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ASYMPTOTIC BEHAVIOUR OF FLOWS ON REDUCIBLE NETWORKS

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Abstract. In this paper we extend some of the previous results for a system of transport equations on a closed network. We consider the Cauchy problem for a flow on a reducible network; that is, a network represented by a diagraph which is not strongly connected. In particular, such a network can contain sources and sinks. We prove well-posedness of the problem with generalized Kirchhoff's conditions, which allow for amplification and/or reduction of the flow at the nodes, on such reducible networks with sources but show that the problem becomes ill-posed if the network has a sink. Furthermore, we extend the existing results on the asymptotic periodicity of the flow to such networks. In particular, in contrast to previous papers, we consider networks with acyclic parts and we prove that such parts of the network become depleted in a finite time, an estimate of which is also provided. Finally, we show how to apply these results to open networks where a portion of the flowing material is allowed to leave the network.

1. **Introduction.** Transport on networks recently has received much attention. In general, the problem consists of a system of first order transport equations posed on the edges of a graph G, coupled by Kirchhoff's type laws at the nodes of G. In particular, in a series of papers, [8, 9, 10, 15, 16, 23], the semigroup approach has been employed to study the well-posedness, long term asymptotics and controllability of flows on network structures. It allowed, in particular, for a development of an elegant theory that relates the long time behaviour of the flow to the graph-theoretic structure of the network.

The problem has been studied in many variants, which are comprehensively discussed in [9]. In this paper we revisit the model discussed in [15]; that is, we consider

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a flow on a finite network and thus, for simplicity, we confine ourselves to the setting of that paper. Our main aim is the extend the results of [15] to reducible networks; that is, networks represented by directed graphs (diagraphs) which are not strongly connected, see e.g. [5]. We mention that a class of not strongly connected diagraphs has been considered in [15] but here we extend the existing theory to cater for diagraphs with acyclic parts. In particular, such diagraphs may have vertices (nodes) with only outgoing edges as well as vertices with only incoming edges. According to the graph theoretical terminology, [13], such vertices are called, respectively, sources and sinks.

Moreover, we allow for amplification and/or absorption of the flow at the nodes (which makes the problem different from that considered in [16], where only absorption was allowed along the edges). We note that allowing for the amplification of the flow makes the problem non-dissipative and requires a different than in [15, 16] approach to the proof of well-posedness, based on the theory of resolvent positive operators. Using this approach, in Section 2, Theorem 2.1, we prove that the problem is well posed if and only if there are no nodes with only incoming edges. We emphasize that we consider the Kirchhoff conditions in the correct form, balancing the flow in the nodes, in contrast to the conditions considered in [15, 16] which only balanced the densities at the nodes (note, that the correct Kirchhoff's conditions were used already in [14]).

Further we study the long time behaviour of the semigroup under the assumption of [15] that the velocities along the edges are rational multiples of a single real number. Under this so-called $LD_{\mathbb{Q}}$ condition, it was proven in [15] that on a strongly connected (as well as on not strongly connected but containing no sources) network, the flow is asymptotically periodic in the sense that it exponentially converges to a periodic flow on invariant strongly connected components of the network. If $LD_{\mathbb{Q}}$ condition is not satisfied, the flow is not asymptotically periodic, [16], and hence we shall not consider such a situation here.

As we mentioned above, the main contribution of this part of the paper, extending the results of [15], is to consider the case when there is an acyclic part of G generated by nodes with no incoming edges (sources). Following and slightly refining the approach of [7, 8, 15], in Section 3 we transform the original problem to an equivalent one, posed on an extended graph with unit velocity along each edge. For such a graph, in Section 4 we propose a modification of the topological sorting algorithm, [6] (or the acyclic ordering algorithm, [5]). While in the existing literature the algorithm is used to provide an ordering of an acyclic graph, we show that it can be used to simultaneously separate the acyclic and the cyclic parts in both diagraph and its line graph. We note that the cyclic part may be not strongly connected (such as in [15]). Though the results of Section 4 are of graph theoretical nature, they have a direct impact on the dynamics of the flow by allowing for a separation of its nilpotent part. In Section 5, by systematically using the adjacency matrix of the line graph of G and the representation of the semigroup as in [7, 8], we show that the acyclic part of the network becomes depleted in a finite time (for which we provide an estimate) and, at the same time, we recover in a more elementary and constructive way Theorem 4.10 of [15] for the cyclic reducible part of the graph. Finally, in Section 6 we discuss a case of on open network where a certain amount of the flowing substance is allowed to leave the network at a sink. We show how such a problem can be incorporated into the theory developed earlier in the paper by constructing a virtual enlargement of the original network and illustrate this approach with an example.

2. **Well-posedness of the problem.** We use the notation consistent with that in the above mentioned papers. The network under consideration is represented by a simple directed graph $G = (V(G), E(G)) = (\{v_1, \dots, v_n\}, \{e_1, \dots, e_m\})$ with n vertices $v_1, \dots, v_n$ and m edges (arcs), $e_1, \dots, e_m$. We suppose that G is connected but not necessarily strongly connected. Each edge is normalized so as to be identified with $[0, 1]$ with the head at 0 and the tail at 1. We use standard terminology from graph theory, as in [5, 6], and only define the terms which are of basic relevance in this paper. The outgoing incidence matrix, $\Phi^- = (\phi^-_{ij})_{1\le i\le n, 1\le j\le m}$, and the incoming incidence matrix, $\Phi^+ = (\phi^+_{ij})_{1\le i\le n, 1\le j\le m}$, of this graph are defined, respectively as

$$\phi^-_{ij} = \begin{cases} 1 & \text{if } v_i \overset{e_j}{\to} \\ 0 & \text{otherwise.} \end{cases} \qquad \phi^+_{ij} = \begin{cases} 1 & \text{if } \overset{e_j}{\to} v_i \\ 0 & \text{otherwise.} \end{cases}$$

If the vertex v_i has more than one outgoing edge, we place a non negative weight w_{ij} on the outgoing edge e_j such that for this vertex v_i,

$$\sum_{j\in E_i} w_{ij} = 1,$$

where E_i is defined by saying that $j \in E_i$ if the edge e_j is outgoing from v_i. Naturally, $w_{ij} = 1$ if $E_i = \{j\}$ and, to shorten notation, we adopt the convention that $w_{ij} = 1$ for any j if $E_i = \emptyset$. Then the weighted outgoing incidence matrix, Φ^-_w, is obtained from Φ^- by replacing each nonzero ϕ^-_{ij} entry by w_{ij}. If each vertex has an outgoing edge, then Φ^-_w is row stochastic, hence $\Phi^- \left(\Phi^-_w\right)^T = I_n$ (where the superscript T denotes the transpose). The (weighted) adjacency matrix $\mathbb{A} = (a_{ij})_{1\le i,j\le n}$ of the graph is defined be taking $a_{ij} = w_{jk}$ if there is e_k such that $v_j \overset{e_k}{\to} v_i$ and 0 otherwise, that is, $\mathbb{A} = \Phi^+(\Phi^-_w)^T$. An important role is played by the line graph Q of G. To recall $Q = (V(Q), E(Q)) = (E(G), E(Q))$, where

$$E(Q) = \{uv;\ u, v \in E(G), \text{the head of } u \text{ coincides with the tail of } v\} = \{\varepsilon_j\}_{1\le j\le k}.$$

By $\mathbb{B}$ we denote the weighted adjacency matrix for the line graph, that is,

$$\mathbb{B} = (\Phi^-_w)^T \Phi^+. \tag{1}$$

If there is an outgoing edge at each vertex, then from the definition of $\mathbb{B}$, we see that it is column stochastic. A vertex v will be called a source if there are no incoming edges towards it and a sink if there are no edges outgoing from it.

We are interested in a flow on a closed network G. Then the standard assumption is that the flow satisfies a generalization of the Kirchhoff law at the vertices

$$\sum_{j=1}^{m} \phi^-_{ij} c_j u_j(1, t) = \sum_{j=1}^{m} \phi^+_{ij} c_j u_j(0, t), \quad t > 0, i \in 1, \dots, n, \tag{2}$$

which, in this context, is the conservation of mass law: the total inflow of mass per unit time equals the total outflow at each node (vertex) of the network. Note that due to definitions of the matrices Φ^- and Φ^+, the summation on the right hand side is over all incoming edges of the vertex v_i and on the left hand side over all outgoing edges of v_i.

Remark 1. It should be noted that the Kirchhoff law used in [15, Eqn. (3)] and [16, Eqn. (3)] is correct only in the special case when the speed of the flow is constant along each edge and the same for each edge, as it balances the densities and not the flows. This resulted in an awkward result, [15, Proposition 2.5], that a semigroup satisfying supposedly the Kirchhoff conservation of mass law fails to be conservative.

Let $u_j(x,t)$ be the density of particles at position x and at time $t \geq 0$ flowing along edge e_j for $x \in [0,1]$. The particles on e_j are assumed to move with velocity $c_j > 0$ which is constant for each j. We consider a generalized Kirchhoff's law by allowing for decrease/amplification of the flow at the entrances and exits at each vertex. Then, following [15], we describe the flow by the system

$$\begin{cases} \partial_t u_j(x,t) & = c_j \partial_x u_j(x,t), \quad x \in (0,1), \quad t \geq 0, \\ u_j(x,0) & = f_j(x), \\ \phi^-_{ij} \xi_j c_j u_j(1,t) & = w_{ij} \sum\limits_{k=1}^{m} \phi^+_{ik}(\gamma_k c_k u_k(0,t)). \end{cases} \tag{3}$$

where $\gamma_j > 0$ and $\xi_j > 0$ are the absorption/amplification coefficients at, respectively, the head and the tail of the edge e_j. If $\gamma_j = \xi_j = 1$ for all $j = 1, \cdots, m$, then the boundary conditions simply describe the Kirchhoff law at the vertices. Note that we could simplify the problem by introducing $U_j = \xi_j c_j u_j$. However, this would make further norm estimates less explicit.

