NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2014.9.97

Volume **9**, Number **1**, March **2014** pp. **97–110**

CHARACTERISTIC HALF SPACE PROBLEM FOR THE BROADWELL MODEL

Linglong Du

Department of Mathematics, National University of Singapore
Block S17, 10 Lower Kent Ridge Road
119076, Singapore

(Communicated by Shih-Hsien Yu)

Abstract. We study an initial boundary value problem for the Broadwell model in half space. The Green's function for the initial boundary value problem is decomposed into two parts: one is the Green's function for the initial value problem, we call it the fundamental solution for the whole space; the other is the convolution of this fundamental solution with full boundary data. A new approach to obtain the full boundary data is established here. Finally, a nonlinear time-asymptotic stability of an equilibrium state is proved.

1. **Introduction.** Most of the interesting phenomena such as thermal transpiration, curvature effect, in the rarefied gas are related to the presence of a physical boundary. They can only be understood with the knowledge of the interaction of the fluid waves with boundary layer. There have been many essential progress on the boundary value problems for rarefied gas, numerical computations by Sone et.al. in[7]; analytical studies, particularly structure of Green's function in [1], [2], [3], [5]. In this paper, our effort is to study the initial-boundary value problem of fundamental equation for the rarefied gas in a new aspect. It is partly motivated by [6], where the dissipative system was studied.

The most basic initial-boundary value problem in the rarefied gas is the one-dimensional Broadwell model given incoming boundary condition $b_+(t)$:

$$\begin{cases} \partial_t \tilde{F} + V\partial_x \tilde{F} = Q(\tilde{F}), \quad x>0, \quad t>0, \\ \tilde{F}(x,0) = \tilde{I}_0(x), \\ (1,0,0)\tilde{F}(0,t) = \tilde{b}_+(t), \end{cases} \tag{1}$$

where

$$\tilde{F}(x,t) = \begin{pmatrix} \tilde{f}_+(x,t) \\ \tilde{f}_0(x,t) \\ \tilde{f}_-(x,t) \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q(\tilde{F}) = \begin{pmatrix} \frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_- \\ -(\frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_-) \\ \frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_- \end{pmatrix}.$$

Here the unknown functions $\tilde{f}_+, \tilde{f}_0, \tilde{f}_-$ represent the mass densities for the gas particles moving in $x-$ direction with constant speed 1, 0 and -1 respectively. The

---

2010 *Mathematics Subject Classification.* Primary: 82C40.

Key words and phrases. Broadwell model, Green's function, characteristic boundary, fundamental pair, Incoming-outgoing map.

The author is supported by the MOE TIER II grant R-146-000-146-112.

part $\partial_t \tilde{F} + V\partial_x \tilde{F}$ represents the free transport mechanism in the gas flow, $Q(\tilde{F})$ models the collision mechanism.

Even for smooth, compatible initial and boundary data, there usually exists singularity in the solution around the boundary. The classical energy method is not enough to study the nonlinearity due to the boundary singularity. So we need to consider the pointwise estimates of the Green's function for the linearized problem and then close the nonlinearity.

For the collision operator Q, the equilibrium states $\tilde{F}$ are the positive-valued vector solutions of $Q(\tilde{F}) = 0$. Furthermore, an absolute equilibrium state M satisfies

$$M = (1/6, 1/3, 1/6)^t.$$

We are interested in the structure of solutions close to this absolute equilibrium state M, i.e., the initial data $\tilde{I}_0(x)$ is a small perturbation around M.

The linearized equation around the absolute equilibrium state M is

$$\partial_t F + V\partial_x F = LF, \tag{2}$$

with the linearized collision operator L:

$$L = -\frac{1}{6}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Since L is symmetric, we have the following macro-micro decomposition $(\mathbf{P_0}, \mathbf{P_1})$ introduced in [2]:

$$\begin{aligned} &I = \mathbf{P_0} + \mathbf{P_1}, \quad \mathbf{P_0} \perp \mathbf{P_1}, \quad I \ \text{ is the identity matrix}, \\ &\mathbf{P_0} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad \mathbf{P_1} = \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{3}$$

This macro-micro decomposition $(\mathbf{P_0}, \mathbf{P_1})$ satisfies:

$$\begin{aligned} \mathbf{P_0} L &= L\mathbf{P_0} = 0, \\ \mathbf{P_1} L &= L = L\mathbf{P_1}. \end{aligned}$$

The Green's function $\mathbb{G}_b(x, y, t, \tau) = \mathbb{G}_b(x, y, t - \tau)$ for the linearized initial boundary value problem of Broadwell model is a 3×3 matrix-valued function which satisfies:

$$\begin{cases} \partial_t \mathbb{G}_b + V\partial_x \mathbb{G}_b = L\mathbb{G}_b, \quad x, \quad y > 0, \quad t > \tau, \\ \mathbb{G}_b(x, y, 0) = \delta(x - y)I, \\ (1, 0, 0)\mathbb{G}_b(0, y, t, \tau) = (0, 0, 0). \end{cases} \tag{4}$$

To construct $\mathbb{G}_b(x, y, t)$, we define a fundamental pair$(\mathbb{G}, \mathbb{H})$:

$$\begin{cases} \mathbb{G} = \mathbb{G}(x - y, t), \\ \mathbb{H}(x, y, t) = \mathbb{G}_b(x, y, t) - \mathbb{G}(x - y, t). \end{cases} \tag{5}$$

Here the first part is the fundamental solution of the linearized initial problem of Broadwell model for the whole space, which is also a 3×3 matrix valued function satisfying:

$$\begin{cases} \partial_t \mathbb{G} + V\partial_x \mathbb{G} = L\mathbb{G}, \quad x \in \mathbb{R}, \quad t > 0, \\ \mathbb{G}(x, 0) = \delta(x)I. \end{cases}$$

