocities inspired by the original Vicsek's one [29]. Interactions among particles are specified by a pairwise potential in such a way that the velocity of any given particle is updated to the weighted average velocity of all those particles interacting with it. The weights are given in terms of the interaction rate function. The interaction is not of mean field type and the system is non-Hamiltonian. When the size of the system is fixed, we show the existence of an invariant manifold in the phase space and prove its exponential asymptotic stability. In the kinetic limit we show that the particle density satisfies a nonlinear kinetic equation of Vlasov type, under suitable conditions on the interaction. We study the qualitative behaviour of the solution and we show that the Boltzmann-Vlasov entropy is strictly decreasing in time.

1. **Introduction.** The analysis of a network of a large number of coordinated self propelled particles (agents) is a sub discipline of control theory which has seen a rapid development during the last decade [8, 30, 21, 13, 4, 14, 12]. This is due to its several potential application in understanding the collective behavior in biological systems (for example fish schools and bird flocks) [13], computer science [25, 8], engineering [21, 14, 12], economy [18] and social sciences [30, 4]. Explaining the emergence of these coordinated movements in terms of microscopic decisions of each individual member of a network is a hot matter of research in the natural sciences. To model the particle self-organized behavior one assigns to each particle

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a simple communication/interaction rule in order for the whole system to dynamically reproduce, in a given regime of the model's parameters, specific phase space patterns.

The emergence of phase space patterns persistent in time described by a large connected cluster of coherently moving particles is called flocking or swarming (also schooling or herd behavior). Basic models of flocking behavior generally follow three simple rules: 1) separation, that is to avoid crowding neighbors (usually modeled by short range repulsive interactions); 2) alignment, i.e. to steer towards average heading of neighbors; 3) cohesion, i.e. to steer towards the average position of neighbors (usually modeled by long range attractive interactions).

The seminal work in the direction of modeling flocking behavior is the one of Vicsek et al. [29]. They proposed a model of N interacting particles located on a two-dimensional torus of diameter D. The velocity of each given particle belongs to the unit circle and at each time step its direction is updated at the empirical average of the velocity's directions of all the particles lying in a neighborhood of radius 1 from the given one, including itself, plus a random perturbation. Particles positions are then updated according to their velocity. Computer simulations proved that, when the particle density $\frac{N}{D^2}$ is sufficiently high and the noise intensity sufficiently small, the distribution of the velocities of the particles concentrates around the velocity of the barycenter of the system, although this is not a quantity preserved by the dynamics.

We propose a simple model of continuous time noiseless multi-agent evolution closely inspired to the original Vicsek's one. The particles interact (communicate) with each other trough a pairwise interaction potential in such a way that the velocity of any given particle is updated to the weighted average velocity of all those particles communicating with it. This choice makes the interaction not of mean field type. Furthermore, the system is non Hamiltonian. As a result, there is a tendency of neighboring particles to align their velocities. This is the crucial element in the mechanism of the emergency of a coherent motion.

For what concerns flocking behaviour our model takes into account alignment and cohesion, but violates the separation rule since the particles can overlap.

We prove for such model two type of results. First, we analyze the N particle dynamics in $\mathbb{R}^d$. We show that there exists an invariant manifold in the phase space and prove exponential asymptotic stability of the invariant manifold when the initial conditions for particles dynamics are suitably chosen. This implies that the system, under the chosen initial conditions, will reach a state of flocking. Then, we study the kinetic limit ($N \to \infty$) of the system. Since the interaction is not of mean field type care needs to be taken in the definition of the velocity field in the phase space and consequently in the evolution of the particle density. We explain in more details how to deal with these difficulties in Section 4. We prove that the particle density satisfies a Boltzmann-Vlasov equation when the particles are confined on a torus and subject to a *short-range potential* of Gaussian type. Similar result holds in $\mathbb{R}^d$ when the interaction among particles is given by a suitable regularization of a finite range potential. We further show that the Boltzmann-Vlasov entropy is strictly decreasing in time. As a consequence, one can argue that, even if the initial distribution of the particles is absolutely continuous w.r.t. Lebesgue measure, the limit density distribution is singular w.r.t. Lebesgue measure. This is consistent with what one expects from the model. For time long enough the position and velocity particle distribution will concentrate on specific phase-space patterns.

A continuous time version of Vicsek's model, as well as its stochastic counterpart driven by the Brownian motion, has been proposed in [15] and the corresponding kinetic equations heuristically derived and studied. In fact, at present time, to our knowledge, a rigorous derivation and analysis of Vicsek's model kinetics, as well as hydrodynamics, is lacking.

Another basic model for flocking is the Cucker-Smale one [14]. In this and related models [17, 3] the variation in time of the momentum of a given particle is the weighted sum of the differences between the particle's momentum and those of the other system's components, with weights depending of the relative distances among particles divided by the total number of particles N. It is worth notice that, for all these models, the interaction between two given particles is of order $1/N$, therefore when the size of the system becomes large, particles tend to decorrelate. On the contrary, in the original Vicsek's model, the interaction between a given couple of particles is of order one. Moreover, Cucker-Smale dynamics preserves the velocity of the barycenter, which is not the case for Vicsek's. The order of the interaction with respect to the size of the system is the peculiar feature distinguishing Vicsek's from Cucker-Smale algorithm. Therefore, in our opinion, variants of the Cucker-Smale momenta updating rule taking into account only the differences among the directions of the momenta of the particles, rather than those of the momenta as vectors, are somewhat improperly ascribed to variants of the Vicsek's model [7]. Cucker-Smale and related models have been more deeply investigated in the mathematical literature and their mean-field limit equations rigorously derived and studied in [20, 19, 11, 9, 3] in the noiseless case and in [6, 7] in the stochastic case driven by Brownian motion. Moreover, the hydrodynamics equations for these models have also been rigorously studied but formally derived [20, 10, 9].

Recently, a model analogous to the one we propose in this paper, but with communication rate function restricted to the Cucker-Smale model one has been introduced and analysed in [23]. The authors prove that the strategy originally proposed to study the emergence of flocking behaviour for a system of self-propelled particle updating their velocity with the standard Cucker-Smale algorithm also applies to this case with the same restrictions on the decay of the communication rate function. It turns out that the model we propose in the present paper is more general than the one proposed in [23], since includes also sufficiently smooth compactly supported communication rate functions, and so are the results about the emergence of flocking behaviour for the particle system.

The plan of the paper is the following. In Section 2 we describe the model, set the notations and present the main results. In Section 3 we analyze the system when the number of particles is fixed. In Section 4 we analyze the system when the number of particles goes to infinity. In the appendix we collect proofs of results used along the previous sections.

2. Description of the model, notation and results.

2.1. Notations.

Given $x \in \mathbb{R}^d, d \geq 1$, we denote by x^i its i-th component, $i = 1, .., d$, with respect to the canonical basis $(e_1, .., e_d)$. For any $x, y \in \mathbb{R}^d$ we set $x \cdot y := \sum_{i=1}^d x^i y^i$ to be the scalar product between x and y. Hence, we denote by $|x| := \sqrt{x \cdot x}$ the associated Euclidean norm and by $B_r(x) := \{y \in \mathbb{R}^d : |y - x| \leq r\}$ the ball of radius $r > 0$ centered at x and $B_r := B_r(0)$. Furthermore we set $\|x\|_\infty := \max_{i=1,..,d} |x^i|$.

Given an integer $N \geq 2$, we denote a point in $\mathbb{R}^{Nd}$ by $\mathbf{x} := (x_1, .., x_N) \in \mathbb{R}^{Nd}$, its norm by $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{y}}$, where $\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{N} x_i \cdot y_i$. We denote by $\mathbf{B}_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^{Nd} : |\mathbf{y} - \mathbf{x}| \leq r\}$ the ball of radius $r > 0$ centered at $\mathbf{x}$.

Partial derivative w.r.t. any component x^i of $x \in \mathbb{R}^d$ will be denoted by ∂_{x^i}, so that ∇_x stands for $(\partial_{x^1}, .., \partial_{x^d})$ while, for any $\mathbf{q} \in \mathbb{R}^{Nd}$, we set $\nabla_{\mathbf{q}} := (\nabla_{q_1}, .., \nabla_{q_N})$.

Moreover, we denote by $\mathcal{L}_n(\mathbb{R})$ the space of linear operators from $\mathbb{R}^n$ to itself and by $\|\cdot\|$ and $\|\cdot\|_\infty$ the operator norm induced by respectively the Euclidean and the supremum norm. In particular $\mathbb{I}_n, \mathbf{0}_n \in \mathcal{L}_n(\mathbb{R})$ denote respectively the identity and the null operator.

2.2. **The model.** Let $N \geq 2$ be an integer. We consider N particles of unitary mass in $\mathbb{R}^d$ evolving according to the equations:

$$\begin{cases} \frac{dq_i(t)}{dt} = p_i(t) \\ \frac{dp_i(t)}{dt} = \frac{\sum_{j=1}^{N} U(q_i(t)-q_j(t))(p_j(t)-p_i(t))}{\sum_{j=1}^{N} U(q_i(t)-q_j(t))} \ , \ i = 1, .., N \\ q_i(0) = q_i^0 \ ; \ p_i(0) = p_i^0 \end{cases} \tag{1}$$

where, for $i = 1, .., N$, $(q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^d$, (q_i^0, p_i^0) are the initial conditions and U is a pairwise interaction. We assume that $U(\cdot)$ is a spherically symmetric positive function, sufficiently smooth, with support the ball of radius R centred at zero and so that $U(0) > 0$. This implies that the denominator in the second equation of (1) is always strictly positive. The choice of R does not play any particular role in the analysis. Without loss of generality we assume that $\int U(x)dx = 1$ and $\sup_{x \in \mathbb{R}^d} U(x) = U(0)$. Namely, for agent-based models it is reasonable to assume that self-interaction is stronger than the interactions between two different particles. A simple example to have in mind for the potential U is $U(x) = C(d)(1-|x|)\mathbb{1}_{B_1}(x)$, where $C(d)$ is taken such that $\int U(x)dx = 1$ or smoother versions of this. In this example, the particle q_i interacts only with particles at distance 1. The vector field in (1) is Lipschitz, therefore the existence and the uniqueness of the solution is granted at least for short time. Since the vector field increases at most linearly in $w = (\mathbf{q}, \mathbf{p})$ the solution $w(t, w^0)$ with initial datum w^0 exists and it is unique for all $t \geq 0$.

To derive the kinetic limit results, the interaction must satisfy further requirements which will be presented and discussed in the following.

2.2.1. *Flocking.* Given a particle configuration $\mathbf{q} \in \mathbb{R}^{Nd}$ we introduce the notion of communication graph. We use only basic definition of graph theory useful to define the *flocking behaviour* for the system (1). We refer the reader to basic textbooks such as [5] for an account on this subject.

Definition 2.1. Given a particle configuration $\mathbf{q} \in \mathbb{R}^{Nd}$, we define the *communication graph* $\mathcal{G}(\mathbf{q}) := (\mathcal{V}(\mathbf{q}), \mathcal{E}(\mathbf{q}))$, where the set of vertices $\mathcal{V}(\mathbf{q}) = \{q_1, \ldots, q_N\}$ is the collection of the N points of $\mathbb{R}^d$ associated to $\mathbf{q}$ and

$$\mathcal{E}(\mathbf{q}) := \{(q, q') \in \mathcal{V}(\mathbf{q}) \times \mathcal{V}(\mathbf{q}) : U(q - q') > 0\} \tag{2}$$

is the set of edges.

Two vertices q and q' are said to be connected if there are $q_1, \ldots, q_k$ vertices in $\mathcal{V}(\mathbf{q}), k \in \{2, \ldots, N\}$, such that $q_1 = q, q_k = q'$ and $U(q_i - q_{i+1}) > 0$, for

$i = 1, \ldots, k-1$. The graph $\mathcal{G}(\mathbf{q})$ is said to be connected if any two of its vertices are connected[1]. We will set $\mathcal{V}_t := \mathcal{V}(\mathbf{q}(t))$ and $\mathcal{G}(t) := \mathcal{G}(\mathbf{q}(t))$.

Definition 2.2. The system (1) with initial conditions $w^0 = (\mathbf{q}^0, \mathbf{p}^0)$ is said to exhibit a *flocking behavior* if there exists $v \in \mathbb{R}^d$ such that, for any $\epsilon > 0, \exists T_\epsilon > 0 : \forall t > T_\epsilon$,

- $p_i\left(t, w^0\right) \in B_\epsilon(v), \forall i = 1, .., N$;
- the communication graph $\mathcal{G}(t)$ is connected.

We remark that our definition of emergence of flocking behaviour differs from the one given for models with long interaction (e.g. Cucker-Smale model [19], [11]). In the latter case the communication graph is always connected, while this is not true for short range interactions. Therefore we have in Definition 2.2 two conditions, one on the particle velocities and the other on the particle positions.

2.3. **Results for finite size system.** Let $\mathcal{I}$ be the $(N+1)d$ linear manifold

$$\mathcal{I} = \cup_{\{v \in \mathbb{R}^d\}} \mathcal{I}(v) \ , \tag{3}$$

where

$$\mathcal{I}(v) = \{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN} : p_i = v, i = 1, .., N\} \ . \tag{4}$$

It is immediate to see that $\mathcal{I}$ is invariant for the evolution (1). Namely, if the initial data belong to $\mathcal{I}(v)$ the particles evolve independently one from the other with constant velocity v. The only critical point of the system (1) is $(\mathbf{0}, \mathbf{0})$. We denote, for any $w \in \mathbb{R}^{2dN}$,

$$\operatorname{dist}(w, \mathcal{I}) = \inf_{w^0 \in \mathcal{I}} |w - w^0| \tag{5}$$

and by $w(t, w^1)$ the solution at time t of (1) starting from $w^1 \in \mathbb{R}^{2dN}$. We have the following results.

