NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2014.9.33
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Volume **9**, Number **1**, March **2014** pp. **33–64**

ASYMPTOTIC SYNCHRONOUS BEHAVIOR OF KURAMOTO TYPE MODELS WITH FRUSTRATIONS

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(Communicated by Benedetto Piccoli)

Abstract. We present a quantitative asymptotic behavior of coupled Kuramoto oscillators with frustrations and give some sufficient conditions for the parameters and initial condition leading to phase or frequency synchronization. We consider three Kuramoto-type models with frustrations. First, we study a general case with nonidentical oscillators; i.e., the natural frequencies are distributed. Second, as a special case, we study an ensemble of two groups of identical oscillators. For these mixture of two identical Kuramoto oscillator groups, we study the relaxation dynamics from the mixed stage to the phase-locked states via the segregation stage. Finally, we consider a Kuramoto-type model that was recently derived from the Van der Pol equations for two coupled oscillator systems in the work of Lück and Pikovsky [27]. In this case, we provide a framework in which the phase synchronization of each group is attained. Moreover, the constant frustration causes the two groups to segregate from each other, although they have the same natural frequency. We also provide several numerical simulations to confirm our analytical results.

1. **Introduction.** The purpose of this paper is to study the dynamic interplay between distinct natural frequencies (*intrinsic frustration*) and phase shift in interactions (*interaction frustration*) among a finite population of Kuramoto oscillators. More precisely, we present several sufficient conditions for the complete (frequency) synchronization in terms of the initial phase diameter, the (interaction) frustration, and the coupling strength. Synchronization is ubiquitous in various disciplines such as physics, biology, chemistry, and the social sciences [33] and recent applications on power system [12, 14]. However, rigorous mathematical treatments of synchronization were initiated only a few decades ago by two pioneers; namely, Winfree [38] and

2010 *Mathematics Subject Classification.* 92D25, 34C15, 76N10.

Key words and phrases. Kuramoto model, bicluster configurations, frustration, synchronization.

The work of S.-Y. Ha was supported by the National Research Foundation of Korea (NRF-2011-0015388). The work of Z. Li was supported by the Korea Research Foundation (KRF-2009-0093137) and the Research Funds for Settled Postdoctors of Heolongjiang Province of China.

Kuramoto [23, 24], who introduced simple ODE models for limit-cycle oscillators. Kuramoto and Sakaguchi [34] proposed a variant of the Kuramoto model in which the coupling function incorporated frustration (phase shift) so that richer dynamical phenomena would be observed than that with no frustration. Let $\theta_i = \theta_i(t)$ be the phase of the i-th Kuramoto oscillator, and let α be the frustration between the oscillators, then the dynamics of Kuramoto oscillators is governed by the following ODE system:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i + \alpha), \qquad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}, \tag{1}$$

where Ω_i, K and N denote a natural frequency of the i-th oscillator, the positive coupling strength, and the number of oscillators, respectively. Each natural frequency is a random variable extracted from some given density function $g = g(\Omega)$.

Note that the R.H.S. of (1) is Lipschitz continuous, so the well-posedness of the system (1) is well known from the Cauchy-Lipschitz theory. Thus, what matters about the solutions is the dynamic behavior such as the relaxation process, the shape of phase-locked states, and the existence of global attractors, etc. In general, the frustration hinders synchronization, so coupling strength greater than that of the original Kuramoto model without frustration is needed to guarantee global synchronization. Note that the use of frustration is needed for modeling real physical and biological systems, but it causes numerous mathematical difficulties in analyzing the synchronization. Daido [9] observed that frustration is common in disordered interactions. By varying the value of α in a series of numerical simulations, Zheng [39] investigated how frustration can induce a desynchronization of oscillators. Recently, the effect of frustration has also been intensively studied in relation to networks of oscillators [31, 32, 36]. Levnajić [26] introduced the notion of link frustration to characterize and quantify the dynamical states of networks. Although these studies have been conducted for systems with frustration, they were mostly based on the *numerical* approach. As far as the authors know, the rigorous study of the finite population of Kuramoto oscillators with frustration is very rare. Note that the conservation law, discussed in Section 2.1, is crucial in the mathematical study of the original Kuramoto model. However, as an important reason for the mathematical interest in frustration, it leads to the lack of conservation law, even for identical oscillators (see Example 2.1). Thus, the standard energy method [6, 7, 16, 17, 18, 20] for the original Kuramoto model, based on the conservation law, cannot be applied in our setting. Hence, analyzing the large-time behavior of physical systems without conservation laws is challenging itself, and to the best knowledge of the authors, there are so far no general tools for dealing with such systems without conservation laws.

In this paper, we consider three Kuramoto-type models with finite population of oscillators under frustration. First, we study the general case of nonidentical Kuramoto oscillators (1) with natural frequencies that are distributed. Second, as a special case, we deal with an ensemble (1) consisting of two groups of identical oscillators. When two identical Kuramoto oscillator groups are mixed, the whole configuration evolves into the segregated state and then asymptotically toward the phase-locked state. Finally, we consider a special Kuramoto-type model that was recently derived from the Van der Pol equations for two coupled oscillator systems in the work of Lück and Pikovsky [27] where a thermodynamic limit based on the order parameter was studied. The main contribution of this work is to present some explicit sufficient conditions on the parameters and initial configurations to reach the complete synchronization for each of the three models.

The rest of this paper is organized as follows: Section 2 reviews the standard Kuramoto model, recalls several definitions of the synchronization, and presents our three Kuramoto-type models that will be investigated in later sections. Section 3 studies the synchronization estimates for the general Kuramoto model with frustration. Section 4 considers an ensemble of two groups of identical Kuramoto oscillators. In this case, we show the detailed relaxation process toward the bi-clusters. Section 5 deals with a Kuramoto-type model with bipartite interactions; i.e., the interactions occur between oscillators from distinct groups. Finally, Section 6 is devoted to a summary of our results and some unresolved questions for investigation in the future.

2. **Problem formulation and preliminaries.**

In this section, we briefly review the original Kuramoto model without frustrations and present a formulation of the problem. As an example, we present an exact synchronization estimate for the two-oscillator system.

2.1. **The Kuramoto model.**

In this subsection, we briefly discuss the Kuramoto model without frustration. Consider an ensemble of weakly coupled Kuramoto oscillators whose phase is governed by the following first-order ODE system:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \dots, N. \tag{2}$$

For the details of the system (2), we refer the readers to survey papers and books [1, 3, 23, 35]. Particularly, Ermentrout [15] found a critical coupling at which all oscillators become phase-locked, independent of the number of oscillators. The linear stability of phase-locked state has been studied using tools such as a Lyapunov functional, spectral graph theory, and control theory [2, 4, 10, 22, 28, 29, 30, 37]. Moreover, some complete frequency synchronization estimates have been provided for the Kuramoto model (2) in [7, 8, 11, 13, 17, 19]; in particular, the transition and relaxation stages have been studied in [7] for initial phase configurations with diameters greater than $\pi/2$. For other interesting issues on Kuramoto's conjecture, we also refer to [5, 21].

Set

$$\Omega_c := \frac{1}{N}\sum_{i=1}^{N}\Omega_i, \qquad \theta_c(t) := \frac{1}{N}\sum_{i=1}^{N}\theta_i(t).$$

It is easy to see from (2) that, for any distribution of natural frequencies, the average phase rotates on the unit circle with a constant average natural frequency Ω_c:

$$\frac{d\theta_c}{dt} = \Omega_c, \quad \text{i.e.,} \quad \theta_c(t) = \theta_c(0) + t\Omega_c, \quad t \geq 0.$$

On the other hand, the fluctuations $(\hat{\theta}_i, \hat{\Omega}_i) := (\theta_i - \theta_c,\ \Omega_i - \Omega_c)$ satisfy equations of the same form:

$$\dot{\hat{\theta}}_i = \hat{\omega}_i = \hat{\Omega}_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\hat{\theta}_j - \hat{\theta}_i),$$

with the additional algebraic constraints

$$\sum_{i=1}^{N} \hat{\theta}_i = 0, \qquad \sum_{i=1}^{N} \hat{\Omega}_i = 0. \tag{3}$$

The above conservation laws (3) for the Kuramoto model without frustration are crucially used in its rigorous studies [7, 8, 11, 13, 17, 20]. We next recall for Kuramoto-type oscillator models the definitions of a few synchronization concepts and the definition of collisions, as these will be used throughout this paper.

Definition 2.1. Let $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$ be the ensemble phase of Kuramoto oscillators.

1. The Kuramoto ensemble asymptotically exhibits complete phase synchronization if and only if the relative phase differences go to zero asymptotically:
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0, \qquad \forall\, i \neq j.$$
2. The Kuramoto ensemble asymptotically exhibits complete frequency synchronization if and only if the relative frequency differences go to zero asymptotically:
$$\lim_{t\to\infty} |\omega_i(t) - \omega_j(t)| = 0, \qquad \forall\, i \neq j.$$
3. The dynamical state $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$ asymptotically approaches the phase-locked state if and only if each relative phase difference goes to a constant as $t \to \infty$; i.e.,
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = \theta_{ij}, \quad \text{for all} \quad i, j \in \{1, \dots, N\}.$$
4. Two oscillators i and j have a collision at time t if and only if their relative phase difference $\theta_i - \theta_j$ becomes zero at time t:
$$\theta_i(t) = \theta_j(t).$$

2.2. **Three Kuramoto-type models with frustrations.** In this subsection, we describe three Kuramoto-type models with frustrations, which are investigated in later sections. The first two models are concerned with nonidentical oscillators with (interaction) frustrations. We only consider the case that frustration α lies in the interval $(-\pi/2, \pi/2)$. Figure 1 shows that when $\alpha = \pi/2$, oscillators approach each other some what, after that, they push each other away. It is difficult to expect the synchronization. Model A corresponds to the system (1) in general form, and Model B corresponds to the special case of (1) in which there are only two distinct natural frequencies. A different ensemble of two groups of oscillators was considered in [25] where the natural frequencies are distributed but the coupling strengths between groups are different. Model C deals with a Kuramoto-type model with identical oscillators on a bipartite graph; i.e., interactions arise only between members from different groups.

- Model A: (Uniform frustration)
$$\dot{\theta}_i = \Omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i + \alpha), \quad i = 1, \dots, N.$$
- Model B: (Interaction of two identical oscillator groups under frustration)
$$\dot{\theta}_i = \Omega_1 + \frac{K}{2N} \Big[\sum_{j=1}^{N} \sin(\theta_j - \theta_i + \alpha) + \sum_{j=1}^{N} \sin(\phi_j - \theta_i + \alpha) \Big], \quad i = 1, \dots, N,$$

$$\dot{\phi}_i = \Omega_2 + \frac{K}{2N}\Big[\sum_{j=1}^{N}\sin(\phi_j - \phi_i + \alpha) + \sum_{j=1}^{N}\sin(\theta_j - \phi_i + \alpha)\Big], \quad i = 1, \dots, N.$$

- Model C: (Identical oscillators on a bipartite graph)

$$\begin{aligned}
\dot{\theta}_i &= \frac{\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - 2\theta_i), && i = 1, \dots, N_1,\\
\dot{\phi}_i &= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(2\theta_j - \phi_i + \alpha), && i = 1, \dots, N_2,
\end{aligned}$$

where $\mu \in (0, 1)$ is a positive constant. This model was recently derived from the Van der Pol equations for two coupled oscillators system in the work of Lück and Pikovsky [27] .

As examples, we consider some simple situations to illustrate how the frustration causes the system to deviate from the original Kuramoto model (2).

Example 2.1. Consider the system of two oscillators with frustration and identical natural frequencies $\Omega_1 = \Omega_2 = 0$:

$$\begin{aligned}
\frac{d\theta_1}{dt} &= \frac{K}{2}\sin(\theta_2 - \theta_1 + \alpha), \quad t > 0,\\
\frac{d\theta_2}{dt} &= \frac{K}{2}\sin(\theta_1 - \theta_2 + \alpha).
\end{aligned}$$

As in Subsection 2.1, we introduce the mean values of the phase and instantaneous frequency:

$$\theta_c := \frac{\theta_1 + \theta_2}{2}, \quad \omega_c := \frac{\omega_1 + \omega_2}{2}.$$

Then, the mean phase satisfies

$$\begin{aligned}
\frac{d\theta_c}{dt} &= \frac{K}{2}\Big[\sin(\theta_2 - \theta_1 + \alpha) + \sin(\theta_1 - \theta_2 + \alpha)\Big]\\
&= K\cos(\theta_1 - \theta_2)\sin\alpha.
\end{aligned}$$

Thus, we do not have conservation of total phase.