Remark 2. We observe that the boundary condition in (3) takes a special form if v_i is either a sink or a source. If it is a sink, then $E_i = \emptyset$ and, by the convention above,

$$0 = \sum_{k=1}^{m} \phi^+_{ik}(\gamma_k c_k u_k(0,t)), \quad t > 0, \tag{4}$$

and

$$\phi^-_{ij} \xi_j c_j u_j(1,t) = 0, \quad t > 0, j = 1, \dots, m, \tag{5}$$

if it is a source. Clearly the last condition is nontrivial only if $j \in E_i$ as then $\phi^-_{ij} \neq 0$.

Let us denote $\mathbb{C} = \operatorname{diag}(c_j)_{1\leq j\leq m}, \mathbb{K} = \operatorname{diag}(\xi_j)_{1\leq j\leq m}$ and $\mathbb{G} = \operatorname{diag}(\gamma_j)_{1\leq j\leq m}$. We consider (3) as an Abstract Cauchy Problem

$$\mathbf{u}_t = A\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{f}, \tag{6}$$

in $X = (L_1([0,1]))^m$, where A is the realization of the expression $\mathsf{A} = (c_j \partial_x)_{1\leq j\leq m}$ on the domain

$$D(A) = \{\mathbf{u} \in (W^1_1([0,1]))^m; \ \mathbf{u} \text{ satisfies the boundary conditions in (3)}\}. \tag{7}$$

Theorem 2.1. *The following conditions are equivalent:*

1. $(A, D(A))$ *generates a* C_0 *semigroup;*
2. $\Phi^- : \mathbb{R}^m \to \mathbb{R}^n$ *is surjective;*
3. *Each vertex of* G *has an outgoing edge.*

Proof. 2. $\Leftrightarrow$ 3. If Φ^- is not surjective, then its row rank is less than n which means that the rows are not linearly independent, that is,

$$0 = \sum_{i=1}^{n} \beta_i (\Phi^-)_i \tag{8}$$

for some constants $\{\beta_i\}_{1\leq i\leq n}$ not all equal to 0, where $(\Phi^-)_i$ denotes the ith row of Φ^-. However, by construction, for each i the row $(\Phi^-)_i$ contains at least one entry 1 unless there is no outgoing edge from the vertex v_i. Thus, for (8) to hold, there must be a zero row in Φ^- and thus a vertex in G with no outgoing edge. Conversely, if a vertex v_i has no outgoing edge, then $(\Phi^-)_i$ is a row of zeroes and therefore Φ^- is not surjective.

1. $\Rightarrow$ 3. Assume that there is a semigroup $(T_A(t))_{t\geq 0}$ generated by A and consider a classical solution $\mathbf{u}(t) = T_A(t)\mathbf{f}$ with $\mathbf{f} \in D(A)$. Suppose that a vertex, say, v_i has no outgoing edge. Then, by (4),

$$0 = \sum_{k=1}^{m} \phi^+_{ik}\gamma_k c_k u_k(0,t), \quad t > 0.$$

In particular, $u_k(x,t) = f_k(x + c_k t)$ for $0 \leq x + c_k t \leq 1$ so $u_k(0,t) = f(c_k t)$ for $0 \leq t \leq \frac{1}{c_k}$. Thus

$$0 = \sum_{k=1}^{m} \phi^+_{ik}\gamma_k c_k f_k(c_k t), \quad 0 \leq t \leq c^{-1} := \min\{c_k^{-1}\}.$$

There is a sequence $(\mathbf{f}^r)_{r\in\mathbb{N}}$, $\mathbf{f}^r \in D(A)$, approximating $\mathbf{1} = (1,1,\ldots,1)$ in X. Then

$$\begin{aligned} 0 \quad &\leq \quad \|\mathbf{1}|_{(0,c^{-1})} - (f_k^r(c_k\cdot))_{1\leq k\leq m}\|_X = \sum_{k=1}^{m}\frac{1}{c_k}\int_0^{c^{-1}c_k} |1 - f_k^r(z)|dz \\ &\leq \quad \sum_{k=1}^{m}\frac{1}{c_k}\int_0^{1} |1 - f_k^r(z)|dz \to 0 \end{aligned}$$

as $r \to \infty$. Since $X-$convergence implies convergence almost everywhere of a subsequence, we have

$$0 = \sum_{k=1}^{m} \phi^+_{ik}\gamma_k c_k$$

almost everywhere on $(0, c^{-1})$, and thus everywhere. Since, however, the graph is connected and we assumed that there is no outgoing edge at v_i, there must be an incoming edge and thus at least one term of the sum is positive while all other terms are nonnegative. Thus, if there is a vertex with no outgoing edge, then the set of initial conditions satisfying the boundary conditions is not dense in X and thus $(A, D(A))$ cannot generate a C_0-semigroup.

Finally, suppose that Φ^- is surjective. Then it has at least one nonzero entry in each row. Since these rows are linearly independent, its a full row rank matrix. So the system of equations $\Phi^-\mathbf{x} = \mathbf{y}$ is consistent for any vector $y \in \mathbb{R}^m$. First we note that

$$D(A) = \{\mathbf{u} \in (W_1^1([0,1]))^m;\ \mathbf{u}(1) = \mathbb{K}^{-1}\mathbb{C}^{-1}\mathbb{B}\mathbb{G}\mathbb{C}\mathbf{u}(0)\}, \tag{9}$$

where $\mathbb{B}$ is the adjacency matrix defined in (1). The proof of this fact is analogous to that of [8, Proposition 3.1] and thus is omitted. Clearly, $(C_0^\infty((0,1)))^m \subset D(A)$ and hence $D(A)$ is dense in X. Let us consider the resolvent equation for A. We have to solve

$$\lambda u_j - c_j\partial_x u_j = f_j, \quad j = 1,\ldots,m, \quad x \in (0,1),$$

with $\mathbf{u} \in D(A)$. Integrating, we find the general solution

$$c_j u_j(x) = c_j e^{\frac{\lambda}{c_j}x} v_j + \int_x^1 e^{\frac{\lambda}{c_j}(x-s)} f_j(s)ds, \tag{10}$$

where $\mathbf{v} = (v_1, \ldots, v_m)$ is an arbitrary vector. Let $\mathbb{E}_\lambda(s) = \operatorname{diag}\left(e^{\frac{\lambda}{c_j}s}\right)_{1\leq j\leq m}$. Then (10) takes the form

$$\mathbb{C}\mathbf{u}(x) = \mathbb{C}\mathbb{E}_\lambda(x)\mathbf{v} + \int_x^1 \mathbb{E}_\lambda(x-s)\mathbf{f}(s)ds.$$

To determine $\mathbf{v}$ such that $\mathbf{u} \in D(A)$, we use the boundary conditions. At $x = 1$ we obtain

$$\mathbb{C}\mathbf{u}(1) = \mathbb{C}\mathbb{E}_\lambda(1)\mathbf{v}$$

and at $x = 0$

$$\mathbb{C}\mathbf{u}(0) = \mathbb{C}\mathbf{v} + \int_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds$$

so that

$$\mathbb{K}\mathbb{C}\mathbb{E}_\lambda(1)\mathbf{v} = \mathbb{K}\mathbb{C}\mathbf{u}(1) = \mathbb{B}\mathbb{G}\mathbb{C}\mathbf{u}(0) = \mathbb{B}\mathbb{G}\left(\mathbb{C}\mathbf{v} + \int_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds\right),$$

which can be written as

$$(\mathbb{I} - \mathbb{E}_\lambda(-1)\mathbb{C}^{-1}\mathbb{K}^{-1}\mathbb{B}\mathbb{G}\mathbb{C})\mathbf{v} = \mathbb{E}_\lambda(-1)\mathbb{C}^{-1}\mathbb{K}^{-1}\mathbb{B}\mathbb{G}\int_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds.$$

Since the norm of $\mathbb{E}_\lambda(-1)$ can be made as small as one wishes by taking large λ, we see that $\mathbf{v}$ is uniquely defined by the Neumann series provided λ is sufficiently large and hence the resolvent of A exists. We need to find an estimate for it. First we observe that the Neumann series expansion ensures that A is a resolvent positive operator and hence the norm estimates can be obtained for nonnegative entries. Next, we recall that $\mathbb{B}$ is column stochastic, that is, each column sums to 1. Adding together the rows in

$$\mathbb{K}\mathbb{C}\mathbb{E}_\lambda(1)\mathbf{v} = \mathbb{B}\mathbb{G}\mathbb{C}\mathbf{v} + \mathbb{B}\mathbb{G}\int_0^1 E_\lambda(-s)\mathbf{f}(s)ds,$$

we obtain

$$\sum_{j=1}^m \xi_j c_j e^{\frac{\lambda}{c_j}} v_j = \sum_{j=1}^m \gamma_j c_j v_j + \sum_{j=1}^m \gamma_j \int_0^1 e^{-\frac{\lambda}{c_j}s} f_j(s)ds.$$

By (10), we can evaluate, for $j \in \{1, \ldots, m\}$,

$$\int_0^1 u_j(x)dx \quad = \quad v_j \int_0^1 e^{\frac{\lambda}{c_j}x}dx + \frac{1}{c_j}\int_0^1\int_x^1 e^{\frac{\lambda}{c_j}(x-s)} f_j(s)ds$$

$$= \frac{v_j c_j}{\lambda}\left(e^{\frac{\lambda}{c_j}} - 1\right) + \frac{1}{\lambda}\int_0^1 \left(1 - e^{-\frac{\lambda}{c_j}s}\right) f_j(s)ds$$

hence, introducing a weighted space X_Ξ with the norm $\|\mathbf{u}\|_\Xi = \sum_{j=1}^m \xi_j \|u_j\|_{L_1([0,1])}$, we have

$$\|\mathbf{u}\|_\Xi \tag{11}$$
$$= \sum_{j=1}^m \xi_j \int_0^1 u_j(x)dx = \frac{1}{\lambda}\sum_{j=1}^m \xi_j v_j c_j \left(e^{\frac{\lambda}{c_j}} - 1\right) + \frac{1}{\lambda}\sum_{j=1}^m \xi_j \int_0^1 \left(1 - e^{-\frac{\lambda}{c_j}s}\right) f_j(s)ds$$
$$= \frac{1}{\lambda}\sum_{j=1}^m c_j v_j(\gamma_j - \xi_j) + \frac{1}{\lambda}\sum_{j=1}^m (\gamma_j - \xi_j)\int_0^1 e^{-\frac{\lambda}{c_j}s} f_j(s)ds + \frac{1}{\lambda}\sum_{j=1}^m \xi_j \int_0^1 f_j(s)ds.$$

We consider three cases (the first one essentially coincides with [8, Proposition 3.3]).

