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SPARSE STABILIZATION OF DYNAMICAL SYSTEMS DRIVEN BY ATTRACTION AND AVOIDANCE FORCES

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Abstract. Conditional self-organization and pattern-formation are relevant phenomena arising in biological, social, and economical contexts, and received a growing attention in recent years in mathematical modeling. An important issue related to *optimal government strategies* is how to design external parsimonious interventions, aiming at enforcing systems to converge to specific patterns. This is in contrast to other models where the players of the systems are allowed to interact *freely* and are supposed autonomously, either by game rules or by embedded decentralized feedback control rules, to converge to patterns. In this paper we tackle the problem of designing optimal centralized feedback controls for systems of moving particles, subject to mutual attraction and repulsion forces, and friction. Under certain conditions on the attraction and repulsion forces, if the total energy of the system, composed of the sum of its kinetic and potential parts, is below a certain critical threshold, then such systems are known to converge autonomously to the stable configuration of keeping confined and collision avoiding in space, uniformly in time. If the energy is above such a critical level, then the space coherence can be lost. We show that in the latter situation of lost self-organization, one can nevertheless steer the system to return to stable energy levels by feedback controls defined as the minimizers of a certain functional with ℓ_1-norm penalty and constraints. Additionally we show that the optimal strategy in this class of controls is necessarily *sparse*, i.e., the control acts on at most one agent at each time. This is another remarkable example of how *homophilious* systems, i.e., systems where agents tend to be strongly more influenced by near agents than far ones, are naturally prone to sparse stabilization, explaining the effectiveness of parsimonious interventions of governments in societies.

Contents

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1. Introduction.

Self-organization in social interactions is a fascinating mechanism which inspired the mathematical modeling of multi-agent interactions to lead to coherent global behaviors, with applications in the study of biological, social, and economical phenomena. Two main machanisms are considered in such models to drive the dynamics. The first, which takes inspiration, e.g., from physics laws of motion, is based on binary forces encoding observed "first principles" of biological, social, or economical interactions. As it is very hard to be exhaustive in accounting all the developments of this very fast growing field, we refer to [4, 6, 19, 24] for recent surveys, where both particle and macroscopic models are reviewed. The second mechanism is based on evolutive games, where the dynamics is driven by the simultaneous optimization of costs by the players, perhaps subjected to selection, from game theoretic models of evolution [16] to mean-field differential games, introduced by Lasry and Lions [18], and independently in the optimal control community under the name Nash Certainty Equivalence (NCE) within the work [17], later greatly popularized, e.g., within consensus problems, for instance in [20, 21]. The common view point of these branches of mathematical modeling of multi-agent systems is that the dynamics is based on the *free* interaction of the agents and the wished phenomenon to be described is their self-organization in terms of the formation of higher level complex, macroscopic patterns. Perhaps opposite to this enthusiastic view about self-organization, we wish to modify this paradigm, by allowing possible external interventions on the system. Let us motivate our point of view by focusing our attention on consensus emergence and coherence in a society.

There are many mechanisms of promotion of coherence in a society, for instance the *heterophilia*, i.e., the tendency to bond more with those who are different rather than those who are similar, as it has been pointed out in the recent work [19]. But also in *homophilious* societies more influenced by local similarities, global self-organization can be expected as soon as enough initial coherence is reached. A relatively simple, but also very instructive mathematical description of such a situation is given by the model by Cucker and Smale [10, 11], see also the generalizations in [15], where consensus emergence is shown to be conditional to initial conditions of coherence whenever the system is predominantly homophilious. However, it is common experience that coherence in a homophilious society can be lost, leading sometimes to dramatic consequences, questioning strongly the role and the effectiveness of governments, which may be called to restore social stability. In the recent work [3] the authors explored the conditions under which Cucker-Smale systems can be stabilized by means of centralized feedback controls, revealing a remarkable fact: if the interactions in a social system are *homophilious*, hence in principle prone to split consensus and lose coherence, *parsimonious* stabilization strategies are not only *unconditionally* effective (i.e., the successful stabilization does not depend on the initial conditions of the system), but also optimal in terms of leading in the shortest time to global consensus, with the minimal amount of external intervention. In other words, if, on the one side, the homophilious character of a society plays against its coherence, on the other side, it plays at its advantage if we allow for *sparse* external intervention. We refer to [22] for other controllability results of multi-agent systems, where a graph-theoretic point of view is taken to consider the modeling of the dynamics of an interacting group driven by a small group of *fixed* leaders, as a generalization of the seminal work of Tanner [23]. Let us stress that our point of view is slightly different, as we do not fix the leaders and we allow the possible exchange in time of them, although promoting instantaneously their small number. The key concept which has to be firmly taken into account in this paper is the promotion of sparse controls. As observed in [3, Section 6], the unconditional sparse stabilization and global sparse controllability results can be easily adapted to any system of the type

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{v} \\ \dot{\boldsymbol{v}} = -L^a(\boldsymbol{x})\boldsymbol{v} + \boldsymbol{u}, \end{cases} \tag{1}$$

where the main state of the group of N agents is given by the N-uple $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$. Similarly for the consensus parameters $\boldsymbol{v} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_N)$. The vector $\boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_N)$ represents the external controls.

The space of main states and the space of consensus parameters is $(\mathbb{R}^d)^N$ for both; the matrix $L^a(\boldsymbol{x})$ is the Laplacian[1] of the $N \times N$ adjacency matrix $(a(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)/N)_{i,j=1}^N$, where $a \in C^1([0, +\infty))$ is a *nonincreasing, positive, and bounded function*. The Laplacian $L^a(\boldsymbol{x})$ is an $N \times N$ matrix acting on $(\mathbb{R}^d)^N$, and verifies $L^a(\boldsymbol{x})(\boldsymbol{v}, \dots, \boldsymbol{v}) = 0$ for every $\boldsymbol{v} \in \mathbb{R}^d$.

Let us however stress that such results have more far reaching potential, as they can address also situations which do not match the structure (1), such as the Cucker and Dong model of cohesion and avoidance [9], where the system has actually the form

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{v} \\ \dot{\boldsymbol{v}} = (L^f(\boldsymbol{x}) - L^a(\boldsymbol{x}))\boldsymbol{x} + \boldsymbol{u}, \end{cases} \tag{2}$$

where $L^a(\boldsymbol{x})$ and $L^f(\boldsymbol{x})$ are graph-Laplacians associated to competing avoidance and cohesion forces respectively. We shall introduce this system in more detail below. Similar models considering attraction, repulsion and other effects, such as alignment or self-drive, appear in the recent literature and they seem effectively describing realistic situations in nature of conditional pattern formation, see, e.g., some of the most related contributions [5, 7, 8, 12]. Under certain conditions on the attraction and repulsion forces, if the total energy of the system (2), composed of the sum of its kinetic and potential parts, is below a certain critical threshold, then such systems are known to converge autonomously to the stable configuration of keeping confined and collision avoiding in space, uniformly in time, see [9, Theorem

[1] Given a real $N \times N$ matrix $A = (a_{ij})_{i,j}$ and $v \in (\mathbb{R}^d)^N$ we denote by Av the action of A on $(\mathbb{R}^d)^N$ by mapping v to $(a_{i1}v_1 + \cdots + a_{iN}v_N)_{i=1,\dots,N}$. Given a nonnegative symmetric $N \times N$ matrix $A = (a_{ij})_{i,j}$, the *Laplacian* L of A is defined by $L = D - A$, with $D = \mathrm{diag}(d_1, \dots, d_N)$ and $d_k = \sum_{j=1}^N a_{kj}$.

2.1]. If the energy is above such a critical level, then the space coherence can be lost. And this introduces us to the main scope of this paper. We show that in the latter situation of lost self-organization, one can nevertheless steer the system (2) to return to stable energy levels by feedback controls. These controls are taken as the minimizers of a specific functional with ℓ_1-norm penalty and constraints, a feature which, additionally to the stabilization effect, let us show that the optimal strategy is necessarily *sparse*, i.e., the control acts on at most one agent at each time. This is another remarkable example beside the one already presented in [3] of how certain social systems are naturally prone to sparse stabilization, explaining the effectiveness of parsimonious interventions of governments in societies. Let us stress that, although here we follow a conceptually similar path as in [3], the structural differences between the models (1) and (2) are such that, not only the analysis in the present paper requires certain nontrivial a priori estimates for stability, collected below in Section 3, which were not necessary for (1), but also our final result turns out to differ substantially. In particular, while the stabilization of the Cucker and Smale model is *unconditional*, i.e., it does not depend on the initial conditions, for the Cucker and Dong model our analysis guarantees stabilization only within certain total energy levels, which is suggesting that also stabilization can be conditional. However, our numerical experiments reported below suggest that it is possible to exceed such a sufficient upper energy barrier to stabilization, but it is unclear whether this is just a matter of fortunate choices of good initial conditions or we can actually have a broader stabilization range than the one analytically derived. This leaves us still with the open question of whether such systems are unconditionally sparse stabilizable. Let us stress that this present paper addresses exclusively feedback controls for the purpose of stabilization. However, we mention that in [3] also the formulation of a finite horizon sparse optimal control problem for the Cucker and Smale model was considered. To deal with the sparse optimal control of a very large number of agents, in [14] mean-field sparse optimal control problems have been introduced and analyzed in a rather general setting, which includes both (1) and (2) as possible starting particle formulations.

The paper is organized as follows. In Section 2 we recall in more detail the model of Cucker and Dong. In particular, we review the main result concerning conditional convergence to stable configurations of cohesion and collision avoidance. We conclude the section by introducing the natural form of a sparse feedback control to be used to stabilize the system when it is above the critical energy threshold. In Section 3 we analyze a *sampling-and-hold* strategy based on the introduced sparse feedback control and we collect several a priori statements on the behavior of the system subjected to such sampled control. In Section 4 we show that the sampled sparse control will be able to steer the system to the stable total energy level, provided that the initial total energy is below a certain upper threshold, and the initial mean velocity of the system is non zero. In Section 5 we show that there exist solutions of the controlled Cucker and Dong system as limits for the sampling time going to zero and we characterize them in terms of certain variational principles involving $\ell_1^N - \ell_2^d$-norm penalties, describing the limit controls. The end of this section is of utmost relevance, as it is a fundamental justification for the choice of sparse controls. There we show that sparse controls are optimal in terms of maximazing the rate of convergence to return to stable total energy levels. In other words it is a mathematical confirmation that for certain social systems parsimonious controls can be mostly effective. We conclude with Section 6 which collects several numerical experiments, illustrating in details our analytical findings.

2. **Preliminaries.**

2.1. **The model.**

The Cucker-Dong model introduced and analyzed in [9] is given by the following system of differential equations, where $i = 1, \dots, N$:

$$\begin{aligned} \dot{\boldsymbol{x}}_i(t) &= \boldsymbol{v}_i(t) \\ \dot{\boldsymbol{v}}_i(t) &= -b_i(t)\boldsymbol{v}_i(t) + \sum_{j=1}^{N} a\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2\right)(\boldsymbol{x}_j - \boldsymbol{x}_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} f\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2\right)(\boldsymbol{x}_i - \boldsymbol{x}_j), \end{aligned} \tag{3}$$

This model describes the dynamics of N particles/agents with main state $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$ and consensus parameters $\boldsymbol{v} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_N)$, which can be considered space and velocity variables in group motion dynamics. The evolution is governed by an *attraction* force, modeled by a function $a : \mathbb{R} \to [0, +\infty)$, which is, for some fixed constant $H > 0$ and $\beta \geq 0$, of the form

$$a(r) = \frac{H}{(1+r)^{\beta}},$$

though in general any Lipschitz-continuous, nonincreasing function with maximum in $a(0)$ will work. This force is counteracted by a *repulsion* determined by a locally Lipschitz continuous or C^1, nonincreasing function $f : (0, \infty) \to [0, \infty)$. We request that

$$\int_{\delta}^{+\infty} f(r)\, dr < \infty \text{ for every } \delta > 0, \tag{4}$$

$$\int_{0}^{+\infty} f(r)\, dr = +\infty. \tag{5}$$

A typical example of such a function is $f(r) = r^{-p}$ for every $p > 1$. The reason of these requests will be clear after Remark 2. The uniformly continuous, bounded functions $b_i : [0, \infty) \to [0, \Lambda]$, $i = 1, \dots, N$, where $\Lambda \geq 0$, are interpreted as a friction which helps the system to stay confined.

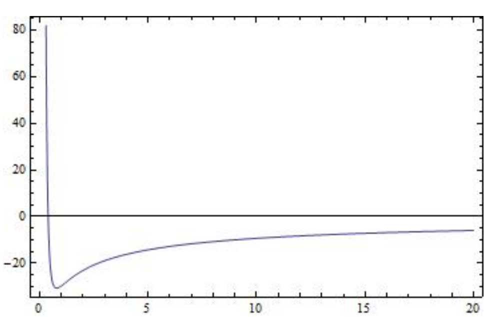

FIGURE 1. Sum of the attraction and repulsion forces governed by the function $h(r) = f(r) - a(r)$ of the distance $r > 0$. The parameters here are $H = 50$, $\beta = 0.7$, and $p = 4$.