The second part $\mathbb{H}(x, y, t)$ satisfies the following system:

$$\begin{cases} \partial_t \mathbb{H} + V\partial_x \mathbb{H} = L\mathbb{H}, \quad x > 0, \quad t > 0, \\ \mathbb{H}(x, y, 0) = 0, \\ (1, 0, 0)\mathbb{H}(0, y, t) = -(1, 0, 0)\mathbb{G}(-y, t). \end{cases} \tag{6}$$

By applying the first Green's identity to (6), we have the representation for the second part:

$$\mathbb{H}(x, y, t) = \int_0^t \mathbb{G}(x, t - \tau)V\mathbb{H}(0, y, \tau)d\tau. \tag{7}$$

Because of the pointwise description of Green's function $\mathbb{G}(x, t)$ obtained in [2], the representation (7) yields a pointwise description of the interaction of the boundary data and the propagation of the interior fluid waves. However, the representation demands the full boundary data $\mathbb{H}(0, y, \tau)$, while physically the boundary data are given only for particle moving in $x-$ direction with speed 1. The global boundary data $\mathbb{H}(0, y, t)$ can be obtained through Fourier transformation and wellposedness of the half space problem. One can apply complex analysis to yield the exponential sharp global estimates of the boundary data.

Once the Green's function for the initial-boundary problem (4) is obtained, we can get the representation for the solution of initial-boundary problem (1). As is usually done, the solution F is written as

$$F = M + W.$$

The boundary value $b_+(t)$, for simplicity, is assumed to be part of the absolute equilibrium state M. To illustrate the wave propagation properties of the solution, we assume the initial perturbation $W_0(x) \equiv \tilde{I_0}(x) - M$ satisfying $\|W_0(x)\| \le \epsilon e^{-\sigma|x|}$, with $\epsilon \ll 1$. Then the perturbation satisfies

$$\begin{cases} \partial_t W + V\partial_x W - LW = Q(W), \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ W(x, 0) = W_0(x), \\ (1, 0, 0)W(0, t) = 0. \end{cases} \tag{8}$$

Now we state the main theorem in our paper:

Theorem 1.1. *There exists a constant C, such that the solution $W(x, t)$ for (8) satisfies*

$$\begin{aligned} \|W(x, t)\| \quad &\le \quad C\epsilon \left(\frac{e^{-\frac{|x - \frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right) \\ &+ C\epsilon \begin{cases} \frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}}, & \textit{for} \quad x \in (0, \frac{1}{\sqrt{3}}t - \sqrt{t}), \\ 0, & \textit{for} \quad x \in (\frac{1}{\sqrt{3}}t - \sqrt{t}, \infty). \end{cases} \end{aligned}$$

The rest of the paper is as follows. In section 2, we prepare the full boundary data through Fourier transformation and wellposedness of the half space problem. In section 3, we will review the Green's function for the linearized initial value problem, then construct the Green's function for the initial-boundary problem by using the aforementioned fundamental pair $(\mathbb{G},\mathbb{H})$. In the section 4, we prove the main nonlinear stability theorem. We list some useful lemmas for the nonlinear wave interactions in Appendix. This problem is very interesting due to a particular fact that the speed of the boundary coincides with one speed of the transport matrix. The resonance between particles and boundary can be clearly realized by the Green's function we constructed. The analysis in this paper also provides a unified tool for studying the initial-boundary value problem for this kinetic equation with differential physical characteristics, as compared to the previous works in [1], [2], [3].

2. **Incoming-outgoing map.** To obtain full boundary $\mathbb{H}(0,y,\tau)$ in (7), we use the Laplace transform to construct a map $(1,0,0)\mathbb{H}(0,y,\tau)\to(0,0,1)\mathbb{H}(0,y,\tau)$. For convenience, we denote

$$\begin{aligned}\mathbb{H}_+(x,y,t)&\equiv(1,0,0)\mathbb{H}(x,y,t),\\ \mathbb{H}_0(x,y,t)&\equiv(0,1,0)\mathbb{H}(x,y,t),\\ \mathbb{H}_-(x,y,t)&\equiv(0,0,1)\mathbb{H}(x,y,t),\\ \mathbb{H}(x,y,t)&=\begin{pmatrix}\mathbb{H}_+(x,y,t)\\ \mathbb{H}_0(x,y,t)\\ \mathbb{H}_-(x,y,t)\end{pmatrix}\end{aligned}$$

For a function $y(t)$ defined for $t\ge 0$, its Laplace transform and inverse Laplace transform are defined as follows:

$$\begin{aligned}Y(s)&=\int_0^\infty e^{-st}y(t)dt,\\ y(t)&=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}e^{st}Y(s)ds,Re(s)=\gamma,\\ &\qquad\text{s.t. }\ \gamma\ \text{ is greater than the real part of all singularities of }\ Y(s).\end{aligned}$$