(a) $\gamma_j \le \xi_j$ for $j = 1, \dots, m$. Let us consider the iterates in the Neumann series for $\mathbf{v}$, $(\mathbb{E}_\lambda(-1)\mathbb{C}^{-1}\mathbb{K}^{-1}\mathbb{B}\mathbb{G}\mathbb{C})^n$. Using the fact that $\mathbb{C}, \mathbb{G}$ and $\mathbb{K}$ are diagonal so that they commute, we find

$$\mathbb{E}_\lambda(-1)\mathbb{C}^{-1}\mathbb{K}^{-1}\mathbb{B}\mathbb{G}\mathbb{C} \le (\mathbb{C}\mathbb{K})^{-1}\mathbb{E}_\lambda(-1)\mathbb{B}(\mathbb{C}\mathbb{K}).$$

Since $\mathbb{B}$ is (column) stochastic, its spectral radius satisfies $r(\mathbb{E}_\lambda(-1)\mathbb{C}^{-1}\mathbb{K}^{-1}\mathbb{B}\mathbb{G}\mathbb{C}) < 1$ for any $\lambda > 0$. Hence $R(\lambda, A)$ is defined and positive for any $\lambda > 0$. Under the assumption of this item, by dropping two first terms in the second line, (11) gives

$$\|\mathbf{u}\|_\Xi \le \frac{1}{\lambda}\sum_{j=1}^m \xi_j \int_0^1 f_j(s)ds = \frac{1}{\lambda}\|\mathbf{f}\|_\Xi, \quad \lambda > 0.$$

Since $D(A)$ is dense in X, $(A, D(A))$ generates a positive semigroup of contractions in X.

(b) $\gamma_j \ge \xi_j$ for $j = 1, \dots, m$ and $\gamma_j > \xi_j$ for at least one j. Then (11) implies that for some $\lambda > 0$ and $c = 1/\lambda$ we have

$$\|R(\lambda, A)\mathbf{f}\|_\Xi \ge c\|\mathbf{f}\|_\Xi$$

and, by density of $D(A)$, the application of the Arendt-Batty-Robinson theorem, [2, 3], gives the existence of a positive semigroup generated by A in X_Ξ. Since, however, the norm $\|\cdot\|_\Xi$ and the standard norm $\|\cdot\|$ are equivalent, we see that A generates a positive semigroup in X.

(c) $\gamma_j \le \xi_j$ for $j \in I_1$ and $\gamma_j > \xi_j$ for $j \in I_2$, where $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1, \dots, m\}$. Let $\mathbb{L} = \mathrm{diag}(l_j)$ where $l_j = \xi_j$ for $j \in I_1$ and $l_j = \gamma_j$ for $j \in I_2$. Then

$$\mathbb{E}_\lambda(-1)\mathbb{C}^{-1}\mathbb{K}^{-1}\mathbb{B}\mathbb{G}\mathbb{C} \le (\mathbb{C}\mathbb{K})^{-1}\mathbb{E}_\lambda(-1)\mathbb{B}(\mathbb{C}\mathbb{L}).$$

Thus, if we denote by $A_{\mathbf{L}}$ the operator given by the expression A restricted to

$$D(A) = \{\mathbf{u} \in (W_1^1([0,1]))^m;\ \mathbf{u}(1) = \mathbb{K}^{-1}\mathbb{C}^{-1}\mathbb{B}\mathbb{L}\mathbb{C}\mathbf{u}(0)\},$$

we see that

$$0 \le R(\lambda, A) \le R(\lambda, A_{\mathbf{L}}) \tag{12}$$

for any λ for which $R(\lambda, A_{\mathbf{L}})$ exists. But, by item (b), $A_{\mathbf{L}}$ generates a positive semigroup and thus satisfies the Hille-Yosida estimates. Since clearly (12) yields $R^k(\lambda, A) \le R^k(\lambda, A_{\mathbf{L}})$ for any $k \in \mathbb{N}$, for some $\omega > 0$ and $M \ge 1$ we have

$$\|R^k(\lambda, A)\| \le \|R^k(\lambda, A_{\mathbf{L}})\| \le M(\lambda - \omega)^{-k}, \qquad \lambda > \omega$$

and hence we obtain the generation of a semigroup by A.

Thus the proof of the theorem is complete. □

Corollary 1. *If $\xi_j = 1$ and $\gamma_j \le 1$ for $j = 1, \ldots, m$, then the semigroup $(T_A(t))_{t\ge0}$ is contractive in X. In particular, if $\gamma_j = \xi_j = 1$ for $j = 1, \ldots, m$, then $(T_A(t))_{t\ge0}$ is stochastic (that is, contractive and conservative on nonnegative data).*

Remark 3. We would like to point out that, in contrast with the better known diffusion on networks, transport is a first order problem in the spatial variable and thus the orientation of the network plays an important role. In particular, as with scalar transport problems, typically the boundary condition is posed upstream since specifying the value of the flow downstream typically leads to an ill posed problem.

This observation is related to the negative result of Theorem 2.1 that the flow problem of a network with a sink is ill-posed in the sense of semigroup theory. We emphasize that this is due to the fact that we adopted the graph theoretical definition of the sink, see e.g. [13, p. 337]; that is, sink is any vertex with no outgoing edges. Thus, we attempted to impose a boundary condition downstream, at the end of the edge.

From the physical point of view, we consider a flow on a closed network with no accumulation of material at the vertices. Thus, our result correctly reflects the impossibility of the situation in which material is flowing into a vertex but is neither accumulated nor removed from it. An extension of this result is offered in Section 6.

3. **Reduction to the constant speed case.** In the remaining part of the paper we adopt the assumption from [15, p. 150] that the speeds c_j, $j = 1, \ldots, m$, are linearly dependent over the field of rational numbers $\mathbb{Q}$. In other words,

$$\exists_{c\in\mathbb{R}} \forall_{j=1,\ldots,m}\ \frac{c}{c_j} = l_j \in \mathbb{N}. \tag{13}$$

By the rescaling of time as $\tau = ct$ and introducing the new spatial variable $y = l_j x$ for each j, we convert the problem (3) to

$$\left\{\begin{array}{lll} \partial_\tau u_j(y,\tau) & = \partial_y u_j(y,\tau), & y \in (0, l_j), \quad \tau \ge 0, \\ u_j(y,0) & = f_j(y), & \\ \phi^-_{ij}\xi_j c_j u_j(l_j,\tau) & = w_{ij} \sum\limits_{k=1}^m \phi^+_{ik}(\gamma_k c_k u_k(0,\tau)). & \end{array}\right. \tag{14}$$

In the above problem the speed on each edge is the same, equal to 1, but the lengthes of intervals are different. We shall enlarge the graph G to a larger graph $\mathcal{G}$ in such a way that the problem again is posed on unit intervals. Roughly speaking, since the lengthes of the intervals $(0, l_j)$ are integer, we subdivide each interval $(0, l_j)$ into l_j intervals of unit length, which then become new edges of $\mathcal{G}$, and create new vertices at the dividing points. Then, a function on $(0, l_j)$ is identified with a set of l_j functions which are defined on these new edges and which are continuous across vertices joining adjacent edges; these conditions become the Kirchhoff conditions at the newly created vertices. The whole process is explicit; full details of construction can be found in [20].

Let the graph $\mathcal{G}$ has $\chi(m)$ edges and $\kappa(m)$ vertices. It can be proved that the above construction provides an isomorphism $\mathbb{S} : (L_1([0,1]))^m \to (L_1([0,1]))^{\chi(m)}$:

$$\mathbf{U} = \mathbb{S}\mathbf{u},$$

$\mathbf{U} = (U_1, \dots, U_{\chi(m)}), \mathbf{u} = (u_1, \dots, u_m)$, such that $\mathbf{u}$ is a solution of (3) if and only if $\mathbf{U}$ satisfies

$$\begin{cases} \partial_\tau U_j(y,\tau) & = \partial_y U_j(y,\tau), \quad y \in (0,1), \quad \tau \geq 0, \\ U_j(y,0) & = F_j(y), \\ \widetilde{\phi}^-_{ij}\xi_j c_j U_j(1,\tau) & = \widetilde{w}_{ij} \sum\limits_{k=1}^{\chi(m)} \widetilde{\phi}^+_{ik}(\gamma_k c_k U_k(0,\tau)), \end{cases} \tag{15}$$

where $\widetilde{\Phi}^-$, Φ^+ and $\widetilde{w}_{ij}$ are, respectively, the incoming and outgoing matrices and the weights, appropriately constructed for the extended graph $\mathcal{G}$. Using Theorem 2.1, we obtain that there exists a positive C_0-semigroup in $\mathcal{X} = (L_1([0,1]))^{\chi(m)}$, say, $(\mathcal{T}_{\mathcal{A}}(t))_{t\geq 0}$ generated by an operated $\mathcal{A}$ which is defined analogously to A, see (7). It is also clear that $(\mathcal{T}_{\mathcal{A}}(t))_{t\geq 0}$ is stochastic, just as $(T_A(t))_{t\geq 0}$ is, if $\gamma_j = \xi_j = 1$ for $j \in \{1, \dots, m\}$.