Example 2.2. Consider a system of two coupled oscillators:

$$\begin{aligned}
\frac{d\theta_1}{dt} &= \Omega_1 + \frac{K}{2}\sin(\theta_2 - \theta_1 + \alpha), \quad t > 0,\\
\frac{d\theta_2}{dt} &= \Omega_2 + \frac{K}{2}\sin(\theta_1 - \theta_2 + \alpha),
\end{aligned} \tag{4}$$

where the natural frequencies Ω_i and frustration α are assumed to satisfy

$$\Omega_1 > \Omega_2, \qquad |\alpha| < \frac{\pi}{2}.$$

To reduce the number of equations, we introduce the following differences:

$$\theta := \theta_1 - \theta_2, \quad \Omega := \Omega_1 - \Omega_2.$$

Then, the system (4) becomes a single equation for the differences:

$$\frac{d\theta}{dt} = \Omega - K\cos\alpha\sin\theta. \tag{5}$$

Note that the interaction frustration clearly reduces the coupling strength; i.e., as the frustration increases, the synchronizability decreases. We easily see that to obtain complete frequency synchronization we need to satisfy the condition

$$K \geq \frac{\Omega}{\cos\alpha}.$$

In this case, there are two equilibriums θ_{e1} and θ_{e2}:

$$\sin\theta_{ei} = \frac{\Omega}{K\cos\alpha}.$$

In the following three sections, we derive sufficient conditions leading to complete synchronization of the aforementioned three models.

3. Synchronization estimate for Model A.

In this section, we study the synchronizability of the Model A introduced in the previous section.

Recall that the Kuramoto model with a uniform interaction frustration is given by

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i + \alpha), \quad t \geq 0, \quad i = 1, \ldots, N, \quad |\alpha| < \frac{\pi}{2}. \tag{6}$$

So far, the sufficient conditions for the Kuramoto model ($\alpha = 0$) have been extensively studied in literature [6, 7, 8, 11, 13, 16, 17, 18, 20]. Our synchronization analysis relies on two steps. First, we show the existence of a positively invariant region so that within a finite time the initial configurations are confined to the interval with length $\pi/2$. Second, we derive a Gronwall-type inequality for the frequency diameter that is subsequently defined to conclude the complete frequency synchronization.

3.1. Existence of a trapping region.

In this subsection, we study the existence of a positively invariant region for the Kuramoto model. For this, we follow the approach of Choi *et al.* [7] and introduce some notation: for a given time $t \geq 0$,

$$\begin{aligned}
\theta_m(t) &:= \min_{1\leq i\leq N}\theta_i(t), \qquad \theta_M(t) := \max_{1\leq i\leq N}\theta_i(t),\\
\omega_m(t) &:= \min_{1\leq i\leq N}\omega_i(t), \qquad \omega_M(t) := \max_{1\leq i\leq N}\omega_i(t),\\
D(\theta(t)) &:= \theta_M(t) - \theta_m(t), \quad D(\omega(t)) := \omega_M(t) - \omega_m(t).\\
D(\Omega) &: = \max_{1\leq i,j\leq N}|\Omega_i - \Omega_j|, \quad K_{ef} := \frac{D(\Omega)}{1-\sin|\alpha|},
\end{aligned}$$

where $\omega_i := \dot{\theta}_i$ is the instantaneous frequency of the ith oscillator, and the coupling strength K_{ef} coincides with K_e in the absence of frustration.(see Remark 3.3 for details.) Note that $D(\theta(t))$ is Lipschitz continuous and differentiable except at times of collision between the extremal phases and their neighboring phases.

Lemma 3.1. (Existence of a positively invariant set) *Suppose that the natural frequencies, coupling strength, and initial data satisfy*

$$D(\Omega) > 0, \quad K \geq K_{ef}, \quad 0 < D(\theta^0) < D_*^\infty - |\alpha|,$$

where D_*^∞ is the unique solution of the following equation:

$$\sin x = \frac{D(\Omega) + K\sin|\alpha|}{K}, \quad x \in \left(\frac{\pi}{2}, \pi\right).$$

Then, for the global solution to (6)*, we have*

$$\sup_{t\geq 0} D(\theta(t)) \leq D_*^\infty - |\alpha|.$$

Proof. We define a set $\mathcal{T}$ and its supremum $T^* \in [0, \infty]$:

$$\mathcal{T} := \big\{T \geq 0 \mid D(\theta(t)) < D_*^\infty - |\alpha|, \ \forall\, t \in [0, T)\big\}, \quad T^* := \sup \mathcal{T}.$$

Since $D(\theta^0) < D_*^\infty - |\alpha|$ and $D(\theta(t))$ is continuous, there exists $T > 0$ such that

$$D(\theta(t)) < D_*^\infty - |\alpha|, \quad \forall\, t \in [0, T).$$

Hence, the set $\mathcal{T}$ is not empty. We now claim that

$$T^* = +\infty.$$

Suppose not; i.e., $T^* < \infty$. Then, we have

$$D(\theta(T^*)) \geq D_*^\infty - |\alpha|, \quad \text{and} \quad D(\theta(t)) < D_*^\infty - |\alpha|, \quad \forall\, t \in [0, T^*).$$

On the other hand, we have for $t \in [0, T^*)$,

$$\begin{aligned}
\frac{d}{dt} D(\theta(t)) &\leq \Omega_M - \Omega_m + \frac{K}{N} \sum_{j=1}^N \Big[\sin(\theta_j - \theta_M + \alpha) - \sin(\theta_j - \theta_m + \alpha)\Big] \\
&\leq D(\Omega) + \frac{K \cos\alpha}{N} \sum_{j=1}^N \Big[\sin(\theta_j - \theta_M) - \sin(\theta_j - \theta_m)\Big] \\
&\quad + \frac{K \sin\alpha}{N} \sum_{j=1}^N \Big[\cos(\theta_j - \theta_M) - \cos(\theta_j - \theta_m)\Big].
\end{aligned}$$

We consider two cases according to the sign of α.

• Case 1: $\alpha \in [0, \frac{\pi}{2})$. In this case, we have

$$\begin{aligned}
&\frac{d}{dt} D(\theta(t)) \\
&\qquad \leq D(\Omega) + \frac{K \cos\alpha \sin D(\theta)}{N D(\theta)} \sum_{j=1}^N \Big[(\theta_j - \theta_M) - (\theta_j - \theta_m)\Big] \\
&\qquad\quad + \frac{K \sin\alpha}{N} \sum_{j=1}^N \Big[1 - \cos D(\theta)\Big] \\
&\qquad = D(\Omega) - K\Big[\sin\big(D(\theta) + \alpha\big) - \sin\alpha\Big] \\
&\qquad = D(\Omega) - K\Big[\sin\big(D(\theta) + |\alpha|\big) - \sin|\alpha|\Big].
\end{aligned}$$

• Case 2: $\alpha \in (-\frac{\pi}{2}, 0)$. In this case, we have

$$\begin{aligned}
&\frac{d}{dt}D(\theta(t)) \\
&\qquad \le D(\Omega) + \frac{K\cos\alpha \sin D(\theta)}{ND(\theta)} \sum_{j=1}^{N} \Big[(\theta_j - \theta_M) - (\theta_j - \theta_m)\Big] \\
&\qquad\quad + \frac{K \sin\alpha}{N} \sum_{j=1}^{N} \Big[\cos D(\theta) - 1\Big] \\
&\qquad = D(\Omega) - K\Big[\sin\left(D(\theta) - \alpha\right) + \sin\alpha\Big] \\
&\qquad = D(\Omega) - K\Big[\sin\left(D(\theta) + |\alpha|\right) - \sin|\alpha|\Big].
\end{aligned}$$

Here we used the condition:

$$\sin(\theta_j - \theta_M) \le \frac{\sin D(\theta)}{D(\theta)}(\theta_j - \theta_M), \quad \sin(\theta_j - \theta_m) \ge \frac{\sin D(\theta)}{D(\theta)}(\theta_j - \theta_m),$$

and

$$\cos(\theta_j - \theta_M), \cos(\theta_j - \theta_m) \le 1, \quad \cos(\theta_j - \theta_m), \cos(\theta_j - \theta_M) \ge \cos D(\theta).$$

Then we have

$$\frac{d}{dt}D(\theta(t)) \le D(\Omega) + K\sin|\alpha| - K\sin(D(\theta(t)) + |\alpha|).$$

Since $D(\theta(t)) + |\alpha| < D_*^\infty < \pi$ for $t \in [0, T^*)$, we obtain

$$\dot{D}(\theta(t)) \le D(\Omega) + K\sin|\alpha| - \frac{K\sin D_*^\infty}{D_*^\infty}(D(\theta(t)) + |\alpha|).$$

We now use the above inequality to observe that

$$D(\theta(t)) \le \Big(D(\theta^0) - D_*^\infty + |\alpha|\Big)e^{-\frac{K\sin D_*^\infty}{D_*^\infty}t} + D_*^\infty - |\alpha|, \quad t \in [0, T^*).$$

This implies

$$\begin{aligned}
D_*^\infty - |\alpha| \le D(\theta(T^*)) &\le \Big(D(\theta^0) - D_*^\infty + |\alpha|\Big)e^{-\frac{K\sin D_*^\infty}{D_*^\infty}T^*} + D_*^\infty - |\alpha| \\
&< D_*^\infty - |\alpha|,
\end{aligned}$$

which is a contradiction. Hence, $T^* = \infty$. □

Remark 3.1. The arguments used in the proof of Lemma 3.1 also imply that $D(\theta(t))$ satisfies

$$\dot{D}(\theta) \le D(\Omega) + K\sin|\alpha| - K\sin\big(D(\theta) + |\alpha|\big), \text{ a.e. } t, \tag{7}$$

where $\dot{D}(\theta(t)) := \frac{d}{dt}D(\theta(t))$.

3.2. **Entrance to the exponential stability regime.** In this subsection, we study the transition of the phase ensemble to the exponential stability regime within a finite time. We introduce a reference angle D^∞:

$$D^\infty \in \left(0, \frac{\pi}{2}\right), \qquad \sin D^\infty = \frac{D(\Omega) + K\sin|\alpha|}{K}. \tag{8}$$

Remark 3.2. 1. Note that D^∞ is the dual angle of D_*^∞ in Lemma 3.1 satisfying

$$\sin D^\infty = \sin D_*^\infty = \frac{D(\Omega) + K\sin|\alpha|}{K}.$$

2. Since

$$\sin D^\infty = \frac{D(\Omega)}{K} + \sin|\alpha| > \sin|\alpha|,$$

we have $D^\infty > |\alpha|$.

3. The inequality (7) implies that $D(\theta(t))$ is decreasing if

$$D(\theta(t)) \in \big(D^\infty - |\alpha|, D_*^\infty - |\alpha|\big).$$

Proposition 3.2. *Suppose that the natural frequencies, coupling strength and initial data satisfy*

$$D(\Omega) > 0, \quad K > K_{ef}, \quad 0 < D(\theta^0) < D_*^\infty - |\alpha|.$$

Let $\theta(t)$ *be the global solution to* (6). *Then for any* $0 < \epsilon \ll 1$, *there exists* $t_0 = t_0(\epsilon) > 0$ *such that*

$$D(\theta(t)) \le D^\infty - |\alpha| + \epsilon, \quad \textit{for} \quad t \ge t_0.$$

In particular, we can choose ϵ *so small that* $D(\theta(t)) + |\alpha| \le D^\infty + \epsilon < \frac{\pi}{2}$ *for* $t \ge t_0$.