Let $\mathbb{L}_s$ and $\mathbb{L}_\xi$ denote the Laplace transform with respect to time variable t and space variable x respectively. Take the Laplace transform of the first equation of (6) in the x and t variables:

$$\begin{cases}s\mathbb{J}[\mathbb{H}_+](\xi,y,s)+\xi\mathbb{J}[\mathbb{H}_+](\xi,y,s)=\mathbb{L}_s[\mathbb{H}_+](0,y,s)\\ -\frac{1}{6}(\mathbb{J}[\mathbb{H}_+](\xi,y,s)-\mathbb{J}[\mathbb{H}_0](\xi,y,s)+\mathbb{J}[\mathbb{H}_-](\xi,y,s)),\\ s\mathbb{J}[\mathbb{H}_0](\xi,y,s)=\frac{1}{6}(\mathbb{J}[\mathbb{H}_+](\xi,y,s)-\mathbb{J}[\mathbb{H}_0](\xi,y,s)+\mathbb{J}[\mathbb{H}_-](\xi,y,s)),\\ s\mathbb{J}[\mathbb{H}_-](\xi,y,s)-\xi\mathbb{J}[\mathbb{H}_-](\xi,y,s)=-\mathbb{L}_s[\mathbb{H}_-](0,y,s)\\ -\frac{1}{6}(\mathbb{J}[\mathbb{H}_+](\xi,y,s)-\mathbb{J}[\mathbb{H}_0](\xi,y,s)+\mathbb{J}[\mathbb{H}_-](\xi,y,s)),\end{cases}\tag{9}$$

here $\mathbb{J}[\mathbb{H}]\equiv\mathbb{L}_s[\mathbb{L}_\xi[\mathbb{H}]],s>0,\xi>0$.

From (9), we have

$$\begin{cases}\mathbb{J}[\mathbb{H}_+](\xi,y,s)=\frac{\mathbb{L}_s[\mathbb{H}_+](0,y,s)-s\mathbb{J}[\mathbb{H}_0](\xi,y,s)}{\xi+s},\\ \mathbb{J}[\mathbb{H}_-](\xi,y,s)=\frac{\mathbb{L}_s[\mathbb{H}_-](0,y,s)+s\mathbb{J}[\mathbb{H}_0](\xi,y,s)}{\xi-s}.\end{cases}$$

We substitute these two representations into the second equation of (9) to get

$$\mathbb{J}[\mathbb{H}_0](\xi,y,s)=\frac{1}{6s+1}\frac{(\xi-s)\mathbb{L}_s[\mathbb{H}_+](0,y,s)+(\xi+s)\mathbb{L}_s[\mathbb{H}_-](0,y,s)}{\xi^2-\frac{6s^3+3s^2}{6s+1}}.$$

Denote $\lambda_1(s) \equiv \sqrt{\frac{6s^3+3s^2}{6s+1}}$, then

$$\mathbb{J}[\mathbb{H}_0](\xi, y, s) = \frac{Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s)}{\xi - \lambda_1} + \frac{Res_{\xi=-\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s)}{\xi + \lambda_1}, \tag{10}$$

here $Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s)$ means the residue of function $\mathbb{J}[\mathbb{H}_0](\xi, y, s)$ at λ_1. Take the inverse Laplace transform of (10) with respect to space variable ξ:

$$\mathbb{L}_s[\mathbb{H}_0](x, y, s) = e^{\lambda_1 x} Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s) + e^{-\lambda_1 x} Res_{\xi=-\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s).$$

The wellposedness of a differential equation imposes the solution $\mathbb{L}_S[\mathbb{H}_0](x, y, s)$ decays to zero as $x \to \infty$. This implies that $Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s) = 0$, which yields the following relationship:

$$\begin{aligned}\mathbb{L}_s[\mathbb{H}_-](0, y, s) &= -\frac{\lambda_1 - s}{\lambda_1 + s}\mathbb{L}_s[\mathbb{H}]_+ y(0, y, s) \\ &= -(6s + 2 - \sqrt{(6s+3)(6s+1)})\mathbb{L}_s[\mathbb{H}_+](0, y, s). \end{aligned}\tag{11}$$

By the inverse Laplace transform of (11), we finally get the Incoming-outgoing Map formula:

$$\begin{aligned}\mathbb{H}_-(0, y, t) &= -6\partial_t\mathbb{H}_+(0, y, t) - 2\mathbb{H}_+(0, y, t) \\ &\quad + 6\partial_t * \frac{e^{-1/2t}}{2\sqrt{\pi t}} * \partial_t * \frac{e^{-1/6t}}{2\sqrt{\pi t}} * \mathbb{H}_+(0, y, t). \end{aligned}\tag{12}$$

Here, instead of studying the inverse Laplace transform of $\sqrt{s}$, we consider $\frac{\sqrt{s}}{s} = \frac{1}{\sqrt{s}}$ by the usual inversion formula of the Laplace transform:

$$L(t) \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{1}{\sqrt{s}}ds = \frac{1}{2\pi\sqrt{s}}\int_{-\infty}^{\infty} e^{is}\frac{1}{\sqrt{is}}ds = \frac{1}{2\sqrt{\pi t}}.$$

The inversion of the Laplace transform has the property that division in s corresponds to differentiation in t.

3. **Construct the Green's function $\mathbb{G}_b(x, y, t)$.** The fundamental pair (5) yields the decomposition $\mathbb{G}_b(x, y, t) = \mathbb{G}(x-y, t) + \mathbb{H}(x, y, t)$. Firstly, we recall the following theorem in [2] on Green's function $\mathbb{G}(x, t)$ for the initial value problem.

Theorem 3.1. *There exists a positive constant C such that*

$$\begin{aligned}&\left\| \mathbb{G}(x,t) - e^{-t/6}\begin{pmatrix} \delta(x-t) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & \delta(x+t) \end{pmatrix} \right\| \\ &\le C\left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} \right) + Ce^{-(|x|+t)/C}, \\ &\textit{for all} \quad x \in \mathbb{R}, \quad t > 0. \end{aligned}\tag{13}$$

To get the pointwise estimate for the function $\mathbb{H}(x, y, t)$, we also need the following lemma:

Lemma 3.2. *The following estimates hold:*

$$\|\mathbb{G}_+(-y,\tau)\| \le O(1)\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+e^{-(|y|+\tau)/C}\right), \tag{14}$$

$$\begin{aligned}&\|\partial_\tau\mathbb{G}_+(-y,\tau)\|\\ \le& O(1)\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{(1+\tau)}+\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{(1+\tau)}\right)(1+\frac{1}{\sqrt{1+\tau}})+O(1)e^{-(|y|+\tau)/C},\end{aligned} \tag{15}$$

$$\begin{aligned}&\left\|\partial_\tau * \frac{e^{-1/2\tau}}{2\sqrt{\pi\tau}} * \partial_\tau * \frac{e^{-1/6\tau}}{2\sqrt{\pi\tau}} * \mathbb{G}_+(-y,\tau)\right\|\\ \le\ & O(1)\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}(1+\tau)}+\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}(1+\tau)}+e^{-(|y|+\tau)/C}\right).\end{aligned} \tag{16}$$

Proof. The fist two inequalities are straightforward. For the third one, we just apply Lemma 4.1 in the Appendix when $z=0$. ☐

Now we have the following theorem:

Theorem 3.3. *The Green's function for the linearized initial boundary value problem of Broadwell model satisfies the following estimate:*

$$\begin{aligned}&\left\|\mathbb{G}_b(x,y,t)-e^{-t/6}\begin{pmatrix}\delta(x-y-t)&0&0\\0&\delta(x-y)&0\\0&0&\delta(x-y+t)\end{pmatrix}\right\|\\ \le\ & O(1)\left[e^{-(|x|+|y|+t)/C}+\frac{e^{-\frac{|x-y+\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x-y-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}\right]\end{aligned}$$

Remark 1. The last term represents the reflections at the boundary of waves with negative speed $-\frac{1}{\sqrt{3}}$ to waves with positive speed $\frac{1}{\sqrt{3}}$.

Proof. Denote the 0th order Particle-Wave decomposition for $\mathbb{G}(x,t)$ as

$$\mathbb{G}(x,t)=e^{-t/6}\begin{pmatrix}\delta(x-t)&0&0\\0&\delta(x)&0\\0&0&\delta(x+t)\end{pmatrix}+\mathbb{W}_0(x,t).$$

From (7), (12) and Theorem 3.1 we have

$$\begin{aligned}&\mathbb{H}(x,y,t)=\int_0^t \mathbb{W}_0(x,t-\tau)V\mathbb{H}(0,y,\tau)d\tau\\ &+\int_0^t e^{-(t-\tau)/6}\begin{pmatrix}\delta(x-(t-\tau))&0&0\\0&\delta(x)&0\\0&0&\delta(x+t-\tau)\end{pmatrix}V\mathbb{H}(0,y,\tau)d\tau.\end{aligned}$$

For the first term, using Lemma 3.2 we have the following estimate

$$
\begin{aligned}
&\left\|\int_0^t \mathbb{W}_0(x,t-\tau)V\mathbb{H}(0,y,\tau)d\tau\right\| \\
\leq & O(1)\int_0^t \|\mathbb{W}_0(x,t-\tau)\|\cdot(\|\mathbb{H}_+(0,y,\tau)\|+\|\mathbb{H}_-(0,y,\tau)\|)d\tau \\
\leq & O(1)\int_0^t \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+(t-\tau))}}}{\sqrt{1+t-\tau}}+\frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}}+e^{-(|x|+t-\tau)/C}\right) \\
&\cdot\left(\|\mathbb{G}_+(-y,\tau)\|+\left\|\partial_t\mathbb{G}_+(-y,\tau)+\partial_\tau * \frac{e^{-1/2\tau}}{2\sqrt{\pi\tau}} * \partial_\tau * \frac{e^{-1/6\tau}}{2\sqrt{\pi\tau}} * \mathbb{G}_+(-y,\tau)\right\|\right)d\tau \\
\leq & O(1)\int_0^t \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+(t-\tau))}}}{\sqrt{1+t-\tau}}+\frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}}+e^{-(|x|+t-\tau)/C}\right) \\
&\cdot\left(\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}\right)+e^{-(|y|+\tau)/C}\right)d\tau \\
\leq & O(1)\left[e^{-(|x|+|y|+t)/C}+\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}\right]
\end{aligned} \tag{17}
$$

For the second term, we have

$$
\begin{aligned}
&\left\|\int_0^t e^{-\frac{t-\tau}{6}}\begin{pmatrix}\delta(x-(t-\tau)) & 0 & 0\\ 0 & \delta(x) & 0\\ 0 & 0 & \delta(x+t-\tau)\end{pmatrix}V\mathbb{H}(0,y,\tau)d\tau\right\| \\
\leq \quad & O(1)\int_0^t e^{-\frac{t-\tau}{6}}[(\delta(x-(t-\tau))\,\|\mathbb{G}_+(-y,\tau)\|+\delta(x+(t-\tau))\,\|\mathbb{G}_-(-y,\tau)\|]d\tau \\
\leq \quad & O(1)\int_0^t e^{-\frac{t-\tau}{6}}\delta(x-(t-\tau))\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+e^{-(|y|+\tau)/C}\right)d\tau \\
\leq \quad & O(1)e^{-x/C}\left(\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}(t-x)|^2}{C(1+(t-x))}}}{\sqrt{1+(t-x)}}+e^{-(|y|+(t-x))/C}\right) \\
&\leq O(1)e^{-(x+y+t)/C}.
\end{aligned} \tag{18}
$$

From (17), (18) and (5) we get the pointwise estimate for $\mathbb{G}_b(x,y,t)$. □

4. **Nonlinear stability of an absolute equilibrium state.** (Proof of the main theorem)

For such a problem (8), the solution can be represented by the Green's function $\mathbb{G}_b(x,y,t)$:

$$
W(x,t) = \int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy+\int_0^t\int_0^\infty \mathbb{G}_b(x,y,t-\tau)Q(W)(y,\tau)dyd\tau
$$

$$= \int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy + \int_0^t\int_0^\infty \mathbb{G}_b(x,y,t-\tau)\mathbf{P_1}Q(W)(y,\tau)dyd\tau,$$

here $\mathbf{P_1}$ is the micro part of macro-micro decomposition (3).