Thus we have the similarity relation

$$T_A(t)\mathbf{u} = \mathbb{S}^{-1}\mathcal{T}_{\mathcal{A}}(ct)\mathbb{S}\mathbf{u} \tag{16}$$

and, in particular $\mathbb{S}^{-1}D(\mathcal{A}) \subset D(A)$ as well as $\mathbb{S}D(A) \subset D(\mathcal{A})$.

The motivation behind the representation (16) is [20, Proposition 4.5.1] (which is an extension of [8, Proposition 20]) which states that for any $\mathbf{U} \in \mathcal{X}$

$$([T_{\mathcal{A}}(t)]\mathbf{U})(x) = [\mathcal{P}^r\mathbf{U}](t+x-r), \quad r \in \mathbb{N}_0, \quad 0 \leq t+x-r \leq 1, \tag{17}$$

where, for $\mathbf{U} \in \mathcal{X}$ and the matrix $\mathbb{P} = \mathbb{C}^{-1}\mathbb{K}^{-1}\mathbb{B}\mathbb{G}\mathbb{C}$, we define

$$[\mathcal{P}\mathbf{U}](x) = \mathbb{P}\mathbf{U}(x), \quad \text{for a.a. } x \in [0,1];$$

that is, $\mathcal{P}$ is the operator in $\mathcal{X}$ induced by the pointwise multiplication by $\mathbb{P}$. In particular, if we consider the standard Kirchhoff's conditions; that is, without any amplification or absorption at the nodes, then, due to $\mathbb{K} = \mathbb{G} = \mathbb{I}$, we have $\mathbb{P}^n = \mathbb{C}^{-1}\mathbb{B}^n\mathbb{C}$ and thus the representation (17) reduces the analysis of $T_{\mathcal{A}}$ to that of the iterates of $\mathbb{B}$.

4. **Properties of the graph G.** Though the results of the section are of purely graph theoretical nature, they determine the structure (21) of the matrix $\mathbb{B}$ and hence allow for separation of the nilpotent and asymptotically periodic parts of the flow.

Let $G = (V(G), E(G)) = (\{v_1, \dots, v_n\}, \{e_1, \dots, e_m\})$ be a digraph. Following the results of the previous section, we assume that the digraph G is connected (but not necessarily strongly connected) and each vertex is the tail of some edge. In other words, some vertices can be sources, with only outgoing edges, but none can be a sink, with only incoming edges. Let us denote by $V_0(G)$ the set of all sources in G. As we have seen, an important role is played by the matrix $\mathbb{B}$ which is the weighted adjacency matrix of the line digraph of G, $Q = L(G)$. Let us introduce the map

$$\Phi_G : E(G) \to V(Q) \tag{18}$$

by defining $\Phi_G(e) = \bar{e}$, where $\bar{e}$ is the vertex in Q corresponding to the edge e in G. Clearly, Φ_G is a one-to-one map between $V(G)$ and $E(Q)$. Similarly, we introduce a multifunction

$$\Psi_G : V(G) \setminus V_0(G) \to 2^{E(Q)}$$

which assigns to a vertex v the set of edges generated by v in Q. Clearly, none $v \in V_0(G)$ generates an edge in Q. Since there are no sinks and no isolated vertices (by connectedness), any $v \in V(G) \setminus V_0(G)$ gives rise to at least one edge in Q. On the other hand, the inverse of Ψ_G is a map; that is, if $\Psi_G(v) \cap \Psi_G(v') \neq \emptyset$, then $v = v'$. Indeed, if ε is generated by a vertex $v \in G$, then there is an incoming edge u and an outgoing edge v such that the vertices $\bar{u}$ and $\bar{v}$ are incident to ε. Since an edge has only one tail and one head, if ε was generated by another vertex $v' \in V(G)$, then v' would be the head of u and the tail v, which is impossible.

These considerations yield, in particular, that the set $V_0(Q)$ of sources in Q is given by $V_0(Q) = \Phi_G(\{e_j\}_{j \in J_0})$, where $e_j, j \in J_0$, is an edge in G with the tail in $V_0(G)$. Now, let us consider the induced subdigraph Q_2 of Q consisting of all cycles of Q and all pathes joining the cycles. Further, let $Q_1 = Q - V(Q_2)$ be the subdigraph obtained by deleting Q_2 from Q, that is, Q_1 is the subdigraph induced by $V(Q) \setminus V(Q_2)$. By definition, Q_1 is acyclic and $V_0(Q) \subset V(Q_1)$. Let us further define $Q_C = \{\bar{e}_j\}_{j \in J_C}$ to be the cut-set separating Q_1 and Q_2. In other words, $\bar{e}_j \in Q_C$ if an only if its tail is in $V(Q_1)$ and its head is in $V(Q_2)$. We note that $E(Q) = E(Q_1) \cup E(Q_2) \cup Q_C$. We can provide an alternative description of Q_1 using the concept of the topological sorting [6, pp. 402-403] or, in a slightly different formulation, [5, pp. 175–176]. Following [6], we begin with an arbitrary vertex in $V_0(Q)$ and label it v_1. Next, we delete it from Q, find a vertex in $V_0 - \{v_1\}$ and label it v_2. We repeat the procedure till either every vertex is labeled or until we find a subdigraph Q' with $V_0(Q') = \emptyset$. The idea of the topological sorting is to order vertices of any diagraph in such a way that any (directed) path goes through vertices with increasing indices. It is known, [6, Theorem 14-4], that the vertices of a diagraph can be arranged in a topological order if an only if it is acyclic.

If Q' is a subdigraph of Q which was not ordered by the topological sorting argument, then it must contain a cycle. In fact, we have

Lemma 4.1. *The subdiagraph Q' does not depend on the execution of the algorithm and $Q' = Q_2$.*

Proof. Denote by $\bar{Q}$ the acyclic graph obtained by an execution of the topological sorting algorithm. First, it is clear that $V_0(Q) \subset \bar{Q}$. Next, we observe that if $v \in \bar{Q}$, then any path ending at v must be in $\bar{Q}$. Indeed, to belong to $\bar{Q}$, v had to be included in $\bar{Q}$ at, say, step i. This means that each edge with the head at v was deleted at an earlier step $k < i$. This could happen only if the adjacent vertex was included into $\bar{Q}$ at step k. Thus, all adjacent vertices are in Q' and the statement follows by induction. From this property it follows that if $v \in V(Q_2)$, then $v \notin V(\bar{Q})$. Indeed, such a v either belongs to a cycle or to a path between two cycles but, by the previous argument, in each case we would have a cycle contained in $\bar{Q}$. Similarly, if $v \in Q_1$, then $v \in \bar{Q}$. Indeed, if not, then there is an incoming edge with the tail at some v'. The vertex v' cannot be in $\bar{Q}$ as in such a case the edge from v' to v would have been deleted earlier. By induction, we construct a reverse path originating at v which is outside $\bar{Q}$. Since the graph is finite, the path cannot continue forever and terminates at a vertex x such that for any adjacent vertex, either all incoming edges have been included in the path in one of the previous steps, or there are no incoming edges. However, in the first case, there would be a cycle in Q_1 and in the other there would be $v \in V_0(Q) \setminus \bar{Q}$. In each case we arrived at a contradiction. Hence $Q' = Q_2$ and $\bar{Q} = Q_1$. □

Let us relate the above structure to the original graph G. Clearly, we can construct the subdigraphs G_1 and G_2 of G in the same way as the subdiagraphs Q_1 and Q_2 for the line graph Q. We prove

Lemma 4.2.

$$\Phi_G(E(G_2)) = V(Q_2), \qquad \Phi_G(E(G_1) \cup G_C) = V(Q_1) \tag{19}$$

and

$$\Psi_G(V(G_2)) = E(Q_2) \cup Q_C, \qquad \Psi_G(V(G_1)) = E(Q_1). \tag{20}$$

Proof. First, we see that any cycle in G corresponds to a unique cycle in Q in the following sense. Let $u_{i_1}, e_{i_1}, \ldots, e_{i_{k-1}}, u_{i_k}$ with $u_{i_k} = u_{i_1}$ be a cycle in G_2. Then we consider vertices $\Phi_G(e_{i_j})$ and, since $e_{i_{j-1}}$ and e_{i_j} connect through u_{i_j}, we see that there is an edge in $\Psi_G(u_{i_j})$ with the head at $\Phi_G(e_{i_j})$ and an edge in $\Psi_G(u_{i_{j+1}})$ with the tail at $\Phi_G(e_{i_j})$ (with obvious modification at $u_{i_1}(= u_{i_k})$ so that $\Psi_G(u_{i_1})$ contains an edge connecting $\Phi_G(e_{i_{k-1}})$ and $\Phi_G(e_{i_1})$. Conversely, let $\upsilon_{i_1}, \varepsilon_{i_1}, \ldots, \varepsilon_{i_{l-1}}, \upsilon_{i_l}$ with $\upsilon_{i_1} = \upsilon_{i_l}$ be a cycle in Q_2. Then $\Phi_G^{-1}(\upsilon_{i_1}), \Psi_G^{-1}(\varepsilon_{i_1}), \ldots, \Psi_G^{-1}(\varepsilon_{i_{l-1}}), \Phi_G^{-1}(\upsilon_{i_l})$ with $\Phi_G(\upsilon_{i_1}) = \Phi_G(\upsilon_{i_l})$ is a sequence of edges and vertices in G. However, for ε_{i_j} to be between υ_{i_j} and $\upsilon_{i_{j+1}}$ means that $\varepsilon_{i_j} \in \Psi_G(\upsilon_{i_j})$ which is the head of $e_{i_j} = \Phi_G^{-1}(\upsilon_{i_j})$ and $e_{i_{j+1}} = \Phi_G^{-1}(\upsilon_{i_{j+1}})$. Thus $\Phi_G^{-1}(\upsilon_{i_1}), \Psi_G^{-1}(\varepsilon_{i_1}), \ldots, \Psi_G^{-1}(\varepsilon_{i_{l-1}}), \Phi_G^{-1}(\upsilon_{i_l})$ is a cycle.