Proof. We consider the ordinary differential equation

$$\dot{y} = D(\Omega) + K\sin|\alpha| - K\sin y. \tag{9}$$

Our assumption on K implies that $y_* = D^\infty$ is an equilibrium point for the equation (9). y_* is locally stable because in the neighborhood of y_*, $\frac{dy}{dt} < 0$ for $y > y_*$ and $\frac{dy}{dt} > 0$ for $y < y_*$. Moreover, for any initial value $y(0)$ with $0 < y(0) < D_*^\infty$, the trajectory $y(t)$ monotonically approaches y_*. Therefore, for any $\epsilon > 0$, there exists a time t_0 such that

$$|y(t) - y_*| < \epsilon, \quad \forall\, t \ge t_0.$$

We now apply this analysis on (9) and the principle of comparison with (7) to find

$$D(\theta(t)) + |\alpha| < D^\infty + \epsilon, \quad \forall\, t \ge t_0.$$

That is,

$$D(\theta(t)) < D^\infty - |\alpha| + \epsilon, \quad \forall\, t \ge t_0.$$

□

3.3. **Relaxation estimate.** In this subsection, we provide an estimate of the relaxation toward the phase-locked state. We next derive the Gronwall's differential inequality for the frequency diameter $D(\omega(t))$. We differentiate the system (6) with respect to time t to obtain

$$\frac{d\omega_i}{dt} = \frac{K}{N}\sum_{j=1}^{N} \cos(\theta_j - \theta_i + \alpha)(\omega_j - \omega_i).$$

This yields

$$\begin{aligned}\frac{d}{dt}D(\omega(t)) &= \frac{K}{N}\sum_{j=1}^{N}\Big[\cos(\theta_j-\theta_M^\omega+\alpha)(\omega_j-\omega_M)-\cos(\theta_j-\theta_m^\omega+\alpha)(\omega_j-\omega_m)\Big]\\ &\le \frac{K}{N}\cos\left(D^\infty+\epsilon\right)\sum_{j=1}^{N}\Big[(\omega_j-\omega_M)-(\omega_j-\omega_m)\Big]\\ &= -K\cos\left(D^\infty+\epsilon\right)D(\omega(t)), \qquad t\ge t_0,\end{aligned} \tag{10}$$

where θ_M^ω and θ_m^ω denote the phases of the oscillators, which have the maximum and minimum instantaneous frequencies, respectively. In summary, we have

$$\frac{d}{dt}D(\omega(t)) \le -K\cos\left(D^\infty+\epsilon\right)D(\omega(t)), \qquad t\ge t_0. \tag{11}$$

We are now ready to make the relaxation estimate for Model A.

Theorem 3.3. *Suppose that the natural frequencies, coupling strength, and initial data satisfy*

$$D(\Omega)>0, \quad K>K_{ef}, \quad 0<D(\theta^0)<D_*^\infty-|\alpha|.$$

Then, for any $0<\epsilon\ll 1$ *with* $D^\infty+\epsilon<\frac{\pi}{2}$, *there exists* $t_0>0$ *such that*

$$D(\omega(t_0))e^{-K(t-t_0)} \le D(\omega(t)) \le D(\omega(t_0))e^{-K\cos(D^\infty+\epsilon)(t-t_0)}, \quad t\ge t_0.$$

Proof. By Proposition 3.2, there exists some $t_0>0$ such that

$$|\theta_j(t)-\theta_i(t)| \le D(\theta(t)) \le D^\infty-|\alpha|+\epsilon, \qquad \forall\, t\ge t_0.$$

• (Upper bound estimate): We combine the above result with (11) to find

$$D(\omega(t)) \le D(\omega(t_0))\exp\Big[-K\cos\big(D^\infty+\epsilon\big)(t-t_0)\Big], \quad t\ge t_0.$$

Hence, the frequency diameter decays exponentially to zero.

• (Lower bound estimate): We use $\cos\theta\le 1$ in (10) to find

$$D(\omega(t)) \ge D(\omega(t_0))\exp[-K(t-t_0)], \quad t\ge t_0.$$

□

Remark 3.3. The rigorous study of the finite population Kuramoto model without frustration ($\alpha=0$) was carried out in [7, 13, 17], where the critical coupling can be given by $K_e=D(\Omega)$. This is consistent with the result in Theorem 3.3 since $K_{ef}|_{\alpha=0}=D(\Omega)$. Moreover, the critical coupling K_{ef} increases as the strength of frustration becomes larger. Therefore, our result extends the previous results on complete synchronization to the Kuramoto model with frustration and exhibits how the frustration hinders the synchronization and influences the critical coupling.

3.4. **Numerical examples.** In this subsection, we present several numerical examples and compare the simulation results with the analytical results in the previous subsection.

For the simulation, we used the fourth-order Runge-Kutta method and employed the parameters

$$N=100, \quad K=20.$$

The natural frequencies Ω_i were randomly chosen from the interval $(-1,1)$ so that

$$D(\Omega)=1.9867, \quad \Omega_c=0.$$

The initial configuration θ_i^0 was randomly chosen from the interval $(0, \pi - 2 \times 0.7)$ satisfying

$$D(\theta^0) = 1.7130.$$

For a fixed initial configuration, we will compare the case $\alpha = 0$ with other cases (i.e., $\alpha = 0.3, 0.7$) to observe the effects of the frustrations. Note that $D(\theta^0)$ satisfies our assumptions in Theorem 3.3. The initial and limiting phase configurations are displayed in Figure 2(a) and (b).

Although we do not have the optimal decay rate for $D(\omega(t))$ in Theorem 3.3, the upper bound estimate for $D(\omega(t))$ suggests that the decay rate is proportional to $\cos(D^\infty + \epsilon) \approx \cos D^\infty$. Recall that the reference angle D^∞ is defined by (8). We observe that as α increases, $\cos D^\infty$ decrease, i.e., the decay rate may decrease, which can be seen in Figure 2(c).

4. **Synchronization estimates for Model B.** In this section, we consider the interaction of two identical oscillator groups with different natural frequencies. In the absence of frustration, this case has already been considered by Ha and Kang [16], who found that the resulting phase-locked states are configurations of two-point clusters and the relaxation speed can be exponential or algebraic depending on the size of K compared to the size of the intrinsic frustration $D(\Omega)$. More precisely, we consider the ensemble of mixed Kuramoto oscillators with two distinct natural frequencies. In this case, the plausible scenario has two stages. First the ensemble will evolve from the mixed phase into the segregated phase; i.e., two identical oscillator groups will separate into two groups that will each have the same natural frequency. Then, these two groups will evolve to become the point clusters, respectively.

Let $\{\theta_i\}_{i=1}^N$ and $\{\phi_i\}_{i=1}^N$ be the identical oscillator groups with natural frequencies Ω_1 and Ω_2, respectively. In this situation, the dynamics of the oscillator groups is governed by the following coupled system:

$$\begin{aligned}
\dot{\theta}_i &= \Omega_1 + \frac{K}{2N}\Big[\sum_{j=1}^N \sin(\theta_j - \theta_i + \alpha) + \sum_{j=1}^N \sin(\phi_j - \theta_i + \alpha)\Big], \quad i = 1, \dots, N, \\
\dot{\phi}_i &= \Omega_2 + \frac{K}{2N}\Big[\sum_{j=1}^N \sin(\phi_j - \phi_i + \alpha) + \sum_{j=1}^N \sin(\theta_j - \phi_i + \alpha)\Big], \quad i = 1, \dots, N,
\end{aligned} \tag{12}$$

subject to initial data and frustration,

$$(\theta_i, \phi_i)(0) = (\theta_i^0, \phi_j^0), \qquad |\alpha| < \frac{\pi}{2}, \tag{13}$$

where $K > 0$ denotes the coupling strength and N denotes the number of oscillators in each group. Without loss of generality, we assume that

$$\Omega_1 > \Omega_2.$$

As an extreme case, we consider the situation where the identical oscillators with the same natural frequencies are collapsed to a single phase at $t = 0$; i.e.,

$$\theta_i^0 = \theta_1^0, \qquad \text{and} \qquad \phi_i^0 = \phi_1^0, \quad i = 2, \dots, N.$$

Then, by the uniqueness of the solution to the system (12), we have

$$\theta_1(t) = \dots = \theta_N(t), \quad \phi_1(t) = \dots = \phi_N(t), \quad t \ge 0.$$

We set

$$D(\Omega) = \Omega_1 - \Omega_2, \quad \text{and} \quad \Delta(t) := \theta_1(t) - \phi_1(t), \quad t \ge 0.$$

From (12), we have

$$\dot{\theta}_1 = \Omega_1 + \frac{K}{2}\sin\alpha + \frac{K}{2N}\sum_{j=1}^{N}\sin(-\Delta+\alpha),$$

$$\dot{\phi}_1 = \Omega_2 + \frac{K}{2}\sin\alpha + \frac{K}{2N}\sum_{j=1}^{N}\sin(\Delta+\alpha).$$

We subtract the second equation from the first equation to obtain

$$\dot{\Delta} = D(\Omega) - K\cos\alpha\sin\Delta, \tag{14}$$

which is a standard Adler's equation that we already obtained as (5).

4.1. **Existence of a trapping region.** In this subsection, we will find a trapping region for the system (12). We set

$$\begin{aligned}
\theta_m(t) &:= \min_{1\le i\le N}\theta_i(t), \qquad \theta_M(t) := \max_{1\le i\le N}\theta_i(t),\\
\phi_m(t) &:= \min_{1\le i\le N}\phi_i(t), \qquad \phi_M(t) := \max_{1\le i\le N}\phi_i(t),\\
D(\theta(t)) &:= \theta_M(t)-\theta_m(t), \quad D(\phi(t)) := \phi_M(t)-\phi_m(t),\\
D(\theta,\phi)(t) &:= \max_{1\le i\le N}\{\theta_i,\phi_i\} - \min_{1\le i\le N}\{\theta_i,\phi_i\}.
\end{aligned} \tag{15}$$

We also set

$$\Delta(t) := \theta_m(t) - \phi_M(t).$$

It is easy to see that if $\Delta(t)\ge 0$, then

$$D(\theta,\phi)(t) = D(\theta(t)) + D(\phi(t)) + \Delta(t).$$

That is, the sum of the three partial diameters is exactly the total diameter of the phases $\{\theta(t),\phi(t)\}$. The next lemma demonstrates a kind of monotonicity property among Kuramoto oscillators with the same natural frequency.

Lemma 4.1. *Let $\{\theta_i,\phi_i\}$ be the global solution to the system (12), with*

$$\phi_1^0 \le \phi_2^0 \le \cdots \le \phi_N^0.$$

Then for all $t>0$, we have

$$\phi_1(t) \le \phi_2(t) \le \cdots \le \phi_N(t).$$

Proof. It is sufficient to show that

$$\phi_i^0 \le \phi_j^0 \quad \Longrightarrow \quad \phi_i(t) \le \phi_j(t),\ t>0.$$

Suppose that there exists some time $t_0>0$ such that $\phi_i(t_0)=\phi_j(t_0)$. Then, by the uniqueness of the solution, we observe that $\phi_i(t)=\phi_j(t)$, $\forall\, t\ge t_0$. Moreover, owing to the analyticity of $\{\theta_i,\phi_i\}$, we can derive $\phi_i(t)=\phi_j(t)$, $\forall\, t\ge 0$. Hence, we have

$$\phi_i^0 < \phi_j^0 \quad \Longrightarrow \quad \phi_i(t) < \phi_j(t),\ t>0.$$

□

Remark 4.1. The same result holds for the group $\{\theta_i\}$ with natural frequency Ω_1. Lemma 4.1 means that no collision occurs in either group during the evolution process, unless the oscillators initially have the same phase. Hence, the oscillators that take the maximum and minimum phases in each group are fixed forever. This implies that the extremal phases $\theta_M(t)$, $\theta_m(t)$, $\phi_m(t)$, $\phi_M(t)$ and partial diameters $D(\theta(t))$, $D(\phi(t))$ are analytic functions.