At first glance it may seem perhaps a bit cumbersome to consider a rather arbitrary splitting of the force into two terms governed by the functions a and f instead of considering more naturally a unique function $h(r) = f(r) - a(r)$ of the distance $r > 0$, as depicted in Figure 1. However, as we shall clarify later, the interplay of the polynomial decay of the function h to infinity and its singularity at 0 is fundamental in order to be able to characterize the confinement and collision avoidance of the dynamics, and such a splitting, emphasizing the individual role of these two properties, will turn out to be useful in our statements. As a matter of fact, several forces in nature do have similar behavior, for instance the van der Waals forces are governed by Lennard-Jones potentials for which $h(r) = \frac{\sigma_f}{r^{13}} - \frac{\sigma_a}{r^7}$, for suitable positive constants σ_f and σ_a. For such forces the polynomial decay corresponds to the parameter $\beta = 7$, hence a rather fast decay, which is actually supercritical ($\beta > 1$) and it makes the confinement of the system conditional to the initial total energy level (see Theorem 2.1 below).

It is easily seen how the above model can be rewritten as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -L(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{v}b, \end{aligned}$$

where $(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and

$$L(\boldsymbol{x}) := L^a(\boldsymbol{x}) - L^f(\boldsymbol{x}) \tag{6}$$

is the difference between the Laplacians of the matrices $[a\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2\right)]_{i,j=1}^N$ and $[f\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2\right)]_{i,j=1}^N$, respectively, and $\boldsymbol{v}b := [\boldsymbol{v}_i b_i]_{i=1}^N$ for $b = [b_1, \ldots, b_N]$. Notice that, in contrast to the Cucker-Smale system (1), now the Laplacians are acting on the variable $\boldsymbol{x}$ and not anymore on the variable $\boldsymbol{v}$, mixing the dynamics of the two components of the state.

To quantify the behavior of the system we introduce a quantity called the *total energy* which includes the kinetic and potential energies; for all $(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ we define

$$E(\boldsymbol{x}, \boldsymbol{v}) := \sum_{i=1}^N \|\boldsymbol{v}_i\|^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int_0^{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2} a(r)dr + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int_{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2}^\infty f(r)dr. \tag{7}$$

If $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ describes a trajectory in time, $E(t)$ will stand for the *total energy function* $E(\boldsymbol{x}(t), \boldsymbol{v}(t))$.

Remark 1. Every term appearing in the definition of $E(\boldsymbol{x}, \boldsymbol{v})$ is nonnegative (and well-defined by (4)), and hence we are allowed to bound from above every term appearing in the expression of $E(\boldsymbol{x}, \boldsymbol{v})$ (as $\|\boldsymbol{v}_i\|^2$ or $\frac{1}{2}\int_{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2}^\infty f(r)dr$) by $E(\boldsymbol{x}, \boldsymbol{v})$ itself.

The total energy $E(t) = E(\boldsymbol{x}(t), \boldsymbol{v}(t))$ is a Lyapunov functional for the system (3) and, provided we are in presence of no friction at all (i.e., $\Lambda = 0$), it is a conserved quantity. Since the proof of this result, already derived in [9], will be helpful later on, we will report it in full details.

Proposition 1. *For every $t \geq 0$, we have*

$$\frac{d}{dt}E(t) = -2\sum_{i=1}^N b_i \|\boldsymbol{v}_i\|^2.$$

Hence, if $\Lambda = 0$ *then* $\frac{dE}{dt} \equiv 0$.

Proof. Let us compute

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\sum_{i=1}^{N}\|\boldsymbol{v}_i\|^2 + \sum_{i,j=1}^{N} a(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)\langle \boldsymbol{x}_i - \boldsymbol{x}_j, \boldsymbol{v}_i - \boldsymbol{v}_j\rangle \\ &\quad - \sum_{i,j=1}^{N} f(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)\langle \boldsymbol{x}_i - \boldsymbol{x}_j, \boldsymbol{v}_i - \boldsymbol{v}_j\rangle. \end{aligned} \tag{8}$$

The first term of the sum above is

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{N}\|\boldsymbol{v}_i\|^2 &= 2\sum_{i=1}^{N}\langle \dot{\boldsymbol{v}}_i, \boldsymbol{v}_i\rangle \\ &= -2\sum_{i=1}^{N} b_i\|\boldsymbol{v}_i\|^2 - \sum_{i,j=1}^{N} a(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)\langle \boldsymbol{x}_i - \boldsymbol{x}_j, \boldsymbol{v}_i - \boldsymbol{v}_j\rangle \\ &\quad + \sum_{i,j=1}^{N} f(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)\langle \boldsymbol{x}_i - \boldsymbol{x}_j, \boldsymbol{v}_i - \boldsymbol{v}_j\rangle, \end{aligned} \tag{9}$$

which, plugging (9) into (8), yields

$$\frac{dE}{dt} = -2\sum_{i=1}^{N} b_i\|\boldsymbol{v}_i\|^2.$$

So, if $\Lambda = 0$ then $b_i \equiv 0$ for every $i = 1, \dots N$, and thus $\frac{dE}{dt} \equiv 0$, as stated. □

If the attraction force at far distance is very strong (for $\beta \leq 1$), despite an initial high level kinetic energy and repulsion potential energy, perhaps due to a space compression of the group of particles, the dynamics is guaranteed to keep confined and collision avoiding in space at all times. If the attraction force is instead weak at far distance, i.e., $\beta > 1$ is supercritical, then confinement and collision avoidance turn out to be properties of the dynamics at all times, only conditionally to *initial* low levels of kinetic energy and repulsion potential energy, meaning that the particles should not be initially too fast and too close to each other. This latter condition is formulated in terms of a total energy critical threshold

$$\vartheta := (N-1)\int_0^\infty a(r)dr. \tag{10}$$

This fundamental dichotomy of the dynamics has been characterized within the following result from [9].

Theorem 2.1. *Consider a population of* N *agents modeled by system* (3) *with initial position satisfying* $\|\boldsymbol{x}_i(0) - \boldsymbol{x}_j(0)\|^2 > 0$ *for all* $i \neq j$ *and*

$$E(0) < \frac{1}{2}\int_0^{+\infty} f(r)dr. \tag{11}$$

Then there exists a unique solution $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ *of system* (3) *with initial state* $(\boldsymbol{x}(0), \boldsymbol{v}(0))$. *Moreover if one of the two following hypotheses holds:*

1. $\beta \leq 1$;
2. $\beta > 1$ *and* $E(0) < \vartheta$.

Then the population is cohesive and collision-avoiding, i.e., there exist two constants B_0 *and* $b_0 > 0$ *such that, for all* $t \geq 0$

$$b_0 \leq \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| \leq B_0 \text{ for all } 1 \leq i \neq j \leq N. \tag{12}$$

Remark 2. In the case $N = 1$ we are dealing with a single agent with initial position $\boldsymbol{x}(0)$ and initial speed $\boldsymbol{v}(0)$, so by definition $E(0) = \|\boldsymbol{v}(0)\|^2$. Hence condition (11) becomes

$$\|\boldsymbol{v}(0)\| < \frac{1}{2}\int_0^{+\infty} f(r)dr.$$

Since a single particle satisfies trivially condition (12) of Theorem 2.1, no matter how its initial velocity is, and it is governed by a system like (3) *for every* $\beta \geq 0$, then in order Theorem 2.1 to be valid for single agents we have to require that f satisfies condition (5), that is

$$\int_0^{+\infty} f(r)\ dr = \infty.$$

Motivated by Theorem 2.1, we will call the *consensus energy sublevel* or *consensus region* the set

$$C := \{w \in \mathbb{R} \mid w \leq \vartheta\}$$

We will say that the system (3) is *in the consensus region at time* t if $E(t) \in C$.

Let us stress the fact that the word *consensus* must be intended here as a stable *cohesion and collision-avoiding* dynamics, in the spirit of the conclusion of Theorem 2.1. This is in contrast with the common meaning of the word *consensus* in literature, which describes a situation where all the agents move according to the same velocity vector. We point out that our definition of *consensus* does not imply this particular feature, but it is rather intended to make a parallel between Theorem 2.1 and [10, Theorem 2], as already done by Cucker and Dong in [9].

It is an obvious corollary of Theorem 2.1 the fact that if system (3) is in the consensus region at time T, for some $T \geq 0$, then condition (12) is fulfilled for every $t \geq T$.

This brings us eventually to the main scope of this paper. We shall investigate, especially in the case where $\beta > 1$ and $E(0) > \vartheta$, if and under which conditions it is possible to define a control, modeling an external policy maker, able to enforce the system, initially not yet in the consensus region at $t = 0$, to be in the consensus region in finite time. Notice that requiring condition $\beta > 1$ means that the tendency to a confinement is mainly a *local* property, i.e., the dynamics is *homophilious*, the agents bond more with those who are closer rather than those who are far. Inspired by the recent results in [3], stating that homophilious dynamics, despite being prone to produce group splitting and lost confinement, are also very sensitive to parsimonious controls, we shall seek specifically for our feedback control to be *sparse*, i.e., with at most one non-zero active component at every instant. We also prove that this kind of controls are particularly easy to implement in numerical simulations and enjoys optimality properties in terms of fastest convergence to consensus region, i.e., every other deterministic control satisfying condition (13) stated below takes necessarily a larger time with respect to the sparse control to stabilize the system within the consensus region (see Theorem 5.5).

2.2. **Introducing the control.** The control for the system (3) shall be an external field $\boldsymbol{u} : [0, +\infty) \to (\mathbb{R}^d)^N$ such that each component is a measurable functions which satisfies the $\ell_1^N - \ell_2^d$-norm constraint

$$\sum_{i=1}^{N} \|\boldsymbol{u}_i(t)\| \leq M \tag{13}$$

for every $t \geq 0$, for a given positive constant M. This bound models the limited resources given to an external policy maker to influence instantaneously the dynamics. Thus, the time evolution of the state of each agent is given now by the following system, where $i = 1, \ldots, N$:

$$\begin{aligned} &\dot{\boldsymbol{x}}_i(t) = \boldsymbol{v}_i(t) \\ &\dot{\boldsymbol{v}}_i(t) = -b_i(t)\boldsymbol{v}_i(t) + \sum_{j=1}^{N} a\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2\right)(\boldsymbol{x}_j - \boldsymbol{x}_i) + \sum_{\substack{j=1 \\ i\neq j}}^{N} f\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2\right)(\boldsymbol{x}_i - \boldsymbol{x}_j) + \boldsymbol{u}_i, \end{aligned} \tag{14}$$

For this system we define again the total energy function E and the threshold ϑ as in (7) and (10), respectively.

We should design the control to act until $E(T) < \theta$ at some finite time T, and then it should be turned off. Since we start from $E(0) > \theta$, then it is necessary that our control forces the total energy to decrease, for instance by ensuring $\frac{dE}{dt} < 0$. Hence, the following technical result helps us to identify accordingly the form of the control. For now we assume that a Filippov solution of exists [13], later we shall prove it.

Lemma 2.2. *Suppose there exists a solution of the system* (14) *and let E be the total energy function associated to it. Then, for every $t \geq 0$ we have*

$$\frac{d}{dt}E(t) \quad = \quad -2\sum_{i=1}^{N} b_i\|\boldsymbol{v}_i\|^2 + 2\sum_{i=1}^{N}\langle \boldsymbol{u}_i(t), \boldsymbol{v}_i(t)\rangle. \tag{15}$$

Proof. The identity (8) in the proof of Proposition 1 is still valid for a solution to (14), while (9) changes as follows

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^{N}\|\boldsymbol{v}_i\|^2 \quad &= \quad -2\sum_{i=1}^{N} b_i\|\boldsymbol{v}_i\|^2 - \sum_{i,j=1}^{N} a(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)\langle \boldsymbol{x}_i - \boldsymbol{x}_j, \boldsymbol{v}_i - \boldsymbol{v}_j\rangle \\ &\quad + \sum_{i,j=1}^{N} f(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)\langle \boldsymbol{x}_i - \boldsymbol{x}_j, \boldsymbol{v}_i - \boldsymbol{v}_j\rangle + 2\sum_{i=1}^{N}\langle \boldsymbol{u}_i, \boldsymbol{v}_i\rangle, \end{aligned}$$

because of the control term. Inserting this expression into (8) we get

$$\frac{dE}{dt} \quad = \quad -2\sum_{i=1}^{N} b_i\|\boldsymbol{v}_i\|^2 + 2\sum_{i=1}^{N}\langle \boldsymbol{u}_i, \boldsymbol{v}_i\rangle.$$

□

From expression (15), it is clear that the only way our control can act on E in order to push it below the threshold is not anymore according to the mutual distances of the agents or the main state variables $\boldsymbol{x}$, but according to the velocities $\boldsymbol{v}$. We focus accordingly to the following family of controls.