Let $e \in E(G_2)$. Then $\upsilon = \Phi_G(e)$ is a vertex. If e is on a cycle, then υ will be on a cycle and thus in $V(Q_2)$. Let now e belong to a path joining two cycles in G_2. This path consists of alternating vertices and edges, say, $u_{i_1}, e_{i_1}, \ldots, e_{i_{k-1}}, u_{i_k}$ with $e = e_{i_r}$ for some $1 < r < k$ and u_{i_1} and u_{i_k} belonging to cycles. Let now u' be the vertex in a cycle, which is adjacent to u_{i_1}; that is, the edge $u'u_{i_1}$ is on the cycle. Then $\Phi_{G_2}(u'u_{i_1})$ is a vertex in the corresponding cycle of Q_2 and $\Psi_G(u_{i_1})$ consists of (at least) two edges, one continuing in the cycle and one branching out to the vertex $\Phi_G(e_{i_1})$. Then, because this is a path, $\Psi_G(u_{i_2})$ contains an edge from $\Phi_G(e_{i_1})$ to $\Phi_G(e_{i_2})$ and so on to $\Phi_G(e_{i_r})$. Analogous argument can be made at u_{i_k} with the only change that $\Psi_G(u_{i_k})$ consists of at least two edges, one on the cycle in Q_2 and the other incoming from $\Phi_G(e_{i_{k-1}})$. Hence, $\Phi_G(e_{i_r})$ is on a path joining two cycles and thus belongs to Q_2.

Next, let us take a vertex υ on a path joining two cycles in Q_2. As above, we can trace the path back to a vertex υ_{i_1} on a cycle and forward to a vertex υ_{i_l}, also on a cycle; that is $\upsilon_{i_1}, \varepsilon_{i_1} \ldots, \varepsilon_{i_{l-1}}, \upsilon_{i_l}$ with $\upsilon = \upsilon_{i_s}$ for some $1 < s < l$. The vertex υ_{i_1} has at least two outgoing edges, say, ε' in the cycle and ε_{i_1} towards υ_{i_s}. But then $\varepsilon', \varepsilon_{i_1} \in \Psi_{G_2}(v)$ for some vertex v in a cycle in G_2 as otherwise the edge $e = \Phi_{G_2}^{-1}(\upsilon_{i_1})$ would have two heads. Now, similarly to the argument above, $\upsilon_{i_2} = \Phi_G(e)$ with e having its tail at v and the head at $\Psi_G^{-1}(\varepsilon_{i_2})$. Continuing, we extend the path in G_2 to $\Phi_G^{-1}(\upsilon_{i_s})$. On the other side, a similar argument holds for υ_{i_l}, only here υ_{i_l} has at least two incoming edges (one on the cycle and $\varepsilon_{i_{l-1}}$ from the path) and at least one outgoing edge (on the cycle). Hence $\upsilon_{i_l} = \Phi_{G_2}(e')$ with e' on a cycle in G_2, while the two incoming edges belong to $\Psi_{G_2}(v')$ with the vertex v', also on the cycle, is the tail of e' and the head of $\Phi_G^{-1}(\upsilon_{i_{l-1}})$ cycle. Thus v' is terminal for the path passing through υ_{i_s}. Thus we obtained the first equality in

(19). Since $E(G) = E(G_2) \cup E(G_1) \cup G_C, V(Q) = V(Q_1) \cup V(Q_2)$, where the sums are disjoint and Φ_G is a bijection, we obtain the second equality in (19).

Equation (20) can be proved in an analogous way. □

5. **Long term behaviour of the semigroup.** Our objective is to study the long term behaviour of the semigroup $(T_A(t))_{t\geq 0}$ generated by A in $X = (L_1([0,1]))^m$, defined by (6) and (7), under the assumptions that $\gamma_i = \xi_i = 1$, $i = 1, \ldots, m$ and (13). The semigroup describes a mass conserving transport on the graph G with n vertices and m edges. By Corollary 1, $(T_A(t))_{t\geq 0}$ is conservative on nonnegative data (stochastic).

The analysis is carried out by transforming the problem to that for the semigroup $(\mathcal{T}_{\mathcal{A}}(t))_{t\geq 0}$ in $\mathcal{X} = (L_1([0,1]))^{\mathrm{m}}$, where $\mathrm{m} = \chi(m)$, defined by (16) (or through (15)), on the extended graph $\mathcal{G}$ with m edges and $\mathrm{n} = \kappa(m)$ vertices. As noted earlier, $(\mathcal{T}_{\mathcal{A}}(t))_{t\geq 0}$ is also conservative. To analyse the long term asymptotics of $(\mathcal{T}_{\mathcal{A}}(t))_{t\geq 0}$, we use (17). Since $\mathbb{P}^n = \mathbb{C}^{-1}\mathbb{B}^n\mathbb{C}$, we can focus on the iterates of $\mathbb{B}$, where $\mathbb{B}$ is the weighted adjacency matrix of the line graph $\mathcal{Q}$ of $\mathcal{G}$, and apply the theory developed in the previous section, with the terminology and notation referring to $\mathcal{G}$ and $\mathcal{Q}$ and their subgraphs. Thus, we denote

$$\widehat{\mathcal{T}}_{\mathcal{A}}(t) = \mathcal{C}\mathcal{T}_{\mathcal{A}}(t)\mathcal{C}^{-1},$$

the semigroup similar to $(\mathcal{T}_{\mathcal{A}}(t))_{t\geq 0}$, defined by the iterates of $\mathbb{B}$.

It is known, see e.g. [17], that an adjacency matrix of a digraph is irreducible if and only if the digraph is strongly connected. Our interest lies, in general, in connected but not strongly connected diagraphs. Then, by simultaneous permutation of rows and columns (equivalent to renumbering of vertices), the adjacency matrix $\mathbb{B}$ can be put in the so-called normal form, [12, p. 90]. Moreover, by the topological sorting of $\mathcal{Q}$ (or by [6, Theorem 9-16]), we see that in the part of $\mathbb{B}$ corresponding to $\mathcal{Q}_1$, the acyclic part of $\mathcal{Q}$, the only nonzero entries can occur below the diagonal. Thus, reordering the vertices so that the sources in $\mathcal{Q}_1$ appear first, followed by other vertices from $\mathcal{Q}_1$, and noting that the sources correspond to zero rows, we obtain the representation

$$\mathbb{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbb{B}_{2,1} & \mathbb{B}_2 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{B}_3 & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ \mathbb{B}_{g-1,1} & \cdots & \cdots & \mathbb{B}_{g-1,g-2} & \mathbb{B}_{g-1} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbb{B}_{g,1} & \cdots & \cdots & \mathbb{B}_{g,g-2} & \mathbb{B}_{g,g-1} & \mathbb{B}_g & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbb{B}_{g+1,1} & \cdots & \cdots & \mathbb{B}_{g+1,g-2} & \mathbb{B}_{g+1,g-1} & \mathbf{0} & \mathbb{B}_{g+1} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{B}_{s,1} & \cdots & \cdots & \mathbb{B}_{s,g-2} & \mathbb{B}_{s,g-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{B}_s \end{pmatrix}. \quad (21)$$

To explain the structure of the bottom right part of $\mathbb{B}$, we recall that $\mathbb{B}$ is column stochastic and hence it has a positive left eigenvector, which can be taken to be $\mathbf{1}_{\mathrm{m}}$, where, for any $k \in \mathbb{N}$, we denote

$$\mathbf{1}_k = (\underbrace{1, 1, \cdots, 1}_{k \text{ times}}).$$

Using [12, Theorem 6, Chapter III] and the analysis of [4, Section 4] we find that for some $g > 2$ the spectral radii $\rho(\mathbb{B}_l)$ must satisfy $\rho(\mathbb{B}_l) = 1$ for $g \leq l \leq s$, $\rho(\mathbb{B}_l) < 1$

for $2 < l < g$ and $\mathbb{B}_{l,k} = \mathbf{0}$ for $g+1 \le l \le s$ and $g \le k \le l-1$ (and certainly, $\mathbb{B}_{l,k} = \mathbf{0}$ above the (block) diagonal). The matrix $\mathbb{B}_2$ is nilpotent, the matrices $\mathbb{B}_l$ for $3 \le l \le g-1$, called transient, are either irreducible matrices of dimension $n_l \times n_l$, or zero matrices of dimension 1×1 (scalars) while for $g \le l \le s$, $\mathbb{B}_l$ are called an ergodic. Since we assumed that there are no loops in G, there are no loops in $\mathcal{G}$ and therefore no loops in $\mathcal{Q}$ and thus all diagonal entries in $\mathbb{B}$ are zero.