Theorem 2.3. *The manifold $\mathcal{I}$ is stable for the evolution* (1).

This means that, for any $\epsilon > 0$, there exists $\delta(\epsilon) \leq \epsilon$ such that, for all initial data $w^0 \in \mathbb{R}^{2dN}$ satisfying $\operatorname{dist}(w^0, \mathcal{I}) \leq \delta(\epsilon)$, then $\operatorname{dist}(w(t, w^0), \mathcal{I}) \leq \epsilon$ for all $t \geq 0$. Stability of the manifold $\mathcal{I}$ does not imply that the system exhibits a flocking behaviour when starting from w^0. Theorem 2.3 is quite easy to show, see for the proof Corollary 1.

Next, we show a stronger result. Assume that at initial time the particle positions are chosen so that the communication graph is connected and their velocities are conveniently taken; then, at later times, the particles will not split into non interacting groups and the velocity of each one converges exponentially fast to a velocity vector which is the same for all the N particles. In other words, the system exhibits a flocking behaviour, see Definition 2.2.

Theorem 2.4. *Let $w^0 = (\mathbf{q}^0, \mathbf{p}^0) \in \mathcal{I}$ and assume that the communication graph $\mathcal{G}\left(\mathbf{q^0}\right)$ is connected. There exist three positive constants $r_0 = r_0(w^0), T = T\left(w^0\right)$, $\epsilon_0 = \epsilon_0\left(w^0\right)$ and a set $\mathcal{B}(r_0, \epsilon_0, w^0) \subset \mathbb{R}^{2Nd}$, such that, for any initial datum $w^1 \in \mathcal{B}(r_0, \epsilon_0, w^0)$*

$$\operatorname{dist}\left(w(t, w^1), \mathcal{I}\right) \leq \epsilon_0(w^0) e^{-t\frac{\log 2}{T}} \ . \tag{6}$$

The proof of Theorem 2.4 is presented in Section 3.

[1]Since U is spherically symmetric, the communication graph is undirected. Hence, in this case, the usual notions of strongly connected graph and connected graph coincide.

2.4. **Results for infinite size system.** The main difficulty in deriving the kinetic limit, from system (1) is that the interaction, see (1), is not mean field, i.e. it is not divided by N, the total particle number. This creates problems in the definition of the evolution of the particle density since the velocity field in the phase space may be ill defined if further assumptions on the interaction U and on the configuration space are not taken into account. We will discuss this point extensively in Section 4. We overcome these difficulties in two ways. The first way is adding $\epsilon > 0$, which will be kept fixed, to the denominator of the second equation of (1). We keep the interaction U of compact support and assume for definiteness $\sup_x |\nabla U(x)| \leq 1$. We will refer to the system (1) modified in such a way as $\epsilon-$*regularized system*. A second way is to confine the system (1) in the torus of linear size $D > 0, \mathcal{T}_D$, and taking interactions U verifying the following assumptions.

Definition 2.5. Assumptions on the interaction Let $\tilde{U} : \mathbb{R}^d \to \mathbb{R}_+$ be such that either

$$\sup_{x\in\mathbb{R}^d} \left|\nabla \log \tilde{U}(x)\right| \leq K \tag{7}$$

or

$$\tilde{U}(x) = \frac{1}{(2\pi R^2)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2R^2}} . \tag{8}$$

We then define U to be the periodization on the torus $\mathcal{T}_D$ of one of the previous $\tilde{U}$:

$$U(x) = \sum_{n\in\mathbb{Z}^d} \tilde{U}(x + nD). \tag{9}$$

Remark 1. The assumption (7) is quite strong. An interaction $\tilde{U}$ verifying this assumption should decay for $|x|$ large as $e^{-\frac{|x|}{R}}$, for some $R > 0$. Interactions with compact support do not satisfy this assumption as well as the interaction (8).

Next, we define the space of measures and the metric we will be using. We denote by $\mathcal{M}$ the space of probability measures on $(X \times B_1, \mathcal{B}(X \times B_1))$, where the symbol X stands either for $\mathbb{R}^d$ or for the torus $\mathcal{T}_D$, B_1 denotes the ball of radius 1 in $\mathbb{R}^d$ and $\mathcal{B}(X \times B_1)$ is the Borel σalgebra on $X \times B_1$. We will prove that there is no loss of generality to confine the velocity in a bounded set and for definiteness we identify this set with B_1. We will be using the same notations either to denote the space of probability measure on $\mathcal{T}_D \times B_1$ or the space of probability measure on $\mathbb{R}^d \times B_1$, unless we will have the need to distinguish between the two configuration spaces in which case we will use the notation $\mathcal{M}(X \times B_1)$. In this space we introduce the bounded Lipschitz distance $d_{b\mathcal{L}}$ defined as follows. The $d_{b\mathcal{L}}$ distance between two measures μ and ν in $\mathcal{M}$ is given by

$$d_{b\mathcal{L}}(\mu, \nu) = \sup_{g\in\mathcal{D}} \left| \int g(x, v)\mu(dx, dv) - \int g(x, v)\nu(dx, dv) \right| , \tag{10}$$

where

$$\mathcal{D} := \left\{ g \mid g : X \times B_1 \to [0, 1] \ ; \ |g(x, v) - g(y, w)| \leq \sqrt{|v - w|^2 + |x - y|^2} \right\} . \tag{11}$$

The metric $d_{b\mathcal{L}}$ generates the weak* topology[2] on $\mathcal{M}$: for a sequence $\mu^N \in \mathcal{M}$ and $\mu \in \mathcal{M}$

[2] We refer the reader to [22] for an account on the notion of weak convergence of measures and to [28] for the relation between the bounded Lipschitz distance and the Kantorovich-Rubinstein (Wasserstein) distance.

$$\lim_{N\to\infty} d_{b\mathcal{L}}(\mu^N,\mu)=0$$

is equivalent to

$$\lim_{N\to\infty}\int g(w)\mu^N(dw)=\int g(w)\mu(dw)\,, \tag{12}$$

for all bounded and continuous function g on X. In the following we denote the convergence in (12) by $\mu^N \overset{w}{\Longrightarrow} \mu$.

For $(q_j,p_j)\in\mathbb{R}^{2d}, j=1,..,N$, we denote by μ^N the empirical measure

$$\mu^N(dx,dv):=\frac{1}{N}\sum_{j=1}^{N}\delta(q_j-x)\delta(p_j-v)dxdv, \tag{13}$$

where $\delta(x-y)dx$ is the Dirac measure at $y\in\mathbb{R}^d$. Hence, μ_t^N denotes the empirical measure (13) when the $((q_j(t),p_j(t)),\ j=1,..,N$, are the solutions of (1). In this case we say that μ_t^N is the empirical measure at time t associated to $w(t,w^0)$, where $w^0=(\mathbf{q}^0,\mathbf{p}^0)$. Given a smooth function g on $X\times B_1$ and $\mu\in\mathcal{M}$ we denote by

$$\mu(g)=\int_{X\times B_1} g(x,v)\mu(dx,dv) \tag{14}$$

and

$$(U\star\mu)(x)=\int_{X\times B_1} U(x-y)\mu(dy,du).$$

We have the following main results.

Theorem 2.6. *Let $w^0=(\mathbf{q}^0,\mathbf{p}^0)\in(\mathcal{T}_D\times B_1)^N$ and μ_t^N, $t\geq 0$, be the empirical measure associated to $w(t,w^0)$, the solution of (1) with U chosen as Definition 2.5. Let $\mu_0\in\mathcal{M}$ be such that*

$$\lim_{N\to\infty} d_{b\mathcal{L}}(\mu_0^N,\mu_0)=0. \tag{15}$$

Then, there exists $\mu_t\in\mathcal{M}$ such that

$$\lim_{N\to\infty} d_{b\mathcal{L}}(\mu_t^N,\mu_t)=0, \tag{16}$$

where μ_t is the measure solution of the following equation

$$\frac{\partial(\mu_t(g))}{\partial t}=\mu_t(v\cdot\nabla_x g)+\mu_t(M(\cdot,\cdot,\mu_t)\cdot\nabla_v g)\ ,\forall g\in\mathcal{D}\ , \tag{17}$$

and for $\nu\in\mathcal{M}$,

$$\mathcal{T}_D\times B_1\ni(x,v)\longmapsto M(x,v,\nu):=\left(\frac{\int_{\mathcal{T}_D\times B_1}U(x-y)u\nu(dy,du)}{\int_{\mathcal{T}_D\times B_1}U(x-y)\nu(dy,du)}\right)-v\in\mathbb{R}^d\ . \tag{18}$$

The next result establishes that under regularity assumptions on the initial measure μ_0 and on the interaction U the solution μ_t of (17) is regular as well.

Theorem 2.7. *Take U as in Definition 2.5. If $\mu_0(dx,dv)=f_0(x,v)dxdv$, then $\mu_t(dx,dv)=f_t(x,v)dxdv$ and f_t is the weak solution of*

$$\frac{\partial}{\partial t}f_t(x,v)+v\cdot\nabla_x f_t(x,v)+\nabla_v\cdot[M(x,v,f_t)f_t(x,v)]=0\ . \tag{19}$$

Furthermore, if $f_0\in C^k(X\times B_1), k\geq 1$, and $U\in C^k(X)$ then $f_t\in C^k(X\times B_1)$.

In Section 4, see Remark 5 and Remark 6, we will show that Theorem 2.6 and Theorem 2.7 hold also for the $\epsilon-$ *regularized system* (1) when considering the configuration space X to be either $\mathbb{R}^d$ or $\mathcal{T}_D$ and, for any $\nu \in \mathcal{M}$, $M(\cdot,\cdot,\nu)$ is replaced by

$$X \times B_1 \ni (x,v) \longmapsto M_\epsilon(x,v,\nu) := \left(\frac{\int_{X\times B_1} U(x-y) u \nu(dy,du)}{\int_{X\times B_1} U(x-y)\nu(dy,du) + \epsilon} \right) - v \in \mathbb{R}^d \ . \tag{20}$$

The results are shown adapting to our context the method reported in Spohn's book [26, Section5] (see also Neunzert [24] and Dobrushin [16]) and some classical tools of dynamical systems. The main difference between the case considered here and the one presented in [26] is that, in our case, the dependence of $M(\cdot,\cdot,\nu)$ from ν is not linear. We are able to overcome this problem when the denominator of $M(\cdot,\cdot,\nu)$ is strictly bigger than a positive number. This is the case when the U in $M(\cdot,\cdot,\nu)$ is chosen as in Definitions 2.5. Notice that the denominator in $M_\epsilon(\cdot,\cdot,\nu)$ is always strictly bigger than ϵ.

The existence and the uniqueness of the measure solution of equation (17) is given in Theorem 4.4. The existence of weak and strong solutions of (19) follows from Theorem 4.5. The qualitative behaviour of the solution of equation (19) is analyzed in Subsection 4.1. In particular, in Lemma 4.7, we show that the Boltzmann-Vlasov entropy is strictly decreasing in time.

3. Particle dynamics.

In the following we analyze the evolution of N particles according equations (1). In this section N is kept fixed, so we omit in the notation to write explicitly the dependence on N.

3.1. Stability.

We first notice that if the velocities of the particles at time zero are bounded, that is, for all $i = 1,..,N$, $p_i^0 \in B_r$ for some $r > 0$, then they will lie in B_r for later times. In fact we have the following result:

Lemma 3.1. *For any $i = 1,..,N$, assume that $p_i(0) \in B_r$. Then, $p_i(t) \in B_r$, for all $t > 0$.*

Proof. Assume, without loss of generality that $r = 1$ and that there is a t^* such that there is at least one $p_i(t^*)$ such that $|p_i(t^*)| = 1$ and $|p_j(t^*)| \le 1$ for $j \ne i$. Then

$$\frac{1}{2}\frac{d}{dt}|p_i(t^*)|^2 = \frac{\sum_{j=1}^N U(q_i(t^*) - q_j(t^*))\,[p_j(t^*) - p_i(t^*)] \cdot p_i(t^*)}{\sum_{j=1}^N U(q_i(t^*) - q_j(t^*))} \le 0 \ . \tag{21}$$

$\square$

Remark 2. The result of Lemma 3.1 holds for any positive smooth interaction U, regardless of its support. In particular, it holds if U does not have compact support.

Next result shows that if at time $t = 0$ the particle velocity vector is close to its mean velocity vector, then, at any further time t, it will always remain close to the mean initial velocity vector. Let $\Omega \in \mathcal{L}_{Nd}$ be the operator such that

$$\mathbb{R}^{Nd} \ni \mathbf{x} \longmapsto \Omega\mathbf{x} \in \mathbb{R}^{Nd} \ , \tag{22}$$

where $\Omega\mathbf{x}$ is the vector in $\mathbb{R}^{Nd}$ whose component are the vectors $(\Omega x)_i = \frac{1}{N}\sum_{j=1}^N x_j \in \mathbb{R}^d, \forall i = 1,..,N$. Notice that by definition Ω is the orthogonal projector on $\left\{\mathbf{x} \in \mathbb{R}^{Nd} : x_1 = \cdots = x_N\right\}$.

Theorem 3.2. *Let $w(t, w^0) = (\mathbf{q}(t), \mathbf{p}(t))$ be the solution of (1) at time t starting from $w^0 = (\mathbf{q}^0, \mathbf{p}^0) \in \mathbb{R}^{2Nd}$. Given $\epsilon > 0$, assume that $|\mathbf{p}^0 - \Omega\mathbf{p}^0| < \epsilon$. Then*

$$|\mathbf{p}(t) - \Omega\mathbf{p}^0| \leq \epsilon, \qquad \forall t \geq 0 \ . \tag{23}$$

Proof. Denote by $v_i(t) := p_i(t) - \left(\Omega p^0\right)_i \in \mathbb{R}^d, i = 1, .., N$, then proceed as in the proof of Lemma 3.1. □

Note that, for any $w \in \mathbb{R}^{2Nd}$,

$$\operatorname{dist}(w, \mathcal{I}) = \inf_{w^0 \in \mathcal{I}} |w - w^0| = \inf_{\{\mathbf{p}^0 \in \mathbb{R}^{Nd} : w = (\mathbf{q}^0, \mathbf{p}^0) \in \mathcal{I}\}} |\mathbf{p} - \mathbf{p}^0| = |\mathbf{p} - \Omega\mathbf{p}| \ , \tag{24}$$

where Ω is the operator defined in (22). From Theorem 3.2 one deduces that the invariant manifold $\mathcal{I}$ is stable for the evolution (1).