Lemma 4.2. *Let $(\theta,\phi) = (\theta(t),\phi(t))$ be the smooth solution to (12)-(13) satisfying*

$$D(\theta,\phi)(0) \le R, \quad \text{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then, we have

$$\frac{d}{dt}D(\theta(t))\Big|_{t=0} < 0, \qquad \frac{d}{dt}D(\phi(t))\Big|_{t=0} < 0.$$

Proof. (1) We use (12) and the definition of $D(\theta(t))$ to obtain

$$\begin{aligned}
\frac{d}{dt}D(\theta(t)) &= \dot\theta_M(t) - \dot\theta_m(t) \\
&= \frac{K}{2N}\sum_{j=1}^N \Big\{ \sin(\theta_j - \theta_M + \alpha) - \sin(\theta_j - \theta_m + \alpha) \Big\} \\
&\quad + \frac{K}{2N}\sum_{j=1}^N \Big\{ \sin(\phi_j - \theta_M + \alpha) - \sin(\phi_j - \theta_m + \alpha) \Big\} \\
&= -\frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^N \cos\Big(\frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} + \alpha\Big) \\
&\quad - \frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^N \cos\Big(\frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} + \alpha\Big).
\end{aligned}$$

Note that at $t=0$ we have

$$\begin{aligned}
-\frac{R}{2} \le -\frac{D(\theta)}{2} \le &\frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} \le \frac{D(\theta)}{2} \le \frac{R}{2}, \\
-R \le \phi_j - \theta_M \le &\frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} \le \phi_j - \theta_m \le R.
\end{aligned}$$

This yields

$$\frac{d}{dt}D(\theta(t))\Big|_{t=0} < 0.$$

(2) We also have a similar result for $D(\phi(t))$. Note that

$$\begin{aligned}
\frac{d}{dt}D(\phi(t)) &= -\frac{K}{N}\sin\frac{D(\phi)}{2}\sum_{j=1}^N \cos\Big(\frac{\phi_j - \phi_M}{2} + \frac{\phi_j - \phi_m}{2} + \alpha\Big) \\
&\quad - \frac{K}{N}\sin\frac{D(\phi)}{2}\sum_{j=1}^N \cos\Big(\frac{\theta_j - \phi_M}{2} + \frac{\theta_j - \phi_m}{2} + \alpha\Big).
\end{aligned}$$

Since

$$\begin{aligned}
-\frac{R}{2} \le -\frac{D(\phi)}{2} \le &\frac{\phi_j - \phi_M}{2} + \frac{\phi_j - \phi_m}{2} \le \frac{D(\phi)}{2} \le \frac{R}{2}, \quad t = 0, \\
-R \le \theta_j - \phi_M \le &\frac{\theta_j - \phi_M}{2} + \frac{\theta_j - \phi_m}{2} \le \theta_j - \phi_m \le R,
\end{aligned}$$

we have

$$\frac{d}{dt}D(\phi(t))\big|_{t=0} < 0.$$

□

Proposition 4.3. *Suppose that the coupling strength K satisfies*

$$K > \frac{D(\Omega)}{1 - \sin|\alpha|},$$

and let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to (12)-(13) satisfying

$$D(\theta, \phi)(0) \leq R, \quad \textit{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then we have the following two assertions:

(1) $D(\theta, \phi)(t) \leq \begin{cases} R, & \textit{if } R \geq D^\infty - |\alpha|, \\ D^\infty - |\alpha|, & \textit{if } R < D^\infty - |\alpha|, \end{cases} \quad \forall\, t \geq 0;$

(2) $D(\theta(t)) \leq D(\theta^0)$, *and* $D(\phi(t)) \leq D(\phi^0), \quad \forall\, t \geq 0.$

Here, D^∞ is given by (8).

Proof. (i) Since our system (12) is a special case for the system (6), we can use the analysis in Section 3, in particular, Remark 3.2 (3) and the analysis in Proposition 3.2 to see the first assertion.
(ii) For the non-increasing property of the phase diameters $D(\theta(t))$ and $D(\phi(t))$, we use Lemma 4.2 and standard continuation arguments. □

Remark 4.2. The assertion (1) is equivalent to $D(\theta, \phi)(t) \leq \max\{R,\, D^\infty - |\alpha|\}$, $\forall\, t \geq 0$. Thus, with loss of generality we may simply say $D(\theta, \phi)(t) \leq R$, since we can choose $R = D^\infty - |\alpha|$ if $D(\theta, \phi)(0) \leq D^\infty - |\alpha|$.

4.2. **From mixed stage to segregated stage.** In this subsection, we present the emergence from the mixture of oscillators to the segregation phase consisting of two identical Kuramoto oscillator groups without overlapping.

Lemma 4.4. *Let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to (12)-(13) satisfying*

$$D(\theta, \phi)(0) \leq R, \quad \textit{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then there exists a time t_0 such that

$$\Delta(t_0) > 0.$$

Proof. We consider three cases depending on the initial phases:

$$\Delta(0) > 0, \quad \Delta(0) = 0, \quad \text{and} \quad \Delta(0) < 0.$$

• Case 1 ($\Delta(0) > 0$): In this case, we have nothing to prove.

• Case 2 ($\Delta(0)=0$): By definition of Δ, we have

$$\begin{aligned}\frac{d}{dt}\Delta(t) &= D(\Omega) + \frac{K}{2N}\sum_{j=1}^N \sin(\theta_j-\theta_m+\alpha) + \frac{K}{2N}\sum_{j=1}^N \sin(\phi_j-\theta_m+\alpha)\\ &\quad - \frac{K}{2N}\sum_{j=1}^N \sin(\phi_j-\phi_M+\alpha) - \frac{K}{2N}\sum_{j=1}^N \sin(\theta_j-\phi_M+\alpha)\\ &= D(\Omega) + \frac{K}{2N}\sum_{j=1}^N \Big[\sin(\theta_j-\theta_m+\alpha)-\sin(\theta_j-\phi_M+\alpha)\Big]\\ &\quad + \frac{K}{2N}\sum_{j=1}^N \Big[\sin(\phi_j-\theta_m+\alpha)-\sin(\phi_j-\phi_M+\alpha)\Big]\\ &= D(\Omega) + \frac{K}{N}\sum_{j=1}^N \cos\Big(\frac{\theta_j-\theta_m}{2}+\frac{\theta_j-\phi_M}{2}+\alpha\Big)\sin\Big(\frac{\phi_M-\theta_m}{2}\Big)\\ &\quad + \frac{K}{N}\sum_{j=1}^N \cos\Big(\frac{\phi_j-\theta_m}{2}+\frac{\phi_j-\phi_M}{2}+\alpha\Big)\sin\Big(\frac{\phi_M-\theta_m}{2}\Big)\\ &= D(\Omega)\\ &\quad - \frac{2K}{N}\sin\Big(\frac{\Delta}{2}\Big)\sum_{j=1}^N \cos\Big(\frac{\theta_j-\theta_m}{2}+\frac{\phi_j-\phi_M}{2}+\alpha\Big)\cos\Big(\frac{\theta_j-\phi_j}{2}\Big).\end{aligned} \tag{16}$$

It is easy to see from $\Delta(0)=0$ that

$$\dot{\Delta}(0) = D(\Omega) > 0.$$

Hence, there exists a small $t_0>0$ such that $\Delta(t_0)>0$.

• Case 3 ($\Delta(0)<0$): Suppose there is no such t, i.e.,

$$\Delta(t) < 0, \quad \forall\, t \le 0. \tag{17}$$

Note that

$$\begin{gathered}-\frac{D(\phi)}{2}+\alpha \le \frac{\theta_j-\theta_m}{2}+\frac{\phi_j-\phi_M}{2}+\alpha \le \frac{D(\theta)}{2}+\alpha,\\ \Delta \le \theta_j-\phi_j \le D(\theta,\phi).\end{gathered}$$

Since $D(\theta), D(\phi) \le D(\theta,\phi) \le R$ and $|\Delta| \le D(\theta,\phi)$, in (16) we obtain

$$\begin{aligned}\dot{\Delta} &\ge D(\Omega) - 2K\sin\frac{\Delta}{2}\cos\Big(\frac{R}{2}+|\alpha|\Big)\cos\frac{R}{2}\\ &\ge D(\Omega) - \frac{2K}{\pi}\cos\frac{R}{2}\cos\Big(\frac{R}{2}+|\alpha|\Big)\Delta,\end{aligned}$$

where we used $\sin x \le \frac{2}{\pi}x,\ x\in\big(-\frac{\pi}{2},0\big)$. Since $R<\frac{\pi}{2}-|\alpha|$ and $|\alpha|<\frac{\pi}{2}$, we note that $\frac{R}{2}+|\alpha|<\frac{\pi}{2}$. Therefore we have

$$\begin{aligned}\Delta(t) &\ge \frac{\pi D(\Omega)}{2K}\sec\frac{R}{2}\sec\Big(\frac{R}{2}+|\alpha|\Big)\\ &\quad + \Big(\Delta(0) - \frac{\pi D(\Omega)}{2K}\sec\frac{R}{2}\sec\big(\frac{R}{2}+|\alpha|\big)\Big)\\ &\quad \times \exp\Big\{-\frac{\pi D(\Omega)t}{2K}\sec\frac{R}{2}\sec\Big(\frac{R}{2}+|\alpha|\Big)\Big\}.\end{aligned}$$

This yields

$$\lim_{t\to\infty} \Delta(t) \geq \frac{\pi D(\Omega)}{2K} \sec\frac{R}{2} \sec\Big(\frac{R}{2} + |\alpha|\Big) > 0.$$

Hence there exists a sufficiently large t_0 such that

$$\Delta(t_0) > \frac{\pi D(\Omega)}{4K} \sec\frac{R}{2} \sec\Big(\frac{R}{2} + |\alpha|\Big) > 0.$$

This is contradictory to our assumption (17). □

Proposition 4.5. *Suppose that the coupling strength K satisfies*

$$K > \frac{D(\Omega)}{1 - \sin|\alpha|},$$

and let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to (12)-(13) satisfying

$$D(\theta, \phi)(0) \leq R, \quad \textit{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then, there exists a finite-time t_0 such that group difference $\Delta(t)$ is uniformly bounded below for $t \geq t_0$:

$$\Delta(t) := \theta_m(t) - \phi_M(t) \geq \min\big\{\Delta(t_0), \frac{D(\Omega)}{K}\big\}, \quad \forall\, t \geq t_0.$$

Proof. By Lemma 4.4, there exists a time t_0 such that $\Delta(t_0) > 0$. Define a set $\mathcal{T}$ and its supremum $T^* \in [t_0, \infty]$:

$$\mathcal{T} := \Big\{T \geq t_0 \ : \ \Delta(t) > 0,\ \forall\, t \in [t_0, T)\Big\}, \qquad T^* := \sup \mathcal{T}.$$

Since $\Delta(t_0) > 0$, by the continuity of $\Delta(t)$, there exists $T > 0$ such that

$$\Delta(t) > 0, \quad \forall\, t \in [t_0, T).$$

Hence $T \in \mathcal{T}$, that is, the set $\mathcal{T}$ is not empty. We now claim:

$$T^* = +\infty.$$

Suppose not, i.e., $T^* < \infty$. Then we have

$$\lim_{t\to T^*-} \Delta(t) = 0,$$

and

$$\Delta(t) > 0, \quad \forall\, t \in [t_0, T^*).$$

We now use (16) again to see that

$$\begin{aligned}\dot{\Delta}(t) &\geq D(\Omega) - \frac{2K}{N} \sin\Big(\frac{\Delta}{2}\Big) \sum_{j=1}^{N} \cos\Big(\frac{\Delta}{2}\Big) \\ &= D(\Omega) - K \sin\Delta, \quad \text{for} \quad t \in [t_0, T^*),\end{aligned}$$

where we used

$$\cos\Big(\frac{\theta_j - \theta_m}{2} + \frac{\phi_j - \phi_M}{2} + \alpha\Big) \leq 1, \quad \cos\Big(\frac{\theta_j - \phi_j}{2}\Big) \leq \cos\Big(\frac{\Delta}{2}\Big).$$

Then we have

$$\Delta(t) \geq \Delta(t_0) e^{-K(t-t_0)} + \frac{D(\Omega)}{K}\Big(1 - e^{-K(t-t_0)}\Big), \quad \forall\, t \in [t_0, T^*). \tag{18}$$

This implies

$$0 = \lim_{t\to T^*-} \Delta(t) \geq \Delta(t_0) e^{-K(T^*-t_0)} + \frac{D(\Omega)}{K}\Big(1 - e^{-K(T^*-t_0)}\Big) > 0,$$

which gives a contradiction. Hence $T^* = \infty$. Then we recall (18) to see that

$$\Delta(t) \geq C := \min\Big\{\Delta(t_0), \frac{D(\Omega)}{K}\Big\}, \quad \forall\, t \geq t_0.$$

□

Remark 4.3. By Lemma 4.4 and Proposition 4.3, if $\Delta(0) \leq 0$, after a finite-time t_0, $\Delta(t_0)$ becomes positive and the trapping condition on the phase diameter is still valid. Then we can regard $(\theta,\phi)(t_0)$ as a new initial configuration. In the next subsection, without loss of generality we will assume initially $\Delta(0) > 0$.