Hence, indices $(1, \dots n_1)$ correspond to the sources and $(n_1+1, \dots, n_1+n_2)$ correspond to the other vertices in $\mathcal{Q}_1$. In particular, the only non-zero entries of $\mathbb{B}_2$ may appear below the diagonal. By Lemma 4.2, the states $1, \dots, n_1+n_2$ in $\mathbb{B}$ exactly correspond to the edges in $\mathcal{G}_1$ and the cut-set between $\mathcal{G}_1$ and $\mathcal{G}_2$. The square nonzero irreducible matrices $\mathbb{B}_3, \dots, \mathbb{B}_{g-1}$ correspond to the strong components of $\mathcal{Q}_2$ for which either there is no path from $\mathcal{Q}_1$ but there is an outflow (such as $\mathbb{B}_3$) or there is both inflow from $\mathcal{Q}_1$ and an outflow (such as B_{g-1}). Finally, the irreducible matrices $\mathbb{B}_r$ with $r \ge g$ correspond to strong components for which there are no outgoing paths (the so called terminal (invariant) strong components of $\mathcal{G}$, [5, p. 17] and [15, Definition 4.9]). The above structure induces also the form of the right Perron eigenvectors of $\mathbb{B}$. Again, following e.g. [4, Section 4], we see that the right Perron eigenspace is spanned by vectors

$$\mathbf{N}_g = (0, \dots, 0, \mathbf{n}^g, 0, \dots, 0), \dots, \mathbf{N}_s = (0, \dots, 0, 0, \dots, 0, \mathbf{n}^s), \tag{22}$$

where $\mathbf{n}_l > 0$ is the Perron eigenvector, $\mathbb{B}_l \mathbf{n}^l = \mathbf{n}^l$, $l = g, \dots, s$. Furthermore, since $\mathbb{B}$ is column stochastic, the matrices $\mathbb{B}_r$ with $r \ge g$ are column stochastic with $1 \in \sigma(\mathbb{B}_r)$, $r \ge g$, being the dominant (but not necessarily strictly dominant) Perron eigenvalue for each of them.

Further, we have $\rho(\mathbb{B}_l) < 1$ for $3 \le l < g$ and the left Perron eigenvectors of $\mathbb{B}$ corresponding to 1 are spanned by $\mathbf{v}_g, \cdots, \mathbf{v}_s$ where

$$\mathbf{v}_g = (\mathbf{y}_1^g, \mathbf{y}_2^g, \dots, \mathbf{y}_{g-1}^g, \mathbf{v}^g, 0, \dots, 0), \dots, \mathbf{v}_s = (\mathbf{y}_1^s, \mathbf{y}_2^s, \dots, \mathbf{y}_{g-1}^s, 0, \dots, 0, \mathbf{v}^s). \tag{23}$$

Here $\mathbf{v}^l = \mathbf{1}_{n_l}, g \le l \le s$, and, inductively,

$$\mathbf{y}_h^l = \left[\sum_{j=h+1}^{g-1} \mathbf{y}_j^l \mathbb{B}_{j,h} + \mathbf{v}^l \mathbb{B}_{l,h} \right] (I - \mathbb{B}_h)^{-1} \tag{24}$$

for any $1 \le h \le g-1$ and $l = g, \cdots, s$. The vectors in (24) are all positive, by e.g. [21, Theorem 2.1]. We assume that $\mathbf{N}_l$ are normalized so as

$$\mathbf{v}_l \cdot \mathbf{N}_l = 1, \quad l = g, \dots, s. \tag{25}$$

Let the number of distinct eigenvalues of $\mathbb{B}$ be ν and $k(\lambda)$ be the algebraic multiplicity of $\lambda \in \sigma(\mathbb{B})$. Due to the block triangular structure of $\mathbb{B}$, $\sigma(\mathbb{B})$ consists of eigenvalues of $\mathbb{B}_l$, $3 \le l \le s$ and 0. As stated above, the dominant, unitary eigenvalues only can come from $\sigma(\mathbb{B}_l)$, $g \le l \le s$ and, by [17, Chapter 8, p. 696], they are semisimple. Since each $\mathbb{B}_l$ is irreducible, 1 is a semisimple eigenvalue of $\mathbb{B}$ of multiplicity $s-g+1$. Further, if we denote by d_l the index of imprimitivity of $\mathbb{B}_l$; that is, the number of distinct unitary eigenvalues of $\mathbb{B}_l$, then the eigenvalues are of the form $\lambda_l^k = e^{2\pi i k/d_l}$, $k = 0, 1, \dots, d_l - 1$, and each of them is simple, [17, Chapter 8, p. 676]. Further, denote $Z = \{\lambda \in \sigma(\mathbb{B}_l), 3 \le l \le s, |\lambda| < 1\}$.

Theorem 5.1. *There is a decomposition*

$$\mathcal{X} = \mathcal{X}_g \oplus \dots \oplus \mathcal{X}_s \oplus \mathcal{Y}_e \oplus \mathcal{Y}_i \tag{26}$$

such that

: *(i) the spaces* $\mathcal{X}_l$, $l = g, \ldots, s$, $\mathcal{Y}_e, \mathcal{Y}_i$, *are invariant under* $(\widehat{\mathcal{T}}_{\mathcal{A}}(t))_{t\geq 0}$*;*
: *(ii)* $(\widehat{\mathcal{T}}_{\mathcal{A}}(t)|_{\mathcal{X}_l})_{t\geq 0}$ *is periodic with period* d_l, $l = g, \ldots, s$*;*
: *(iii)* $(\widehat{\mathcal{T}}_{\mathcal{A}}(t)|_{\mathcal{Y}_e})_{t\geq 0}$ *is exponentially stable of type* $0 > \omega > \max\{\ln|\lambda|;\ \lambda \in Z\}$*;*
: *(iv)* $(\widehat{\mathcal{T}}_{\mathcal{A}}(t)|_{\mathcal{Y}_i})_{t\geq 0}$ *is nilpotent and*

$$\widehat{\mathcal{T}}_{\mathcal{A}}(t)|_{\mathcal{Y}_i} = 0 \quad \text{for } t \geq n_2. \tag{27}$$

Proof. Let us denote by $\mathbb{F}_\lambda$ the spectral projection onto the (generalized) eigenspace of $\mathbb{B}$, corresponding to the eigenvalue λ. Using the spectral properties of $\mathbb{B}$, for any $\mathbf{u} \in \mathbb{R}^{\mathrm{m}}$ we can write

$$\mathbb{B}^r\mathbf{u} = \sum_{l=g}^{s}\sum_{k=0}^{d_{l-1}} \lambda_l^{kr}\mathbb{F}_{\lambda_l^k}\mathbf{u} + \sum_{\lambda\in Z}\lambda^r\mathbf{p}_\lambda(r)\mathbb{F}_\lambda\mathbf{u} + \mathbb{B}_0^r\mathbf{u}, \quad r \in \mathbb{N}, \tag{28}$$

where $\mathbf{p}_\lambda(r)$ is a matrix valued polynomial in r of order not exceeding $k(\lambda)$ and

$$\mathbb{B}_0 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbb{B}_{2,1} & \mathbb{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{29}$$

corresponds to the vertices in $\mathcal{Q}_1$. Clearly, $\mathbb{B}_0$ is a nilpotent matrix with index not exceeding n_2, that is

$$\mathbb{B}_0^r = \mathbf{0}, \quad r \geq n_2. \tag{30}$$

Let us recall that n_2 is the number of vertices in $\mathcal{Q}_1$ which are not sources; that is, by (19), n_2 is the number of edges in $\mathcal{G}_1$ plus the number of edges in the cut between $\mathcal{G}_1$ and $\mathcal{G}_2$ minus the number of edges emanating from the sources in $\mathcal{G}$ (which equals $\sum_{v\in V_0(\mathcal{G})} d^+(v)$). We observe that (21) induces a decomposition of $\mathbb{R}^{\mathrm{m}}$ into $\mathbb{R}^{n_1+n_2}$ and $\mathbb{R}^{m'}$ with $m' = \mathrm{m} - (n_1 + n_2)$. Since 0 is the only eigenvalue of a nilpotent matrix, the projectors $\mathbb{F}_{\lambda_l^k}, l = g, \ldots, s, k = 0, \ldots, d_l - 1$, and $\mathbb{F}_\lambda, \lambda \in Z$ act into $\mathbb{R}^{m'}$.

Following earlier calculations, (22) and (23), the spectral projection $\mathbb{F}_{\lambda_l^0}$ corresponding to $1 = \lambda_l^0 \in \sigma(\mathbb{B}_l)$, $l = g, \ldots, s$, is given by

$$\mathbb{F}_{\lambda_l^0}\mathbf{u} = (\mathbf{v}_l \cdot \mathbf{u})\mathbf{N}_l, \quad \mathbf{u} \in \mathbb{R}^{\mathrm{m}}. \tag{31}$$

The spectral projections $\mathbb{F}_{\lambda_l^k}$ have an analogous form, $\mathbb{F}_{\lambda_l^k}\mathbf{u} = (\mathbf{e}^*_{\lambda_l^k} \cdot \mathbf{u})\mathbf{e}_{\lambda_l^k}$, where $\mathbf{e}_{\lambda_l^k}$ is the right eigenvector of $\mathbb{B}_l$ corresponding to λ_l^k and $\mathbf{e}^*_{\lambda_l^k}$ is the left eigenvector of $\mathbb{B}$ for the same eigenvalue, normalized so as $\mathbf{e}^*_{\lambda_l^k} \cdot \mathbf{e}_{\lambda_l^k} = 1$. Recalling (17) and (28), for $\mathbf{u} \in \mathcal{X}$, we have

$$\begin{aligned}[\widehat{\mathcal{T}}_{\mathcal{A}}(t)\mathbf{u}](x) &= [\mathcal{B}^r\mathbf{u}](t+x-r) = \sum_{l=g}^{s}\sum_{k=0}^{d_{l-1}} \lambda_l^{kr}[\mathcal{F}_{\lambda_l^k}\mathbf{u}](t+x-r) \\ &\quad + \sum_{\lambda\in Z}\lambda^r[\mathbf{p}_\lambda(r)\mathcal{F}_\lambda\mathbf{u}](t+x-r) + [\mathcal{B}_0^r\mathbf{u}](t+x-r),\end{aligned} \tag{32}$$