Definition 2.3. Let $(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and $0 \le \varepsilon \le \frac{M}{E(0)}$. We define the *sparse feedback control* $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{v}) = [\boldsymbol{u}_1(\boldsymbol{x}, \boldsymbol{v}), \ldots, \boldsymbol{u}_N(\boldsymbol{x}, \boldsymbol{v})]^T \in (\mathbb{R}^d)^N$ associated to $(\boldsymbol{x}, \boldsymbol{v})$ as

$$\boldsymbol{u}_i(\boldsymbol{x}, \boldsymbol{v}) = \begin{cases} -\varepsilon E(\boldsymbol{x}, \boldsymbol{v}) \frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|} & \text{if } i = \widehat{\iota}(\boldsymbol{x}, \boldsymbol{v}) \\ 0 & \text{if } i \neq \widehat{\iota}(\boldsymbol{x}, \boldsymbol{v}) \end{cases}$$

where $\widehat{\iota}(\boldsymbol{x}, \boldsymbol{v})$ is the minimum index such that

$$\|\boldsymbol{v}_{\widehat{\iota}(\boldsymbol{x},\boldsymbol{v})}\| = \max_{j=1,\ldots,N} \|\boldsymbol{v}_j\|.$$

Whenever the point $(\boldsymbol{x}, \boldsymbol{v})$ is a point of a curve $(\boldsymbol{x}(t), \boldsymbol{v}(t)) : [0, +\infty) \to (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, i.e. $(\boldsymbol{x}, \boldsymbol{v}) = (\boldsymbol{x}(t), \boldsymbol{v}(t))$ for some $t \in [0, +\infty)$, we will replace everywhere $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{u}(\boldsymbol{x}(t), \boldsymbol{v}(t))$ with $\boldsymbol{u}(t)$.

The parameter ε will help us to tune the control in order to ensure the convergence to the consensus region. Moreover, whenever it is clear from the context, we will omit the time (or the space) dependency of $\widehat{\iota}$.

Remark 3. Definition 2.3 makes sense if $\|\boldsymbol{v}_{\widehat{\iota}(t)}(t)\| \neq 0$ for at least almost every $t \geq 0$. Notice that, if the latter condition were not holding, then $\boldsymbol{v}_i(t) = 0$ for all $i = 1, \ldots, N$ and for all $t \geq 0$, hence $\dot{\boldsymbol{v}}_i(t) = 0$ for all $i = 1, \ldots, N$ and for all $t \geq 0$, and the configuration of the system would be in a steady state and no control would be needed.

Remark 4. Besides this latter observation on the well-posedness of Definition 2.3, later we show that actually $\|\boldsymbol{v}_{\widehat{\iota}(t)}(t)\|$ keeps bounded away from 0 in the interval of time where we intend the control to act. In particular in the proof of Theorem 4.1 we prove that if the average speed at time 0 is nonzero, i.e., $\|\overline{\boldsymbol{v}}(0)\| > 0$, and $E(0)$ is not "too far" from the threshold (in a precise sense), then we have $\|\boldsymbol{v}_{\widehat{\iota}(t)}(t)\| > 0$ and $E(t) \le E(0)$ for all t such that the control is needed. Hence, for the choice of $\boldsymbol{u}$ as in Definition 2.3 with parameter $\varepsilon \le \frac{M}{E(0)}$ we have the validity of the constraint (13).

3. Sample-controlled Cucker-Dong systems.

The main goal of this section is to introduce the basic notions of *sampling solution* and *sampling control*, and investigate some properties of controlled Cucker-Dong systems associated with this kind of control strategy. This will pave the way to an existence result for systems modeled by (14) but associated to a class of controls containing properly the one introduced in Definition 2.3.

3.1. Sampling solutions.

Definition 3.1. Let $U \subseteq \mathbb{R}^m$, $f : \mathbb{R}^n \times U \mapsto \mathbb{R}^n$ be continuous and locally Lipschitz in $\boldsymbol{x}$ uniformly on a compact subset of $\mathbb{R}^n \times U$. Given a function $\boldsymbol{u} : \mathbb{R}^n \to U$, $\tau > 0$, and $\boldsymbol{x}_0 \in \mathbb{R}^n$ we define the *sampling solution associated to the sampling time* τ of the differential system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

as the piecewise C^1 function $\boldsymbol{x} : [0, T] \to \mathbb{R}^n$ solving, recursively for $k \geq 0$, the equations

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}(k\tau)), \qquad t \in [k\tau, (k+1)\tau]$$

using as initial value $\boldsymbol{x}(k\tau)$, the endpoint of the solution of the preceding interval, and starting with $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

As anticipated, in the following we focus on systems like (14) associated to the piecewise constant controls with sampling time $\tau > 0$ given by

$$\widetilde{\boldsymbol{u}}_i(t) = \boldsymbol{u}(n\tau) \text{ for every } t \in [n\tau, (n+1)\tau]. \tag{16}$$

where $\boldsymbol{u}$ is a control satisfying Definition 2.3.

We shall show that Cucker-Dong systems associated to these controls possess several monotonicity properties, which will let us conclude that each of these systems satisfying certain *regularity conditions* can be driven in the consensus region in finite time.

3.2. **Properties of controlled Cucker-Dong systems.** Lemma 2.2, together with (16), gives us the following crucial fact concerning the growth of the energy function. Indeed, we can estimate from above the energy function inside every interval $[n\tau, (n+1)\tau]$ by a quadratic function on τ.

Lemma 3.2. *For all $\tau > 0$, for all $n \geq 0$ and for all s, t such that $n\tau \leq s \leq t \leq (n+1)\tau$,*

$$\sqrt{E(t)} \leq \sqrt{E(s)} + \varepsilon E(n\tau)(t-s) \tag{17}$$

Proof. From (15), for all $n\tau \leq t < (n+1)\tau$ we get

$$\begin{aligned} E'(t) &\leq 2\sum_{i=1}^{N}\langle \widetilde{\boldsymbol{u}}_i(t), \boldsymbol{v}_{\hat{\imath}}(t)\rangle \\ &= -2\varepsilon E(n\tau)\frac{\langle \boldsymbol{v}_{\hat{\imath}}(n\tau), \boldsymbol{v}_{\hat{\imath}}(t)\rangle}{\|\boldsymbol{v}_{\hat{\imath}}(n\tau)\|} \\ &\leq 2\varepsilon E(n\tau)\|\boldsymbol{v}_{\hat{\imath}}(t)\| \\ &\leq 2\varepsilon E(n\tau)\sqrt{E(t)} \end{aligned}$$

from Cauchy-Schwarz inequality and the fact that $\|\boldsymbol{v}_i(t)\| \leq \sqrt{E(t)}$ for all i. Hence, integrating between s and t, where $n\tau \leq s \leq t \leq (n+1)\tau$, we get the desired inequality. □

A useful corollary of the lemma above tells us that, if the energy does not increase in the interval $[0, n\tau]$, then the agents does not collide until $t = (n+1)\tau$, i.e., we are able to bound from below the mutual distances of the agents of system (14) on the whole interval $[0, (n+1)\tau]$.

Corollary 1. *There are sufficiently small $\tau_0 > 0$ and $\omega > 0$ such that, for all $\tau < \tau_0$ there exists $d_0 > \omega$ satisfying*

$$\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| \geq d_0 \text{ for all } i, j = 1, \ldots, N, \text{ and } i \neq j. \tag{18}$$

for all $t \in [0, \tau]$. Moreover, if E is nonincreasing in $[0, n\tau]$ ($n \geq 0$), then $d_0 \geq \omega$ can be chosen such that (18) is true for all $t \in [0, (n+1)\tau]$.

Proof. Hypothesis (11) ensures us that there are sufficiently small $\tau_0 > 0$ and $\omega > 0$ such that

$$E(0) < -[(\sqrt{E(0)} + \varepsilon E(0)\tau_0)^2 - E(0)] + \frac{1}{2}\int_{\omega}^{\infty} f(r)dr \tag{19}$$

Take $\tau < \tau_0$; then, from estimate (17) we have that, for all $t \in [0, \tau]$,

$$\sqrt{E(t)} \leq \sqrt{E(0)} + \varepsilon E(0)\tau, \tag{20}$$

so, plugging (20) into (19), we get

$$E(t) < \frac{1}{2}\int_{\omega}^{\infty} f(r)dr.$$

From the definition of E, it follows that, for all $i, j = 1, \dots, N$,

$$\int_{\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|^2}^{\infty} f(r)dr < \int_{\omega}^{\infty} f(r)dr;$$

the thesis then follows from the fact that f is nonincreasing.

Now suppose that E is nonincreasing in $[0, n\tau]$. Using this fact and estimate (17), we get that (20) holds for all $t \in [0, (n+1)\tau]$, and from the same argument as above, we obtain $d_0 \geq \omega$ satisfying (18) for all $t \in [0, (n+1)\tau]$. □

Corollary 2. *If E is nonincreasing in $[0, n\tau]$ $(n \geq 0)$, then the function $L(\boldsymbol{x}(t))$ defined in (6) is bounded from above for every $t \in [0, (n+1)\tau]$.*

Proof. Since the function a is bounded from above by $a(0)$ and, by Corollary 1, $f(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|^2)$ is bounded from above by $f(\omega^2)$ for every $t \in [0, (n+1)\tau]$, then $L(\boldsymbol{x}(t))$ is bounded from above in the whole interval $[0, (n+1)\tau]$. □

The following lemma, again a corollary of Lemma 3.2, says that the mutual distances of the agents grows at most quadratically in time.

Lemma 3.3. *Fix $\tau > 0$ and suppose that*

$$\|\boldsymbol{x}_i(0) - \boldsymbol{x}_j(0)\| \leq C_0\sqrt{E(0)} \text{ for all } i \neq j,$$

for some $C_0 > 0$. Then we get

$$\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| \quad \leq \quad (\tau D(\tau) + C_0)\sqrt{E(0)} \text{ for all } i \neq j \tag{21}$$

for all $t \in [0, \tau]$, where

$$D(\tau) = \frac{M}{\sqrt{E(0)}}\tau + 2.$$

Moreover, for all $k \in \mathbb{N}$, if E is nonincreasing in $[0, k\tau]$ then for all $t \in [k\tau, (k+1)\tau]$

$$\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| \quad \leq \quad ((k+1)\tau D(\tau) + C_0)\sqrt{E(0)} \text{ for all } i \neq j. \tag{22}$$

Proof. The following inequalities follow easily from the equations in (14), the definition of the control, and the estimate (17): for all $k\tau \leq s \leq t \leq (k+1)\tau$

$$\begin{aligned}
\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| &\leq \int_s^t \|\boldsymbol{v}_i(\theta) - \boldsymbol{v}_j(\theta)\|d\theta + \|\boldsymbol{x}_i(s) - \boldsymbol{x}_j(s)\| \\
&\leq \int_s^t (\|\boldsymbol{v}_i(\theta)\| + \|\boldsymbol{v}_j(\theta)\|)d\theta + \|\boldsymbol{x}_i(s) - \boldsymbol{x}_j(s)\| \\
&\leq 2\int_s^t \sqrt{E(\theta)}d\theta + \|\boldsymbol{x}_i(s) - \boldsymbol{x}_j(s)\| \\
&\leq 2\int_s^t (\sqrt{E(s)} + \varepsilon E(n\tau)(\theta - s))d\theta + \|\boldsymbol{x}_i(s) - \boldsymbol{x}_j(s)\| \\
&= 2\sqrt{E(s)}(t-s) + \varepsilon E(n\tau)(t-s)^2 + \|\boldsymbol{x}_i(s) - \boldsymbol{x}_j(s)\|,
\end{aligned}$$

which, taking $s = k\tau$, yields

$$\begin{aligned}
\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| &\leq 2\sqrt{E(k\tau)}(t - k\tau) + \varepsilon E(k\tau)(t-k\tau)^2 + \|\boldsymbol{x}_i(k\tau) - \boldsymbol{x}_j(k\tau)\| \\
&\leq 2\sqrt{E(k\tau)}\tau + \varepsilon E(k\tau)\tau^2 + \|\boldsymbol{x}_i(k\tau) - \boldsymbol{x}_j(k\tau)\|
\end{aligned} \tag{23}$$

Thus, if $k = 0$

$$\begin{aligned}\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| &\leq 2\sqrt{E(0)}\tau + \varepsilon E(0)\tau^2 + \|\boldsymbol{x}_i(0) - \boldsymbol{x}_j(0)\| \\ &\leq (\tau(\varepsilon\sqrt{E(0)}\tau + 2) + C_0)\sqrt{E(0)}\tau \\ &\leq (\tau D(\tau) + C_0)\sqrt{E(0)}\tau.\end{aligned}$$

Now let us prove by induction on k that, if $t \in [k\tau, (k+1)\tau]$ then (22) holds assuming that E is nonincreasing in $[0, k\tau]$. If $k = 0$, then it follows from (21); so suppose (22) holds for all $j \leq k$. Then from (23) we get,

$$\begin{aligned}\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| &\leq 2\sqrt{E(k\tau)}\tau + \varepsilon E(k\tau)\tau^2 + \|\boldsymbol{x}_i(k\tau) - \boldsymbol{x}_j(k\tau)\| \\ &\leq 2\sqrt{E(0)}\tau + \varepsilon E(0)\tau^2 + (k\tau D(\tau) + C_0)\sqrt{E(0)} \\ &\leq ((k+1)\tau D(\tau) + C_0)\sqrt{E(0)}.\end{aligned}$$

□

The last property of the systems under investigation is the fact that if at time $t = 0$ the norm of the average velocity of the system, i.e., of the quantity

$$\overline{\boldsymbol{v}}(t) = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{v}_i(t),$$

is larger than a positive scalar η, then the norm of $\overline{\boldsymbol{v}}(t)$ is bounded from below by η until time $t = T^\star(\eta)$, defined below.