4.3. **Formation of the two-point cluster configuration.** In this subsection, we study the asymptotic formation of two-point cluster configurations from initial configurations satisfying some conditions. For this, we derive nonlinear Gronwall-type inequalities for $D(\theta)$, $D(\phi)$, and Δ.

Lemma 4.6. *Let $(\theta,\phi) = (\theta(t),\phi(t))$ be the smooth solution to (12)–(13) satisfying*

$$\Delta(0) > 0, \quad D(\theta,\phi)(0) \leq R, \quad \textit{for some constant } R < \tfrac{\pi}{2} - |\alpha|.$$

Then, $D(\theta(t)), D(\phi(t))$, and $\Delta(t)$ satisfy the following system of differential inequalities:

$$\begin{aligned}
\dot{D}(\theta(t)) &\leq -\frac{K\cos\alpha}{\pi}D(\theta(t)) + \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta(t) + D(\phi(t)) + |\alpha|\big)\Big]D(\theta(t))^2,\\
\dot{D}(\phi(t)) &\leq -\frac{K\cos\alpha}{\pi}D(\phi(t)) + \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta(t) + D(\theta(t)) + |\alpha|\big)\Big]D(\phi(t))^2,\\
\dot{\Delta}(t) &\leq D(\Omega) - \frac{2K\cos\alpha}{\pi}\Delta(t) + \frac{K}{2}\Big[\sin(D(\theta(t)) + \alpha) + \sin(D(\phi(t)) - \alpha)\Big].
\end{aligned}$$

Proof. (i) From the proof of Lemma 4.2, we already know that

$$\begin{aligned}
\frac{d}{dt}D(\theta(t)) = &-\frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} + \alpha\Big)\\
&-\frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} + \alpha\Big).
\end{aligned}$$

Note that

$$\begin{aligned}
-\frac{D(\theta)}{2} &\leq \frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} \leq \frac{D(\theta)}{2},\\
-\Delta - D(\phi) - \frac{D(\theta)}{2} &\leq \frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} \leq -\Delta - \frac{D(\theta)}{2}, \quad \frac{D(\theta)}{2} + |\alpha| < \frac{\pi}{2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{d}{dt}D(\theta(t)) &\leq -K\sin\frac{D(\theta)}{2}\cos\Big(\frac{D(\theta)}{2} + |\alpha|\Big)\\
&\quad - K\sin\frac{D(\theta)}{2}\cos\Big(\Delta + D(\phi) + \frac{D(\theta)}{2} + |\alpha|\Big)\\
&= -\frac{K}{2}\Big[\cos\alpha + \cos\big(\Delta + D(\phi) + |\alpha|\big)\Big]\sin D(\theta)
\end{aligned}$$

$$
\begin{aligned}
&+ K\Big[\sin|\alpha| + \sin\big(\Delta + D(\phi) + |\alpha|\big)\Big]\sin^2\frac{D(\theta)}{2} \\
&\le -\frac{K}{\pi}\Big[\cos\alpha + \cos\big(\Delta + D(\phi) + |\alpha|\big)\Big]D(\theta) \\
&+ \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta + D(\phi) + |\alpha|\big)\Big]D(\theta)^2,
\end{aligned}
$$

where we used the relations

$$
\sin x \ge \frac{2}{\pi}x, \quad 2\sin^2\frac{x}{2} = 1 - \cos x \le x^2 \quad \text{for} \quad x \in \Big[0, \frac{\pi}{2}\Big].
$$

To obtain the upper bound, we used the inequality $\cos(\Delta + D(\phi) + |\alpha|) > 0$.
(ii) The result is derived in the same manner as that of (i) above. We use

$$
\begin{aligned}
-\frac{D(\phi)}{2} &\le \frac{\phi_j - \phi_M}{2} + \frac{\phi_j - \phi_m}{2} \le \frac{D(\phi)}{2}, \\
\Delta + \frac{D(\phi)}{2} &\le \frac{\theta_j - \phi_M}{2} + \frac{\theta_j - \phi_m}{2} \le \Delta + D(\theta) + \frac{D(\phi)}{2},
\end{aligned}
$$

to obtain

$$
\begin{aligned}
\frac{d}{dt}D(\phi(t)) &\le -K\sin\frac{D(\phi)}{2}\cos\Big(\frac{D(\phi)}{2} + |\alpha|\Big) \\
&\quad - K\sin\frac{D(\phi)}{2}\cos\Big(\Delta + D(\theta) + \frac{D(\phi)}{2} + |\alpha|\Big) \\
&= -\frac{K}{2}\Big[\cos\alpha + \cos\big(\Delta + D(\theta) + |\alpha|\big)\Big]\sin D(\phi) \\
&\quad + K\Big[\sin|\alpha| + \sin\big(\Delta + D(\theta) + |\alpha|\big)\Big]\sin^2\frac{D(\phi)}{2} \\
&\le -\frac{K}{\pi}\Big[\cos\alpha + \cos\big(\Delta + D(\theta) + |\alpha|\big)\Big]D(\phi) \\
&\quad + \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta + D(\theta) + |\alpha|\big)\Big]D(\phi)^2.
\end{aligned}
$$

(iii) Note that

$$
-\Delta - D(\phi) \le \phi_j - \theta_m \le -\Delta, \quad \Delta \le \theta_j - \phi_M \le \Delta + D(\theta). \tag{19}
$$

By the definition of $\Delta(t)$ and the relations (19), we obtain

$$
\begin{aligned}
\frac{d}{dt}\Delta(t) &= \dot\theta_m(t) - \dot\phi_M(t) \\
&= D(\Omega) + \frac{K}{2N}\sum_{j=1}^N \sin(\theta_j - \theta_m + \alpha) + \frac{K}{2N}\sum_{j=1}^N \sin(\phi_j - \theta_m + \alpha) \\
&\quad - \frac{K}{2N}\sum_{j=1}^N \sin(\phi_j - \phi_M + \alpha) - \frac{K}{2N}\sum_{j=1}^N \sin(\theta_j - \phi_M + \alpha) \\
&\le D(\Omega) + \frac{K}{2}\sin(D(\theta) + \alpha) + \frac{K}{2}\sin(-\Delta + \alpha) \\
&\quad - \frac{K}{2}\sin(-D(\phi) + \alpha) - \frac{K}{2}\sin(\Delta + \alpha),
\end{aligned}
$$

where we used the monotonicity of $\sin x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We then have the result

$$\begin{aligned}\frac{d}{dt}\Delta(t) &\le D(\Omega) - K\cos\alpha\sin\Delta + \frac{K}{2}\sin(D(\theta)+\alpha) + \frac{K}{2}\sin(D(\phi)-\alpha)\\ &\le D(\Omega) - \frac{2K\cos\alpha}{\pi}\Delta + \frac{K}{2}\Big[\sin(D(\theta)+\alpha) + \sin(D(\phi)-\alpha)\Big].\end{aligned}$$

□

We next present an elementary estimate for a system of differential inequalities.

Lemma 4.7. *Let X, Y, and Z be differentiable functions satisfying the following system of differential inequalities: for $t \ge t_0$,*

$$\begin{aligned}&\dot X \le -cX + \Big(\beta + \gamma\sin(Y+Z+|\alpha|)\Big)X^2,\\ &\dot Y \le -cY + \Big(\beta + \gamma\sin(X+Z+|\alpha|)\Big)Y^2,\\ &\dot Z \le -2cZ + \delta + \frac{K}{2}\Big(\sin(X+\alpha) + \sin(Y+\alpha)\Big),\\ &X(t_0) = X_0, \quad Y(t_0) = Y_0,\end{aligned} \tag{20}$$

where c, β, γ, and δ are positive constants. If $X_0, Y_0 < \frac{c}{\beta+\gamma}$, then there exist $t_1 \ge t_0 > 0$ and positive constants $C(X_0), C(Y_0)$ such that

$$\begin{aligned}&X(t) \le \frac{c}{\beta+\gamma+C(X_0)e^{c(t-t_0)}}, \quad t \ge t_0,\\ &Y(t) \le \frac{c}{\beta+\gamma+C(Y_0)e^{c(t-t_0)}}, \quad t \ge t_0,\\ &Z(t) \le Z(t_1)e^{-2c(t-t_1)} + \frac{\delta}{c}\Big(1 - e^{-2c(t-t_1)}\Big), \quad t \ge t_1.\end{aligned}$$

Proof. • (Estimate of X): We use $\sin(Y+Z+|\alpha|) \le 1$ to obtain

$$\dot X \le X\left(-c + (\beta+\gamma)X\right).$$

This yields

$$X(t) \le \frac{c}{\beta+\gamma+C(X_0)e^{c(t-t_0)}}, \quad t \ge t_0, \tag{21}$$

where $C(X_0)$ is given as follows:

$$C(X_0) := \Big|\frac{c-(\beta+\gamma)X_0}{X_0}\Big|.$$

• (Estimate of Y): Similarly, we have

$$Y(t) \le \frac{c}{\beta+\gamma+C(Y_0)e^{c(t-t_0)}}, \quad t \ge t_0. \tag{22}$$

• (Estimate of Z): We use the elementary relation

$$\begin{aligned}|\sin(X+\alpha) + \sin(Y-\alpha)| &= 2\Big|\sin\Big(\frac{X+Y}{2}\Big)\cos\Big(\frac{X-Y}{2}+\alpha\Big)\Big|\\ &\le |X| + |Y|,\end{aligned}$$

to derive

$$\dot Z \le -2cZ + \delta + \frac{K}{2}(|X|+|Y|).$$

On the other hand, it follows from (21) and (22) that there exists $t_1 \ge t_0$ such that

$$|X(t)| + |Y(t)| \le \frac{2\delta}{K}, \quad t \ge t_1.$$

Thus, we have

$$\dot{Z} \le -2cZ + 2\delta, \quad t \ge t_1.$$

This yields

$$Z(t) \le Z(t_1)e^{-2c(t-t_1)} + \frac{\delta}{c}\Big(1 - e^{-2c(t-t_1)}\Big), \quad t \ge t_1.$$

□

Remark 4.4. From the result of Lemma 4.7, we see that

$$X(t) = \mathcal{O}(1)e^{-ct}, \quad Y(t) = \mathcal{O}(1)e^{-ct}, \quad \text{as } t \to \infty \quad \text{and} \quad \lim_{t\to\infty} Z(t) \le \frac{\delta}{c}.$$

Theorem 4.8. *Let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to (12)–(13) satisfying*

$$\Delta(0) > 0, \quad D(\theta,\phi)(0) \le R, \quad \textit{for some constant } R < \tfrac{\pi}{2} - |\alpha|.$$

Then, $D(\theta)$, $D(\phi)$, and Δ satisfy the following estimates:

$$(i)\ D(\theta(t)) = \mathcal{O}(1)\exp\Big(-\frac{K\cos\alpha}{\pi}t\Big), \quad \textit{as } t \to \infty.$$

$$(ii)\ D(\phi(t)) = \mathcal{O}(1)\exp\Big(-\frac{K\cos\alpha}{\pi}t\Big).$$

$$(iii)\ \lim_{t\to\infty} \Delta(t) \le \frac{\pi D(\Omega)}{K\cos\alpha}.$$

Proof. In Lemma 4.6, we set

$$X(t) := D(\theta(t)), \quad Y(t) := D(\phi(t)), \quad Z(t) := \Delta(t),$$
$$c := \frac{K\cos\alpha}{\pi}, \quad \beta := \frac{K\sin|\alpha|}{2}, \quad \gamma := \frac{K}{2}, \quad \delta := D(\Omega).$$

To apply Lemma 4.7, we only need to verify the initial conditions, i.e.,

$$X_0, Y_0 < \frac{c}{\beta+\gamma} = \frac{2\cos\alpha}{\pi(\sin|\alpha|+1)}. \tag{23}$$

We only show the part for X_0, and the other one for Y_0 is similar.