Corollary 1. *For any $\epsilon > 0$ let $B(\epsilon, \mathcal{I}) = \left\{w \in \mathbb{R}^{2Nd} : \operatorname{dist}(w, \mathcal{I}) \leq \epsilon\right\}$ be a neighborhood of radius ϵ of $\mathcal{I}$. Let $w(t, w^0)$ be the solution of (1) at time t starting from $w^0 = (\mathbf{q}^0, \mathbf{p}^0) \in B(\epsilon, \mathcal{I})$. Then*

$$\operatorname{dist}\left(w(t, w^0), \mathcal{I}\right) \leq 2\epsilon, \qquad \forall t > 0 \ . \tag{25}$$

Proof. By (24) we have

$$\operatorname{dist}\left(w(t, w^0), \mathcal{I}\right) = |\mathbf{p}(t) - \Omega\mathbf{p}(t)| \leq |\mathbf{p}(t) - \Omega\mathbf{p}^0| + |\Omega\mathbf{p}(t) - \Omega\mathbf{p}^0| \ . \tag{26}$$

By definition of Ω, see (22),

$$|\Omega\mathbf{p}(t) - \Omega\mathbf{p}^0| = |\Omega(\mathbf{p}(t) - \Omega\mathbf{p}^0)| \leq |\mathbf{p}(t) - \Omega\mathbf{p}^0| \ . \tag{27}$$

Hence, by Theorem 3.2,

$$\operatorname{dist}\left(w(t, w^0), \mathcal{I}\right) \leq 2|\mathbf{p}(t) - \Omega\mathbf{p}^0| \leq 2\epsilon \ , \qquad \forall t \geq 0 \ . \tag{28}$$

□

3.2. **Asymptotic stability.** To prove Theorem 2.4 we rewrite the non linear system (1) as follows:

$$\begin{cases} \begin{pmatrix} \frac{d\mathbf{q}(t)}{dt} \\ \frac{d\mathbf{p}(t)}{dt} \end{pmatrix} = C\left(\mathbf{q}(t)\right) \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} \\ \mathbf{q}(0) = \mathbf{q}^0, \mathbf{p}(0) = \mathbf{p}^0 \end{cases} \tag{29}$$

where

$$\mathbb{R}^{Nd} \ni \mathbf{q} \longmapsto C\left(\mathbf{q}\right) := \begin{pmatrix} \mathbf{0}_{Nd} & \mathbb{I}_{Nd} \\ \mathbf{0}_{Nd} & L\left(\mathbf{q}\right) \end{pmatrix} \in \mathcal{L}_{2Nd}\left(\mathbb{R}\right) \ , \tag{30}$$

$$L\left(\mathbf{q}\right) := A\left(\mathbf{q}\right) - \mathbb{I}_{Nd} \tag{31}$$

and $A(\mathbf{q})$ is the linear operator valued function so defined

$$\mathbb{R}^{Nd} \ni \mathbf{q} \longmapsto A(\mathbf{q}) := \begin{bmatrix} a_{1,1}(\mathbf{q})\mathbb{I}_d & a_{1,2}(\mathbf{q})\mathbb{I}_d & \dots & a_{1,N}(\mathbf{q})\mathbb{I}_d \\ a_{2,1}(\mathbf{q})\mathbb{I}_d & a_{2,2}(\mathbf{q})\mathbb{I}_d & \dots & a_{2,N}(\mathbf{q})\mathbb{I}_d \\ a_{N,1}(\mathbf{q})\mathbb{I}_d & \dots & a_{N,N-1}(\mathbf{q})\mathbb{I}_d & a_{N,N}(\mathbf{q})\mathbb{I}_d \end{bmatrix} \in \mathcal{L}_{Nd}\left(\mathbb{R}\right) \tag{32}$$

$$a_{i,j}(\mathbf{q}) := \frac{U(q_i - q_j)}{\sum_{k=1}^{N} U(q_i - q_k)} \ , \qquad j = 1, .., N, \quad i = 1, .., N. \tag{33}$$

Remark 3. Notice that for $\mathbf{q} \in \mathbb{R}^{Nd}$

$$a_{i,j}(\mathbf{q}) = a_{i,j}(\mathbf{q} + \Omega\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{Nd}, \qquad j = 1, .., N, \quad i = 1, .., N \tag{34}$$

and

$$\sum_{j=1}^{N} a_{i,j}(\mathbf{q}) = 1. \tag{35}$$

These two properties are important when studying the spectrum of $C(\mathbf{q})$ for a fixed value of $\mathbf{q}$.

3.2.1. *Spectral analysis of* $C(\mathbf{q})$. Let $\mathbf{q} \in \mathbb{R}^{Nd}$ be fixed. The eigenvalues of $C(\mathbf{q})$ are the roots of the characteristic equation

$$\mathrm{Det}\left[C(\mathbf{q}) - \lambda \mathbb{I}_{2Nd}\right] = (-\lambda)^{Nd} \,\mathrm{Det}\left[L(\mathbf{q}) - \lambda \mathbb{I}_{Nd}\right] = 0. \tag{36}$$

We need then to study the spectrum of $L(\mathbf{q})$ and therefore, by (31) the spectrum of $A(\mathbf{q})$. To do this it is convenient to introduce the tensor space $\mathbb{R}^N \otimes \mathbb{R}^d$. We denote by $\mathcal{F}$ the isomorphism

$$\mathbb{R}^{Nd} \ni \mathbf{x} \longrightarrow \mathcal{F}(\mathbf{x}) := \sum_{i=1}^{N} \sum_{j=1}^{d} x_i^j e_i \otimes e_j \in \mathbb{R}^N \otimes \mathbb{R}^d \ , \tag{37}$$

such that $\mathcal{F}(\mathbf{x})_{i,j} = x_i^j$, $i = 1, .., N$ and $j = 1, .., d$.

To ease the notation we omit in the following to write the dependence on $\mathbf{q}$ if no confusion arises. We therefore set $A := A(\mathbf{q})$. One obtains immediately that $A : \mathbb{R}^{Nd} \longrightarrow \mathbb{R}^{Nd}$ acts on $\mathbb{R}^N \otimes \mathbb{R}^d$ as follows

$$\tilde{A} \otimes \mathbb{I}_d : \mathbb{R}^N \otimes \mathbb{R}^d \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^d \ , \tag{38}$$

where, by (33), setting $a_{i,j} := a_{i,j}(\mathbf{q})$,

$$\tilde{A} := \begin{bmatrix} a_{1,1} & a_{1,2} & & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & & \dots & a_{2,N} \\ a_{N,1} & & \dots & a_{N,N-1} & a_{N,N} \end{bmatrix} \ . \tag{39}$$

Namely, one has that

$$\left(\tilde{A} \otimes \mathbb{I}_d\right) \mathcal{F}(\mathbf{x}) = \mathcal{F}(A\mathbf{x}) \ . \tag{40}$$

Furthermore, denoting by $\Sigma(A) \subset \mathbb{C}$ the spectrum of A,

$$\Sigma(A) = \Sigma(\tilde{A} \otimes \mathbb{I}_d) = \Sigma(\tilde{A})\Sigma(\mathbb{I}_d)^{3} \ . \tag{41}$$

Since the only eigenvalue of $\mathbb{I}_d$ is 1 with multiplicity d, the problem is reduced to study the spectrum of $\tilde{A}$. The matrix $\tilde{A}$ is a (right) stochastic matrix, that is it has non-negative entries and, by (35), $\sum_{j=1}^{N} a_{i,j} = 1$, $\forall i = 1, .., N$. Then, if it is irreducible one can apply the Perron-Frobenius Theorem.

Recall that a matrix $D \in \mathcal{L}_n(\mathbb{R})$ with non-negative entries is said to be irreducible if there exists an integer m such that D^m has strictly positive entries. We have the following.

---

[3] If $Z := \{z_1, .., z_n\}$ and $W := \{w_1, .., w_m\}$ are two discrete subsets of $\mathbb{C}$ we denote by

$$ZW := \{z_i w_j \in \mathbb{C} : i = 1, .., n \ ; \ j = 1, .., m\} .$$

Lemma 3.3. *Let $\tilde{A}(\mathbf{q})$, $\mathbf{q} \in \mathbb{R}^{Nd}$, be irreducible. Then 1 is the maximum eigenvalue and all the other eigenvalues $\lambda(\mathbf{q}) \in \mathbb{C}$ are strictly smaller in absolute value of 1, i.e. $|\lambda(\mathbf{q})| < 1$. The eigenspace associated to the eigenvalue 1 is one dimensional and it is generated by the eigenvector η, $\eta_i = \frac{1}{\sqrt{N}}$ for $i = 1, .., N$. There are no other positive eigenvectors except multiples of η.*

Proof. Because for any $\mathbf{q} \in \mathbb{R}^{Nd}$, $\|\tilde{A}(\mathbf{q})\|_\infty \leq \max_{i=1,..,N} \sum_{j=1}^{N} a_{i,j}(\mathbf{q}) = 1$, we have that the maximum eigenvalue is 1 and any other eigenvalue $\lambda(\mathbf{q}) \in \mathbb{C}$ is strictly smaller in absolute value of 1. By Perron Frobenius Theorem the maximum eigenvalue is simple and the associated positive eigenvector is η with $\eta_i = \frac{1}{\sqrt{N}}$ for $i = 1, .., N$. □

It is possible to show, assuming that $\tilde{A}(\mathbf{q})$ is irreducible, that the spectrum of $\tilde{A}(\mathbf{q})$ is indeed real, although this information is not relevant for the proofs of the results.

Remark 4. For any $\mathbf{q} \in \mathbb{R}^{Nd}$, $\tilde{A}(\mathbf{q})$ represents the transition matrix for the Markov chain with state space $\mathcal{S}_N := \{1, ..., N\}$. By (33) we have that $\tilde{A}(\mathbf{q})$ is reversible w.r.t. the probability distribution $\{\mu_i(\mathbf{q})\}_{i\in\mathcal{S}_N}$ such that $\forall i \in \mathcal{S}_N$,

$$\mu_i(\mathbf{q}) := \frac{\sum_{j=1}^{N} U(q_i - q_j)}{\sum_{i,j=1}^{N} U(q_i - q_j)} > 0 \ , \tag{42}$$

(for an account on reversible Markov chains we refer the reader to and [27]). Let $\mathbb{H}_N(\mathbf{q})$ be the space $\mathbb{R}^N$ equipped with the scalar product

$$\mathbb{R}^N \times \mathbb{R}^N \ni (f, g) \longmapsto \langle f, g \rangle_{\mathbf{q}} := \sum_{i\in\mathcal{V}_N} \mu_i(\mathbf{q}) g_i f_i \in \mathbb{R} \ . \tag{43}$$

It is easy to verify that $\tilde{A}(\mathbf{q})$ is selfadjoint on $\mathbb{H}_N(\mathbf{q})$, hence the eigenvalues of $\tilde{A}(\mathbf{q})$ are real.

Lemma 3.4. *For any $\mathbf{q} \in \mathbb{R}^{Nd}$, such that $\tilde{A}(\mathbf{q})$ is irreducible, let $A(\mathbf{q})$ be the matrix as in (32). We have that $1 \in \Sigma(A(\mathbf{q}))$ is the maximum eigenvalue. The associated eigenspace is the d-dimensional manifold $\{\mathbf{p} \in \mathbb{R}^{Nd} : p_i = v,\ i = 1, .., N; v \in \mathbb{R}^d\}$. All the other eigenvalues $\lambda(\mathbf{q}) \in \Sigma(A(\mathbf{q}))$ are such that $|\lambda(\mathbf{q})| < 1$.*

Proof. It is an immediate consequence of (41) and Lemma 3.3. □

We have finally the following result.