Lemma 3.4. *Suppose that $\|\overline{\boldsymbol{v}}(0)\| > 0$ and let η be a positive estimate from below of $\|\overline{\boldsymbol{v}}(0)\|$, i.e., $\|\overline{\boldsymbol{v}}(0)\| \geq \eta > 0$. Define the following auxiliary time horizon:*

$$T^\star(\eta) := \frac{\|\overline{\boldsymbol{v}}(0)\|^2 - \eta^2}{2\sqrt{E(0)}\left(\Lambda\sqrt{E(0)} + \frac{M}{N}\right)}$$

Furthermore, let $\varepsilon \leq \frac{M}{E(0)}$, fix $\tau > 0$ and $n \leq [T^\star(\eta)/\tau]$. Then, if E is nonincreasing in $[0, n\tau]$ we have

$$\|\overline{\boldsymbol{v}}(t)\| \geq \eta, \tag{24}$$

for every $t \in [0, n\tau]$.

Proof. Let us compute, for all $t \in [k\tau, (k+1)\tau)$ with $k \leq n$, the derivative of $\|\overline{\boldsymbol{v}}(t)\|^2$.

$$\begin{aligned}\frac{d}{dt}\|\overline{\boldsymbol{v}}(t)\|^2 &= 2\left\langle \dot{\overline{\boldsymbol{v}}}(t), \overline{\boldsymbol{v}}(t)\right\rangle \\ &= -\frac{2}{N}\sum_{j=1}^{N} b_j(t)\left\langle \boldsymbol{v}_j(t), \overline{\boldsymbol{v}}(t)\right\rangle - \frac{2\varepsilon E(k\tau)}{N}\frac{\left\langle \boldsymbol{v}_{\widehat{\iota}(k\tau)}(k\tau), \overline{\boldsymbol{v}}(t)\right\rangle}{\|\boldsymbol{v}_{\widehat{\iota}(k\tau)}(k\tau)\|} \\ &\geq -2\Lambda\|\boldsymbol{v}_{\widehat{\iota}(t)}(t)\|^2 - \frac{2\varepsilon E(k\tau)}{N}\|\boldsymbol{v}_{\widehat{\iota}(t)}(t)\|,\end{aligned}$$

using the fact that $\|\boldsymbol{v}_{\widehat{\iota}(t)}(t)\| \geq \max\{\|\boldsymbol{v}_j(t)\|, \|\overline{\boldsymbol{v}}(t)\|\}$ for all t and for all $j = 1, \ldots, N$. Now we use again the inequality $\|\boldsymbol{v}_i(t)\| \leq \sqrt{E(t)}$, valid for all i, to obtain

$$\begin{aligned}\frac{d}{dt}\|\overline{\boldsymbol{v}}(t)\|^2 &\geq -2\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\|\left[\Lambda\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\| + \frac{\varepsilon E(k\tau)}{N}\right]\\ &\geq -2\sqrt{E(t)}\left[\Lambda\sqrt{E(t)} + \frac{\varepsilon E(k\tau)}{N}\right]\\ &\geq -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{\varepsilon E(0)}{N}\right]\\ &\geq -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{M}{N}\right]\end{aligned}$$

since E is nonincreasing in $[0, k\tau]$ for all $k \leq n$ and $\varepsilon \leq \frac{M}{E(0)}$. Furthermore, since the last quantity is constant, by integrating between $k\tau$ and t we get

$$\|\overline{\boldsymbol{v}}(t)\|^2 \geq -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{M}{N}\right](t - k\tau) + \|\overline{\boldsymbol{v}}(k\tau)\|^2. \tag{25}$$

for every $t \in [k\tau, (k+1)\tau]$. We will now prove by induction on $k \leq n$ that

$$\|\overline{\boldsymbol{v}}(t)\|^2 \geq -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{M}{N}\right]t + \|\overline{\boldsymbol{v}}(0)\|^2. \tag{26}$$

for every $t \in [k\tau, (k+1)\tau]$. This is straightforward for $k = 0$, so suppose (26) holds for $k \leq n$ and let us prove it for $k+1$. From (25) and from the inductive hypothesis we get

$$\begin{aligned}\|\overline{\boldsymbol{v}}(t)\|^2 &\geq -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{M}{N}\right](t - k\tau) + \|\overline{\boldsymbol{v}}(k\tau)\|^2\\ &\geq -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{M}{N}\right](t - k\tau)\\ &\quad -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{M}{N}\right]k\tau + \|\overline{\boldsymbol{v}}(0)\|^2\\ &= -2\sqrt{E(0)}\left[\Lambda\sqrt{E(0)} + \frac{M}{N}\right]t + \|\overline{\boldsymbol{v}}(0)\|^2.\end{aligned}$$

Hence, if $t \leq n\tau \leq [T^\star(\eta)/\tau]\tau \leq T^\star(\eta)$, we get

$$\|\overline{\boldsymbol{v}}(t)\|^2 \geq \eta^2.$$

□

Remark 5. We can get a larger interval of validity of (24) by using in the proof above the upper bound $\|\boldsymbol{v}_i(t)\|^2 \leq E(t) - E_p(t)$ instead of $\|\boldsymbol{v}_i(t)\|^2 \leq E(t)$, where

$$E_p(t) := \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^N \int_0^{\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|^2} a(r)dr + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^N \int_{\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|^2}^\infty f(r)dr$$

is the *potential part of* $E(t)$. Then, given $b, B \geq 0$ such that $b \leq \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|^2 \leq B$ for every $i \neq j$ and $t \geq 0$, we can estimate $E_p(t)$ from below by

$$S(b, B) := \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^N \int_0^b a(r)dr + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^N \int_B^\infty f(r)dr.$$

Thus, given b and B as above we get the finer estimate $\|\boldsymbol{v}_i(t)\|^2 \leq E(t) - S(b,B)$, and if we replace $\|\boldsymbol{v}_i(t)\|^2 \leq E(t)$ with it we get

$$T^\star(\eta, b, B) := \frac{\|\overline{\boldsymbol{v}}(0)\|^2 - \eta^2}{2\sqrt{E(0) - S(b,B)}\left(\Lambda\sqrt{E(0) - S(b,B)} + \frac{M}{N}\right)}.$$

Moreover, with the choice $b = 0$ and $B = +\infty$ we get back $T^\star(\eta, b, B) = T^\star(\eta)$, since $S(b,B) = 0$.

For the convenience of the reader, we now list all the highlighted properties of controlled Cucker-Dong systems which will be useful in the proof of the main theorem of the following section:

(1): linear growth estimate of $\sqrt{E(t)}$ (Lemma 3.2);
($2^\dagger$): the distance $\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|$ can be estimated from below by a constant $d_0 \geq \omega$ (Corollary 1);
($3^\dagger$): quadratic growth estimate from above of $\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|$ (Lemma 3.3);
(4): if $\|\overline{\boldsymbol{v}}(0)\| \geq \eta > 0$ then there exists $T > 0$ such that $\|\overline{\boldsymbol{v}}(t)\| \geq \eta$ for every $t \leq T$ (Lemma 3.4), provided that E is not increasing in $[0, T]$.

Each condition marked by a $\dagger$ holds in each starting interval $[0, \tau]$ and in every interval $[0, (n+1)\tau]$ provided that E is nonincreasing in $[0, n\tau]$.

4. **Piecewise-constant sparse controls are effective.** In this section we will prove that our sampling strategy is indeed able to steer Cucker-Dong systems into the consensus region, provided they are not too *unruly*, from the properties listed in the section above.

Since in what follows, η will always be a fraction of $\|\overline{\boldsymbol{v}}(0)\|$, i.e., $\eta = \frac{1}{q}\|\overline{\boldsymbol{v}}(0)\|$ for some $q > 1$, we will set

$$T^\star(q) := T^\star\left(\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\|\right) = \frac{\|\overline{\boldsymbol{v}}(0)\|^2}{2\sqrt{E(0)}\left(\Lambda\sqrt{E(0)} + \frac{M}{N}\right)}\left(1 - \frac{1}{q^2}\right).$$

We can now present the main result of this section.

Theorem 4.1. *Fix $M > 0$. Let $(\boldsymbol{x}_0, \boldsymbol{v}_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that the following hold:*

(a): $0 < \|\boldsymbol{x}_{0i} - \boldsymbol{x}_{0j}\| \leq C_0\sqrt{E(0)}$ *for all $i \neq j$,*
(b): $\|\overline{\boldsymbol{v}}(0)\| > 0$,

and suppose there exists a $q > 1$ satisfying

(c):
$$E(0)\ln\left(\frac{E(0)}{\vartheta}\right) < \frac{2M}{q}\|\overline{\boldsymbol{v}}(0)\|\,T^\star(q).$$

Then there exist $\tau_0 > 0$, $\omega > 0$ and $\varepsilon > 0$ such that for every $\tau \leq \tau_0$

$$\frac{1}{2T^\star(q)\left(\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\| - \mu(\tau, q)\right)}\ln\left(\frac{E(0)}{\vartheta}\right) \leq \varepsilon \leq \frac{M}{E(0)} \tag{27}$$

holds, where for the sake of brevity we have set

$$\mu(\tau, q) = M\Lambda\tau^2 + \left(M + \Lambda\sqrt{E(0)} + (a(0) + f(\omega^2))\left((T^\star(q) + \tau)D(\tau) + C_0\right)N\sqrt{E(0)}\right)\tau. \tag{28}$$

Moreover, the sampling solution of system (14) associated with the control $\boldsymbol{u}$ *of Definition 2.3 with control parameter* ε *satisfying (27), the sampling time* $\tau \leq \tau_0$ *and initial datum* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *reaches the consensus region in finite time* $T^\star(q)$*.*

Proof. Let $q > 1$ such that it satisfies condition (c) of the statement, and take τ_0 and ω sufficiently small to allow us to apply Corollary 1 and to satisfy the inequality

$$\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau_0, q) \geq \frac{E(0)}{2M\, T^\star(q)} \ln\left(\frac{E(0)}{\vartheta}\right). \tag{29}$$

Notice that the existence of such τ_0 and ω follows from condition (c). Furthermore, (29) grants the existence of an $\varepsilon > 0$ for which (27) is satisfied and implies that the following inequalities are true:

$$\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau_0, q) + T^\star(q) D(\tau_0) > 0, \tag{30}$$

$$\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau_0, q) > 0. \tag{31}$$

Moreover, all the above inequalities are satisfied if we substitute τ_0 by any $\tau \leq \tau_0$.

Now, denote with $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ the solution of (14) associated with the control $\boldsymbol{u}$ with ε as in (27), the sampling time $\tau \leq \tau_0$ and initial datum $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. As already pointed out in the proof of Lemma 3.2, for every $t \in [n\tau, (n+1)\tau)$

$$E'(t) \leq -2\varepsilon E(n\tau) \frac{\langle \boldsymbol{v}_{\hat{\iota}}(n\tau), \boldsymbol{v}_{\hat{\iota}}(t)\rangle}{\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\|}.$$