• Case 1: $\dfrac{2\cos\alpha}{\pi(\sin|\alpha|+1)} > R$. Then by Proposition 4.3 we immediately have

$$D(\theta(t)) \le D(\theta,\phi)(t) \le R < \frac{2\cos\alpha}{\pi(\sin|\alpha|+1)}, \quad t \ge 0.$$

• Case 2: $\dfrac{2\cos\alpha}{\pi(\sin|\alpha|+1)} \le R$. In this case, we have

$$D(\theta^0) \ge \frac{2\cos\alpha}{\pi(\sin|\alpha|+1)}.$$

However, the proof of Lemma 4.2 tells us that, as long as $D(\theta(t)) \in \Big(\frac{2\cos\alpha}{\pi(\sin|\alpha|+1)}, R\Big)$ the derivative $\dfrac{d}{dt}D(\theta(t))$ has a negative upper bound. Thus, $D(\theta(t))$ must decrease to $\dfrac{2\cos\alpha}{\pi(\sin|\alpha|+1)}$ at some time t_*. Choosing $t_0 > t_*$, we have

$$D(\theta(t_0)) < \frac{2\cos\alpha}{\pi(\sin|\alpha|+1)},$$

since $\dfrac{d}{dt}D(\theta(t)) < 0$ as in Lemma 4.2.

By the above analysis, (23) is true and we can now apply Lemma 4.7. Then, the desired result is obtained from the direct application of Lemma 4.7. □

4.4. **Numerical examples.** In this subsection, we present the results of numerical simulations and compare these with the analytical results in Section 4.2 and 4.3.

Figure 3 shows the effects of α on the dynamic behavior of θ and ϕ. We consider effects such as preservation of segregated states and relaxation to the two-point clusters. For this simulation, we employed

$$N = 40, \quad K = 10.$$

The initial configurations of θ and ϕ were randomly chosen from $(0, \pi - 2\times 0.8)$, and Ω was chosen from $(0,1)$ so that $\Omega_1 > \Omega_2$. From the numerical simulations, we observed that the segregated states are robust, as expected from Proposition 4.5. We can see the effect of α on the asymptotic phase of θ and ϕ in Figure 3(c).

Figure 4 shows the effect of K for fixed N and α. In this simulation, we used

$$N = 40, \quad \alpha = 0.8,$$

and the initial configurations θ^0, ϕ^0 and Ω was the same as for Figure 3.

We see in Figure 4 that the value of Δ is affected by that of K. Based on these two simulations, we conclude that the behavior of θ and ϕ is asymptotically like that for the two-oscillators system.

5. **Synchronization estimates for Model C.** In this section, we present synchronization estimates for Model C, which is

$$\begin{aligned}
\dot\theta_i &= \frac{\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - 2\theta_i), \qquad i = 1,\dots,N_1.\\
\dot\phi_i &= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(2\theta_j - \phi_i + \alpha), \qquad i = 1,\dots,N_2.
\end{aligned} \tag{24}$$

Owing to the symmetry $(\theta,\phi,\alpha)\to(-\theta,-\phi,-\alpha)$, without loss of generality, we can assume that $\alpha \in (0, \frac{\pi}{2})$. To transform the system (24) into the familiar form, we introduce a new variable

$$\tilde\theta_i := 2\theta_i.$$

Then, the system (24) can be rewritten as

$$\begin{aligned}
\dot{\tilde\theta}_i &= \frac{2\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - \tilde\theta_i), \qquad i = 1,\dots,N_1,\\
\dot\phi_i &= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(\tilde\theta_j - \phi_i + \alpha), \qquad i = 1,\dots,N_2,
\end{aligned}$$

Throughout this section, we still use θ_i to denote $\tilde\theta_i$; i.e., we have

$$\begin{aligned}
\dot\theta_i &= \frac{2\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - \theta_i), \qquad i = 1,\dots,N_1,\\
\dot\phi_i &= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(\theta_j - \phi_i + \alpha), \qquad i = 1,\dots,N_2,
\end{aligned} \tag{25}$$

Note that there are no interactions between duplicate groups, so the interactions occur only for different groups. Thus, the system (25) can be regarded as the Kuramoto model on a bipartite graph, which can be applied in political science.

Example 5.1. As a simple example, we consider a two-oscillator system consisting of θ_1 and ϕ_1; i.e.,

$$\begin{aligned}\frac{d\theta_1}{dt} &= 2\mu\sin(\phi_1-\theta_1),\\ \frac{d\phi_1}{dt} &= (1-\mu)\cos\alpha\sin(\theta_1-\phi_1)+(1-\mu)\sin\alpha\cos(\theta_1-\phi_1).\end{aligned}$$

We introduce the difference $\psi := \phi_1-\theta_1$. Then, the system above becomes a single equation for ψ:

$$\begin{aligned}\frac{d\psi}{dt} &= -(2\mu+(1-\mu)\cos\alpha)\sin\psi+(1-\mu)\sin\alpha\cos\psi\\ &= -\sqrt{4\mu^2+4\mu(1-\mu)\cos\alpha+(1-\mu)^2}\sin(\psi-\Phi),\end{aligned}$$

where

$$\Phi = \arctan\left(\frac{(1-\mu)\sin\alpha}{2\mu+(1-\mu)\cos\alpha}\right)\in\left(0,\frac{\pi}{2}\right). \tag{26}$$

Note that, due to the frustration, complete phase synchronization does not occur, even though the oscillators are identical.

5.1. **Existence of a trapping region.** In this subsection, we study the existence of a positive invariant region. We define the extremal phases of each group, the partial diameters, and the total diameter in ways similar to those in (15). We set

$$\Delta(t) := \phi_m(t)-\theta_M(t).$$

Note that the arguments in Lemma 4.1 can be applied to Model C. Thus, the results in Lemma 4.1 and the statement in Remark 4.1 still hold. That is, there are no collisions between the oscillators in each group.

Lemma 5.1. *Suppose that the initial configuration satisfies*

$$D(\theta,\phi)(0) < R, \qquad \textit{for some constant } R < \tfrac{\pi}{2}-\alpha.$$

Then, we have

$$\frac{d^+}{dt}D(\theta(t))\Big|_{t=0}\le 0, \quad \frac{d^+}{dt}D(\phi(t))\Big|_{t=0}\le 0.$$

Proof. By direct calculation, we have

$$\begin{aligned}\frac{d^+}{dt}D(\theta(t))\Big|_{t=0} &= \frac{2\mu}{N_2}\sum_{j=1}^{N_2}\Big[\sin(\phi_j(0)-\theta_M(0))-\sin(\phi_j(0)-\theta_m(0))\Big]\\ &= -\frac{4\mu}{N_2}\sin\frac{D(\theta(0))}{2}\sum_{j=1}^{N_2}\cos\left(\frac{\phi_j(0)-\theta_M(0)}{2}+\frac{\phi_j(0)-\theta_m(0)}{2}\right)\\ &\le -4\mu\cos R\sin\frac{D(\theta(0))}{2}\\ &\le 0,\end{aligned}$$

where we used

$$-R\le\phi_j-\theta_M\le\frac{\phi_j-\theta_M}{2}+\frac{\phi_j-\theta_m}{2}\le\phi_j-\theta_m<R.$$

We obtain the estimate for $D(\phi(t))$ similarly:

$$
\begin{aligned}
&\frac{d^+}{dt} D(\phi(t))\Big|_{t=0} \\
&= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\Big[\sin(\theta_j(0)-\phi_M(0)+\alpha)-\sin(\theta_j(0)-\phi_m(0)+\alpha)\Big] \\
&= -\frac{2(1-\mu)}{N_1}\sin\frac{D(\phi(0))}{2}\sum_{j=1}^{N_2}\cos\left(\frac{\theta_j(0)-\phi_M(0)}{2}+\frac{\theta_j(0)-\phi_m(0)}{2}+\alpha\right) \\
&\le -2(1-\mu)\cos(R+\alpha)\sin\frac{D(\phi(0))}{2} \\
&\le 0.
\end{aligned}
$$

□

Lemma 5.2. *Let $(\theta,\phi)=(\theta(t),\phi(t))$ be the smooth solution to (25) satisfying*

$$\max\big\{|\Delta(0)|,\Phi\big\}+D(\theta(0))+D(\phi(0))<R, \quad \textit{for some constant } R<\tfrac{\pi}{2}-\alpha. \tag{27}$$

Then, we have the following:

(i) $D(\theta,\phi)(t)<R, \quad \forall\, t\ge 0;$

(ii) $D(\theta(t))\le D(\theta(0))$, *and* $D(\phi(t))\le D(\phi(0)), \quad \forall\, t\ge 0.$

Proof. We define a set $\mathcal{T}_1$ and its supremum $T_1^*\in[0,\infty]$:

$$
\begin{aligned}
&\mathcal{T}_1 := \Big\{T\in\mathbb{R}_+ : D(\theta,\phi)(t)<R \ \text{and}\ \Delta(t)<\max\{\Delta(0),\Phi\}+\alpha,\ \forall\, t\in[0,T)\Big\},\\
&T_1^* := \sup\mathcal{T}_1,
\end{aligned}
$$

where Φ is the positive constant defined by (26).

It follows from (27) that we have

$$
\begin{aligned}
&D(\theta,\phi)(0)=\Delta(0)+D(\theta(0))+D(\phi(0))<R, && \text{if } \Delta(0)>0,\\
&D(\theta,\phi)(0)\le D(\theta(0))+D(\phi(0))<R, && \text{if } \Delta(0)<0,
\end{aligned}
$$

and

$$\Delta(0)<\max\{\Delta(0),\Phi\}+\alpha.$$

By the continuity of $D(\theta,\phi)(t)$ and $\Delta(t)$, there exists $T>0$ such that $T\in\mathcal{T}_1$; i.e., the set $\mathcal{T}_1$ is not empty and $T_1^*>0$. We now claim that

$$T_1^*=+\infty.$$

Proof of claim. Suppose not; i.e., $T_1^*<\infty$. Then, at $t=T_1^*$ we have

$$\text{either}\ \ D(\theta,\phi)(T_1^*)=R \ \ \text{or}\ \ \Delta(t)=\max\{\Delta(0),\Phi\}+\alpha, \tag{28}$$

and

$$D(\theta,\phi)(t)<R \ \ \text{and}\ \ \Delta(t)<\max\{\Delta(0),\Phi\}+\alpha \quad \forall\, t\in[0,T_1^*). \tag{29}$$

Then, by Lemma 5.1, we deduce that for all $t\in[0,T_1^*)$ the partial diameters $D(\theta(t))$ and $D(\phi(t))$ are nonexpanding. Hence, we have

$$D(\theta(t))\le D(\theta(0)) \quad\text{and}\quad D(\phi(t))\le D(\phi(0)), \qquad \forall\, t\in[0,T_1^*). \tag{30}$$

Next, we estimate the lower and upper bounds for $\Delta(t)$ in $[0,T_1^*)$. Obviously, we have

$$\frac{d}{dt}\Delta(t)=-\frac{2\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j-\theta_M)+\frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(\theta_j-\phi_m+\alpha).$$

Note that

$$\Delta \le \phi_j - \theta_M \le \Delta + D(\phi), \quad -\Delta - D(\theta) \le \theta_j - \phi_m \le -\Delta, \quad \forall\, t \in [0, T_1^*).$$

We use (29)–(30) to obtain the upper bound

$$\begin{aligned}\dot{\Delta} &\le -2\mu \sin\Delta - (1-\mu)\sin(\Delta - \alpha)\\ &= -(2\mu + (1-\mu)\cos\alpha)\sin\Delta + (1-\mu)\sin\alpha\cos\Delta\\ &= -A\sin\Delta + B\cos\Delta\\ &= -\sqrt{A^2+B^2}\sin(\Delta - \Phi).\end{aligned}$$