Theorem 3.5. *For any $\mathbf{q} \in \mathbb{R}^{Nd}$, such that $\tilde{A}(\mathbf{q})$ is irreducible, let $C(\mathbf{q})$ be defined in (30). We have that $0 \in \Sigma(C(\mathbf{q}))$. The $(N+1)d$ dimensional manifold $\mathcal{I}$ defined in (3) is the eigenspace associated to the eigenvalue 0. All the other eigenvalues of $C(\mathbf{q})$ have real part strictly negative.*

Proof. From (36) and Lemma 3.4 we deduce that $0 \in \Sigma(C(\mathbf{q}))$ and all other eigenvalues have real part strictly negative. It is immediate to see that the algebraic multiplicity of 0 is $Nd + d$. The $(N+1)d$-dimensional manifold $\mathcal{I}$ defined in (3) is the associated eigenspace. Namely, if $w \in \mathcal{I}$ then $C(\mathbf{q})w \in \mathcal{I}$. From this one deduces that $\mathcal{I}$ is an eigenspace for the matrix $C(\mathbf{q})$. Moreover, since the kernel of $C^2(\mathbf{q})$ is $\mathcal{I}$, we get that $\mathcal{I}$ is the eigenspace associated to the eigenvalue 0. □

We denote by $\alpha(\mathbf{q})$ the spectral gap of the matrix $C(\mathbf{q})$, that is

$$\alpha(\mathbf{q}) := \min \{|\mathrm{Re}(\lambda(\mathbf{q}))| : \lambda(\mathbf{q}) \in \Sigma(C(\mathbf{q})),\ \mathrm{Re}(\lambda(\mathbf{q})) < 0\} \ . \tag{44}$$

Let $\mathbf{q} \in \mathbb{R}^{Nd}$ such that $\tilde{A}(\mathbf{q})$ is irreducible. By Theorem 3.5, $\mathcal{I}$ is the eigenspace associated to the 0 eigenvalue of $C(\mathbf{q})$ for any $\mathbf{q}$. We can therefore decompose $\mathbb{R}^{2Nd}$ as follows:

$$\mathbb{R}^{2Nd} = W(\mathbf{q}) \oplus \mathcal{I} \tag{45}$$

in such a way that $W(\mathbf{q})$ and $\mathcal{I}$ are eigenspaces of $C(\mathbf{q})$ and denote by $\Pi(\mathbf{q})$ the projection operator

$$\Pi(\mathbf{q}) : \mathbb{R}^{2Nd} \to W(\mathbf{q}) \ . \tag{46}$$

3.2.2. *Asymptotic analysis.* Let $w^0 = (\mathbf{q}^0, \mathbf{p}^0) \in \mathcal{I}$ be such that $\tilde{A}(\mathbf{q}^0)$ is irreducible and let us set, for any $r > 0$ and $\epsilon > 0$,

$$\tilde{\mathcal{B}}(r, \epsilon, w^0) := \{w = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2Nd} : |\mathbf{q} - \mathbf{q}^0| \leq r \ ; \ |\mathbf{p} - \mathbf{p}^0| \leq \epsilon\} \ . \tag{47}$$

Denote by r_0 the biggest value of r such that, for any $w = (\mathbf{q}, \mathbf{p}) \in \tilde{\mathcal{B}}(r_0, \epsilon, w^0)$, $\tilde{A}(\mathbf{q})$ is still irreducible and $\alpha(\mathbf{q}) \geq \frac{1}{2}\alpha(\mathbf{q}^0)$. We set

$$\mathcal{B}(r_0, \epsilon, w^0) := \left\{ w = (\mathbf{q}, \mathbf{p}) \in \tilde{\mathcal{B}}(r_0, \epsilon, w^0) : \alpha(\mathbf{q}) \geq \frac{1}{2}\alpha(\mathbf{q}^0) \right\} \ . \tag{48}$$

The existence of r_0 is granted since, by assumption, $\tilde{A}(\mathbf{q}^0)$ is irreducible and U is smooth. To apply the spectral results obtained for $C(\mathbf{q})$ ($\mathbf{q}$ fixed) to the nonlinear system (29) we write

$$C(\mathbf{q}(t)) = C(\mathbf{q}(0)) + \Gamma(\mathbf{q}(t)) \ , \tag{49}$$

where

$$\Gamma(\mathbf{q}(t)) := \begin{pmatrix} \mathbf{0}_{Nd} & \mathbf{0}_{Nd} \\ \mathbf{0}_{Nd} & B(\mathbf{q}(t)) \end{pmatrix} \ , \tag{50}$$

and

$$B(\mathbf{q}(t)) := A(\mathbf{q}(t)) - A(\mathbf{q}(0)) \ . \tag{51}$$

Next we estimate the norm of $B(\mathbf{q}(t))$.

Lemma 3.6. *Let $(\mathbf{q}(t), \mathbf{p}(t))$ be the solution of (29) starting from the initial data $(\mathbf{q}^0, \mathbf{p}^0)$. We have*

$$\begin{aligned} \|B(\mathbf{q}(t))\| \leq 2N \frac{\sup_{x\in\mathbb{R}^d} |\nabla U(x)|}{U(0) + (N-1)\eta(\mathbf{q}(t), \mathbf{q}^0)} \times \\ \times \sup_{i,k\in\{1,..,N\}} \left| -(q_i^0 - q_k^0) + q_i(t) - q_k(t) \right| \ , \end{aligned}$$

where $\eta(\mathbf{q}(t), \mathbf{q}^0) \geq 0$ is defined in (147).

We defer the proof of this result to the appendix. In the proof of (2.4) we will use Lemma 3.6 taking $\eta(\mathbf{q}(t), \mathbf{q}^0) = 0$.

Proof of Theorem (2.4). For any $s > 0$, we define

$$\tilde{\mathcal{Q}}(s, w^0) := \{w = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2Nd} : |[\mathbb{I}_{Nd} - \Omega] \left(\mathbf{q} - \mathbf{q}^0\right)| \leq s\} \ , \tag{52}$$

where Ω is the operator defined in (22). Denote by s_0 the largest value of s such that, for any $w = (\mathbf{q}, \mathbf{p}) \in \tilde{\mathcal{Q}}(s_0, w^0)$, $\tilde{A}(\mathbf{q})$ is still irreducible and $\alpha(\mathbf{q}) \geq \frac{1}{4}\alpha(\mathbf{q}^0)$. Such a value s_0 exists since $\tilde{A}(\mathbf{q}^0)$ is irreducible and U is smooth. Let us set

$$\mathcal{Q}(s_0, w^0) := \left\{ w = (\mathbf{q}, \mathbf{p}) \in \tilde{\mathcal{Q}}(s_0, w^0) : \alpha(\mathbf{q}) \geq \frac{1}{4}\alpha(\mathbf{q}^0) \right\} \ . \tag{53}$$

We have that
$$\mathcal{B}(r_0,\epsilon,w^0)\subset\mathcal{Q}(s_0,w^0)\ ,\qquad \forall\epsilon>0\ . \tag{54}$$
Namely we have that $s_0\geq r_0$ since requirement (48) is stronger than (53) and
$$|[\mathbb{I}_{Nd}-\Omega](\mathbf{q}-\mathbf{q}^0)|\leq|\mathbf{q}-\mathbf{q}^0|\leq r_0\ . \tag{55}$$
Let $w(t,w^1)=(\mathbf{q}(t,w^1),\mathbf{p}(t,w^1))$ be the solution of system (29) starting from an initial datum $w^1\in\mathcal{B}(r_0,\epsilon,w^0)$ and let $t^*(w_1)>0$ be the first exit time of $w(t,w^1)$ from $\mathcal{Q}(s_0,w^0)$. If $w(t,w^1)\in\mathcal{Q}(s_0,w^0)$ for all $t\geq 0$, then we set $t^*(w^1)=\infty$. Next we analyze the solution for $t<t^*(w^1)$ and we will show that $t^*(w^1)=\infty$ for any initial datum $w^1\in\mathcal{B}(r_0,\epsilon,w^0)$, provided that ϵ in (47) is suitably chosen. Let us define
$$\xi(t):=\Pi(\mathbf{q}(t,w^1))w(t,w^1)\ , \tag{56}$$
$$\chi(t):=\left(\mathbb{I}_{2Nd}-\Pi(\mathbf{q}(t,w^1)\right)w(t,w^1)\ ,\quad t<t^*(w^1)\ . \tag{57}$$
By construction $\chi(t)\in\mathcal{I},\xi(t)\in W(\mathbf{q}(t,w^1))$. We then have
$$\begin{aligned}\frac{d}{dt}\xi(t)&=\left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right)w(t,w^1)+\Pi(\mathbf{q}(t))\frac{d}{dt}w(t,w^1)\\&=\left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right)w(t,w^1)+\Pi(\mathbf{q}(t))C(\mathbf{q}(t))w(t,w^1)\ .\end{aligned} \tag{58}$$
Taking into account that $w(t,w^1)=\xi(t)+\chi(t)$ we get
$$\frac{d}{dt}\xi(t)=\left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right)\xi(t)+\left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right)\chi(t)+\Pi(\mathbf{q}(t))C(\mathbf{q}(t))\xi(t)\ . \tag{59}$$
Since for any given $w\in\mathcal{I}$, by the definition $\Pi(\mathbf{q}(t))$, we have $\frac{d}{dt}\Pi(\mathbf{q}(t))w=0$ and $C(\mathbf{q}(t))$ and $\Pi(\mathbf{q}(t))$ commute, we obtain
$$\frac{d}{dt}\xi(t)=\left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right)\xi(t)+C(\mathbf{q}(t))\xi(t)\ . \tag{60}$$
Setting
$$C(\mathbf{q}(t))=C(\mathbf{q}(0))+\Gamma(\mathbf{q}(t))\ , \tag{61}$$
where $\Gamma(\mathbf{q}(t))$ is defined in (50), we get
$$\frac{d}{dt}\xi(t)=\left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right)\xi(t)+C(\mathbf{q}(0))\xi(t)+\Gamma(\mathbf{q}(t))\xi(t)\ . \tag{62}$$
By the formula of variation of constants:
$$\xi(t)=e^{C(\mathbf{q}(0))t}\xi(0)+\int_0^t e^{C(\mathbf{q}(0))(t-s)}\left\{\left(\frac{d}{ds}\Pi(\mathbf{q}(s))\right)\xi(s)+\Gamma(\mathbf{q}(s))\xi(s)\right\}ds. \tag{63}$$
Performing the exponential of the matrix $C(\mathbf{q}(0))$ one needs to take into account that, because of the possible presence of Jordan blocks, powers of t might appear. We control such terms paying $e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}$ and multiplying the remaining exponential by a constant $D(C(\mathbf{q}(0)))$ which depends only on $C(\mathbf{q}(0))$. Since $\mathbf{q}(0)\in\mathbf{B}_{r_0}\left(\mathbf{q}^0\right)$, which is a compact set in $\mathbb{R}^{Nd}$, we denote by $D_0:=\sup_{\mathbf{q}\in\mathbf{B}_{r_0}(\mathbf{q}^0)}D(C(\mathbf{q}))$, which depends only on $C(\mathbf{q}^0)$ and r_0. Therefore, we get
$$\begin{aligned}|\xi(t)|\leq&D_0e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}|\xi(0)|+\\&+D_0\int_0^t e^{-\frac{1}{2}\alpha(\mathbf{q}(0))(t-s)}\left\{|\left(\frac{d}{ds}\Pi(\mathbf{q}(s))\right)\xi(s)|+|\Gamma(\mathbf{q}(s))\xi(s)|\right\}ds\ .\end{aligned}$$

Next we estimate $\left\|\frac{d}{dt}\Pi(\mathbf{q}(t))\right\|$. Let $\Pi(\mathbf{q}(t)) = \{\pi_{i,j}(\mathbf{q}(t)) \otimes \mathbb{I}_d\}_{i,j=1,..,,N}$, we then have

$$\begin{aligned}\frac{d}{dt}\Pi(\mathbf{q}(t)) &= \left\{\nabla_{\mathbf{q}(t)}\pi_{i,j}(\mathbf{q}(t)) \cdot \mathbf{p}(t) \otimes \mathbb{I}_d\right\}_{i,j=1,..,,N} \\ &= \left\{\nabla_{\mathbf{q}(t)}\pi_{i,j}(\mathbf{q}(t)) \cdot [\mathbf{p}(t) - \Omega\mathbf{p}(t)] \otimes \mathbb{I}_d\right\}_{i,j=1,..,,N} .\end{aligned} \tag{64}$$

The last equality holds since, by (34), $\Pi(\mathbf{q}(t)) = \Pi(\mathbf{q}(0)), \forall t \in \mathbb{R}$, when $\mathbf{q}(t)$ is the evolution given by the flow on the invariant manifold, i.e. when $\mathbf{p}(t) = \Omega\mathbf{p}(t)$. We get by Corollary 1

$$\begin{aligned}\left\|\frac{d}{dt}\Pi(\mathbf{q}(t))\right\| &\leq \sup_{\{\mathbf{q}\in\mathbb{R}^{Nd}\ :\ w=(\mathbf{q},\mathbf{p})\in\mathcal{Q}(s_0,w^0)\}} \sup_{i=1,..,N} \sum_{j=1}^{N} |\nabla_q \pi_{i,j}(\mathbf{q})|\, |\mathbf{p}(t) - \Omega\mathbf{p}(t)| \\ &\leq D'(s_0)\epsilon , \qquad \forall t \in [0, t^*(w^1)) ,\end{aligned} \tag{65}$$

where $D'(s_0) > 0$. Furthermore, by (50) and Lemma 3.6, we have

$$\begin{aligned}\|\Gamma(\mathbf{q}(t))\| = \|B(\mathbf{q}(t))\| \leq & 2\frac{N}{U(0)} \sup_{x\in\mathbb{R}^d} |\nabla U(x)| \\ & \max_{1\leq i,k\leq N} \left|-(q_i^1 - q_k^1) + q_i(t) - q_k(t)\right|\end{aligned}$$

and, by Theorem 3.2, for $i, k = 1, .., N$,

$$\begin{aligned}\left|q_i(t) - q_k(t) - (q_i^1 - q_k^1)\right| &= \int_0^t |p_i(s') - p_k(s')|\, ds' \\ &= \int_0^t \left|p_i(s') - p_i^0 + p_k^0 - p_k(s')\right| ds' \leq 2\epsilon t,\end{aligned} \tag{66}$$

where $p_i^0 = p_k^0$ since $(\mathbf{q}^0, \mathbf{p}^0) \in \mathcal{I}$. Thus, setting $D_1 := 2\frac{\sup_{x\in\mathbb{R}^d}|\nabla U(x)|}{U(0)}$, $\forall t \in [0, t^*(w^1))$ we obtain

$$\begin{aligned}|\xi(t)| \leq & D_0 e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}|\xi(0)| + \\ & + D_0\epsilon \int_0^t e^{-\frac{1}{2}\alpha(\mathbf{q}(0))(t-s)} \left\{[D'(s_0) + 2ND_1 s]\, |\xi(s)|\right\} ds .\end{aligned} \tag{67}$$