We define

$$\phi(t) = E(n\tau) \frac{\langle \boldsymbol{v}_{\hat{\iota}}(n\tau), \boldsymbol{v}_{\hat{\iota}}(t)\rangle}{\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\|},$$

and we study its derivative with respect to $t \in [n\tau, (n+1)\tau)$. Notice that if E is nonincreasing in $[0, n\tau]$ and if $n\tau \leq T^\star(q)$, then

$$\phi(nt) = E(n\tau)\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\| \geq \frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| E(n\tau)$$

holds because we are under the hypothesis of Lemma 3.4 and we can use the estimate (24). We have the following estimate from below

$$\begin{aligned}
\phi'(t) &= E(n\tau) \frac{\langle \boldsymbol{v}_{\hat{\iota}}(n\tau), \dot{\boldsymbol{v}}_{\hat{\iota}}(t)\rangle}{\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\|} \\
&= \frac{E(n\tau)}{\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\|}\Bigg[- b_{\hat{\iota}}(t)\langle \boldsymbol{v}_{\hat{\iota}}(n\tau), \boldsymbol{v}_{\hat{\iota}}(t)\rangle + \sum_{j=1}^N a\left(\|\boldsymbol{x}_{\hat{\iota}}(t) - \boldsymbol{x}_j(t)\|^2\right) \langle \boldsymbol{v}_{\hat{\iota}}(n\tau), \boldsymbol{x}_j(t) - \boldsymbol{x}_{\hat{\iota}}(t)\rangle \\
&\quad + \sum_{j=1}^N f\left(\|\boldsymbol{x}_{\hat{\iota}}(t) - \boldsymbol{x}_j(t)\|^2\right) \langle \boldsymbol{v}_{\hat{\iota}}(n\tau), \boldsymbol{x}_{\hat{\iota}}(t) - \boldsymbol{x}_j(t)\rangle - \frac{\varepsilon E(n\tau)\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\|^2}{\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\|}\Bigg] \\
&\geq E(n\tau)\Bigg[- b_{\hat{\iota}}(t)\|\boldsymbol{v}_{\hat{\iota}}(t)\| - \sum_{j=1}^N a\left(\|\boldsymbol{x}_{\hat{\iota}}(t) - \boldsymbol{x}_j(t)\|^2\right) \|\boldsymbol{x}_j(t) - \boldsymbol{x}_{\hat{\iota}}(t)\| \\
&\quad - \sum_{j=1}^N f\left(\|\boldsymbol{x}_{\hat{\iota}}(t) - \boldsymbol{x}_j(t)\|^2\right) \|\boldsymbol{x}_{\hat{\iota}}(t) - \boldsymbol{x}_j(t)\| - \varepsilon E(n\tau)\Bigg]
\end{aligned}$$

$$\geq -E(n\tau)\left[\Lambda\|\boldsymbol{v}_{\hat{\imath}}(t)\|+\sum_{j=1}^{N}\left(a(0)+f\left(\|\boldsymbol{x}_{\hat{\imath}}(t)-\boldsymbol{x}_j(t)\|^2\right)\right)\|\boldsymbol{x}_j(t)-\boldsymbol{x}_{\hat{\imath}}(t)\|+\varepsilon E(n\tau)\right]$$

$$\geq -E(n\tau)\left[\Lambda\sqrt{E(t)}+\sum_{j=1}^{N}\left(a(0)+f\left(\|\boldsymbol{x}_{\hat{\imath}}(t)-\boldsymbol{x}_j(t)\|^2\right)\right)\|\boldsymbol{x}_j(t)-\boldsymbol{x}_{\hat{\imath}}(t)\|+\varepsilon E(n\tau)\right],$$

having used the Cauchy-Schwarz inequality and the fact that the damping functions b_i and the attraction force function a are bounded from above.

Now, for the sake of compact notation, set

$$d_{i,j}(t) = \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|\,.$$

By the mean value theorem, there exists $\xi_n \in [n\tau, t]$ such that, using the estimate (17) in Lemma 3.2, we get

$$\phi(t) \geq \phi(n\tau) - \tau E(n\tau)\left[\Lambda\sqrt{E(\xi_n)} + \sum_{j=1}^{N}(a(0) + f(d_{\hat{\imath},j}(\xi_n)^2))d_{\hat{\imath},j}(\xi_n) + \varepsilon E(n\tau)\right]$$

$$\geq E(n\tau)\Bigg[\|\boldsymbol{v}_{\hat{\imath}}(n\tau)\| - (\Lambda\varepsilon E(n\tau))\tau^2 -$$

$$-\left(\left(\varepsilon E(n\tau) + \Lambda\sqrt{E(n\tau)}\right) + \sum_{j=1}^{N}(a(0) + f(d_{\hat{\imath},j}(\xi_n)^2))d_{\hat{\imath},j}(\xi_n)\right)\tau\Bigg]$$

$$\geq E(n\tau)\left[\|\boldsymbol{v}_{\hat{\imath}}(n\tau)\| - \beta(n)\tau^2 - \left(\gamma(n) + \sum_{j=1}^{N}\left(a(0) + f(d_{\hat{\imath},j}(\xi_n)^2)\right)d_{\hat{\imath},j}(\xi_n)\right)\tau\right],$$

where we have estimated ε from above by $\frac{M}{E(0)}$ and, again for the sake of brevity, we have set

$$\beta(k) = \frac{\Lambda M E(k\tau)}{E(0)},$$
$$\gamma(k) = \frac{M E(k\tau)}{E(0)} + \Lambda\sqrt{E(k\tau)}.$$

for every $k \in \mathbb{N}$. So in the end we obtained the estimate

$$E'(t) \leq -2\varepsilon E(n\tau)\left[\|\boldsymbol{v}_{\hat{\imath}}(n\tau)\|-\beta(n)\tau^2-\left(\gamma(n)+\sum_{j=1}^{N}(a(0)+f(d_{\hat{\imath},j}(\xi_n)^2))d_{\hat{\imath},j}(\xi_n)\right)\tau\right],$$

which is valid for all $t \in [n\tau,(n+1)\tau]$ where $n\tau \leq T^\star(q)$.

Now, we will prove by induction on n that E is nonincreasing in $[0,(n+1)\tau]$, where $n\tau \leq T^\star(q)$. First of all, if $n = 0$ then $\xi_0 \in [0,\tau]$, and we can use Corollary 1 and Lemma 3.3 to bound $d_{\hat{\imath},j}(\xi_0)$ from below by ω and from above by $(\tau D(\tau) + C_0)\sqrt{E(0)}$. Moreover, since $\tau \leq T^\star(q)$, we can bound $\|\boldsymbol{v}_{\hat{\imath}}(n\tau)\|$ from below by $\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\|$ by Lemma 3.4. So we get

$$\begin{aligned}E'(t) &\leq -2\varepsilon E(0)\left[\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\|-\beta(0)\tau^2-\left(\gamma(0)+(a(0)+f(\omega^2))(\tau D(\tau)+C_0)N\sqrt{E(0)}\right)\tau\right]\\ &= -2\varepsilon E(0)\left[\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\| - \mu(\tau,q) + T^\star(q)D(\tau)\right]\\ &< 0,\end{aligned}$$

by (30).

Hence suppose that E is nonincreasing in $[0, n\tau]$ where $(n-1)\tau \leq T^\star(q)$, and let us prove that it is also true in $[n\tau, (n+1)\tau]$, if $n\tau \leq T^\star(q)$. By the fact that $\xi_n \geq n\tau$ and by the inductive hypothesis, we can use the bounds from Corollary 1, Lemma 3.3, and Lemma 3.4 as before to get

$$\begin{aligned}
E'(t) &\leq -2\varepsilon E(0)\Bigg[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \beta(0)\tau^2 \\
&\quad - \left(\gamma(0) + (a(0)+f(\omega^2))((n+1)\tau D(\tau) + C_0)N\sqrt{E(0)}\right)\tau\Bigg] \\
&\leq -2\varepsilon E(0)\Bigg[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \beta(0)\tau^2 \\
&\quad - \left(\gamma(0) + (a(0)+f(\omega^2))((T^\star(q)+\tau) D(\tau) + C_0)N\sqrt{E(0)}\right)\tau\Bigg] \\
&\leq -2\varepsilon E(0)\left[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau,q)\right] \\
&< 0,
\end{aligned}$$

by condition (31). It is worth noticing that since E is nonincreasing in $[0, n\tau]$ then

$$\beta(n) \leq \beta(0) \text{ and } \gamma(n) \leq \gamma(0).$$

Hence, E is nonincreasing in $[0, [T^\star(q)/\tau]\tau + 1]$. From this fact follows that for all n such that $n\tau \leq T^\star(q)$ and for all $t \in [n\tau, (n+1)\tau]$ we have $E(t) \leq E(n\tau)$, which gives us, together with Corollary 1, Lemma 3.3, and Lemma 3.4, the following:

$$\begin{aligned}
E'(t) &\leq -2\varepsilon E(n\tau)\left[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \beta(n)\tau^2 - \left(\gamma(n) + (a(0)+f(\omega^2))\sum_{j=1}^N d_{\widehat{\imath},j}(\xi_n)\right)\tau\right] \\
&\leq -2\varepsilon E(t)\left[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau,q)\right]. \qquad (32)
\end{aligned}$$

By integrating between 0 and t we obtain

$$E(t) \leq E(0)e^{-2\varepsilon\left[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau,q)\right]t},$$

so if

$$t \geq \frac{1}{2\varepsilon\left[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau,q)\right]}\ln\left(\frac{E(0)}{\vartheta}\right),$$

then $E(t) \leq \vartheta$ and thus the control steers the system into the consensus region in finite time.

Now we have to ensure that the energy level goes below the threshold before inequality (32) does not hold anymore; clearly, this happens if we impose that

$$T^\star(q) \geq \frac{1}{2\varepsilon\left[\frac{1}{q}\left\|\overline{\boldsymbol{v}}(0)\right\| - \mu(\tau,q)\right]}\ln\left(\frac{E(0)}{\vartheta}\right),$$

and this is true if and only if

$$\varepsilon \geq \frac{1}{2T^\star(q)\left[\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\| - \mu(\tau,q)\right]} \ln\left(\frac{E(0)}{\vartheta}\right),$$

which holds because ε was chosen to satisfy (27).

Moreover, condition (13) is also satisfied since, for all $t \in [n\tau,(n+1)\tau]$

$$\begin{aligned}\sum_{i=1}^N \|\boldsymbol{u}_i(t)\| &\leq \frac{\varepsilon E(n\tau)}{\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\|}\|\boldsymbol{v}_{\hat{\iota}}(n\tau)\| \\ &\leq \varepsilon E(0) \\ &\leq M\end{aligned}$$

which follows from (27) and the fact that E is decreasing in the whole time interval of activity of the control. □

We now give a more operative reformulation of condition (c), which makes the implementation of the control easier.

Proposition 2. *Condition* (c) *of Theorem 4.1 is equivalent to the following request:*

(c')**:**

$$\vartheta > E(0)\exp\left(-\frac{2\sqrt{3}}{9}\frac{M\|\overline{\boldsymbol{v}}(0)\|^3}{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)}+\frac{M}{N}\right)}\right).$$

Moreover, if (c') *is satisfied, one can take*

$$q = \frac{2M\|\overline{\boldsymbol{v}}(0)\|^3}{3E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)}+\frac{M}{N}\right)\ln\left(\frac{E(0)}{\vartheta}\right)}.$$

Proof. Remembering that

$$T^\star(q) = \frac{\|\overline{\boldsymbol{v}}(0)\|^2}{2\sqrt{E(0)}\left(\Lambda\sqrt{E(0)}+\frac{M}{N}\right)}\left(1-\frac{1}{q^2}\right),$$

we may rewrite condition (c) of Theorem 4.1 as

$$E(0)\ln\left(\frac{E(0)}{\vartheta}\right) < \frac{M\|\overline{\boldsymbol{v}}(0)\|^3}{\sqrt{E(0)}\left(\Lambda\sqrt{E(0)}+\frac{M}{N}\right)}\left(\frac{q^2-1}{q^3}\right). \tag{33}$$

If we set

$$K = \frac{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)}+\frac{M}{N}\right)}{M\|\overline{\boldsymbol{v}}(0)\|^3}\ln\left(\frac{E(0)}{\vartheta}\right),$$

we obtain from (33) the equivalent polynomial inequality in q

$$Kq^3 - q^2 < -1.$$

We now show that the request $K < \frac{2\sqrt{3}}{9}$ is equivalent to the existence of a $q > 0$ such that $Kq^3 - q^2 < -1$ holds, and additionally one can choose $q > 1$, as in the statement of the proposition. If we study the derivative of the polynomial $\pi(q) = Kq^3 - q^2$ we find that the stationary points of $\pi(q)$ are $q = 0$ (which is a local maximum) and $\overline{q} = \frac{2}{3K}$ (which is a local minimum). Moreover, since the roots of $\pi(q)$ are $q = 0$ and $q = \frac{1}{K}$ then $\pi(q) \leq 0$ if and only if $q \leq \frac{1}{K}$. Hence, there exists a $q > 0$ such that $\pi(q) < -1$ if and only if $\pi(\overline{q}) < -1$. But

$$\begin{aligned}\pi(\overline{q}) &= (K\overline{q} - 1)\overline{q}^2 \\ &= -\frac{4}{27K^2},\end{aligned}$$

so we have $\pi(\overline{q}) < -1$ if and only if

$$K < \frac{2\sqrt{3}}{9}. \tag{34}$$

The bound (34) trivially implies that $\overline{q} > 1$, and furthermore is equivalent to

$$\ln\left(\frac{E(0)}{\vartheta}\right) < \frac{2\sqrt{3}}{9}\frac{M\|\overline{\boldsymbol{v}}(0)\|^3}{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)} + \frac{M}{N}\right)}.$$

From this, it follows that hypothesis (c) of Theorem 4.1 holds if and only if (c') holds. □

Remark 6. Since we were under the hypothesis that the initial energy was *above* the threshold, from condition (c') we get that our control is effective if

$$E(0) > \vartheta > E(0)\exp\left(-\frac{2\sqrt{3}}{9}\frac{M\|\overline{\boldsymbol{v}}(0)\|^3}{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)} + \frac{M}{N}\right)}\right),$$

which means that the threshold should not be too far away from the initial energy.

5. **Existence of a solution for Cucker-Dong systems.** In order to prove an existence result for (14), as already claimed at the beginning of Section 3, we need to enlarge the set of admissible controls we want to work with. Indeed the family of sparse controls introduced in Definition 2.3 is too tight because, as the sampling time goes to 0, we may lose the sparsity we have enforced on our controls, and we may have to drop it to gain convergence for the sequence of controls.

