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CONVERGENCE OF VANISHING CAPILLARITY APPROXIMATIONS FOR SCALAR CONSERVATION LAWS WITH DISCONTINUOUS FLUXES

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Abstract. Flow of two phases in a heterogeneous porous medium is modeled by a scalar conservation law with a discontinuous coefficient. As solutions of conservation laws with discontinuous coefficients depend explicitly on the underlying small scale effects, we consider a model where the relevant small scale effect is dynamic capillary pressure. We prove that the limit of vanishing dynamic capillary pressure exists and is a weak solution of the corresponding scalar conservation law with discontinuous coefficient. A robust numerical scheme for approximating the resulting limit solutions is introduced. Numerical experiments show that the scheme is able to approximate interesting solution features such as propagating non-classical shock waves as well as discontinuous standing waves efficiently.

1. **Introduction.** Scalar conservation laws with spatially dependent and possibly discontinuous coefficients arise in a wide variety of models in physics and engineering [1, 2] and other references therein, for instance in flows in heterogeneous porous media [19], modeling of the clarifier-thickener units in the chemical industry [5] and in traffic flows with heterogeneous surface conditions [21]. The generic form of such equations is given by,

$$\begin{cases}\partial_t u + \partial_x f(k(x), u) = 0, & t > 0, x \in \mathbb{R},\\ u(0,x) = u_0(x), & x \in \mathbb{R}.\end{cases} \tag{1.1}$$

Here, $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is the unknown and $k : \mathbb{R} \to \mathbb{R}$ is the (possibly discontinuous) coefficient with $f : \mathbb{R}^2 \to \mathbb{R}$ being the flux function.

It is well known that solutions of nonlinear conservation laws contain discontinuities in the form of shock waves, even when the initial data u_0 is smooth. Hence,

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solutions to conservation laws are sought in the sense of distributions. Such weak solutions are not unique. Additional admissibility criteria or entropy conditions need to be imposed in order to single out the physically relevant solution [8]. Uniqueness of entropy solutions for scalar conservation laws with a smooth (C^1) coefficient k follows from the seminal result of Kruzkhov [8].

As the conservation law (1.1) is derived by neglecting underlying small scale effects such as diffusion, dispersion, capillarity etc, it is customary to incorporate information about these small scale effects in the entropy conditions in order to single out the physically relevant solutions. Again, for smooth coefficients k, entropy solutions (à la Kruzkhov) encapsulate the vanishing viscosity limit.

The design of suitable entropy conditions for a scalar conservation law with a discontinuous coefficient k has received considerable attention in recent years, see [10, 1, 3, 16, 2, 14, 15, 16] and references therein. In [1], it was pointed out that the scalar conservation law (1.1) with a discontinuous coefficient can have infinitely many L^1 contractive semigroups of solutions. In particular, incorporating different underlying small scale effects in (1.1) with a discontinuous k, can result in different solutions. Thus, physically relevant solutions of scalar conservation laws with discontinuous coefficients *depend* on the underlying small scale effect. This implies that every physical model of interest has to be considered separately and the resulting semigroup of solutions which arise by neglecting the underlying small scale effects has to be characterized. We are interested in one such model in this paper.

A motivating example for scalar conservation laws with discontinuous coefficient is provided by two-phase flow in a heterogeneous porous medium. A brief derivation of this model is presented below.

1.1. **Modeling two phase flow in a heterogeneous porous medium with dynamic capillary pressure.** Consider a one dimensional porous medium with (possibly discontinuous) rock permeability K. We consider the flow of two phases (say water and oil) in this porous medium. Denoting, water and oil saturations as $S^w \in [0,1]$ and $S^o \in [0,1]$, respectively, we see that mass conservation for each phase results in

$$\begin{aligned} S^w_t + v^w_x &= 0, \\ S^o_t + v^o_x &= 0. \end{aligned} \tag{1.2}$$

Here, v^w, v^o are the phase velocities for the water and oil phases, respectively.

As the total saturation $S^w + S^o = 1$, adding the two equations in (1.2) results in

$$v^w + v^o = q, \tag{1.3}$$

with $q = q(t)$ being the total flow rate (specified for instance by boundary conditions).

The phase velocities are modeled by Darcy's law [4]:

$$\begin{aligned} v^w &= -K\lambda^w P^w_x + K\lambda^w \rho^w g, \\ v^o &= -K\lambda^o P^o_x + K\lambda^o \rho^o g. \end{aligned} \tag{1.4}$$

Here, $\lambda^r = \frac{k^r(S^r)}{\mu^r}, r = w, o$, are the phase mobilities, given in terms of the relative permeabilities k^r and the phase viscosities μ^r, $\rho^r, r = w, o$ is the phase density and g is the (constant) acceleration due to gravity.