We note that the solution of the ODE

$$\dot{x} = -\sqrt{A^2+B^2}\sin(x - \Phi), \qquad x(0) = \Delta(0) \in (-\frac{\pi}{2}, \frac{\pi}{2}),$$

monotonically approaches the equilibrium $x_* = \Phi$; therefore, by the principle of comparison we have an upper bound

$$\Delta(t) \le \max\{\Delta(0), \Phi\}, \qquad t \in [0, T_1^*). \tag{31}$$

Similarly, we use (29)–(30) to derive

$$\begin{aligned}\dot{\Delta} &\ge -2\mu\sin(\Delta + D(\phi)) - (1-\mu)\sin\big(\Delta + D(\theta) - \alpha\big)\\ &\ge -2\mu\sin(\Delta + D(\phi(0))) - (1-\mu)\sin\big(\Delta + D(\theta(0)) - \alpha\big)\\ &= -\sin\Delta\Big(2\mu\cos(D(\phi(0))) + (1-\mu)\cos(D(\theta(0)) - \alpha)\Big)\\ &\quad - \cos\Delta\Big(2\mu\sin(D(\phi(0))) + (1-\mu)\sin(D(\theta(0)) - \alpha)\Big)\\ &= -C\sin(\Delta + \hat{\Phi}),\end{aligned} \tag{32}$$

where $C = C(D(\theta(0)), D(\phi(0)), \alpha, \mu)$ is a positive constant and $\hat{\Phi}$ is given by

$$\hat{\Phi} = \arctan\Big[\frac{2\mu\sin(D(\phi(0))) + (1-\mu)\sin(D(\theta(0)) - \alpha)}{2\mu\cos(D(\phi(0))) + (1-\mu)\cos(D(\theta(0)) - \alpha)}\Big] \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

We use the elementary inequality

$$\frac{x_1}{y_1} \le \frac{x_1 + x_2}{y_1 + y_2} \le \frac{x_2}{y_2}, \quad \text{if} \ \ \frac{x_1}{y_1} \le \frac{x_2}{y_2} \ \ \text{and} \ \ y_1,\, y_2 > 0,$$

as well as the monotonicity of $\tan x$, $x \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$, to observe that $\hat{\Phi}$ is located within the interval bounded by $D(\theta(0)) - \alpha$ and $D(\phi(0))$.

Consider the following ODE:

$$\dot{y} = -C\sin(y + \hat{\Phi}), \qquad y(0) = \Delta(0).$$

Its solution $y(t)$ monotonically approaches the equilibrium $y_* = -\hat{\Phi}$; thus, the trajectory $y(t)$ is confined within the interval bounded by $-\hat{\Phi}$ and $\Delta(0)$. We now recall (32) and use the principle of comparison to derive

$$\Delta(t) \ge y(t).$$

Therefore, we have

$$\Delta(t) \ge \min\{-\hat{\Phi},\, \Delta(0)\} \ge \min\big\{\alpha - D(\theta(0)),\, -D(\phi(0)),\, \Delta(0)\big\}. \tag{33}$$

We combine (31) and (33), the estimates of the upper and lower bounds, to obtain

$$-\max\{D(\theta(0)), D(\phi(0)), \Delta(0)\} \le \Delta(t) \le \max\{\Delta(0), \Phi\}, \qquad \forall\, t \in [0, T_1^*). \tag{34}$$

Then, from the continuity we know that the estimate (34) is still valid for $t = T_1^*$; thus,

$$-\max\{D(\theta(0)), D(\phi(0)), \Delta(0)\} \le \Delta(T_1^*) < \max\{\Delta(0), \Phi\} + \alpha. \tag{35}$$

We now combine (30) and (34) to estimate $D(\theta, \phi)(t)$ on $[0, T_1^*)$.

• Case 1 $(\Delta(t) \ge 0)$: In this case, we have

$$D(\theta, \phi)(t) = \Delta(t) + D(\theta(t)) + D(\phi(t)) \le \max\{\Delta(0), \Phi\} + D(\theta(0)) + D(\phi(0)).$$

• Case 2 $(\Delta(t) < 0)$: In this case, we have

$$D(\theta, \phi)(t) \le D(\theta(t)) + D(\phi(t)) \le D(\theta(0)) + D(\phi(0)).$$

We use the continuity of $D(\theta, \phi)(t)$ and combine Cases 1 and 2 at $t = T_1^*$ to observe that

$$D(\theta, \phi)(T_1^*) \le \max\{\Delta(0), \Phi\} + D(\theta(0)) + D(\phi(0)) < R. \tag{36}$$

Note that (35)–(36) contradict (28). Hence, $T_1^* = \infty$, which implies that

$$D(\theta, \phi)(t) < R, \qquad \forall\, t \ge 0.$$

Thus, we have the result(i) desired. To derive the result(ii), we use the result(i) and Lemma 5.1. □

5.2. **Relaxation estimate.** In this subsection, by means of the following theorem, we present an estimate of the relaxation toward the phase-locked states.

Theorem 5.3. *Let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to (25) with initial data satisfying*

$$\max\big\{|\Delta(0)|, \Phi\big\} + D(\theta(0)) + D(\phi(0)) < R, \quad \text{for some constant } R < \tfrac{\pi}{2} - \alpha.$$

Then each group exhibits asymptotic phase synchronization and the two groups will asymptotically become separated. More precisely, $D(\theta(t))$, $D(\phi(t))$ and $\Delta(t)$ satisfy the following estimates:

$$(i)\ D(\theta(t)) \le D(\theta(0)) \exp\left\{-\Big(\frac{4\mu \cos R}{\pi}\Big)t\right\}, \quad t \ge 0.$$

$$(ii)\ D(\phi(t)) \le D(\phi(0)) \exp\left\{-\Big(\frac{2(1-\mu)\cos(R+\alpha)}{\pi}\Big)t\right\}.$$

$$(iii)\ \Delta(t) \to \Phi, \quad \text{as } t \to \infty.$$

Here, Φ is given by (26).

Proof. First, by Lemma 5.2, we have

$$D(\theta, \phi)(t) < R, \quad D(\theta(t)) \le D(\theta(0)), \quad D(\phi(t)) \le D(\phi(0)), \quad \forall\, t \ge 0.$$

(i) In the proof of Lemma 5.2, we used

$$\dot{D}(\theta) \le -4\mu \cos R \sin\frac{D(\theta)}{2} \le -\frac{4\mu}{\pi}\cos R D(\theta),$$

where we used $\sin x \ge \frac{2}{\pi}x$, $x \in \big(0, \frac{\pi}{2}\big)$. Then, the standard Gronwall estimate yields the desired result.

(ii) Similarly, we have

$$\dot{D}(\phi) \le -2(1-\mu)\cos(R+\alpha)\sin\frac{D(\phi)}{2} \le -\frac{2(1-\mu)}{\pi}\cos(R+\alpha)D(\phi).$$

(iii) This assertion follows from the phase synchronization of each group and the analysis in Example 5.1. □

5.3. **Numerical examples.** In this subsection, we provide the results of several numerical simulations and compare these with analytical results in previous subsections.

To produce Figure 5, we used

$$N_1 = 30, \quad N_2 = 70, \quad \alpha = 0.3.$$

The initial configuration of θ^0 and ϕ^0 was chosen randomly from $[0, \pi]$ and $[0, \pi - 2 \times 0.3]$, respectively. We compare the case $\mu = 0.2$ with the case $\mu = 0.8$ with the same initial configuration. In Figure 5(b), we observe that θ and ϕ are synchronized to two distinct point clusters. In Figures 5(c) and 5(d), we observe that if μ is small, then the sub-configuration of ϕ becomes synchronized faster than the sub-configuration of θ, whereas if μ is large, then θ synchronizes faster than ϕ. Note that the group ϕ is under the effect of frustration. Hence, when μ is large, the group ϕ becomes synchronized much slower than the group θ, because of the low coupling strength and the effect of the frustration on the dynamics of ϕ. This is why the difference in relaxation rates in Figure 5(d) is much greater than the difference in relaxation rates in Figure 5(c). Recall that the group distance Δ depends on the frustration α and the coupling strength μ. Thus, in the next two sets of simulations, we investigated the dynamic behavior of Δ with respect to α and μ separately.

Figure 6 shows the relation between α and the group distance Δ. For this simulation, we used

$$N_1 = 70, \quad N_2 = 30, \quad \mu = 0.6,$$

and the initial configurations θ^0 and ϕ^0 were chosen randomly from $[0, \pi - 2 \times 0.9]$. Note that if $\alpha = 0$, then θ and ϕ are completely synchronized, even though there is no coupling strength within the group. The same phenomenon exists in the Kuramoto model for identical oscillators. We see that Δ is monotonically increasing as α increases.

Figure 7 shows the relation between μ and Δ. For this simulation, we used

$$N_1 = 60, \quad N_2 = 40, \quad \alpha = 0.5,$$

and the initial configurations θ^0 and ϕ^0 were chosen ramdomly from$[0, \pi - 2 \times 0.5]$. From the numerical simulations, it is easy to see that the group distance Δ decreases as μ increases.

6. **Conclusion.** We studied the effect of interaction frustration on the complete synchronization of Kuramoto oscillators. In general, interaction frustration hinders the formation of complete (frequency) synchronization. Hence, even for the same initial configuration, we need a larger coupling strength to ensure the synchronization in the presence of nonzero interaction frustration. For more quantitative estimates, we considered three Kuramoto-type models. Our first model is for an ensemble of Kuramoto oscillators with interaction frustration. In this case, we derived some explicit sufficient conditions (Theorem 3.3) for the initial configurations, coupling strength, and frustration that lead to complete frequency synchronization. Although we do not have an optimal rate of decay for the frequency diameter, the upper bound estimate of Theorem 3.3 suggests how the frustration slows down the decay, and this coincides with Figure 2(b). Our second model is a special case of the first model; i.e., the ensemble is simply a mixture of two groups of identical Kuramoto oscillators with distinct natural frequencies. In this case, we showed that the mixed configuration evolves toward two-point cluster configurations exponentially fast. We also estimated the lower and upper bounds on the distance

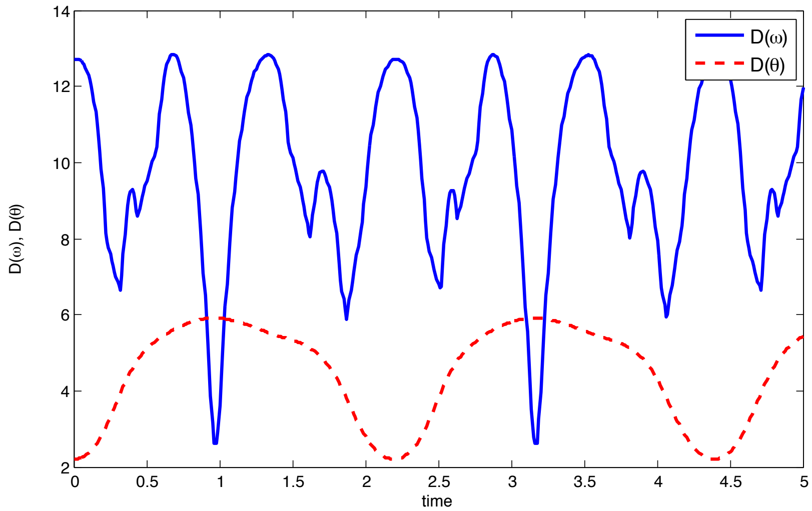

FIGURE 1. The natural frequencies were all same and $\alpha = \frac{\pi}{2}$. Then, we can see that $D(\theta)$ oscillates even though identical oscillators case.

between two point clusters in terms of the system parameter K, the frustration α, and $D(\Omega)$. Our third model is a Kuramoto-type model for identical oscillators on the bipartite graph. In this case, the configuration evolves toward the two-point cluster configuration, and furthermore, we obtained the exact asymptotic diameter of the two-point cluster configuration.