5.1. **Enlarging the set of controls.** In what follows, we fix a Cucker-Dong system (14) satisfying conditions $(a)-(c)$ of Theorem 4.1, and we indicate with $E(0)$ its initial energy and with

$$\gamma_q = \frac{\|\overline{\boldsymbol{v}}(0)\|}{q},$$

where q is as in Proposition 2.

Each value of γ_q yields a partition of $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ into four disjoint sets:

$\mathcal{P}_1$**:** $= \{(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1\leq i\leq N}\|\boldsymbol{v}_i\| < \gamma_q\}$,

$\mathcal{P}_2$**:** $= \{(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1\leq i\leq N}\|\boldsymbol{v}_i\| = \gamma_q$ and $\exists k \geq 1$ and $i_1, \ldots, i_k \in \{1, \ldots N\}$ such that $\|\boldsymbol{v}_{i_1}\| = \ldots = \|\boldsymbol{v}_{i_k}\|$ and $\|\boldsymbol{v}_{i_1}\| > \|\boldsymbol{v}_j\|$ for every $j \notin \{i_1, \ldots, i_k\}\}$,

$\mathcal{P}_3$**:** $= \{(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1\leq i\leq N}\|\boldsymbol{v}_i\| > \gamma_q$ and $\exists! i \in \{1, \ldots N\}$ such that $\|\boldsymbol{v}_i\| > \|\boldsymbol{v}_j\|$ for every $j \neq i\}$,

$\mathcal{P}_4$**:** $= \{(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1\leq i\leq N}\|\boldsymbol{v}_i\| > \gamma_q$ and $\exists k > 1$ and $i_1, \ldots, i_k \in \{1, \ldots N\}$ such that $\|\boldsymbol{v}_{i_1}\| = \ldots = \|\boldsymbol{v}_{i_k}\|$ and $\|\boldsymbol{v}_{i_1}\| > \|\boldsymbol{v}_j\|$ for every $j \notin \{i_1, \ldots, i_k\}\}$,

Definition 5.1. For every $(\boldsymbol{x},\boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ we denote with $U(\boldsymbol{x},\boldsymbol{v}) \subseteq (\mathbb{R}^d)^N$ the set of all vectors $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{v}) = (\boldsymbol{u}_1(\boldsymbol{x},\boldsymbol{v}),\ldots,\boldsymbol{u}_N(\boldsymbol{x},\boldsymbol{v}))$, whose vector entries are of the form

$$\boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{v}) = \begin{cases} -\varepsilon_i E(\boldsymbol{x},\boldsymbol{v})\frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|} & \text{if } \|\boldsymbol{v}_i(t)\| \neq 0 \\ 0 & \text{if } \|\boldsymbol{v}_i(t)\| = 0 \end{cases}$$

where $\varepsilon_i \geq 0$ for every $i = 1,\ldots,N$, and

$$\sum_{i=1}^N \varepsilon_i \leq \frac{M}{E(0)},$$

such that:

- if $(\boldsymbol{x},\boldsymbol{v}) \in \mathcal{P}_1$ then $\varepsilon_i = 0$ for every $i = 1,\ldots,N$;
- if $(\boldsymbol{x},\boldsymbol{v}) \in \mathcal{P}_2$ then indicating with $i_1,\ldots,i_k$ the indexes such that $\|\boldsymbol{v}_{i_1}\| = \ldots = \|\boldsymbol{v}_{i_k}\| = \gamma_q$ and $\|\boldsymbol{v}_{i_1}\| > \|\boldsymbol{v}_j\|$ for every $j \notin \{i_1,\ldots,i_k\}$, we have $\varepsilon_j = 0$ for every $j \notin \{i_1,\ldots,i_k\}$;
- if $(\boldsymbol{x},\boldsymbol{v}) \in \mathcal{P}_3$ then, indicating with i the only index such that $\|\boldsymbol{v}_i\| > \|\boldsymbol{v}_j\|$ for every $j \neq i$, we have $\varepsilon_i = \frac{M}{E(0)}$ and $\varepsilon_j = 0$ for every $j \neq i$;
- if $(\boldsymbol{x},\boldsymbol{v}) \in \mathcal{P}_4$ then, indicating with $i_1,\ldots,i_k$ the indexes such that $\|\boldsymbol{v}_{i_1}\| = \ldots = \|\boldsymbol{v}_{i_k}\|$ and $\|\boldsymbol{v}_{i_1}\| > \|\boldsymbol{v}_j\|$ for every $j \notin \{i_1,\ldots,i_k\}$, we have $\varepsilon_j = 0$ for every $j \notin \{i_1,\ldots,i_k\}$ and $\sum_{\ell=1}^k \varepsilon_{i_\ell} = \frac{M}{E(0)}$.

Remark 7. For every $0 \leq \varepsilon \leq \frac{M}{E(0)}$ and for every $t \geq 0$, the control $\boldsymbol{u}(t)$ introduced in Definition 2.3 associated to ε belongs to $U(\boldsymbol{x}(t),\boldsymbol{v}(t))$ whenever $t \leq T^\star(q)$, since Lemma 3.4 guarantees $\max_{i\leq i\leq N}\|\boldsymbol{v}_i(t)\| \geq \gamma_q$ for every $t \in [0, T^\star(q)]$.

The set $U(\boldsymbol{x},\boldsymbol{v})$ is closed and convex, and, moreover, has the following very elegant alternative variational interpretation.

Proposition 3. *For every $(\boldsymbol{x},\boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and for every $M \geq 0$, set*

$$M(\boldsymbol{x},\boldsymbol{v}) := M\cdot\frac{E(\boldsymbol{x},\boldsymbol{v})}{E(0)}, \qquad K := \left\{\boldsymbol{u} \in (\mathbb{R}^d)^N \mid \sum_{i=1}^N \|\boldsymbol{u}_i\| \leq M(\boldsymbol{x},\boldsymbol{v})\right\},$$

and let $\mathcal{J} : (\mathbb{R}^d)^N \to \mathbb{R}$ be the functional defined by

$$\mathcal{J}(\boldsymbol{u}) = \langle \boldsymbol{v},\boldsymbol{u}\rangle + \gamma_q \sum_{i=1}^N \|\boldsymbol{u}_i\|. \tag{35}$$

Then

$$U(\boldsymbol{x},\boldsymbol{v}) = \operatorname*{argmin}_{\boldsymbol{u}\in K} \mathcal{J}(\boldsymbol{u}).$$

Proof. Without loss of generality one can minimize (35) componentwise, i.e., considering the minimization of the function

$$\boldsymbol{u}_i \mapsto \langle \boldsymbol{v}_i,\boldsymbol{u}_i\rangle + \gamma_q \|\boldsymbol{u}_i\|,$$

for all $i = 1,\cdots,N$ individually. If $\|\boldsymbol{v}_i\| < \gamma_q$ then, by Cauchy-Schwarz, we infer that

$$\langle \boldsymbol{v}_i,\boldsymbol{u}_i\rangle + \gamma_q \|\boldsymbol{u}_i\| \geq 0, \text{ for all } \boldsymbol{u} \in K,$$

and in particular the choice $\boldsymbol{u}_i = 0$ minimizes this component. Otherwise, if $\|\boldsymbol{v}_i\| \geq \gamma_q$ then, due to isotropy of the norm $\|\boldsymbol{u}_i\|$ the minimal direction is the opposite to $\boldsymbol{v}_i$, i.e., for $\boldsymbol{u}_i = -\alpha_i\frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|}$, for some $\alpha_i \geq 0$, or, equivalently, there exists $\varepsilon_i \geq 0$

such that $\alpha_i = \varepsilon_i E(\boldsymbol{x}, \boldsymbol{v})$. Hence, now the problem is how to allocate the weights ε_i in order to keep $\boldsymbol{u}$ within K and simultaneously to minimize $\mathcal{J}$. Clearly, for $\boldsymbol{u}$ to be in K, the choice of ε_i has to be such that $\sum_{\{i:\|\boldsymbol{v}_i\|\geq\gamma_q\}} \varepsilon_i \leq \frac{E(0)}{M}$. To conclude, from a direct check of the individual cases depending on the partition $\mathcal{P}_\ell\ (\boldsymbol{x}, \boldsymbol{v})$, $\ell = 1, \ldots, 4$, we can easily infer that $U(\boldsymbol{x}, \boldsymbol{v})$ is the set of minimizers of $\mathcal{J}$ over the set K. □

Definition 5.2. A set-valued function F from a topological space X to the subset of a real vector space Y is said to be *upper hemicontinuous at* $x_0 \in X$ if and only if for every continuous linear functional $p : Y \to \mathbb{R}$ the function

$$x \longmapsto \sup_{u \in F(x)} p(u)$$

is upper semicontinuous at $x_0 \in X$. F is called *upper hemicontinuous* if it is upper hemicontinuous at every $x_0 \in X$.

Proposition 4. *The function* $U : (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \to (\mathbb{R}^d)^N$ *which associates to every point* $(\boldsymbol{x}, \boldsymbol{v})$ *the set* $U(\boldsymbol{x}, \boldsymbol{v})$ *is upper hemicontinuous.*

Proof. We have to prove that for every continuous linear functional p from $(\mathbb{R}^d)^N$ to $\mathbb{R}$, the function $\sigma : (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \to \mathbb{R}$ defined as follows

$$\sigma :\ (\boldsymbol{x}, \boldsymbol{v})\ \longmapsto \sup_{\boldsymbol{u} \in U(\boldsymbol{x}, \boldsymbol{v})} p(\boldsymbol{u})$$

is upper semicontinuous, i.e. for every $(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{N}$ of $(\boldsymbol{x}, \boldsymbol{v})$ such that $\sigma(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}}) \leq \sigma(\boldsymbol{x}, \boldsymbol{v}) + \varepsilon$ for all $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}}) \in \mathcal{N}$ whenever $\sigma(\boldsymbol{x}, \boldsymbol{v}) > -\infty$, and $\sigma(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}})$ tends to $-\infty$ as $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}})$ tends to $(\boldsymbol{x}, \boldsymbol{v})$ whenever $\sigma(\boldsymbol{x}, \boldsymbol{v}) = -\infty$. First of all, notice that since $U(\boldsymbol{x}, \boldsymbol{v})$ is closed and bounded, then it is compact and hence $\sigma(\boldsymbol{x}, \boldsymbol{v}) > -\infty$.

Hence fix a continuous linear functional p from $(\mathbb{R}^d)^N$ to $\mathbb{R}$, $(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N$ and $\varepsilon > 0$. From Definition 5.1, an entry $\boldsymbol{u}_j$ of $\boldsymbol{u} \in U(\boldsymbol{x}, \boldsymbol{v})$ is zero if and only if $\|\boldsymbol{v}_j\| < \max\{\gamma_q, \max_{i\leq i\leq N} \|\boldsymbol{v}_i\|\}$, so if the latter condition holds, by the continuity of $\|\cdot\|$ we can find a neighborhood $\mathcal{N}$ of $(\boldsymbol{x}, \boldsymbol{v})$ such that for every $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}}) \in \mathcal{N}$ we have

$$\|\widetilde{\boldsymbol{v}}_j\| < \max\left\{\gamma_q, \max_{i\leq i\leq N} \|\widetilde{\boldsymbol{v}}_i\|\right\}.$$

Thus, since given any two vectors $\boldsymbol{u}, \boldsymbol{w} \in U(\boldsymbol{x}, \boldsymbol{v})$, $\boldsymbol{u}_i$ is zero if and only if $\boldsymbol{w}_i$ is, this implies that for every $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}}) \in \mathcal{N}$ and for every $\widetilde{\boldsymbol{u}} \in U(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}})$, the component $\widetilde{\boldsymbol{u}}_i$ is zero only if $\boldsymbol{u}_i$ is zero for some $\boldsymbol{u} \in U(\boldsymbol{x}, \boldsymbol{v})$.

Moreover, if $\|\boldsymbol{v}_{i_1}\| = \ldots = \|\boldsymbol{v}_{i_k}\| = \max_{i\leq i\leq N} \|\boldsymbol{v}_i\|$ and for some $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}}) \in \mathcal{N}$ there exists $1 \leq h \leq k$ such that

$$\max\left\{\left\|\widetilde{\boldsymbol{v}}_{i_{j_1}}\right\|, \ldots, \left\|\widetilde{\boldsymbol{v}}_{i_{j_h}}\right\|\right\} < \left\|\widetilde{\boldsymbol{v}}_{i_{\ell_1}}\right\| = \ldots = \left\|\widetilde{\boldsymbol{v}}_{i_{\ell_{k-h}}}\right\| = \max_{i\leq i\leq N} \|\widetilde{\boldsymbol{v}}_i\|,$$

for $i_{\ell_s} \neq i_{j_r}$, for all $s = 1, \ldots, k-h$ and $r = 1, \ldots, h$, Then $\widetilde{\boldsymbol{u}}_{i_{j_1}} = \ldots = \widetilde{\boldsymbol{u}}_{i_{j_h}} = 0$, while $\widetilde{\boldsymbol{u}}_{i_{\ell_s}} \neq 0$ for every $1 \leq s \leq k-h$, for all $\widetilde{\boldsymbol{u}} \in U(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}})$.