$W(x,t)$ can be solved by a Picard's iteration:

$$\begin{cases} W^{(0)}(x,t) = \int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy, \\ W^{(l)}(x,t) = W^{(0)}(x,t) + \int_0^t\int_0^\infty \mathbb{G}_b(x,y,t-\tau)Q(W^{(l-1)})(y,\tau)dyd\tau, \quad \text{for} \quad l \geq 1. \end{cases}$$

By Theorem 3.3, there exists some constant K_0 such that

$$\left\| \int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy \right\| \leq K_0\epsilon \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right).$$

We make an ansatz assumption:

$$\begin{aligned} \|W(x,t)\| \leq & 2K_0\epsilon \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right) \\ & + 2K_0\epsilon \begin{cases} \frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}} & \text{for} \quad x \in (0, \frac{1}{\sqrt{3}}t - \sqrt{t}), \\ 0 & \text{for} \quad x \in (\frac{1}{\sqrt{3}}t - \sqrt{t}, \infty). \end{cases} \end{aligned} \tag{19}$$

We will prove that the ansatz assumption is true. Note that for $\mathbb{G}(x,t)$:

$$\|\mathbb{G}(x,t)\mathbf{P_1}\| = O(1) \left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{1+t} + \frac{e^{-\frac{(x+\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{1+t} + e^{-(|x|+t)/C} \right).$$

However, the algebraic decaying rate $(1+t)^{-1}$ of $\mathbb{G}(x,y,t)\mathbf{P_1}$ is not necessarily true for $\mathbb{G}_b(x,y,t)\mathbf{P_1}$ globally due to the presence of the boundary.

Proof. **Case 1.** $\{0 < x < \frac{1}{\sqrt{3}}t/2\}$.

We compute $W(x,t)$ by the forward representation:

$$\begin{aligned} & \|W(x,t)\| \\ = & \|\int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy\| + \int_0^t\int_0^\infty \|\mathbb{G}_b(x,y,t-\tau)\mathbf{P_1}Q(W)(y,\tau)\|\, dyd\tau \\ \leq & K_0\epsilon \left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right) \\ & + \int_0^t\int_0^\infty \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \|W(y,\tau)\|^2 dyd\tau \\ & + \int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)} \frac{e^{-\frac{[x+y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}} \|W(y,\tau)\|^2 dyd\tau \end{aligned} \tag{20}$$

Under the ansatz assumption (19), we have:

$$\int_0^t \int_0^\infty \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \|W(y,\tau)\|^2 dy d\tau$$

$$\leq \quad O(1)K_0^2\epsilon^2 \left[\int_0^t \int_0^\infty \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \right.$$

$$\left(\frac{e^{-\frac{(y-\frac{1}{\sqrt{3}}\tau)^2}{C(1+\tau)}}}{1+\tau} + e^{-(|y|+\tau)/C} \right) dy d\tau$$

$$+ \int_0^t \int_0^{\frac{1}{\sqrt{3}}\tau - \sqrt{\tau}} \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right)$$

$$\left. \frac{1}{(|y+\frac{1}{\sqrt{3}}\tau|+1)(|y-\frac{1}{\sqrt{3}}\tau|+1)} dy d\tau \right].$$

By Lemma 4.2 and Lemma 4.3 in the Appendix,

$$\int_0^t \int_0^\infty \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \left(\frac{e^{-\frac{(y-\frac{1}{\sqrt{3}}\tau)^2}{C(1+\tau)}}}{1+\tau} + e^{-(|y|+\tau)/C} \right) dy d\tau$$

$$\leq O(1) \left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right)$$

$$+ O(1) \begin{cases} \frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}} & \text{for } \ x \in (0, \frac{1}{\sqrt{3}}t - \sqrt{t}), \\ 0 & \text{for } \ x \in (\frac{1}{\sqrt{3}}t - \sqrt{t}, \infty). \end{cases}$$

By Lemma 4.4,

$$\int_0^t \int_0^{\frac{1}{\sqrt{3}}\tau - \sqrt{\tau}} \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right)$$

$$\frac{1}{(|y+\frac{1}{\sqrt{3}}\tau|+1)(|y-\frac{1}{\sqrt{3}}\tau|+1)} dy d\tau$$

$$\leq B^{-\frac{1}{\sqrt{3}}}(x,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};C) + B^{\frac{1}{\sqrt{3}}}(x,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};C)$$

$$= O(1) \left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right)$$

$$+O(1) \begin{cases} \frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}} & \text{for } \ x \in (0, \frac{1}{\sqrt{3}}t - \sqrt{t}), \\ 0 & \text{for } \ x \in (\frac{1}{\sqrt{3}}t - \sqrt{t}, \infty). \end{cases}$$

$$
\begin{aligned}
&\int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)}\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}}\|W(y,\tau)\|^2dyd\tau\\
\le&O(1)K_0^2\epsilon^2\int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)}\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}}\left(\frac{e^{-\frac{|y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{1+\tau}+e^{-(|y|+\tau)/C}\right)dyd\tau\\
&+O(1)K_0^2\epsilon^2\int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)}\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}}\frac{1}{(|y+\frac{1}{\sqrt{3}}\tau|+1)(|y-\frac{1}{\sqrt{3}}\tau|+1)}dyd\tau\\
\le&O(1)K_0^2\epsilon^2(t+1)^{-1}. \qquad (21)
\end{aligned}
$$

Here the decaying rate -1 in (21) is obtained by the similar calculations in the proof of Theorem 1.2 in [5].