Given $K \geq \max\{D'(s_0), 2D_1\}$, take $T \in \left(0, t^*(w^1)\right)$. A suitable choice of T will be done later. Then, $\forall t \in [0, T]$,

$$|\xi(t)| \leq D_0 e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}|\xi(0)| + \epsilon D_0 K \{1 + TN\} \int_0^t e^{-\frac{1}{2}\alpha(\mathbf{q}(0))(t-s)}|\xi(s)|ds, . \tag{68}$$

By the Gronwall's inequality we get

$$|\xi(t)| \leq D_0|\xi(0)|e^{-t\left[\frac{1}{2}\alpha(\mathbf{q}(0))-\epsilon\delta\right]} \leq D_0|\xi(0)|e^{-t\left[\frac{1}{8}\alpha(\mathbf{q}^0)-\epsilon\delta\right]} , \quad \forall t \in [0, T] , \tag{69}$$

where we made use of (53) and set $\delta := D_0 K \{1 + NT\}$. Let us choose ϵ such that

$$\frac{1}{16}\alpha(\mathbf{q}^0) \geq \epsilon D_0 K\{1 + NT\} . \tag{70}$$

Then,

$$|\xi(t)| \leq D_0|\xi(0)|e^{-t\frac{1}{16}\alpha(\mathbf{q}^0)} , \quad \forall t \in [0, T] . \tag{71}$$

Since

$$\begin{aligned}\operatorname{dist}\left(w(t,w^1),\mathcal{I}\right) &= \inf_{\tilde{w}\in\mathcal{I}}|w(t,w^1)-\tilde{w}| = \inf_{\tilde{w}\in\mathcal{I}}|\chi(t)+\xi(t)-\tilde{w}| \\ &\le |\xi(t)| + \inf_{\tilde{w}\in\mathcal{I}}|\chi(t)-\tilde{w}| = |\xi(t)|\ ,\end{aligned} \tag{72}$$

we have

$$\operatorname{dist}\left(w(t,w^1),\mathcal{I}\right) \le D_0|\xi(0)|e^{-t\frac{1}{16}\alpha(\mathbf{q}^0)}\ , \quad \forall t\in[0,T]\ . \tag{73}$$

Then, recalling that $\operatorname{dist}\left(w(t,w^1),\mathcal{I}\right) = |\mathbf{p}(t)-\Omega\mathbf{p}(t)|$, and, since for $w^1\in\mathcal{B}(r_0,\epsilon, w^0)$,

$$|\xi(0)| \le D(s_0)\operatorname{dist}\left(w^1,\mathcal{I}\right) \le D(s_0)\epsilon\ , \tag{74}$$

we have

$$|\mathbf{p}(t)-\Omega\mathbf{p}(t)| \le D_0 D(s_0)\epsilon e^{-\frac{1}{16}\alpha(\mathbf{q}^0)t} \le \epsilon(D_0 D(s_0)\vee 1)e^{-\frac{1}{16}\alpha(\mathbf{q}^0)t}\ , \quad \forall t\in[0,T]\ . \tag{75}$$

From this we get

$$\begin{aligned}|[\mathbb{I}_{Nd}-\Omega]\left(\mathbf{q}(t)-\mathbf{q}^0\right)| &\le |[\mathbb{I}_{Nd}-\Omega]\left(\mathbf{q}(0)-\mathbf{q}^0\right)| + \int_0^t |\left[\mathbb{I}_{Nd}-\Omega\right]\mathbf{p}(s)|ds \\ &\le |\mathbf{q}(0)-\mathbf{q}^0| + (D_0D(s_0)\vee 1)\frac{16}{\alpha(\mathbf{q}^0)}\epsilon(1-e^{-\frac{1}{16}\alpha(\mathbf{q}^0)t}) \\ &\le |\mathbf{q}(0)-\mathbf{q}^0| + (D_0D(s_0)\vee 1))\frac{16}{\alpha(\mathbf{q}^0)}\epsilon\ .\end{aligned} \tag{76}$$

Let us choose ϵ such that

$$r_0 + (D_0D(s_0)\vee 1))\frac{16}{\alpha(\mathbf{q}^0)}\epsilon \le \frac{1}{2}s_0\ , \tag{77}$$

and denote this chosen value by $\tilde{\epsilon}_1$. Now we first choose T such that

$$(D_0D(s_0)\vee 1)e^{-\frac{1}{16}\alpha(\mathbf{q}^0)T} = \frac{1}{2}\wedge\frac{1}{2D_0} \tag{78}$$

and denote this chosen value by T_0, then we choose $\tilde{\epsilon}_2$ in such a way that (70) holds with T replaced by T_0. We then set

$$\epsilon_0 := \min\left\{\tilde{\epsilon}_1,\tilde{\epsilon}_2\right\}\ . \tag{79}$$

Notice that, by (77),

$$\tilde{\epsilon}_1\frac{16}{\alpha(\mathbf{q}^0)} \le r_0 + (D_0D(s_0)\vee 1))\frac{16}{\alpha(\mathbf{q}^0)}\epsilon \le \frac{1}{2}s_0 \tag{80}$$

so, since $\alpha\left(\mathbf{q}^0\right) < 1$,

$$\epsilon_0 \le \tilde{\epsilon}_1 \le \frac{1}{32}s_0\ . \tag{81}$$

We remark that the choice of T_0 and ϵ_0 depends on $w^0\in\mathcal{I}$. Therefore, at time T_0 we have

$$|[\mathbb{I}_{Nd}-\Omega]\left(\mathbf{q}(T_0)-\mathbf{q}^0\right)| \le \frac{1}{2}s_0 \tag{82}$$

and

$$\operatorname{dist}\left(w(T_0,w^1),\mathcal{I}\right) = |\mathbf{p}(T_0)-\Omega\mathbf{p}(T_0)| \le \frac{\epsilon_0}{2}\ . \tag{83}$$

We can then repeat the previous argument for the solution of the system (1) starting at time T_0 from the initial datum $(\mathbf{q}(T_0),\mathbf{p}(T_0))$. We need to recall that $\alpha(\mathbf{q}(T_0)) \ge \frac{1}{4}\alpha(\mathbf{q}^0)$. In a similar way we can show that that for $t\in[T_0,2T_0]$,

$$|\mathbf{p}(t)-\Omega\mathbf{p}(t)| \le D_0|\xi(T_0)|e^{-(t-T_0)\frac{1}{16}\alpha(\mathbf{q}^0)}\ , \quad \forall t\in[T_0,2T_0]\ . \tag{84}$$

Therefore, by (78), we have

$$\operatorname{dist}\left(w(2T_0, w(T_0)), \mathcal{I}\right) = |\mathbf{p}(2T_0) - \Omega \mathbf{p}(2T_0)| \le \frac{\epsilon_0}{2^2(D_0 D(s_0) \vee 1)} \le \frac{\epsilon_0}{2^2}\ , \tag{85}$$

and, by (84),

$$\begin{aligned} |[\mathbb{I}_{Nd} - \Omega]\left(\mathbf{q}(t) - \mathbf{q}^0\right)| &\le |[\mathbb{I}_{Nd} - \Omega]\left(\mathbf{q}(T_0) - \mathbf{q}^0\right)| + \int_{T_0}^{t} |[\mathbb{I}_{Nd} - \Omega]\mathbf{p}(s)| ds \\ &\le |[\mathbb{I}_{Nd} - \Omega]\left(\mathbf{q}(T_0) - \mathbf{q}^0\right)| + D_0 |\xi(T_0)| \int_{T_0}^{t} e^{-(s-T_0)\frac{1}{16}\alpha(\mathbf{q}^0)} ds \\ &\le \frac{1}{2}s_0 + \frac{1}{4}s_0\ , \end{aligned} \tag{86}$$

the last inequality being a consequence of (77) and (78). Thus, at time $T_1 = 2T_0$

$$|[\mathbb{I}_{Nd} - \Omega]\left(\mathbf{q}(T_1) - \mathbf{q}^0\right)| \le \frac{1}{2}s_0 + \frac{s_0}{4}\ . \tag{87}$$

Hence, we have that $(\mathbf{q}(T_1), \mathbf{p}(T_1)) \in \mathcal{Q}(s_0, w^0)$. Iterating this procedure m times we get

$$\operatorname{dist}\left(w(T_m, w^1), \mathcal{I}\right) = |\mathbf{p}(T_m) - \Omega \mathbf{p}(T_m)| \le \frac{\epsilon_0}{2^{m+1}}\ , \tag{88}$$

and

$$|[\mathbb{I}_{Nd} - \Omega]\left(\mathbf{q}(T_m) - \mathbf{q}^0\right)| \le s_0 \sum_{k=0}^{m} \frac{1}{2^{k+1}}\ . \tag{89}$$

Since $\sum_{k\ge 1} \frac{1}{2^m} = \frac{1}{2}$ we obtain the thesis of the theorem. $\square$

4. **Kinetic limit: Vlasov type equation.** We study system (1) when the number of particles N goes to infinity and derive the kinetic equation for the density $f_t(x, v)$ of particles at x with velocity v at time t. The heuristic argument goes as following. Let μ_t^N be the empirical measure, see (13), at time t associated to $w(t, w^0)$, solution of the system (1), where $w^0 = (\mathbf{q}^0, \mathbf{p}^0)$, $\|p_j^0\| \le 1, j = 1, .., N$. By Lemma 3.1 and Remark 2 μ_t^N has support on $\mathbb{R}^d \times B_1$, for all $t \ge 0$. Writing the second equation of (1) in term of μ_t^N we get

$$\begin{aligned} \frac{dp_i(t)}{dt} &= \frac{\int_{\mathbb{R}^d \times B_1} U(q_i(t) - y)\left(u - p_i\left(t\right)\right) \mu_t^N(dy, du)}{\int_{\mathbb{R}^d \times B_1} U(q_i(t) - y) \mu_t^N(dy, du)} \\ &=: M(q_i(t), p_i(t), \mu_t^N). \end{aligned} \tag{90}$$

Therefore, the evolution of μ_t^N is given by

$$\frac{\partial(\mu_t^N(g))}{\partial t} = \mu_t^N(v \cdot \nabla_x g) + \mu_t^N(M\left(\cdot, \cdot, \mu_t^N\right) \cdot \nabla_v g)\ , \tag{91}$$

where g is a smooth test function. In the equation (91), N is fixed. To study the limit as $N \to \infty$ we assume that at $t = 0$ there exists $\mu_0 \in \mathcal{M}$ such that

$$\mu_0^N \overset{w}{\Longrightarrow} \mu_0. \tag{92}$$

We want to show that if (92) holds at time $t = 0$, then

$$\mu_t^N \overset{w}{\Longrightarrow} \mu_t, \tag{93}$$

where μ_t is the measure solution of the following equation

$$\frac{\partial(\mu_t(g))}{\partial t} = \mu_t(v \cdot \nabla_x g) + \mu_t(M\left(\cdot, \cdot, \mu_t\right) \cdot \nabla_v g)\ , \tag{94}$$

which is the formal limit of (91). To prove it rigorously one needs to have $M(\cdot,\cdot,\nu)$ well defined and Lipschitz continuous in (x,v) for all $\nu\in\mathcal{M}$. But, already at N finite, the denominator of (90) is equal to zero when the supports of U and μ_t^N are disjoint. To overcome these problems we consider two classes of interaction U. The first one is the class of interactions in Definition 2.5. In this case we define $M(x,v,\nu)$ as in (18). The second one is the class of smooth interactions U with compact support. In this case we fix $\epsilon>0$ and define $M_\epsilon(x,v,\nu)$ as in (20), the $\epsilon-$ regularized system. Hence, in the case U has compact support, we modify the interaction term in such a way that when $\int_{X\times B_1}U(x-y)\nu(dy,du)=0$ then $M_\epsilon(x,v,\nu)=0$, when $\int_{X\times B_1}U(x-y)\nu(dy,du)>\epsilon$ then $M_\epsilon(x,v,\nu)=M(x,v,\nu)+O(\epsilon)$, when $\epsilon>\int_{X\times B_1}U(x-y)\nu(dy,du)>0$ then $M_\epsilon(x,v,\nu)$ is a large perturbation of $M(x,v,\nu)$. It is easy to see that for any measure ν on $X\times B_1$ we have

$$\sup_{(x,v)\in X\times B_1}|M(x,v,\nu)|\le 2\ ,\qquad \sup_{(x,v)\in X\times B_1}|M_\epsilon(x,v,\nu)|\le 2\ . \tag{95}$$

The Lipschitz continuity of $M(\cdot,\cdot,\nu)$ with respect to v follows from the linearity of $M(\cdot,\cdot,\nu)$ as a function of v. The Lipschitz continuity of $M(\cdot,\cdot,\nu)$ with respect to x does not hold in general even if one takes smooth interactions U, due to the presence of the denominator in $M(\cdot,v,\nu)$. In Lemma (4.1), Lemma 4.2 and Lemma 4.3 we show for three different type of interactions, respectively for U defined trough $\tilde U$ as in (7) and (9) and for the $\epsilon-$ regularised system, that $M(\cdot,v,\nu)$ is Lipschiz for all v and ν. We denote by

$$A(\cdot,\nu):=\left(\frac{\int_{\mathcal{T}_D\times B_1}U(x-y)u\nu(dy,du)}{\int_{\mathcal{T}_D\times B_1}U(x-y)\nu(dy,du)}\right). \tag{96}$$

Lemma 4.1. *Let U be the interaction defined on $\mathcal{T}_D$ through the periodization of $\tilde U$ as defined in (7). For $\nu\in\mathcal{M}$ and $A(\cdot,\nu)$ as in (96) we have*

$$\left|A^i(x,\nu)-A^i(z,\nu)\right|\le L\,|x-z|\ ,\qquad x,z\in\mathcal{T}_D,\ i=1,..,d,\qquad L=2K\ . \tag{97}$$

Proof. $\forall i=1,..,d$, we have

$$(\nabla_xA^i)(x,\nu)=\frac{\int\nu(dy,du)\nu(dy',du')u^i\left[\nabla_xU(x-y)U(x-y')-U(x-y)\nabla_xU(x-y')\right]}{[(U\star\nu)(x)]^2}. \tag{98}$$