These two facts together bring us to the conclusion that $U[\mathcal{N}] = U(\boldsymbol{x}, \boldsymbol{v})$, so for every $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}}) \in \mathcal{N}$, $\sigma(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{v}}) = \sigma(\boldsymbol{x}, \boldsymbol{v}) < \sigma(\boldsymbol{x}, \boldsymbol{v}) + \varepsilon$, proving the upper semicontinuity of σ. □

Remark 8. The problem with the family of controls introduced in Definition 2.3 is that if we were to take $U(\boldsymbol{x}, \boldsymbol{v}) = \{\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{v})\}$ instead of the set in Definition 5.1, where $\boldsymbol{u}$ is as in Definition 2.3, then U would not have been upper hemicontinuous.

5.2. **Existence of solutions for the differential inclusion.** We are going to use the following convergence result concerning hemicontinuous set-valued functions.

Theorem 5.3 ([2], Theorem 1 pag.60)**.** *Let F be a proper hemicontinuous map from a Hausdorff locally convex space X to the closed convex subsets of a Banach space Y. Let I be an interval of $\mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be measurable functions from I to X and Y respectively, satisfying*

i. for almost every $t \in I$ and for every neighborhood $\mathcal{N}$ of 0 in $X \times Y$ there exists n_0 such that for every $n \geq n_0$, $y_n(t) \in F(x_n(t)) + \mathcal{N}$;
ii. $(x_n)_{n\in\mathbb{N}}$ converges almost everywhere to a function $x : I \to X$;
iii. $(y_n)_{n\in\mathbb{N}} \subseteq L^1(I, Y)$ and converges weakly to y in $L^1(I, Y)$.

Then for almost every $t \in I$

$$y(t) \in F(x(t)).$$

Theorem 5.4. *For every $M > 0$ and for every initial condition $(\boldsymbol{x}_0, \boldsymbol{v}_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, if system (14) satisfies conditions $(a) - (c)$ of Theorem 4.1 then there exist $q > 1$, $\tau_0 > 0$, $\omega > 0$, and $\varepsilon > 0$ such that*

$$\frac{1}{2T^\star(q)\left(\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\| - \mu(\tau_0, q)\right)} \ln\left(\frac{E(0)}{\vartheta}\right) \leq \varepsilon \leq \frac{M}{E(0)} \tag{36}$$

holds, where $\mu(\tau, q)$ is as in (28).

Moreover, for every $t \leq T^\star(q)$ there exists a solution of

$$\boldsymbol{z}(t) = \boldsymbol{z}_0 + \int_0^t (g(\boldsymbol{z}(s)) + \boldsymbol{u}(\boldsymbol{z}(s)))\, ds, \quad \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v}), \quad g(\boldsymbol{z}) = (\boldsymbol{v}, -L(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{v}b)$$

associated with a control $\boldsymbol{u}$ such that $\boldsymbol{u}(t) \in U(\boldsymbol{z}(t))$ for every $t \leq T^\star(q)$. Here we use the compact notation introduced in (6).

Proof. Apply Theorem 4.1 to get $q > 1$, $\tau_0 > 0$, $\omega > 0$, and $\varepsilon > 0$ satisfying (36). Notice that, if $\tau \leq \tau_0$, then

$$\frac{1}{\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\| - \mu(\tau, q)} < \frac{1}{\frac{1}{q}\|\overline{\boldsymbol{v}}(0)\| - \mu(\tau_0, q)}$$

which implies that ε satisfies also (27) for every $\tau \geq \tau_0$.

Hence, for every $n > 1/\tau_0$, Theorem 4.1 guarantees the existence of a sampling solution $\boldsymbol{z}_n$ of system (14) associated with the feedback control $\boldsymbol{u}$ satisfying Definition 2.3, the sampling time $\tau = 1/n$ and initial datum $\boldsymbol{z}_0 = (\boldsymbol{x}_0, \boldsymbol{v}_0)$. Setting $\widetilde{\boldsymbol{u}}_n(t)$ to be the $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ vector $[0, \ldots, 0, \boldsymbol{u}(\boldsymbol{z}([nt]/n))]^T$, we have that

$$\boldsymbol{z}_n(t) = \boldsymbol{z}_0 + \int_0^t (g(\boldsymbol{z}_n(s)) + \widetilde{\boldsymbol{u}}_n(s))\, ds. \tag{37}$$

By Corollary 2 the function $L(\boldsymbol{x}(t))$ defined in (6) is bounded in the interval $[0, T^\star(q)]$, thus there is a constant α such that the following linear growth estimate for g holds in $[0, T^\star(q)]$:

$$\|g(\boldsymbol{z})\| \leq \alpha(\|\boldsymbol{z}\| + 1).$$

Integrating (37) we get that

$$\|\boldsymbol{z}_n(t)\| \leq \frac{e^{\alpha t}(\alpha\|\boldsymbol{z}_0\| + \alpha + M) - \alpha - M}{\alpha},$$

is true in $[0, T^\star(q)]$, hence the sequence of piecewise C^1 functions $(\boldsymbol{z}_n)_{n\in\mathbb{N}}$ is pointwise equibounded by the function

$$C(t) = \frac{e^{\alpha t}(\alpha\|\boldsymbol{z}_0\| + \alpha + M) - \alpha - M}{\alpha}$$

for every $t \in [0, T^\star(q)]$. Moreover, the sequence $(\boldsymbol{z}_n)_{n\in\mathbb{N}}$ is also equicontinuous in $[0, T^\star(q)]$ since for every $t, s \in [0, T^\star(q)]$ such that $t \leq s$,

$$\|\boldsymbol{z}_n(t) - \boldsymbol{z}_n(s)\| \quad \leq \quad \int_s^t (\|g(\boldsymbol{z}_n(\theta))\| + M)\ d\theta \leq (t-s)(\alpha(C(T^\star(q)) + 1) + M).$$

Thus an application of the Ascoli-Arzelá Theorem yields a subsequence which we rename again as $(\boldsymbol{z}_n)_{n\in\mathbb{N}}$ converging uniformly to a continuous function $\boldsymbol{z} : [0, T^\star(q)] \to (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and, by the continuity of g, $(g(\boldsymbol{z}_n))_{n\in\mathbb{N}}$ converges to $g(\boldsymbol{z})$. Concerning the sequence of controls $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, since each $\boldsymbol{u}_n$ has norm bounded by M, it is a uniformly integrable sequence; hence it satisfies the hypothesis of the Dunford-Pettis Theorem (see, e.g., [1, Theorem 1.38]), which means that, up to subsequences, it converges weakly in L^1 to a function $\boldsymbol{u} \in L^1([0, T^\star(q)])$.

From Proposition 4, by taking $F = U$, $I = [0, T^\star(q)]$, $(x_n)_{n\in\mathbb{N}} = (\boldsymbol{z}_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} = (\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $x = \boldsymbol{z}$ and $y = \boldsymbol{u}$, we are under the hypothesis of Theorem 5.3 and so we can conclude that $\boldsymbol{u}(t) \in U(\boldsymbol{z}(t))$ for every $t \leq T^\star(q)$ (since by Remark 7 we can guarantee $\boldsymbol{u}_n(t) \in U(\boldsymbol{z}_n(t))$ only if $t \leq T^\star(q)$). Since $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges weakly to $\boldsymbol{u}$ in $L^1([0, T^\star(q)])$, in particular we have

$$\lim_{n\to\infty} \int_0^t \boldsymbol{u}_n(s)\ ds = \int_0^t \boldsymbol{u}(s)\ ds$$

for every $t \leq T^\star(q)$. Hence passing to the limit in (37) we conclude the proof. $\square$

Corollary 3. *For every $M > 0$ and every $(\boldsymbol{x}, \boldsymbol{v}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ define*

$$F(\boldsymbol{x}, \boldsymbol{v}) = \{(\boldsymbol{v}, -L(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{v}b + \boldsymbol{u}) \mid \boldsymbol{u} \in U(\boldsymbol{x}, \boldsymbol{v})\}.$$

Then for every $(\boldsymbol{x}_0, \boldsymbol{v}_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ if conditions $(a) - (c)$ of Theorem 4.1 are satisfied, the differential inclusion

$$(\dot{\boldsymbol{x}}, \dot{\boldsymbol{v}}) \in F(\boldsymbol{x}, \boldsymbol{v})$$

with initial condition $(\boldsymbol{x}(0), \boldsymbol{v}(0)) = (\boldsymbol{x}_0, \boldsymbol{v}_0)$ is well-posed as long as $0 \leq t \leq T^\star(q)$.

5.3. **Decay rate estimate for sparse control strategy and instantaneous optimality.** Let us stress very much that in general the existence of a solution associated to a sparse control as in Definition 2.3 does not follow from Corollary 3, as we can say only that solutions exists within the class of feedback controls $U(\boldsymbol{x}, \boldsymbol{v})$, which is much larger and contains nonsparse controls as well. Nevertheless, in what follows, we do suppose there exists a solution to system (14) associated to the control introduced in Definition 2.3 with $\varepsilon = \frac{M}{E(0)}$, and we analyze what could be the corresponding rate of convergence to the consensus region, compared with any other solution within the class of feedback controls $U(\boldsymbol{x}, \boldsymbol{v})$.

Proposition 5. *If $\|\overline{\boldsymbol{v}}(0)\| \geq \eta > 0$ then, for every $0 \leq t \leq T^\star(\eta)$,*

$$E(t) \leq E(0)e^{-\frac{2\eta}{E(0)}Mt}.$$

Proof. Before we start, notice that we can prove a continuous-time analogous of Lemma 3.4 in the same way, thus obtaining the fact that if $\|\overline{\boldsymbol{v}}(0)\| \geq \eta > 0$ then $\|\overline{\boldsymbol{v}}(t)\| \geq \eta$ for every $t \leq T^\star(\eta)$. Using Lemma 2.2 together with this fact we get

$$\begin{aligned}\frac{d}{dt}E(t) &\leq 2\left\langle -\varepsilon E(t)\frac{\boldsymbol{v}_{\widehat{\imath}(t)}(t)}{\left\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\right\|}, \boldsymbol{v}_{\widehat{\imath}(t)}(t)\right\rangle \\ &= -2\varepsilon E(t)\left\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\right\| \\ &\leq -\frac{2\eta M}{E(0)}E(t),\end{aligned}$$

thus integrating between 0 and t we get the desired estimate. □

The next result shows that this decay estimate is the best we can get among the controls introduced in Definition 5.1.

Theorem 5.5. *The feedback control of Definition 2.3 associated to the solution of Theorem 4.1 is an instantaneous minimizer of*

$$\mathcal{D}(t, \boldsymbol{u}) = \frac{d}{dt}E(t)$$

over all possible feedback controls in $U(\boldsymbol{x}(t), \boldsymbol{v}(t))$.

Proof. We have already proven in Lemma 2.2 that

$$\frac{d}{dt}E(t) = -2\sum_{i=1}^{N} b_i(t)\|\boldsymbol{v}_i(t)\|^2 + 2\sum_{i=1}^{N}\langle \boldsymbol{u}_i(t), \boldsymbol{v}_i(t)\rangle,$$

so in order to minimize $\mathcal{D}(t, \boldsymbol{u})$ with respect to $\boldsymbol{u} \in U(\boldsymbol{x}(t), \boldsymbol{v}(t))$ we have to work on the second term. Now, since

$$\sum_{i=1}^{N}\langle \boldsymbol{u}_i(t), \boldsymbol{v}_i(t)\rangle = -E(t)\sum_{i=1}^{N}\varepsilon_i\|\boldsymbol{v}_i(t)\|,$$

minimizing $\mathcal{D}(t, \boldsymbol{u})$ is equivalent to solve

$$\text{maximize } \sum_{i=1}^{N}\varepsilon_i\|\boldsymbol{v}_i(t)\|, \quad \text{subject to } \sum_{i=1}^{N}\varepsilon_i \leq \frac{M}{E(0)}. \tag{38}$$

By definition

$$\begin{aligned}\sum_{i=1}^{N}\varepsilon_i\|\boldsymbol{v}_i(t)\| &\leq \left\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\right\|\sum_{i=1}^{N}\varepsilon_i \\ &\leq \frac{M}{E(0)}\left\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\right\|,\end{aligned}$$

which means that the choice of the control made in Definition 2.3 is a maximizer of (38). Moreover, the solution of the problem is unique whenever $\left\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\right\| > \|\boldsymbol{v}_j(t)\|$ for all $j \neq \widehat{\imath}(t)$. □

Let us briefly comment this last result. Its meaning is that, for the kind of systems we are considering here, a feedback stabilization is most effective if all the attention of the controller is focused on very few, actually in this case only one, agents at each switching time. This also means that, despite the fact that the external policy maker may have few resources at disposal and can allocate them at each time only on very few key players in the system, it is possible to effectively stabilize the dynamics to return to energy levels where the dynamics keeps bounded and collision avoiding. One of the most relevant differences with respect to [3] though is that for the Cucker and Smale model the stabilization can be achieved unconditionally, i.e., independently of the initial conditions. For the Cucker and Dong model analyzed in the present paper, a similar strategy yields only a conditional results, i.e., we obtain stabilization conditionally to an initial energy level

$$\vartheta < E(0) < c\vartheta,$$

for a constant $c > 1$, as stated in condition (c') of Proposition 2. Our numerical experiments, which follow below, suggest that it is possible to exceed such an upper energy barrier, but it is unclear whether this is just a matter of fortunate choices of good initial conditions or we can actually have a broader stabilization range than the one analytically derived in the previous sections.