Case 2. $\{\frac{1}{\sqrt{3}}t/2<x<\frac{1}{\sqrt{3}}t\}$.

Fix (x,t) and treat $\mathbb{G}_b(x,y,t,\tau)$ as an operator-valued function of (y,τ). From a symmetry consideration, we have

$$
\begin{aligned}
\|\mathbb{G}_b(x,y,t,\tau)\mathbf{P_1}\| \quad = \quad & O(1)(\frac{1}{\sqrt{t-\tau+1}}+\frac{1}{\sqrt{x+1}})\left(\frac{e^{-\frac{|x+y+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}}\right)\\
&+O(1)(\frac{e^{-\frac{|x-y+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau}+\frac{e^{-\frac{|x-y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau}+e^{-(x+y+t-\tau)})
\end{aligned}
$$

Using the fact that, in the region $x\in(\frac{1}{\sqrt{3}}t/2,\frac{1}{\sqrt{3}}t)$,

$$\|\mathbb{G}_b(x,y,t,\tau)\mathbf{P_1}\|=\|\mathbb{G}(x,y,t,\tau)\mathbf{P_1}\|,$$

by similar calculation, we can prove that

$$
\begin{aligned}
&\int_0^t\int_0^\infty\mathbb{G}_b(x,y,t-\tau)Q(W)(y,\tau)dyd\tau\\
\le\quad & O(1)K_0^2\epsilon^2(\left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C}\right)+\frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}}).
\end{aligned}
$$

Case 3. $\{x>\frac{1}{\sqrt{3}}t\}$.

$$
\begin{aligned}
&\left\|\int_0^t\int_0^\infty\mathbb{G}_b(x,y,t-\tau)Q(W)(y,\tau)dyd\tau\right\|\\
\le\quad&\int_0^t\int_0^\infty\left(\frac{e^{-\frac{|x-y+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau}+\frac{e^{-\frac{|x-y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau}\right)\|W(y,\tau)\|^2dyd\tau\\
&+\int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)}\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+x)}}\|W(y,\tau)\|^2dyd\tau
\end{aligned}
$$

The calculation for the first term is similar to case 1. To deal with the second term, note that

$$\frac{1}{\sqrt{1+x}}<O(1)\frac{1}{\sqrt{1+t}},$$

we bring out the decaying factor $\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}}$ and the left is $O(1)$, so

$$\begin{aligned}&\left\|\int_0^t\int_0^\infty \mathbb{G}_b(x,y,t-\tau)Q(W)(y,\tau)dyd\tau\right\|\\ \le\ & O(1)K_0^2\epsilon^2\left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C}\right),\end{aligned}$$

and the last case is proved.

Therefore, we verify the ansatz and prove the nonlinear stability theorem. □

Appendix: Nonlinear wave interactions. The first lemma is from [5].

Lemma 4.1. *Suppose that λ, $\mu>0$. Then for given positive constants D_0 and D_1, there exists $D_2>0$ such that for any x, z, $t\ge 0$, and $\alpha\ge 1$,*

$$\int_0^t \frac{e^{-\frac{(x-\lambda(t-\tau))^2}{D_1(t-\tau)}}}{\sqrt{t-\tau+1}}\frac{e^{-\frac{(z-\mu\tau)^2}{D_0(\tau+1)}}}{(\tau+1)^{\frac{\alpha}{2}}}d\tau=O(1)\left(\frac{1}{(z+1)^{\frac{\alpha-1}{2}}}+\frac{1}{(t+1)^{\frac{\alpha-1}{2}}}\right)\frac{e^{-\frac{(x-\lambda t+\frac{\lambda}{\mu}z)^2}{D_2(t+1)}}}{\sqrt{t+1}}.$$

Define

$$I^{\alpha,\beta}(x,t;0,t;\lambda,\mu,D)\equiv\int_0^t\int_{-\infty}^\infty(t-\tau+1)^{-\beta/2}e^{-\frac{[x-y-\lambda(t-\tau)]^2}{D(t-\tau)}}\theta^\alpha(y,\tau;\mu,D)dyd\tau,$$

where

$$\theta^\alpha(y,\tau;\mu,D)=(\tau+1)^{-\alpha/2}e^{-\frac{(y-\mu(\tau+1))^2}{D(\tau+1)}}$$

The following two lemmas are similar to those in [4]. For the sake of completeness, we give a simplified proof.

Lemma 4.2. *The following estimate holds:*

$$I^{2,2}(x,t;0,t;\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D)=O(1)\theta(x,t;\frac{1}{\sqrt{3}},D)$$

Proof. By completing the square for the exponents, we have

$$\begin{aligned}&I^{2,2}(x,t;0,t;\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D)\\ \equiv\ &\int_0^t\int_{-\infty}^\infty\frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{D(t-\tau)}}}{t-\tau+1}\frac{e^{-\frac{(y-\frac{1}{\sqrt{3}}(\tau+1))^2}{D(\tau+1)}}}{\tau+1}dyd\tau\\ =\ &O(1)\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}(t+1))^2}{D(t+1)}}}{\sqrt{t+1}}\int_0^t\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{\tau+1}}d\tau\\ =\ &O(1)\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}(t+1))^2}{D(t+1)}}}{\sqrt{t+1}}\\ &\left(\int_0^{t/2}\frac{1}{\sqrt{t/2+1}}\frac{1}{\sqrt{\tau+1}}d\tau+\int_{t/2}^t\frac{1}{\sqrt{t-\tau+1}}\frac{1}{\sqrt{t/2+1}}d\tau\right)\\ =\ &O(1)\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}(t+1))^2}{D(t+1)}}}{\sqrt{t+1}}\end{aligned}$$