Taking into account that $|u|\le 1$ we have

$$\begin{aligned}\left|(\nabla_xA^i)(x,\nu)\right|&\le 2\frac{\int\nu(dy,du)\nu(dy',du')\,|\nabla_xU(x-y)|\,U(x-y')}{[(U\star\nu)(x)]^2}\\&=2\frac{\int\nu(dy,du)\nu(dy',du')\,|\nabla_xU(x-y)|\,\frac{U(x-y)}{U(x-y)}U(x-y')}{[(U\star\nu)(x)]^2}\\&\le 2\sup_y\frac{|\nabla_xU(x-y)|}{U(x-y)}\le 2K\ .\end{aligned} \tag{99}$$

□

Lemma 4.2. *Let U be the interaction defined on $\mathcal{T}_D$ through the periodization of $\tilde U$ as defined in (9). For $\nu\in\mathcal{M}$ and $A(\cdot,\nu)$ as in (96) we have*

$$|A^i(x,\nu)-A^i(z,\nu)|\le L\,|x-z|\ ,\qquad x,z\in\mathcal{T}_D,\quad i=1,..,d,\qquad L=\frac{D}{R^2}\ . \tag{100}$$

Proof. Let us write

$$\left|A^i(x,\mu) - A^i(z,\mu)\right| = \left|\int_0^1 ds \frac{d}{ds} A^i(sx + (1-s)z, \mu)\right| \le \sup_{s\in[0,1]} \left|\frac{d}{ds} A^i(sx + (1-s)z, \mu)\right| , \qquad i = 1,..,d, \tag{101}$$

and set $x_0 = sx + (1-s)z$. We obtain

$$\frac{d}{ds} A^i(x_0,\mu) = \frac{(x-z)}{R} C_i(x_0, R, \mu)$$

where

$$C_i(x_0, R, \mu) = \frac{\int \frac{(y'-y)}{R} U(x_0 - y')U(x_0 - y) \int (u')^i \mu(dy', du')\mu(dy, du)}{\int U(x_0 - y')U(x_0 - y) \int \mu_t(dy', du')\mu(dy, du)} . \tag{102}$$

Recalling that $|u| \le 1$, we obtain

$$|C(x_0, R, \mu)| \le \frac{D}{R} , \tag{103}$$

since in the torus $|y' - y| \le D$ and the result follows by (101). □

For any $v \in B_1, \nu \in \mathcal{M}, M_\epsilon(\cdot, v, \nu)$ is easily seen to be Lipschitz continuous in X. In fact we have the following:

Lemma 4.3. *Let $\nu \in \mathcal{M}, \epsilon > 0, U(\cdot)$ a smooth interaction whose support contained in a ball of radius R such that $\sup_{x\in B_R} |\nabla U(x)| \le 1$ and $M_\epsilon(\cdot,\cdot,\nu)$ as in (20). Then, for any $v \in B_1, M_\epsilon(\cdot, v, \nu)$ is Lipschitz continuous in X:*

$$\left|M^i_\epsilon(x, v, \nu) - M^i_\epsilon(y, v, \nu)\right| \le L\,|x - y| , \qquad x, y \in X, i = 1,..,d, \qquad L = \frac{2}{\epsilon} . \tag{104}$$

To prove the existence of the solution of (94) we prescribe a curve $t \to \mu_t \in \mathcal{M}$ weakly continuous in t and we consider the following non-autonomous system of ordinary differential equations:

$$\begin{cases} \frac{d}{dt} x(t) = v(t) \\ \frac{d}{dt} v(t) = M(x(t), v(t), \mu_t) \end{cases} . \tag{105}$$

Under the assumption that $M(\cdot,\cdot,\mu_t)$ is Lipschitz continuous in $X \times B_1$ there exists an unique global solution of (105) for any given initial datum. The corresponding time dependent two parameters flow is denoted by $T_{t,s}[\mu.]$. Under this time dependent flow any initial measure evolves as

$$\nu_t = \nu_0 \circ T_{0,t}[\mu.] , \tag{106}$$

where $\nu_0 \circ T_{0,t}[\mu.]$ is the push forward of the measure ν_0 under the flow. For any test function g we have that

$$\nu_t(g) = \nu_0(g \circ T_{t,0}[\mu.]) , \tag{107}$$

where $g \circ T_{t,0}[\mu.]$ is the pull back under the flow of any test functions g. By the existence and uniqueness of the solution of (105) for any initial datum, the inverse flow $(T_{t,s}[\mu.])^{-1}$ is well defined. The equation for the evolution of ν_t, easily derived, is

$$\begin{aligned}\frac{\partial(\nu_t(g))}{\partial t} = \frac{\partial(\nu_0(g\circ T_{t,0}[\mu.]))}{\partial t} &= \nu_0((v\nabla_x g)\circ T_{t,0}[\mu.])+ \\ &+\nu_0((M(x,v,\mu_t)\nabla_v g)\circ T_{t,0}[\mu.]) \\ &= \nu_t(v\cdot\nabla_x g)+\nu_t(M(x,v,\mu_t)\cdot\nabla_v g)\ .\end{aligned} \tag{108}$$

One immediately realizes that proving the existence and uniqueness of the solution of (94) is equivalent to prove the existence of a fixed point for the time dependent flow $\mu_t = \mu_0 \circ T_{0,t}[\mu.]$. This is the content of the next theorem.

Theorem 4.4. *Let U be as in Lemma 4.1 or as in Lemma 4.2 and let $M(\cdot,\cdot,\nu)$ be defined as in (18) for any $\nu \in \mathcal{M}(\mathcal{T}_D \times B_1)$. The equation (94) has an unique solution in the space $\mathcal{M}(\mathcal{T}_D \times B_1)$ if $\mu_0 \in \mathcal{M}(\mathcal{T}_D \times B_1)$. Furthermore, take two solutions of (94), μ_t starting at $\mu_0 = \mu$ and ν_t starting at $\nu_0 = \nu$ then in the bounded Lipschitz distance*

$$d_{b\mathcal{L}}(\nu_t,\mu_t) \le e^{ct} d_{b\mathcal{L}}(\mu,\nu)\ , \tag{109}$$

where c is a constant which depends on the Lipschitz constant of $M(\cdot,\cdot,\nu)$ and on $\inf_{x\in\mathcal{T}_D} U(x) =: a > 0$.

The proof is obtained adapting the method explained in [26, Chapter 5] to our context. To facilitate the reader we report the proof of Theorem 4.4 in the Appendix.

Remark 5. Theorem 4.4 does not hold in $\mathbb{R}^d \times B_1$ when U satisfies Lemma 4.1. Although in this case U is globally Lipschitz continuous in $\mathbb{R}^d$, we are not able to show that $M(x,v,\cdot)$ when $x \in \mathbb{R}^d, v \in B_1$ is Lipschitz continuous with respect to $\nu \in \mathcal{M}$ in the $d_{b\mathcal{L}}$ metric. The theorem applies with obvious modification to the $\epsilon-$regularized system where M is replaced by M_ϵ defined in (20) and holds either for the system defined on $\mathcal{T}_D \times B_1$ or on $\mathbb{R}^d \times B_1$. The constant c in the statement of Theorem 4.4 will then depend on ϵ, the lower bound of the denominator of M_ϵ.

The proof of Theorem 2.6 is an immediate consequence of Theorem 4.4. The validity of Theorem 2.6 for the $\epsilon-$regularized system, where the *local mean velocity increment* is M_ϵ, is immediate as well.

Theorem 4.5. *Let $M(\cdot,\cdot,\mu)$ be as in (18) and assume that $M(\cdot,\cdot,\mu) \in C^1(X \times B_1)$ for $\mu \in \mathcal{M}$. If $\mu_0(dx,dv) = f_0(x,v)dxdv$, then $\mu_t(dx,dv) = f_t(x,v)dxdv$ and f_t is the weak solution of (19). Furthermore, if $f_0 \in C^k(X \times B_1), k \ge 1$, and $M(\cdot,\cdot,\mu) \in C^k(X \times B_1)$ for $\mu \in \mathcal{M}$, then $f_t \in C^k(X \times B_1)$.*

Proof. We start showing that for any given weakly continuous curve $t \to \mu_t \in \mathcal{M}$, if $\nu_0(dx,dv) = q_0(x,v)dxdv$, i.e. absolutely continuous with respect to the Lebesgue measure, then $\nu_t(dx,dv) = q_t(x,v)dxdv$, where

$$\frac{\partial}{\partial t} q_t(x,v) + \nabla_x q_t(x,v)\cdot v + \nabla_v \cdot [M(x,v,\mu_t) q_t(x,v)] = 0\ , \tag{110}$$

and, if $q_0 \in C^k(X \times B_1)$ and $M(\cdot,v,\mu) \in C^k(X \times B_1)$ for any $\mu \in \mathcal{M}$, then $q_t \in C^k(X \times B_1)$. Note that (110) corresponds to a linearization of (19) since $M(x,v,\mu_t)$ does not depend on $q.$ once μ_t is given. In Theorem 4.4 we proved that the fixed point equation $\mu_t = \mu_0 \circ T_{0,t}[\mu.]$ holds. Therefore, by this result and the validity of (110), one immediately obtains that μ_t has density and the thesis of the theorem is proven. We are then left with the proof of (110). Let us set $w = (x, v) \in X \times B_1$. For any test function g we obtain

$$\begin{aligned} \nu_t(g) &= \nu_0 \circ T_{0,t}[\mu.](g) = \nu_0(g \circ T_{t,0}[\mu.]) \\ &= \int_{X\times B_1} \nu_0(dw)(g \circ T_{t,0}[\mu.])(w) = \int_{X\times B_1} q_0(w)\,(g \circ T_{t,0}[\mu.])\,(w)dw \\ &= \int_{X\times B_1} q_0(w) \circ (T_{t,0}[\mu.])^{-1} \mathcal{J}(w, \mu_t) g(w)dw \end{aligned} \tag{111}$$

where $\mathcal{J}(w, \mu_t) = Det\left[\partial.(T_{t,0})[\mu.])^{-1}(w)\right]$ is the Jacobian of the flow $(T_{t,0}[\mu.])^{-1}$ computed in w. Since the divergence of the vector field $(v(s), M(x, v, \mu_s))$ is given by

$$\sum_{i=1}^{d} \left[\frac{\partial v^i}{\partial x^i} + \frac{M^i(x, v, \mu_s)}{\partial v^i} \right] = -d \ , \tag{112}$$

by Liouville Theorem (see [1] or [2]) for any weakly continuous curve $t \to \mu_t \in \mathcal{M}$, we have

$$Det\left[\partial.(T_{t,0})[\mu.])\right](w) = e^{-dt} \ , \qquad \forall w \in X \times B_1 \ , \tag{113}$$

hence,

$$\mathcal{J}(w, \mu_t) = e^{dt} \ , \qquad \forall w \in X \times B_1 \ . \tag{114}$$

Then, from (111) and (114), we obtain

$$\nu_t(g) = \int_{X\times B_1} e^{dt} \left(q_0 \circ (T_{t,0}[\mu.])^{-1}\right)(w) g(w)dw = \int_{X\times B_1} q_t(w) g(w)dw \ , \tag{115}$$

where we denote by

$$q_t(w) := e^{dt} \left(q_0 \circ (T_{t,0}[\mu.])^{-1}\right)(w) \ . \tag{116}$$

Notice that $q_t(w)$ is weakly continuous in time, since $\mu.$ is weakly continuous. Furthermore, if, for $k \geq 1, M(\cdot, \cdot, \mu) \in C^k(X \times B_1), \mu \in \mathcal{M}$ and $q_0 \in C^k(X \times B_1)$, then $q_t \in C^k(X \times B_1)$. Writing $e^{-dt}(q_t \circ T_{t,0}[\mu.])(w) = q_0(w)$ and differentiating with respect to t we get

$$\begin{aligned} &\frac{\partial}{\partial t}\left(e^{-dt} q_t \circ (T_{t,0}[\mu.])(w)\right) = -de^{-dt}(q_t \circ T_{t,0}[\mu.])(w) + \\ &+ e^{-dt} \frac{\partial}{\partial t}(q_t \circ T_{t,0}[\mu.])(w) + e^{-dt} \nabla_x (q_t \circ T_{t,0}[\mu.])(w) \cdot v \circ T_{t,0}[\mu.] + \\ &+ e^{-dt} \nabla_v \left((q_t \circ T_{t,0}[\mu.])(w)\right) \cdot (M(\cdot, \cdot, \mu_t) \circ T_{t,0}[\mu.])(w) = 0 \ . \end{aligned} \tag{117}$$

Multiplying both members of the previous identity for e^{dt} and applying to $(T_{t,0}$ $[\mu.])^{-1}w$ we obtain

$$-\,dq_t(x, v) + \frac{\partial}{\partial t} q_t(x, v) + \nabla_x q_t(x, v) \cdot v + \nabla_v q_t(x, v) \cdot M(x, v, \mu_t) = 0 \ . \tag{118}$$

Notice that this last equation is linear in $q.$ since $\mu.$ is given. Therefore, the equation for f_t is

$$\frac{\partial}{\partial t} f_t(x, v) - df_t(x, v) + \nabla_x f_t(x, v) \cdot v + M(x, v, f_t) \cdot \nabla_v f_t(x, v) = 0 \tag{119}$$

which corresponds to (19). □

Remark 6. Theorem 4.5 holds also for the $\epsilon-$regularized system. In such a case the *local mean velocity increment* is M_ϵ, see (20), and the phase space of the system can be either $\mathbb{R}^d \times B_1$ or $\mathcal{T}_D \times B_1$. The difference from the computations in Theorem 4.5 is that

$$\nabla \cdot M_\epsilon(x,v,\mu_s) = -d\frac{(U \star \mu_s)(x)}{(U \star \mu_s)(x)+\epsilon} := -dh_s^\epsilon(x) \equiv -dh^\epsilon(\mu_s)(x)\ . \tag{120}$$