6. **Numerical experiments.** In this section we will report the results of significant numerical simulations on Cucker-Dong systems in dimension $d = 2$ with and without the use of our sparse control strategy outlined in Definition 2.3. Throughout the section, we will keep fixed the number of agents ($N = 8$), the friction applied ($\Lambda = 0$, i.e. frictionless) and the form of the repulsive function ($f(r) = r^{-p}$). We restrict only to $N = 8$ simply for an easier visualization of the results. This means that we will vary the shape of the function a (i.e., we will act on β), the slope of the repulsion function (changing the value of p) and the maximum amount of strength of the sparse control (the parameter M). The parameter ε is always set equal to $\frac{M}{E(0)}$.

6.1. **First case study: $\beta = 1.1$ and $p = 2$.** Figure 2 displays the spatial evolution and speeds of the agents of a Cucker-Dong system with $\beta = 1.1$ and $p = 2$:

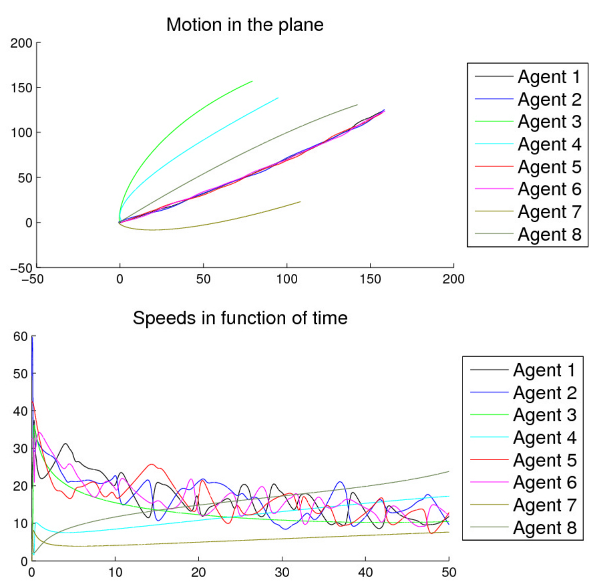

FIGURE 2. Space evolution and speeds of the uncontrolled system.

Though we can not infer the divergence of the system from this finite time simulation, the portrayed situation seems far from going towards a flocking behavior. The only agents which seem to flock are Agent 1, Agent 2, Agent 5 and Agent 6 (resp. black, blue, red and magenta trajectories), as it is also visible by the corresponding speed graph, in which the speed of each agent is adjusted to the one of the other agents.

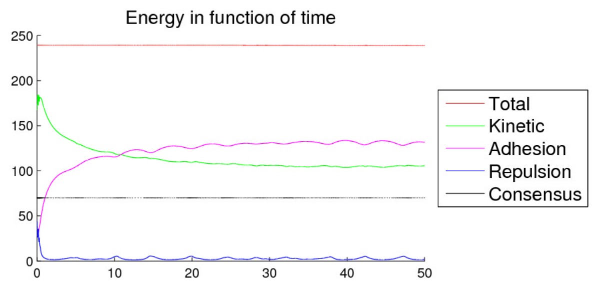

FIGURE 3. Energy profile of the uncontrolled system.

Figure 3 shows that the total energy E (the red line) is constant and far away from the consensus threshold ϑ (black line). The increase in the distances between particles is reflected in an increase in the adhesion potential energy (the one due to a, see (7)) and in a decrease in the repulsive one (due to f).

If instead we apply our sparse control strategy with $M = 35$ on the same system with the same initial conditions, the situation gets immediately far better from a consensus point of view, as we can witness from Figure 4.

The spatial evolution graph shows a braid movement which resembles a pattern near to flocking as it is commonly interpreted. The action of our control is evident from the energy profile of the system, where the total energy is driven below the threshold in a very short time. The fall of the total energy is mainly due to its kinetic part (the green line), which is the only one directly affected by our control strategy. The sharp decrease of the kinetic energy is also witnessed in the graph showing the modulus of the speeds, where, after a quick, strong brake at the beginning, they stabilize at a very low level.

6.2. **Second case study: $\beta = 1.02$ and $p = 1.1$.** The second case study takes into account a system with a weaker communication rate than before ($\beta = 1.02$) and with a different form of the repulsive function ($p = 1.1$), and we apply on it our control strategy with several values for M.

The top-left corner of Figure 5 is the uncontrolled system: it seems legitimate to suppose that flocking is very unlikely to happen in the future, even because of its energy profile graph (top-left corner of Figure 6), which witnesses an increase in the adhesion potential energy, phenomenon associated to an increase in the distance between particles, as already pointed out. To its right there is the spatial evolution graph of the same system but with our sparse control strategy acting with parameter $M = 0.1$, where the agents are starting to converge to consensus, as is also evident in their energy profile.

The two bottom figures of Figure 5 display the action of controls with $M = 1$ and $M = 10$, respectively. It is clear how the situation goes better as M increases,

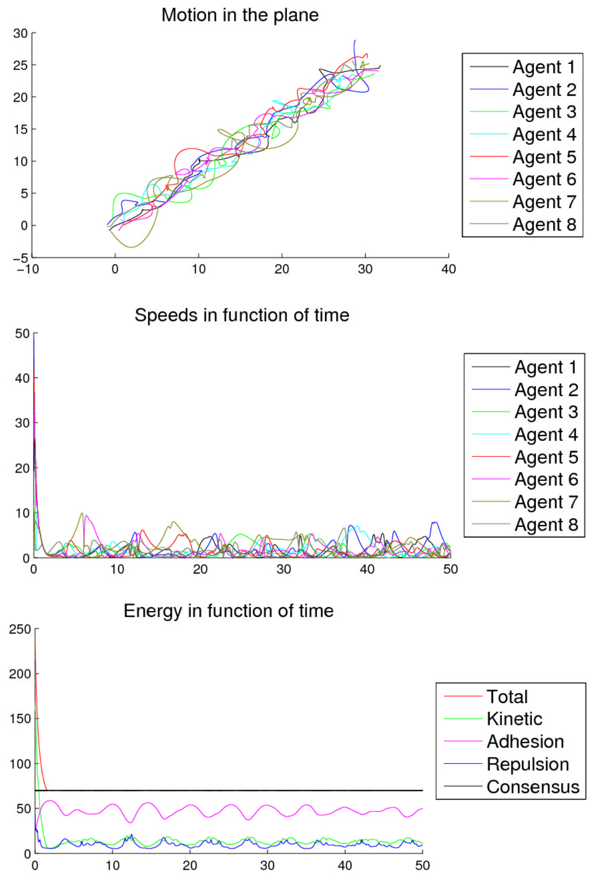

FIGURE 4. Profile of the controlled system.

which is due to the fact that the threshold is reached in shorter time (see bottom graphs of Figure 6).

Figure 6 also clearly confirms the behavior of the decay rate of the energy as a function of M, as predicted by our analytical result of Proposition 5: $E(t)$ decreases as e^{-kMt}, for a certain constant $k > 0$.

It is interesting to notice that convergence to the consensus region occurs even if the hypothesis of Proposition 2 is not met, i.e., ϑ is very far away from $E(0)$, as it is likely to be a sub-optimal sufficient condition. Indeed, in all the case studies above

$$E(0) \exp\left(-\frac{2\sqrt{3}}{9}\frac{M\left\|\overline{\boldsymbol{v}}(0)\right\|^3}{E(0)\sqrt{E(0)}\left(\Lambda\sqrt{E(0)}+\frac{M}{N}\right)}\right) \approx E(0),$$

but, nonetheless, we were able to steer each system to consensus in finite time.

This is probably due to the fact that in all our numerical simulations we have that the speed $\left\|\boldsymbol{v}_{\widehat{\imath}(t)}(t)\right\| > 0$ for every $t \geq 0$, hence Lemma 3.4 is not necessary to infer the effectiveness of our control. This could mean that neither condition (b)

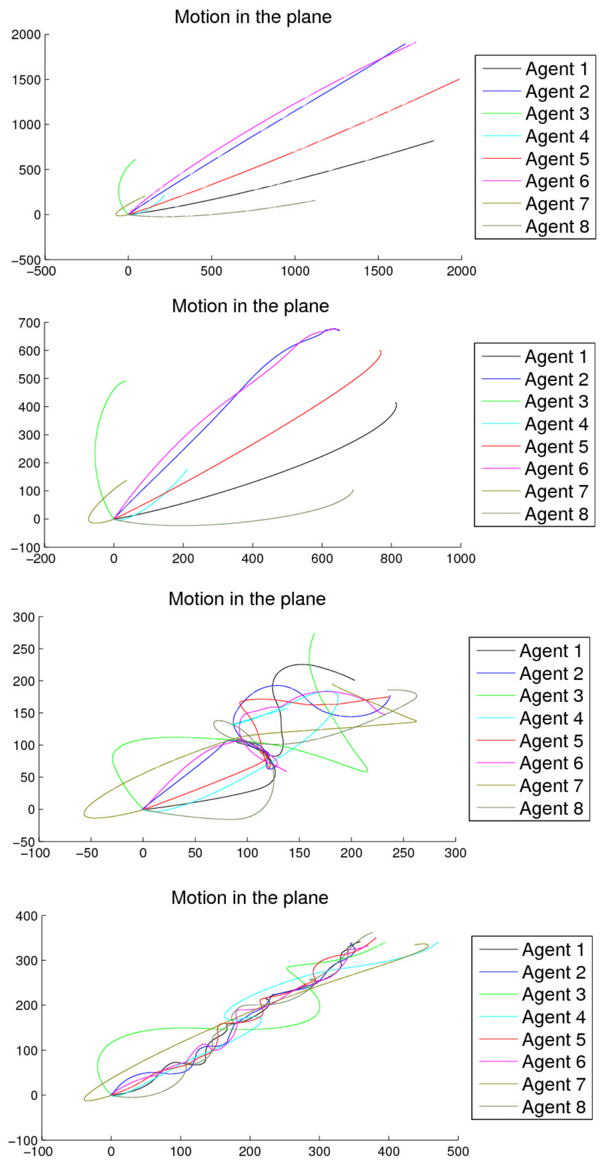

FIGURE 5. Spatial evolutions. From top-left corner to bottom-right: M = 0, M = 0.1, M = 1, M = 10.

nor condition (c) is necessary for the proof of Theorem 4.1, and hence we would have unconditional convergence for our sparse control strategy.

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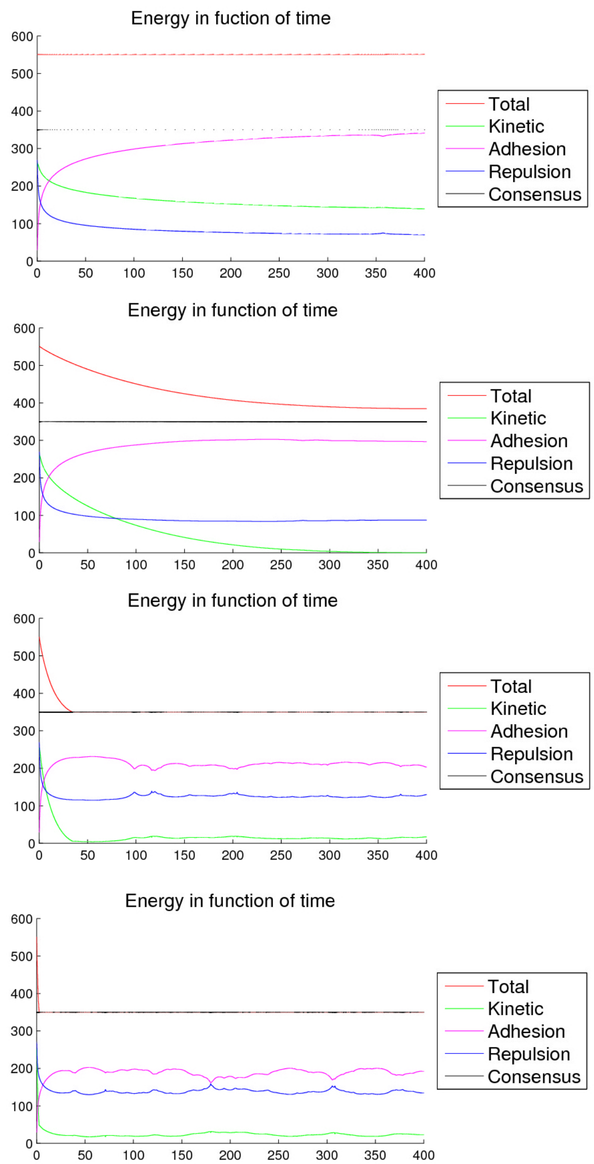

FIGURE 6. Energy profiles. From top-left corner to bottom-right: M = 0, M = 0.1, M = 1, M = 10.

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