□

Lemma 4.3. *The following estimate holds:*

$$
\begin{aligned}
& I^{2,2}(x,t;0,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D)\\
= \;& O(1)(t+1)^{-1/4}(\theta(x,t;\frac{1}{\sqrt{3}},D)+\theta(x,t;-\frac{1}{\sqrt{3}},D))\\
& + \left\{ \begin{array}{l} 0, \quad for \quad x>\frac{1}{\sqrt{3}}(t+1)-\sqrt{t+1},\\ O(1)[\theta(x,t;-\frac{1}{\sqrt{3}},D)\frac{\sqrt{x+\frac{1}{\sqrt{3}}t}}{\sqrt{t+1}}\\ \qquad +(\frac{1}{\sqrt{3}}t-x)^{-1/2}(x+\frac{1}{\sqrt{3}}t)^{-1/2}+\theta(x,t;\frac{1}{\sqrt{3}},D)\frac{\sqrt{\frac{1}{\sqrt{3}}t-x}}{\sqrt{t+1}}],\\ \quad for \quad 0<x<\frac{1}{\sqrt{3}}(t+1)-\sqrt{t+1}. \end{array} \right.
\end{aligned}
\tag{22}
$$

Proof. Note that

$$
\begin{aligned}
& I^{2,2}(x,t;0,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D)\\
\equiv\;& \int_0^t\int_{-\infty}^{\infty}\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{D(t-\tau)}}}{t-\tau+1}\frac{e^{-\frac{(y-\frac{1}{\sqrt{3}}(\tau+1))^2}{D(\tau+1)}}}{\tau+1}dyd\tau\\
=\;& \int_0^t\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{\tau+1}}\frac{e^{-\frac{[x+\frac{1}{\sqrt{3}}(t+1)-\frac{2}{\sqrt{3}}(\tau+1)]^2}{D(t+1)}}}{\sqrt{t+1}}d\tau.
\end{aligned}
$$

We have the following cases:
Case 1. $x+\frac{1}{\sqrt{3}}(t+1)<(t+1)^{1/2}$.
Case 2. $(t+1)^{1/2}<x+\frac{1}{\sqrt{3}}(t+1)<\frac{2}{\sqrt{3}}(t+1)-(t+1)^{1/2}$.
Case 3. $\frac{2}{\sqrt{3}}(t+1)-(t+1)^{1/2}<x+\frac{1}{\sqrt{3}}(t+1)<\frac{2}{\sqrt{3}}(t+1)+(t+1)^{1/2}$.
Case 4. $x+\frac{1}{\sqrt{3}}(t+1)>\frac{2}{\sqrt{3}}(t+1)+(t+1)^{1/2}$.

For case 1, we have

$$
\begin{aligned}
& I^{2,2}(x,t;0,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D)\\
=\;& O(1)\frac{e^{-\frac{[x+\frac{1}{\sqrt{3}}(t+1)]^2}{D(t+1)}}}{\sqrt{t+1}}\int_0^t\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{\tau+1}}e^{-\frac{(\tau+1)^2}{D_1(t+1)}}d\tau.\\
=\;& O(1)\theta(x,t;-\frac{1}{\sqrt{3}},D)[(i)+(ii)+(iii)],
\end{aligned}
$$

where

$$
\begin{aligned}
(i) &= \int_0^{\sqrt{t+1}}\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{\tau+1}}d\tau=O(1)(t+1)^{-1/4}\\
(ii) &= \int_{\sqrt{t+1}}^{t/2}\frac{e^{-\frac{(\tau+1)^2}{D_1(t+1)}}}{\sqrt{t+1}\sqrt{\tau+1}}d\tau=O(1)(t+1)^{-3/4}\\
(iii) &= O(1)\frac{1}{\sqrt{t+1}}\int_{t/2}^t\frac{e^{-\frac{(t+2)^2}{D_2(t+1)}}}{\sqrt{\tau+1}}d\tau=O(1)e^{-\frac{(t+2)^2}{D_2(t+1)}}.
\end{aligned}
$$

D_1, D_2 are constants. Hence, the estimates (22) follows immediately.

For case 2

$$
\begin{aligned}
& I^{2,2}(x,t;0,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D) \\
= \; & O(1)\left[\int_0^{(x+\frac{1}{\sqrt{3}}(t+1))/M} \frac{e^{-\frac{[x+\frac{1}{\sqrt{3}}(t+1)]^2}{D_1(t+1)}}}{(t+1)\sqrt{\tau+1}}d\tau\right. \\
& + \int_{(x+\frac{1}{\sqrt{3}}(t+1))/M}^{t-(\frac{1}{\sqrt{3}}t-x)/M} (\frac{1}{\sqrt{3}}t-x)^{-1/2}(x+\frac{1}{\sqrt{3}}(t+1))^{-1/2}\frac{e^{-\frac{[x+\frac{1}{\sqrt{3}}(t+1)-\frac{2}{\sqrt{3}}(\tau+1)]^2}{D(t+1)}}}{\sqrt{t+1}}d\tau \\
& \left. + \int_{t-(\frac{1}{\sqrt{3}}t-x)/M}^{t} \frac{e^{-\frac{[x-\frac{1}{\sqrt{3}}(t+1)]^2}{D_2(t+1)}}}{\sqrt{t+1}}\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{t+1}}d\tau\right] \\
= \; & O(1)[\theta(x,t;-\frac{1}{\sqrt{3}},D)\frac{\sqrt{x+\frac{1}{\sqrt{3}}t}}{\sqrt{t+1}} + (\frac{1}{\sqrt{3}}t-x)^{-1/2}(x+\frac{1}{\sqrt{3}}t)^{-1/2} \\
& +\theta(x,t;\frac{1}{\sqrt{3}},D)\frac{\sqrt{\frac{1}{\sqrt{3}}t-x}}{\sqrt{t+1}}].
\end{aligned}
$$

Here, D_1, D_2, D, M are constants.