For any weakly continuous curve $t \to \mu_t \in \mathcal{M}$ and any $w \in X \times B_1$, by Liouville Theorem, we have

$$\text{Det}\left[\partial.(T_{t,0}[\mu.])\right](w) = e^{-d\int_0^t ds(h_s^\epsilon \circ T_{s,0}[\mu.])(w)}\ , \tag{121}$$

therefore

$$\mathcal{J}(w,\mu_t) = \text{Det}\left[\partial.[(T_{t,0}[\mu.])^{-1}]\right] = e^{d\int_0^t ds(h_s^\epsilon \circ T_{s,t}[\mu.])(w)}\ . \tag{122}$$

Then, from (111) and (122), we obtain

$$\begin{aligned}\nu_t(g) &= \int_{X\times B_1} e^{d\int_0^t ds(h_s^\epsilon \circ T_{s,t}[\mu.])(w)} \left(q_0 \circ (T_{t,0}[\mu.])^{-1}\right)(w)g(w)dw \\ &= \int_{X\times B_1} q_t(w)g(w)dw\ ,\end{aligned} \tag{123}$$

where

$$q_t(w) := e^{d\int_0^t ds(h_s^\epsilon \circ T_{s,t}[\mu.])(w)} \left(q_0 \circ (T_{t,0}[\mu.])^{-1}\right)(w)\ . \tag{124}$$

Notice that $q_t(w)$ is weakly continuous in time, since $\mu.$ is weakly continuous. Furthermore, if $M_\epsilon(\cdot,\cdot,\mu) \in C^k(X \times B_1), \mu \in \mathcal{M}$ and $q_0 \in C^k(X \times B_1)$, then $q_t \in C^k(X \times B_1)$. Writing

$$e^{-d\int_0^t ds(h_s^\epsilon \circ T_{s,0}[\mu.])(w)}(q_t \circ T_{t,0}[\mu.])(w) = q_0(w) \tag{125}$$

and differentiating with respect to t we get

$$\begin{aligned}&\frac{\partial}{\partial t}\left(e^{-d\int_0^t ds(h_s^\epsilon \circ T_{s,0}[\mu.])(w)}(q_t \circ (T_{t,0}[\mu.])(w)\right) = \\ &\qquad - d(h_t^\epsilon \circ T_{t,0}[\mu.])(w)e^{-d\int_0^t ds(h_s^\epsilon \circ T_{s,0}[\mu.])(w)}(q_t \circ T_{t,0}[\mu.])(w) \\ &\qquad + e^{-d\int_0^t ds(h_s^\epsilon \circ (T_{s,0}[\mu.])(w)}\frac{\partial}{\partial t}(q_t \circ T_{t,0}[\mu.])(w) \\ &\qquad + e^{-d\int_0^t ds(h_s^\epsilon \circ T_{s,0}[\mu.])(w)}\nabla_x q_t(w) \circ T_{t,0}[\mu.] \cdot v \circ T_{t,0}[\mu.] \\ &+ e^{-d\int_0^t ds(h_s^\epsilon (\circ T_{s,0}[\mu.])(w)}\nabla_v\left((q_t \circ T_{t,0}[\mu.])(w)\right) \cdot (M_\epsilon(\cdot,\cdot,\mu_t) \circ T_{t,0}[\mu.])(w) = 0\ .\end{aligned}$$

Multiplying by $e^{d\int_0^t ds(h_s^\epsilon \circ (T_{s,0}[\mu.])(w)}$ and applying to $(T_{t,0}[\mu.])^{-1}w$ we obtain

$$-dh(\mu_t)(x)q_t(x,v)+\frac{\partial}{\partial t}q_t(x,v)+\nabla_x q_t(x,v)\cdot v+\nabla_v q_t(x,v)\cdot M_\epsilon(x,v,\mu_t) = 0, \tag{126}$$

which is linear in $q.$ since $\mu.$ is given. Therefore the equation for f_t is

$$\frac{\partial}{\partial t}f_t(x,v)-dh^\epsilon(f_t)(x)f_t(x,v)+\nabla_x f_t(x,v)\cdot v+M_\epsilon(x,v,f_t)\cdot\nabla_v f_t(x,v) = 0\ , \tag{127}$$

which corresponds to (19) with M replaced by M_ϵ, taking into account the definition of h^ϵ in (120).

4.1. **Qualitative behaviour of the solution of (94).**

Lemma 4.6. *Let M as in (18) and $t \to \mu_t \in \mathcal{M}$ be the solution of (94) with initial datum μ_0. We have*

$$\mu_t(x) = \mu_0(x) + \int_0^t \mu_s(v)ds \ , \tag{128}$$

$$\int_{X\times B_1} |v|^2 \mu_t(dx,dv) \leq \int_{X\times B_1} |v|^2 \mu_0(dx,dv) \ . \tag{129}$$

Proof. By (94) we have

$$\frac{d}{dt}\int_{X\times B_1} x^i \mu_t(dx,dv) = \mu_t(v^i) \ , \qquad i=1,..,d \ , \tag{130}$$

which implies (128). To obtain (129), again from equation (94) we get

$$\begin{aligned}\frac{d}{dt}\int_{X\times B_1} |v|^2 \mu_t(dx,dv) &= \frac{d}{dt}\mu_t(|v|^2) = \mu_t(v\cdot\nabla_x |v|^2) + \mu_t(M(\cdot,\cdot,\mu_t)\cdot\nabla_v |v|^2) \\ &= 2\sum_{i=1}^d \mu_t(M^i(\cdot,\cdot,\mu_t)v^i) \leq 0 \ . \end{aligned}\tag{131}$$

Namely, for $i=1,..,d$, when $M^i(\cdot,\cdot,\mu_t)\neq 0$, we obtain

$$\begin{aligned}\mu_t(M^i(\cdot,\cdot,\mu_t)v^i) &= \int_{X\times B_1} \mu_t(dx,dv) M^i(x,v,\mu_t)v_i \\ &= \int_{X\times B_1} \mu_t(dx,dv)\left(\frac{\int_{X\times B_1} U(x-y)\left(v^iu^i - \left(v^i\right)^2\right)\mu_t(dy,du)}{\int_{X\times B_1} U(x-y)\mu_t(dy,du)}\right) \\ &= \int_{(X\times B_1)^2} \mu_t(dx,dv)\mu_t(dy,du)\frac{U(x-y)v^iu^i}{\int_{X\times B_1} U(x-y)\mu_t(dy,du)} + \\ &- \int_{X\times B_1} \mu_t(dx,dv)\left(v^i\right)^2 \leq 0 \ ,\end{aligned}\tag{132}$$

by Schwartz inequality. □

The Jensen inequality and (129) imply the boundedness of the mean velocity $\mu_t(v)$.

Let f_t be the solution at time t of the equation (19). We denote by $H(f_t)$ the Boltzmann-Vlasov entropy

$$H(f_t) := -\int_{X\times B_1} f_t(x,v)\ln(f_t(x,v))dxdv \ . \tag{133}$$

In the next lemma we show that the Boltzmann-Vlasov entropy $H(f_t)$ is a decreasing function of time. Notice that equation (19) is not time reversible, i.e. invariant under simultaneous reflection $t \to -t$ and $v \to -v$.

Lemma 4.7. *Let $f_\cdot$ be the solution of (19) with M chosen as in (18), then*

$$\frac{d}{dt}H(f_t) = -d \ . \tag{134}$$

Let $f_\cdot^\epsilon$ be the solution of (19) with M replaced by M_ϵ chosen as in (20), then

$$\frac{d}{dt}H(f_t^\epsilon) = -d\int_{X\times B_1} h_t^\epsilon(x)f_t^\epsilon(x,v)dxdv \ , \tag{135}$$

where, as in (120), $h_t^\epsilon = \frac{(U\star f_t^\epsilon)}{(U\star f_t^\epsilon)+\epsilon}$.

Proof. We start showing (134). The proof of (135) is similar and we will only outline the differences.

$$\begin{aligned}\frac{d}{dt}H(f_t) &= -\int_{X\times B_1}\frac{\partial f_t}{\partial t}(x,v)\left[(\ln f_t(x,v))+1\right]dxdv \qquad (136)\\ &= \int_{X\times B_1}(\ln f_t(x,v))\left[v\cdot\nabla_x f_t(x,v)+\nabla_v\cdot M(x,v,t)f_t(x,v)\right]dxdv\ .\end{aligned}$$

Integrating by part the last term in (136) we get

$$\begin{aligned}\frac{d}{dt}H(f_t) &= -\int_{X\times B_1}\nabla_x\left(\ln f_t(x,v)\right)\cdot v f_t(x,v)dxdv \qquad (137)\\ &\quad -\int_{X\times B_1}\nabla_v\left(\ln f_t(x,v)\right)\cdot\left[M(x,v,f_t)f_t(x,v)\right]dxdv\\ &= -\int_{X\times B_1}\nabla_x f_t(x,v)\cdot v dxdv-\int_{X\times B_1}\nabla_v f_t(x,v)\cdot\left[M(x,v,f_t)\right]dxdv\ .\end{aligned}$$

The first integral gives zero contribution since $\int_{X\times B_1} f_t(x,v)dxdv = 1$ for all $t>0$, i.e. $f_t\in L^1(X\times B_1)$. For the second term notice that $\nabla_v\cdot[M(x,v,f_t)] = -d$, therefore

$$\begin{aligned}\int_{X\times B_1}\nabla_v f_t(x,v)\cdot\left[M(x,v,f_t)\right]dxdv &= -\int_{X\times B_1} f_t(x,v)\nabla_v\cdot\left[M(x,v,f_t)\right]dxdv\\ &= d\int_{X\times B_1} f_t(x,v)dxdv = d\ . \qquad (138)\end{aligned}$$

We then obtain (134). To get (135) we proceed in the same way. We need only to modify (138) as

$$\begin{aligned}\int_{X\times B_1}\nabla_v f_t^\epsilon(x,v)\cdot\left[M_\epsilon(x,v,f_t^\epsilon)\right]dxdv &= -\int_{X\times B_1} f_t^\epsilon(x,v)\nabla_v\cdot\left[M_\epsilon(x,v,f_t^\epsilon)\right]dxdv\\ &= d\int_{X\times B_1} h_t^\epsilon(x)f_t^\epsilon(x,v)dxdv\ . \qquad (139)\end{aligned}$$

□

By the above lemma,

$$\lim_{t\to\infty} H(f_t) = -\infty\ . \qquad (140)$$

From this we can deduce that even starting at time $t=0$ from a measure which is absolutely continuous with respect to Lebesgue measure in $X\times B_1$, having therefore finite Boltzmann-Vlasov entropy, at infinity the asymptotic measure is singular with respect to the Lebesgue one. The same conclusions can be also drawn for the $\epsilon-$regularized system.

5. **Appendix.**

5.1. **Proof of Lemma 3.6.** Denote by $b_{i,j}(\cdot)$, for $i=1,\dots,N$ and $j=1,\dots,N$ the elements of the matrix $B(\cdot)=\{b_{i,j}(\cdot)\otimes\mathbb{I}_d\}_{i,j=1,..,N}$ defined in (51). By definition of $B(\mathbf{q}(t))$

$$b_{i,j}(\mathbf{q}(t)) := a_{i,j}(\mathbf{q}(t)) - a_{i,j}(\mathbf{q}^0), \qquad (141)$$

where $\{a_{i,j}(\cdot)\}$ are defined in (33). Writing $a_{i,j}(\mathbf{q}(t))$ as

$$a_{i,j}(\mathbf{q}(t)) = a_{i,j}(\mathbf{q}^0) + \int_0^1 ds \frac{d}{ds} a_{i,j}((1-s)\mathbf{q}^0 + s\mathbf{q}(t)), \tag{142}$$

we have

$$b_{i,j}(\mathbf{q}(t)) = \int_0^1 ds \frac{d}{ds} a_{i,j}((1-s)\mathbf{q}_N^0 + s\mathbf{q}_N(t)) \; . \tag{143}$$

Therefore, setting

$$x_{i,j}(s,t) := (1-s)(q_i^0 - q_j^0) + s\,(q_i(t) - q_j(t)) \;\;, \quad i,j = 1,..,N \;, \tag{144}$$

we have

$$\frac{d}{ds} a_{i,j}((1-s)\mathbf{q}^0 + s\mathbf{q}(t)) = \frac{d}{ds}\left(\frac{U((1-s)(q_i^0 - q_j^0) + s(q_i(t) - q_j(t)))}{\sum_{k=1}^N U((1-s)(q_i^0 - q_k^0) + s(q_i(t) - q_k(t)))}\right) \tag{145}$$

$$\begin{aligned}
&= \frac{d}{ds}\left(\frac{U(x_{i,j}(s,t))}{\sum_{k=1}^N U(x_{i,k}(s,t))}\right) \\
&= \frac{\nabla U(x_{i,j}(s,t)) \cdot \left[-(q_i^0 - q_j^0) + q_i(t) - q_j(t)\right]}{\sum_{k=1}^N U(x_{i,k}(s,t))} \\
&- \frac{U(x_{i,j}(s,t)) \sum_{k=1}^N \nabla U(x_{i,k}(s,t)) \cdot \left[-(q_i^0 - q_k^0) + q_i(t) - q_k(t)\right]}{\left(\sum_{k=1}^N U(x_{i,k}(s,t))\right)^2} \; .
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{j=1}^N |b_{i,j}(\mathbf{q}(t))| &\leq 2 \frac{\sum_{j=1}^N |\nabla U(x_{i,j}(s,t))| \left|-(q_i^0 - q_j^0) + q_i(t) - q_j(t)\right|}{\sum_{k=1}^N U(x_{i,k}(s,t))} \qquad (146) \\
&\leq 2 \frac{N}{U(0) + (N-1)\eta(\mathbf{q}(t), \mathbf{q}^0)} \sup_{x \in \mathbb{R}^d} |\nabla U(x)| \\
&\quad \max_{i,j \in \{1,..,N\}} \left|-(q_i^0 - q_j^0) + q_i(t) - q_j(t)\right| \;,
\end{aligned}$$

where

$$\eta(\mathbf{q}(t), \mathbf{q}^0) = \inf_{s \in [0,1]} \inf_{i,k \in \{1,\ldots,N\}} U((1-s)(q_i^0 - q_k^0) + s\,(q_i(t) - q_k(t))) \geq 0. \tag{147}$$

Since

$$\|B(\mathbf{q}(t))\| \leq \left\|\tilde{B}\left(\mathbf{q}\left(t\right)\right)\right\|_\infty = \max_{i=1,..,N} \sum_{j=1}^N |b_{i,j}(\mathbf{q}(t))| \tag{148}$$

we get the thesis.