for $r \in \mathbb{N}_0,\ 0 \leq t + x - r \leq 1$, where, as before, for the matrix $\mathbb{F}$ acting in $\mathbb{R}^m$, $\mathcal{F}$ denotes the operator in $L_1([0,1])^{\mathrm{m}}$ defined by the pointwise action of $\mathbb{F}$. We define $\mathcal{X}_l = \bigoplus_{k=0}^{d_{l-1}} \mathcal{F}_{\lambda_l^k}\mathcal{X}, l = g, \ldots, s$, $\mathcal{Y}_e = \bigoplus_{\lambda\in Z}\mathcal{F}_\lambda\mathcal{X}$, while by $\mathcal{Y}_i$ we denote the subspace of $\mathcal{X}$ consisting of vector functions with all but the first $n_1 + n_2$ coordinates being zero. By (21) and the definition of Z, the only common element of $\mathcal{Y}_i$ and other spaces is $\mathbf{0}$. Then we observe that, by (30), for any $\mathbf{u} \in \mathcal{Y}_i$ we have $\widehat{\mathcal{T}}_{\mathcal{A}}(t)\mathbf{u} = 0$ for $t \geq n_2$. In other words, for any $\mathbf{u} \in \mathcal{X}$,

$$[\widehat{\mathcal{T}}_{\mathcal{A}}(t)\mathbf{u}]|_{E(\mathcal{G}_1)\cup E_C} = 0$$

for $t \geq n_2$; that is, irrespective of the initial distribution of the mass on the graph, after $t = n_2$ all edges of the acyclic part together with the edges joining the acyclic part with the cyclic part become depleted. The statement (iv) is proved.

To prove (iii), we denote $\tau_r = t - r$ with $0 \leq \tau_r < 1$ and, using boundedness of $\mathcal{F}_\lambda$, we get

$$\begin{aligned}
&\|\lambda^r[\mathbf{p}_\lambda(r)\mathcal{F}_\lambda\mathbf{u}](t+\cdot-r)\|_X \\
&\leq C(|\lambda|^r|\mathbf{p}_\lambda(r)|\int_{\tau_r}^{1}|\mathbf{u}(s)|ds + |\lambda|^{r+1}|\mathbf{p}_\lambda(r+1)|\int_{0}^{\tau_r}|\mathbf{u}(s)|ds) \leq C''\|\mathbf{u}\|_X e^{t\ln\bar{\lambda}}
\end{aligned}$$

for some constants $C, C', C'' > 0$ and $|\lambda| < \bar{\lambda} < 1$, $\lambda \in Z$.

Finally, to prove (ii), let $\mathbf{u} \in \mathcal{X}_l = L_1([0,1], \mathbb{X}_l)$, where $\mathbb{X}_l = Span\{\mathbf{e}_{\lambda_l^k}\}_{k=0,\ldots,d_l-1}$, $l = g, \ldots, s$, and consider

$$[\widehat{\mathcal{T}}_{\mathcal{A}}(t)|_{\mathcal{X}_l}\mathbf{u}](x) := [\widehat{\mathcal{T}}_{\mathcal{A},l}(t)\mathbf{u}](x) = \sum_{k=0}^{d_{l-1}}\lambda_l^{kr}[\mathcal{F}_{\lambda_l^k}\mathbf{u}](t+x-r), \quad 0 \leq t+x-r \leq 1, \tag{33}$$

for $l = g, \ldots, s$. Then $(\widehat{\mathcal{T}}_{\mathcal{A},l}(t)(t))_{t\geq 0}$ extends to a periodic group in $\mathcal{X}_l$ with period equal to the imprimitivity index d_l of $\mathbb{B}_l$. Indeed, to evaluate $\widehat{\mathcal{T}}_{\mathcal{A},l}(t+d_l)$, we must take r' satisfying $0 \leq t + d_l + x - r' \leq 1$; that is, $r' = r + d_l$ and hence

$$\begin{aligned}
[\widehat{\mathcal{T}}_{\mathcal{A},l}(t+d_l)\mathbf{u}](x) &= \sum_{k=0}^{d_{l-1}}\lambda_l^{kr'}[\mathcal{F}_{\lambda_l^k}\mathbf{u}](t+d_l+x-r'), \qquad 0 \leq t+d_l+x-r' \leq 1, \\
&= \sum_{k=0}^{d_{l-1}} e^{\frac{2\pi kri}{d_l}}e^{2\pi ki}[\mathcal{F}_{\lambda_l^k}\mathbf{u}](t+x-r), \qquad 0 \leq t+x-r \leq 1, \\
&= [\widehat{\mathcal{T}}_{\mathcal{A},l}(t)\mathbf{u}](x), \qquad 0 \leq t+x-r \leq 1.
\end{aligned}$$

Thus, the period τ_l of $(\widehat{\mathcal{T}}_{\mathcal{A},l}(t))_{t\geq 0}$ does not exceed d_l. The fact that d_l indeed is the period of $(\widehat{\mathcal{T}}_{\mathcal{A},l}(t))_{t\geq 0}$ can be proved as in [15, Theorem 4.5] or [8, Theorem 24] but their argument use an advanced theory from [19] and the representation (33) gives the same result in a more elementary way. Indeed, taking the Laplace transform of $(\widehat{\mathcal{T}}_{\mathcal{A},l}(t))_{t\geq 0}$ we have, with $\mathbf{u}_k := \mathcal{F}_{\lambda_l^k}\mathbf{u}$ for $\mathbf{u} \in \mathcal{X}_l$,

$$\begin{aligned}
&\int_0^\infty e^{-t\lambda}[\widehat{\mathcal{T}}_{\mathcal{A},l}(t)\mathbf{u}](x)dt \\
&\qquad = \sum_{k=0}^{d_l-1}\left(\int_x^1 e^{-\lambda(s-x)}\mathbf{u}_k(s)ds + \sum_{r=1}^{\infty} e^{\left(\frac{2\pi ki}{d_l}-\lambda\right)r}\int_0^1 e^{-\lambda(s-x)}\mathbf{u}_k(s)ds\right).
\end{aligned}$$

From this expression we see that the resolvent of the generator of $(\widehat{\mathcal{T}}_{\mathcal{A},l}(t))_{t\geq 0}$ has poles at $\lambda \in \mathbb{C}$ which satisfy $e^\lambda = e^{\frac{2\pi ik}{d_l}}$, $k = 0, \ldots, d_l - 1$. On the other hand, by [11, Lemma IV.2.25], if τ is the period of a semigroup, then the resolvent of its generator has poles at $(2\pi i/\tau)\cdot\mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers). Comparing, we find that τ is a multiple of d_l which yields $\tau = d_l$. □

Remark 4. In particular, if $d_l = 1$; that is, if $\mathbb{B}_l$ is primitive, $\mathcal{X}_l$ is one dimensional, spanned by the Perron eigenvector of $\mathbb{B}_l$ (extended by zeroes), $\mathbf{N}_l = (\mathbf{0}, \mathbf{n}^g, \mathbf{0}) = (0\ldots, 0, n^l_{n_1+\ldots+n_{l-1}+1}, \ldots, n^l_{n_1+\ldots+n_l}, 0, \ldots, 0)$ we get $\mathbf{u} = f\mathbf{N}_l$, where $f \in L_1([0,1])$, and

$$[\widehat{\mathcal{T}}_{\mathcal{A},l}(t)\mathbf{u}](x) = f(t+x-r)\mathbf{N}_l, \quad 0 \le t+x-r \le 1, \tag{34}$$

which is a semigroup of period 1.

Remark 5. The estimate (27) can be made more precise. In fact, the nonzero entries of $\mathbb{B}_0^r$ correspond to pathes in $\mathcal{Q}_1$ of length r; that is, consisting of r edges. Hence, $\mathbb{B}_0^r \neq \mathbf{0}$ as long as there is a path of length r in $\mathcal{Q}_1$. Hence,

$$\widehat{\mathcal{T}}_{\mathcal{A}}(t)|_{\mathcal{Y}_i} = 0 \quad \text{for } t \ge k+1,$$

where k is the length of the longest path in the acyclic part of $\mathcal{Q}$ (which could be determined by the topological sorting algorithm). By (19), k also is the length of the longest path in the acyclic part $\mathcal{G}_1$ of $\mathcal{G}$. Clearly, $k+1$ may be equal to n_2 if all vertices of $\mathcal{Q}_1$ are on the longest path in $\mathcal{Q}_1$.

Remark 6. Using the argument of [15, Theorem 4.5], based on [18, Theorem IV.3.3], (but augmented by Lemma 4.2 to ascertain that the cycles in $\mathcal{G}$ correspond to cycles in $\mathcal{Q}$ in a one-to-one way with the same lengthes) we see that the period of $(\widehat{\mathcal{T}}_{\mathcal{A},l}(t))_{t\ge 0}$ equals to the greatest common divisor of the lengthes of cycles composed of edges in $\mathcal{G}$ which are among states $n_1+\ldots+n_{l-1}+1, \ldots, n_1+\ldots+n_l$; that is, the states covered by the matrix $\mathbb{B}_l$, $g \le l \le s$.

To conclude, we return to the original problem for the semigroup $(T_A(t))_{t\ge0}$ on the graph G. Before we formulate the relevant result, we have to introduce a new notation. Consider the diagonal block $\mathbb{B}_l$, $g \le l \le s$, in the adjacency matrix $\mathbb{B}$ of the line graph Q of the diagraph G. Let Q_l be the diagraph whose adjacency matrix is $\mathbb{B}_l$. Since $\mathbb{B}_l$ is irreducible, Q_l is strongly connected. Moreover, Q_l is an invariant strongly connected component of Q. Clearly, there exists a subdigraph G_l of G whose line graph is Q_l. Indeed, by definition, the set of edges of G_l corresponds to the set of vertices of Q_l and each edge in Q_l joins two vertices of Q_l and thus correspond to a vertex in G_l. Moreover, G_l is a strongly connected invariant component of G. Indeed, if there was an outgoing edge from G_l, originating in a vertex in G_l, then this would imply that this vertex would generate an edge in Q outgoing from Q_l.