For case 3

$$
\begin{aligned}
& I^{2,2}(x,t;0,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D) \\
= \; & O(1)\left[\int_0^{t/2}(t+1)^{-1}(\tau+1)^{-1/2}e^{-C(t+1)}d\tau\right. \\
& + \int_{t/2}^{t-\sqrt{t+1}}(t+1)^{-1/4}(t+1)^{-1/2}\frac{e^{-\frac{[x+\frac{1}{\sqrt{3}}(t+1)-\frac{2}{\sqrt{3}}(\tau+1)]^2}{D(t+1)}}}{\sqrt{t+1}}d\tau \\
& \left. + \int_{t-\sqrt{t+1}}^{t}\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{t+1}}(t+1)^{-1/2}d\tau\right] \\
= \; & O(1)(t+1)^{-3/4},
\end{aligned}
$$

Finally, for case 4.

$$
\begin{aligned}
& I^{2,2}(x,t;0,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D) \\
= \; & O(1)\int_0^t \frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{\tau+1}}e^{-\frac{[x-\frac{1}{\sqrt{3}}(t+1)]^2}{D(t+1)}-\frac{(t-\tau)^2}{D_1(t+1)}}\frac{1}{\sqrt{t+1}}d\tau \\
= \; & O(1)\theta(x,t;\frac{1}{\sqrt{3}},D) \\
& \left[\int_0^{t/2}\frac{1}{\sqrt{t+1}}\frac{1}{\sqrt{\tau+1}}e^{-\frac{(t-\tau)^2}{D_1(t+1)}}d\tau + \int_{t/2}^{t-\sqrt{t+1}}\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{t+1}}e^{-\frac{(t-\tau)^2}{D_1(t+1)}}d\tau\right. \\
& \left. + \int_{t-\sqrt{t+1}}^{t}\frac{\sqrt{t-\tau}}{t-\tau+1}\frac{1}{\sqrt{t+1}}d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\theta(x,t;\frac{1}{\sqrt{3}},D)\left[\int_0^{t/2}\frac{1}{\sqrt{t+1}}\frac{1}{\sqrt{\tau+1}}e^{-C(t+1)}d\tau+(t+1)^{-1/4}\right]\\
&= O(1)\theta(x,t;\frac{1}{\sqrt{3}},D)\left[e^{-C(t+1)}+(t+1)^{-1/4}\right].
\end{aligned}
$$

This completes the proof of the lemma. □

Consider the following wave interaction:

$$
\begin{aligned}
&B^{\mu_i}(x,t,\mu_1,\mu_2;C)\\
&\equiv \int_0^t\int_{\mu_1(\tau+1)+\sqrt{\tau+1}}^{\mu_2(\tau+1)-\sqrt{\tau+1}}\frac{e^{-\frac{(x-y-\mu_i(t-\tau+1))^2}{C(t-\tau+1)}}}{t-\tau+1}\frac{1}{(|y-\mu_1\tau|+1)(|\mu_2\tau-y|+1)}dyd\tau,
\end{aligned}
$$

for $i=1,2$ and $\mu_1<\mu_2$. We quote a lemma in [1].

Lemma 4.4. *For some constant $C>0$,*

$$
\begin{aligned}
&B^{\mu_i}(x,t,\mu_1,\mu_2;C)\\
=\;& C\left(\frac{e^{-\frac{(x-\mu_1t)^2}{C(1+t)}}+e^{-\frac{(x-\mu_2t)^2}{C(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C}\right)\\
+\;& C\begin{cases}\frac{1}{\sqrt{(|x+\mu_1t|+1)(|x-\mu_2t|+1)}}, & \textit{for } x\in(\mu_1t+\sqrt{t},\mu_2t-\sqrt{t}),\\ 0, & \textit{for } x\in(0,\mu_1t+\sqrt{t})\cup(\mu_2t-\sqrt{t},\infty).\end{cases}
\end{aligned}
$$

for $i=1,2$.

Acknowledgments. The author would like to express her gratitude to Professor Shih-Hsien Yu for his guidance and suggestions. The author would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] S.-J. Deng, W.-K. Wang and S.-H. Yu, Pointwise convergence to a Maxwellian for a Broadwellw model with a supersonic boundary, *Netw. Heterog. Media*, **2** (2007), 383–395.
[2] C.-Y. Lan, H.-E. Lin and S.-H. Yu, The Green's functions for the Broadwell model in a half space problem, *Netw. Heterog. Media*, **1** (2006), 167–183.
[3] C.-Y. Lan, H.-E. Lin and S.-H. Yu, The Green's functions for the Broadwell model with a transonic boundary, *J. Hyperbolic Differ. Equ.*, **5** (2008), 279–294.
[4] T.-P. Liu, Pointwise convergence to shock waves for viscous conservarion laws, *Commun. Pure Appl. Math.*, **50** (1997), 1113–1182.
[5] T.-P. Liu and S.-H. Yu, Initial-boundary value problem for one-dimensional wave solutions of the Boltzmann equation, *Commun. Pure Appl. Math.*, **60** (2007), 295–356.
[6] T.-P. Liu and S.-H. Yu, On boundary relation for some dissipative systems, *Bull. Inst. Math. Acad. Sin. (N.S.)*, **6** (2011), 245–267.
[7] Y. Sone, *Kinetic Theory and Fluid Dynamics*, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser Boston, Inc., Boston, MA, 2002.

Received July 2013; revised August 2013.

E-mail address: g0901229@nus.edu.sg