5.2. **Proof of Theorem 4.4.** We adapt to our model [26, Theorem 5.1] and divide the proof in two steps.

Step 1. We start proving (109). Assume that ν_t and μ_t solve (167). We have, by the triangular inequality, that

$$\begin{aligned}
d_{b\mathcal{L}}(\nu_t, \mu_t) &= d_{b\mathcal{L}}(\nu_0 \circ T_{0,t}[\nu_\cdot], \mu_0 \circ T_{0,t}[\mu_\cdot]) \qquad (149) \\
&\leq d_{b\mathcal{L}}(\mu_0 \circ T_{0,t}[\nu_\cdot], \mu_0 \circ T_{0,t}[\mu_\cdot]) + d_{b\mathcal{L}}(\mu_0 \circ T_{0,t}[\nu_\cdot], \nu_0 \circ T_{0,t}[\nu_\cdot]) \; .
\end{aligned}$$

Denote by $w := (x, v), V(\mu.)_s(w) := (v(s), A(x(s), \mu_s) - v(s))$ the vector field on the right hand side of (105). The second term can be bounded as

$$d_{b\mathcal{L}}(\mu_0 \circ T_{0,t}[\nu.], \nu_0 \circ T_{0,t}[\nu.]) = e^{Lt} \sup_{f \in \mathcal{D}} \left| \int_{\mathcal{T}_D \times B_1} [d\mu_0 - d\nu_0] \left(e^{-Lt} f \circ T_{t,0}[\nu.]\right) \right| \tag{150}$$
$$\leq e^{Lt} d_{b\mathcal{L}}(\mu_0, \nu_0)$$

where L is the Lipschitz constant of $V(\mu.)_s(\cdot)$. Notice that the Lipschitz bound of $V(\mu.)_s(\cdot)$ can be easily derived from the Lipschitz bound of $A(\cdot, \mu.)$. We get (150) if we can show, since $f \in \mathcal{D}$, that $e^{-Lt} f \circ T_{t,0}[\nu.]$ is Lipschitz continuous with constant one and therefore it belongs to $\mathcal{D}$. Let $w(t) = (x(t), v(t))$ be the solution of (105) with initial condition $w_0 = (x_0, v_0)$ and let $\tilde{w}(t)$ be the solution of (105) with initial condition $\tilde{w}_0 = (\tilde{x}_0, \tilde{v}_0)$, then we need to show that

$$|f(w(t)) - f(\tilde{w}(t))| \leq C(t) |w_0 - \tilde{w}_0| \ , \tag{151}$$

with $C(t) \leq e^{Lt}$. Writing

$$w(t) = w_0 + \int_0^t V(\mu.)_s(w(s)) \tag{152}$$

and

$$\tilde{w}(t) = \tilde{w}_0 + \int_0^t V(\mu.)_s(\tilde{w}(s)) \ , \tag{153}$$

since $f \in \mathcal{D}$, we have

$$|f(w(t)) - f(\tilde{w}(t))| \leq |w(t) - \tilde{w}(t)| \ . \tag{154}$$

Furthermore,

$$|w(t) - \tilde{w}(t)| \leq |w_0 - \tilde{w}_0| + \int_0^t |V(\mu.)_s(w(s)) - V(\mu.)_s(\tilde{w}(s))| ds \tag{155}$$
$$\leq |w_0 - \tilde{w}_0| + L \int_0^t |w(s) - \tilde{w}(s)| ds \ .$$

By the Gronwall's inequality

$$|w(t) - \tilde{w}(t)| \leq e^{Lt} |w_0 - \tilde{w}_0| \tag{156}$$

proving $e^{-Lt} f \circ T_{t,0}[\nu.] \in \mathcal{D}$ and so (150). We are then left with the estimate the other term in (149) which, since $f \in \mathcal{D}$,

$$d_{b\mathcal{L}}(\mu_0 \circ T_{0,t}[\nu.], \mu_0 \circ T_{0,t}[\mu.]) = \sup_{f \in \mathcal{D}} \left| \int_{\mathcal{T}_D \times B_1} d\mu_0 \left\{ f \circ T_{t,0}[\nu.] - f \circ T_{t,0}[\mu.] \right\} \right| \tag{157}$$
$$\leq \int_{\mathcal{T}_D \times B_1} \mu_0 \, (dw) \, |T_{t,0}[\nu.] w - T_{t,0}[\mu.] w| =: \lambda(t)$$

where $T_{t,0}[\nu.]$ and $T_{t,0}[\mu.]$ are both solutions of the equation (105) with the same initial conditions but with different vector fields. We have

$$\begin{aligned}
\lambda(t) &= \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\,|\{T_{t,0}[\nu.]w - T_{t,0}[\mu.]w|\} \\
&= \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t dsV(\nu.)_s(T_{s,0}[\nu.]w) - \int_0^t dsV(\mu.)_s(T_{s,0}[\mu.]w)\right| \\
&\leq \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\nu.)_s(T_{s,0}[\nu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \\
&+ \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\mu.)_s(T_{s,0}[\mu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \ .
\end{aligned} \tag{158}$$

The first term of (158) can be estimated by the Lipschitz property of the vector field

$$\begin{aligned}
&\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\nu.)_s(T_{s,0}[\nu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \\
&\leq L\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\int_0^t ds\,|T_{s,0}[\nu.]w - T_{s,0}[\mu.]w]| \\
&= L\int_0^t ds\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\,|T_{s,0}[\nu.]w - T_{s,0}[\mu.]w]| = L\int_0^t \lambda(s)ds \ .
\end{aligned} \tag{159}$$

For the second term of (158) we have

$$\begin{aligned}
&\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\mu.)_s(T_{s,0}[\mu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \\
&\leq \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\int_0^t ds\,|V(\mu.)_s\,(T_{s,0}[\mu.]w) - V(\nu.)_s\,(T_{s,0}[\mu.]w)| \\
&= \int_0^t ds\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\,|V(\mu.)_s(T_{s,0}[\mu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)| \\
&= \int_0^t ds\int_{\mathcal{T}_D\times B_1} \mu_s(dw)\,|V(\mu.)_s(w) - V(\nu.)_s(w)| \ .
\end{aligned} \tag{160}$$

But,

$$\begin{aligned}
&|V(\mu.)_s(w) - V(\nu.)_s(w)| \leq |A(x,\mu_s) - A(x,\nu_s)| \\
&\leq \left|\frac{\int_{\mathcal{T}_D\times B_1} U(x-y)u\mu_s(dy,du) - \int_{\mathcal{T}_D\times B_1} U(x-y)u\nu_s(dy,du)}{\int_{\mathcal{T}_D\times B_1} U(x-y)\mu_s(dy,du)}\right| \\
&+ \left|\frac{\int_{\mathcal{T}_D\times B_1} U(x-y)\mu_s(dy,du) - \int_{\mathcal{T}_D\times B_1} U(x-y)\nu_s(dy,du)}{\int_{\mathcal{T}_D\times B_1} U(x-y)\mu_s(dy,du)}\right|
\end{aligned} \tag{161}$$

Since for any measure $\nu \in \mathcal{M}$, $\int_{\mathcal{T}_D\times B_1} U(x-y)\nu_s(dy,du) \geq \inf_{x\in\mathcal{T}_D} U(x) = a$, we have

$$\begin{aligned}
&\left|\frac{\int_{\mathcal{T}_D\times B_1} U(x-y)\mu_s(dy,du) - \int_{\mathcal{T}_D\times B_1} U(x-y)\nu_s(dy,du)}{\int_{\mathcal{T}_D\times B_1} U(x-y)\mu_s(dy,du)}\right| \\
&\leq \frac{\sup_{x\in\mathcal{T}_D}|\nabla U(x)| + \sup_{x\in\mathcal{T}_D} U(x)}{a}\, d_{b\mathcal{L}}(\mu_s,\nu_s)
\end{aligned} \tag{162}$$

and

$$\begin{aligned}
&\left|\frac{\int_{\mathcal{T}_D\times B_1} U(x-y)u\mu_s(dy,du) - \int_{\mathcal{T}_D\times B_1} U(x-y)u\nu_s(dy,du)}{\int_{\mathcal{T}_D\times B_1} U(x-y)\mu_s(dy,du)}\right| \\
&\leq \sum_{i=1}^{d}\left|\frac{\int_{\mathcal{T}_D\times B_1} U(x-y)u^i\mu_s(dy,du) - \int_{\mathcal{T}_D\times B_1} U(x-y)u^i\nu_s(dy,du)}{\int_{\mathcal{T}_D\times B_1} U(x-y)\mu_s(dy,du)}\right| \\
&\leq d\frac{\sup_{x\in\mathcal{T}_D}|\nabla U(x)| + \sup_{x\in\mathcal{T}_D} U(x)}{a} d_{b\mathcal{L}}(\mu_s,\nu_s)\ .
\end{aligned} \tag{163}$$

Therefore,

$$\begin{aligned}
|V(\mu_\cdot)_s(w) - V(\nu_\cdot)_s(w)| &\leq 2d\frac{\sup_{x\in\mathcal{T}_D}|\nabla U(x)| + \sup_{x\in\mathcal{T}_D} U(x)}{a} d_{b\mathcal{L}}(\mu_s,\nu_s) \\
&= \frac{c_0}{a} d_{b\mathcal{L}}(\mu_s,\nu_s)\ ,
\end{aligned}$$

where we have set $c_0 := 2d(\sup_{x\in\mathcal{T}_D}|\nabla U(x)| + \sup_{x\in\mathcal{T}_D} U(x))$. It is essential that $a > 0$. This is the case for interactions considered in the Lemmata 4.1 and 4.2 once the system is confined on the torus $\mathcal{T}_D$[4]. Thus, by (158), (159), (160) and (161) we have that

$$\lambda(t) \leq L\int_0^t \lambda(s)ds + \frac{c_0}{a}\int_0^t d_{b\mathcal{L}}(\mu_s,\nu_s)ds\ . \tag{164}$$

Hence, since by (158) $\lambda(0) = 0$ we obtain

$$\lambda(t) \leq \frac{c_0}{a}\int_0^t e^{L(t-s)} d_{b\mathcal{L}}(\mu_s,\nu_s)ds\ . \tag{165}$$

Taking in account (149), (150), (157) and (165) we get

$$d_{b\mathcal{L}}(\nu_t,\mu_t) \leq e^{Lt} d_{b\mathcal{L}}(\mu_0,\nu_0) + \frac{c_0}{a}\int_0^t e^{L(t-s)} d_{b\mathcal{L}}(\mu_s,\nu_s)ds\ . \tag{166}$$

Applying the Gronwall's lemma we get bound (109).

Step 2. To prove the existence of a solution for the fixed point equation

$$\mu_t = \mu_0 \circ T_{0,t}[\mu_\cdot]\ , \tag{167}$$

we use the Banach fixed point theorem. Let μ be the initial condition. To every curve $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}$, $\mu_0 = \mu$, we relate the solution curve

$$[0,T] \ni t \longmapsto \mu \circ T_{0,t}[\mu_\cdot] \in \mathcal{M} \tag{168}$$

Let us denote this map $\mathcal{F} : C_{\mathcal{M}} \to C_{\mathcal{M}}$, where $C_{\mathcal{M}}$ is the space of weakly continuous function $[0,T] \to \mathcal{M}$ with $\mu_0 = \mu$. We equip $C_{\mathcal{M}}$ with the metric

$$d_\alpha(\mu(\cdot),\nu(\cdot)) = \sup_{t\in[0,T]}\left[e^{-\alpha t} d_{b\mathcal{L}}(\nu_t,\mu_t)\right]\ , \tag{169}$$

for some $\alpha > 0$ which will be suitably chosen. Since $(\mathcal{M}, d_{b\mathcal{L}})$ is a complete metric space, so is $(C_{\mathcal{M}}, d_\alpha)$. Now from Step 1 we have

$$d_{b\mathcal{L}}(\nu_t,\mu_t) = d_{b\mathcal{L}}(\mathcal{F}(\mu(\cdot))(t), \mathcal{F}(\nu(\cdot))(t)) \leq \frac{c_0}{a}\int_0^t e^{L(t-s)} d_{b\mathcal{L}}(\mu_s,\nu_s)ds \tag{170}$$

[4]In the case where U is with compact support and M is replaced by M_ϵ we have that

$$\inf_{x\in X}(U \star \nu)(x) + \epsilon \geq \epsilon.$$

In this case X can be either $\mathbb{R}^d$ or $\mathcal{T}_D$.

and therefore

$$d_\alpha(\mathcal{F}(\mu(\cdot))(t), \mathcal{F}(\nu(\cdot))(t)) \leq \frac{c_0}{a(\alpha - L)} d_\alpha(\mu(\cdot), \nu(\cdot)) \tag{171}$$

for $\alpha > L$. By a suitable choice of α this proves that $\mathcal{F}$ is a contraction. □

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