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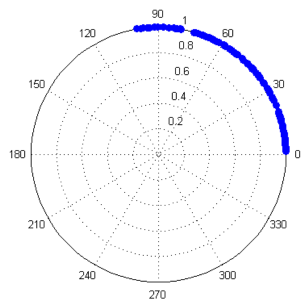
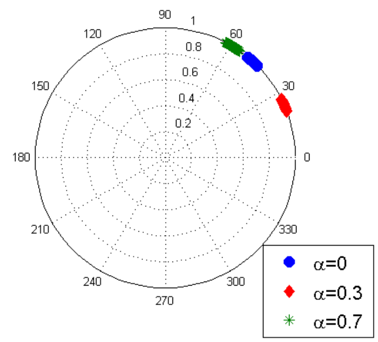
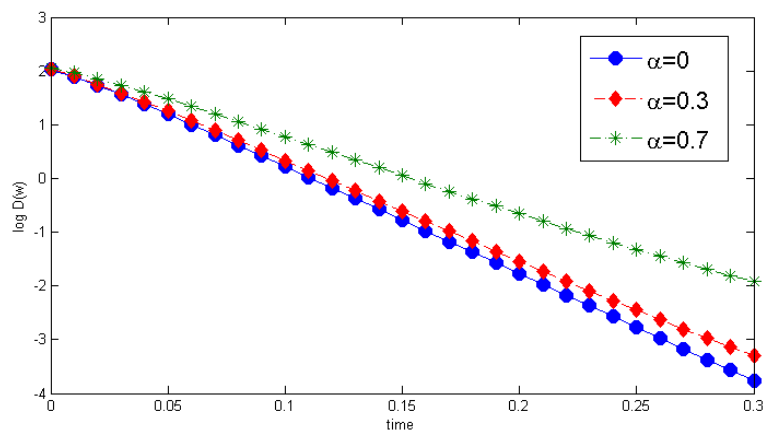


(a) Initial phase configuration



(b) Final phase configuration



(c) $\log D(\omega)$ for $\alpha = 0, 0.3, 0.7$

Figure 2. Model A: The initial and limiting phase configurations are displayed in Figure 2(a) and (b). We observe that as α increases, $\cos D^{\infty}$ decrease, i.e., the decay rate may decrease, which can be seen in Figure 2(c).

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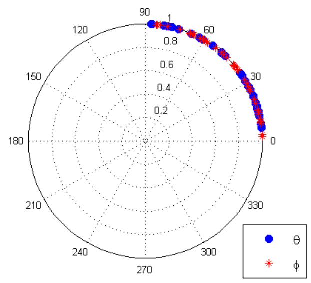
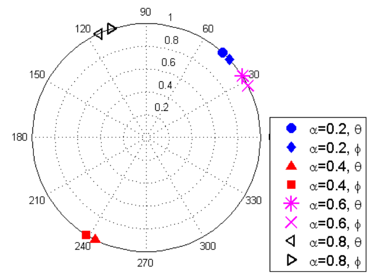
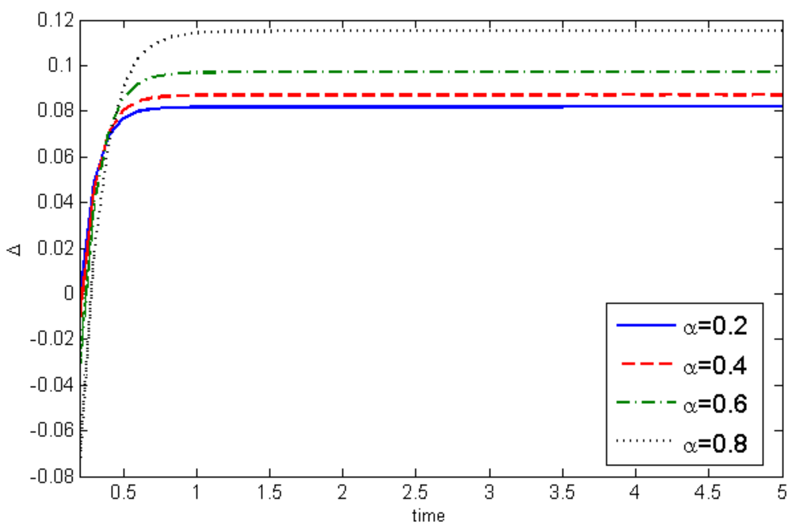


(a) Initial phase configuration



(b) Final phase configuration



(c) Dynamics of Δ for $\alpha = 0.2, 0.4, 0.6, 0.8$

FIGURE 3. Model B: Figure 3 shows the effects of α on the dynamic behavior of θ and ϕ. The initial configurations of θ and ϕ were randomly chosen from $(0, \pi - 2 \times 0.8)$, and Ω was chosen from $(0, 1)$ so that $\Omega_1 > \Omega_2$. From the numerical simulations, we observed that the segregated states are robust. We can see the effect of α on the asymptotic phase of θ and ϕ in Figure 3(c)

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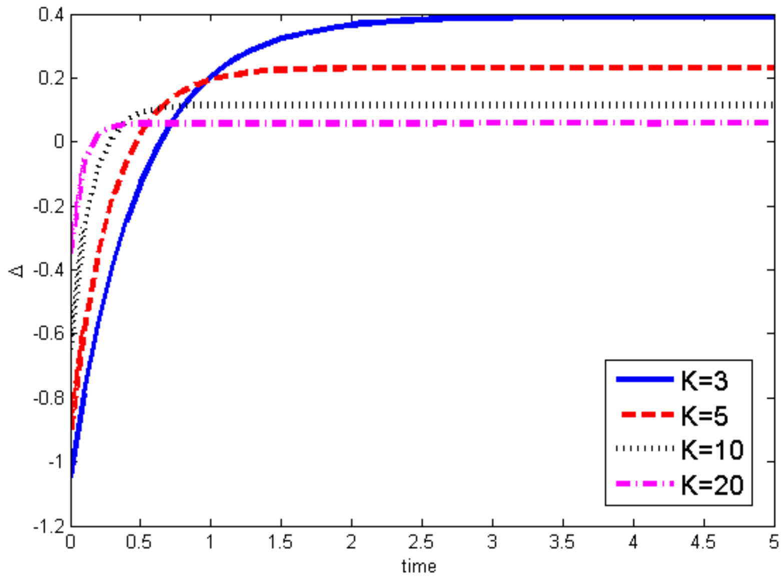

FIGURE 4. Model B: dynamics of Δ for $K = 3, 5, 10, 20$. The initial configurations θ^0, ϕ^0 and Ω were the same as for Figure 3. We can see that the value of Δ is affected by that of K. Based on these two simulations, we conclude that the behavior of θ and ϕ is asymptotically like that for the two-oscillators system.

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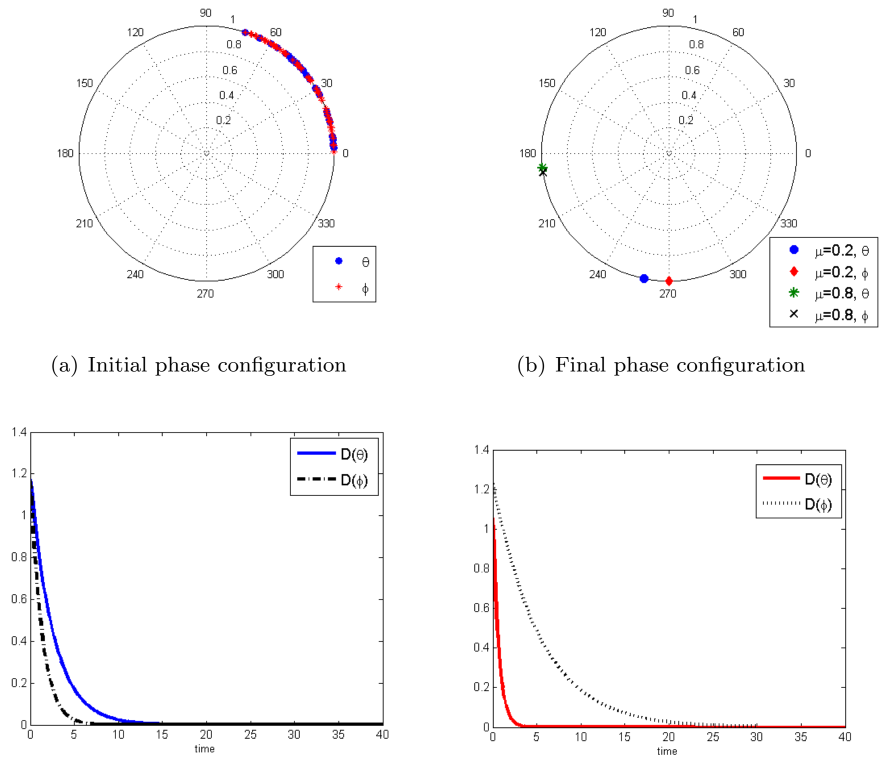

(a) Initial phase configuration (b) Final phase configuration

(c) $\mu = 0.2$ (d) $\mu = 0.8$

FIGURE 5. Model C: dynamics of phases for $\mu = 0.2, 0.8$. The initial configuration of θ^0 and ϕ^0 was chosen randomly from $[0, \pi]$ and $[0, \pi - 2 \times 0.3]$, respectively. We compare the case $\mu = 0.2$ with the case $\mu = 0.8$ with the same initial configuration. In Figure 5(b), we observe that θ and ϕ are synchronized to two distinct point clusters. In Figures 5(c) and (d), we observe that if μ is small, then the sub-configuration of ϕ becomes synchronized faster than the sub-configuration of θ, whereas if μ is large, then θ synchronizes faster than ϕ. Note that the group ϕ is under the effect of frustration.

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Received September 2013; revised February 2014.

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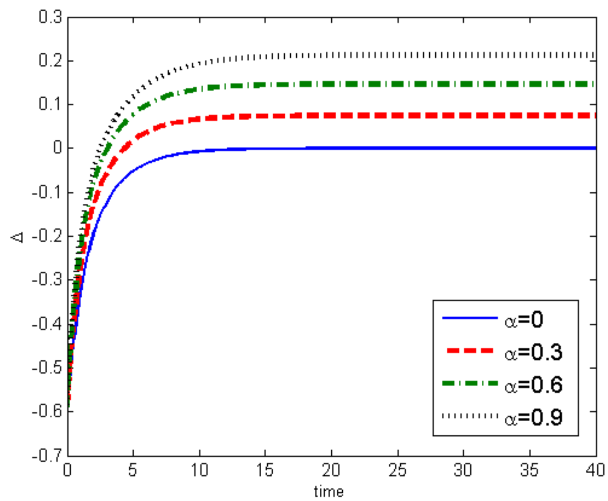

FIGURE 6. Model C: dynamics of Δ for $\alpha = 0, 0.3, 0.6, 0.9$. The initial configurations θ^0 and ϕ^0 were chosen randomly from $[0, \pi - 2 \times 0.9]$. Note that if $\alpha = 0$, then θ and ϕ are completely synchronized, even though there is no coupling strength within the group. We see that Δ is monotonically increasing as α increases.

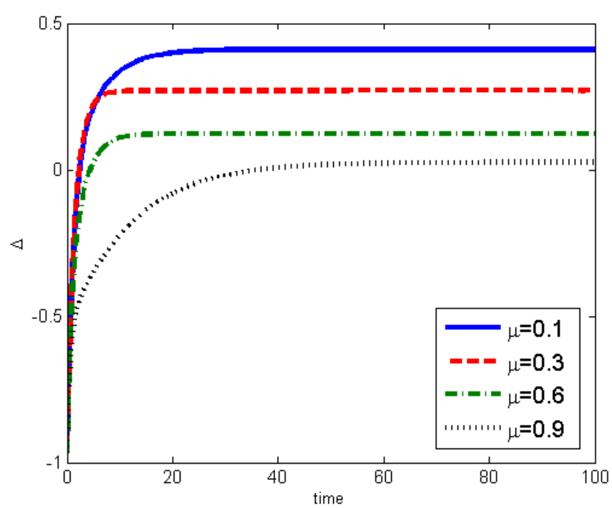

FIGURE 7. Model C: dynamics of Δ for $\mu = 0.1, 0.3, 0.6, 0.9$. The initial configurations θ^0 and ϕ^0 were chosen ramdomly from$[0, \pi - 2 \times 0.5]$. We observe that the group distance Δ decreases as μ increases.