Theorem 5.2. *Assume that the assumption (13) holds. There is a decomposition*

$$X = X_g \oplus \ldots \oplus X_s \oplus Y_e \oplus Y_i \tag{35}$$

such that

- *(i) the subspaces X_l, $l = g, \ldots, s$, Y_e, Y_i are invariant under $(T_A(t))_{t\ge0}$;*
- *(ii) $(T_A(t)|_{X_l})_{t\ge0}$ is periodic with period τ_l,*

$$\tau_l = \frac{1}{c} gcd\left\{c\left(\frac{1}{c_{i_1}} + \ldots + \frac{1}{c_{i_k}}\right), e_{i_1}, \ldots, e_{i_k} \text{ form a cycle in } G_l\right\}, \tag{36}$$

  for $l = g, \ldots, s$;
- *(iii) $(T_A(t)|_{Y_e})_{t\ge0}$ is exponentially stable of type $0 > \omega > \max\{c \ln|\lambda|;\ \lambda \in Z\}$;*

: *(iv)* $(T_A(t)|_{Y_i})_{t\geq 0}$ *is nilpotent and*

$$T_A(t)|_{Y_i} = 0 \quad \text{for } t \geq \kappa, \tag{37}$$

where Y_i *can be identified with* $E(G_1) \cup G_C$ *(recall that* G_C *is the the set of edges between* G_1 *and* G_2*) and*

$$\kappa = \max\left\{\frac{1}{c_{i_1}} + \ldots + \frac{1}{c_{i_k}}; e_{i_1}, \ldots, e_{i_k} \text{ form a path in } G_1 \cup G_C\right\}. \tag{38}$$

Proof. The proof follows from (16) as the transformation of the graph introduced in Section 3 does not change the number of the cycles and does not affect the split between the acyclic and cyclic parts of G but only increases the lengthes of cycles and pathes and rescales time by c. Hence, by (16), the conclusion of Theorem 5.1 holds with the same number of periodic semigroups. However, their periods and the time it takes to deplete $G_1 \cup G_C$ changes. To make it precise, we recall that the cycles in Q_l correspond in a one to one way, including lengthes, to the cycles in G_l, as we mentioned in Remark 6. With this we can argue as in the proof of [15, Theorem 4.5]. We see, that if $e_{i_1}, \ldots, e_{i_k}$ form a cycle in G_l, then the length of the corresponding cycle in $\mathcal{G}$ will be

$$l_{i_1} + \ldots + l_{i_k} = c\left(\frac{1}{c_{i_1}} + \ldots + \frac{1}{c_{i_k}}\right).$$

Thus the period of the $(T_{\mathcal{A},l}(t)(t))_{t\geq 0}$ is the greatest common divisor of all such numbers for $e_{i_1}, \ldots, e_{i_k}$ forming a cycle in G_l. Hence, by (16), the period of $(T_{A,l}(t))_{t\geq 0} = (T_A|_{X_l}(t))_{t\geq 0}$ is given by (36).

Formula (38) follows in a similar way. □

6. **An open network.** The result of Theorem 2.1 that the flow problem of a network with a sink is ill-posed in the sense of semigroup theory may seem counterintuitive. However, this is due to the fact that we adopted the graph theoretical definition of the sink, see e.g. [13, p. 337]; that is, a sink is any vertex with no outgoing edges. Since we considered a flow on a closed network with no accumulation of material in vertices, our result correctly reflects impossibility of the situation in which material is flowing into a vertex but is neither accumulated nor removed from it.

The results of the paper can be extended to a more realistic situation in which the network is open. Let $\{v_i\}_{i\in J_s}$, where $J_s := \{n_1, \ldots, n_k\} \subset \{1, \ldots, n\}$, be the set of sinks of G in the above (graph theoretical) sense. Then we can consider them to be sinks in the sense of the classical theory of flows on networks, [5, p. 97] or [6, p. 385]; that is, the material is flowing outside G from v_i at the rate

$$h_i(\mathbf{u}(t)) = \sum_{j=1}^{m} \phi^+_{ij} c_j u_j(0,t) > 0, \quad t > 0. \tag{39}$$

Such a problem can be fit into the theory developed earlier by using the same idea as when converting dishonest (sub-) Markov process into a honest one, see e.g. [1, Proposition 1.1]. To this end, we enlarge G by appending, to each sink $v_i, i \in J_s$, a directed subgraph $G_i = (\{v^i_1, v^i_2\}, \{e^i_2, e^i_3\})$, where $v^i_1 \overset{e^i_2}{\to} v^i_3$ and $v^i_2 \overset{e^i_3}{\to} v^i_1$ via a bridge e^i_1; that is, $v_i \overset{e^i_1}{\to} v^i_1$. Let us denote such an extended graph by $\widetilde{G}$. Clearly, $\widetilde{G}$ satisfies all assumptions required in Section 4 and, in particular, each $G_i, i \in J_s$ is a terminal (invariant) strong component of $\widetilde{G}$, see [5, p. 17] or [15, Definition 4.9]. This shows that the adjacency matrix $\widetilde{\mathbb{B}}$ of the line graph of $\widetilde{G}$ has the same structure as $\mathbb{B}$ in (21), where all block components $\widetilde{\mathbb{B}}_r$, corresponding to the strong components $G_i, i \in J_s$ can be renumbered so that they occupy the last k positions at the bottom right of $\widetilde{\mathbb{B}}$. Clearly, there can be also other invariant strong components of $\widetilde{G}$, which were also strong invariant components in G.

After these preliminaries we can consider the transport problem (3) on the extended diagraph $\widetilde{G}$. Due to the structure of $\widetilde{\mathbb{B}}$ and the equivalent formulation of the boundary conditions (9), we see that neither the particular structure of the additional graphs G_i, nor the speeds along the edges $e^i_1, e^i_2, e^i_3, i \in J_s$, and the conditions imposed at their endpoints, have any influence on the solutions along the edges of the original graph G. Hence, if we only are interested in the solution in G, Theorem 2.1 is applicable for an arbitrary choice of the boundary conditions on the added edges.

This observation also allows for the application of Theorems 27 and 5.2. Indeed, if the original graph G satisfies assumption (13) then, by choosing the speeds on the edges $e^i_1, e^i_2, e^i_3, i \in J_s$, to be equal to c, we see that the graph $\widetilde{G}$ also satisfies this assumption and, moreover, these edges do not change when we pass to the extended graph $\widetilde{\mathcal{G}}$, see Section 4. Thus, in particular, the flow will be asymptotically periodic in each $G_i, i \in J_s$ (which all are outside G) and the edges incoming to the sinks $v_i, i \in J_s$ will be eventually depleted unless they are in the acyclic part of $\widetilde{G}$ in which case they will be depleted in finite time.

We illustrate these considerations with the following example.

Example 1. Let us consider the transport problem (3) on the graph G and its extension $\widetilde{G}$, depicted on Fig. 1. For simplicity we assume that all coefficients in the problem are 1 and, since there are no vertices with more than one outgoing edge, there is no need to introduce weights.

The line graph of $\widetilde{G}$ is shown on Fig. 2. The adjacency matrix of the line graph of $\widetilde{G}$ is given by

$$\widetilde{\mathbb{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where the 2×2 matrix in the bottom right corner corresponds to the appended cyclic part in $\widetilde{G}$.

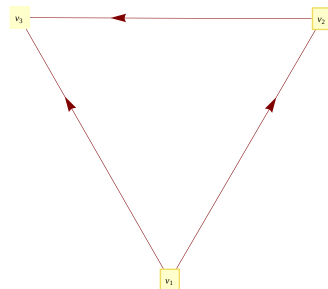
System (3) in the new enumeration can be written as

$$\partial_t u_i = \partial_x u_i, \quad u_i(x, 0) = f_i(x), \qquad i = 1, \ldots, 6,$$

with

$$\begin{aligned} u_1(1,t) &= 0, \quad u_2(1,t) = 0, \quad u_3(1,t) = u_1(0,t), \\ u_4(1,t) &= u_2(0,t) + u_3(0,t), \quad u_5(1,t) = u_4(0,t) + u_6(0,t), \quad u_6(1,t) = u_5(0,t), \end{aligned}$$

where the first row above represents the boundary conditions on the original graph G. We can see that the solution on G, consisting of (u_1, u_2, u_3), is independent of the appended graph and that it will become zero in finite time (at $t = 2$ if $f_1(x) > 0$ for $x \in [0, 1]$).



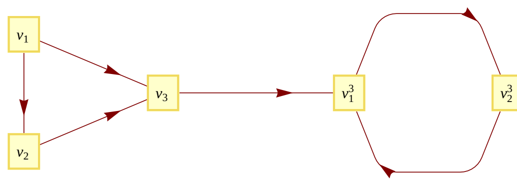


FIGURE 1. Graph G with sink at v_3 (left) and its extended graph $\widetilde{G}$ (right).

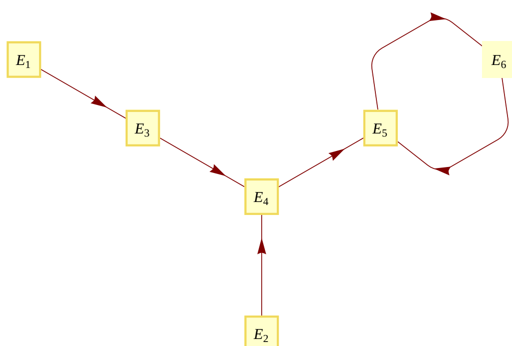


FIGURE 2. The line graph of $\widetilde{G}$ with vertices numbered according to the sorting algorithm of Section 4.

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