The phase pressures are denoted by $P^r, r = w, o$. Introducing the capillary pressure $P^c = P^w - P^o$, adding the two equations in (1.4) and using (1.3), we obtain,

$$-KP^w_x = \frac{q - K(\lambda^w \rho^w + \lambda^o \rho^o)g}{\lambda^w + \lambda^o} - \frac{K\lambda^o}{\lambda^w + \lambda^o} P^c_x. \tag{1.5}$$

Hence, by substituting (1.5) in the first equation of (1.4), we eliminate the phase pressure P^w and obtain,

$$v^w = \left(\frac{q\lambda^w}{\lambda^w + \lambda^o} - \frac{K\lambda^w \lambda^o (\rho^w - \rho^o) g}{\lambda^w + \lambda^o} \right) - \frac{K\lambda^w \lambda^o}{\lambda^w + \lambda^o} P^c_x.$$

Substituting for v^w in the first equation for mass conservation (1.2) results in,

$$S^w_t + \left(\frac{q\lambda^w}{\lambda^w + \lambda^o} - \frac{K\lambda^w \lambda^o (\rho^w - \rho^o) g}{\lambda^w + \lambda^o} \right)_x = \left(\frac{K\lambda^w \lambda^o}{\lambda^w + \lambda^o} P^c_x \right)_x. \tag{1.6}$$

As λ^w, λ^o are given (smooth functions) of S^w, the mass conservation equation is completed once the capillary pressure P^c is specified. In standard models of capillary pressure [4],

$$P^c = P^c(S^w) = \nu P^c(S^w). \tag{1.7}$$

Here, P^c is a smooth monotone increasing function of the saturation and $\nu << 1$ is given in terms of the surface tension and the pore size (see [4] for a detailed description of the capillarity parameter ν). However, this model of the capillary pressure was found to be inadequate in explaining dynamic behavior of the flow. Instead, we follow the model proposed in [12] (see also [11]) and introduce a *dynamic capillary pressure*:

$$P^c = \nu \overline{P}^c(S^w) + \tau \nu^2 S^w_t. \tag{1.8}$$

Here $\overline{P}^c(S^w)$ is a smooth and monotone increasing function of the saturation. Hence, the capillary pressure also depends on the dynamics of moving fronts. We observe that the static capillary pressure function $\overline{P}^c(S^w)$ is the same in the above equation as in (1.7). The main difference is in the addition of the dynamics of moving fronts through a dispersion parameter τ.

The model is completed by substituting the expression of the capillary pressure (1.8) in (1.6) to obtain the following equation:

$$\begin{aligned} S^w_t + &\left(\frac{q\lambda^w}{\lambda^w + \lambda^o} - \frac{K\lambda^w \lambda^o (\rho^w - \rho^o) g}{\lambda^w + \lambda^o} \right)_x \\ &= \nu \left(\frac{K\lambda^w \lambda^o}{\lambda^w + \lambda^o} (\overline{P}^c(S^w))_x \right)_x + \tau \nu^2 \left(\frac{K\lambda^w \lambda^o}{\lambda^w + \lambda^o} S^w_{xt} \right)_x. \end{aligned} \tag{1.9}$$

Furthermore, we observe that the rock permeability K can vary (discontinuously) in space (see [19] and references therein). Our aim is to obtain information about the limit as $\nu \to 0$, even when the coefficient K is discontinuous. Hence, we will recover the physically relevant solutions, obtained using the correct small scale information (including the dynamic capillary pressure).

1.2. **Aims and scope of the current paper.** The main aim of the current paper is to investigate mathematically the flow of two phases in a heterogeneous porous medium, in the limit of vanishing dynamic capillary pressure. For notational simplicity, we rename the water saturation by $S^w = u$ and consider the following

Cauchy problem;

$$\begin{cases} \partial_t u_\nu + \partial_x f(k_\nu, u_\nu) \\ \qquad = \nu \partial_x (g(\ell_\nu, u_\nu)\partial_x u_\nu) + \nu^2 \partial_x (h(m_\nu, u_\nu)\partial^2_{tx} u_\nu), & t > 0, x \in \mathbb{R}, \\ \partial_t k_\nu = \nu \partial^2_{xx} k_\nu, & t > 0, x \in \mathbb{R}, \\ \partial_t \ell_\nu = \nu \partial^2_{xx} \ell_\nu, & t > 0, x \in \mathbb{R}, \\ \partial_t m_\nu = \nu \partial^2_{xx} m_\nu, & t > 0, x \in \mathbb{R}, \\ u_\nu(0,x) = u_{0,\nu}(x), & x \in \mathbb{R}, \\ k_\nu(0,x) = k_{0,\nu}(x), & x \in \mathbb{R}, \\ \ell_\nu(0,x) = \ell_{0,\nu}(x), & x \in \mathbb{R}, \\ m_\nu(0,x) = m_{0,\nu}(x), & x \in \mathbb{R}, \end{cases} \tag{1.10}$$

where $0 < \nu \leq 1$, $u_\nu : [0,\infty) \times \mathbb{R} \to \mathbb{R}$. The case $h \equiv 0$, $g \equiv 1$ has been studied in [7].

By identifying the functions f, g and h with their counterparts in (1.9), we can assume that

(A.1) f, g, $h : \mathbb{R}^2 \to \mathbb{R}$, are smooth functions such that

$$\alpha \leq g(\cdot,\cdot),\ h(\cdot,\cdot) \tag{1.11}$$

for some constant $\alpha > 0$;

(A.2) $f(k,\cdot)$ is assumed to be genuinely nonlinear for every $k \in \mathbb{R}$, namely the map $u \in [0,1] \mapsto f(k,u)$ is not affine on any nontrivial interval for every $k \in \mathbb{R}$.

This assumption is satisfied by most physically relevant relative permeability functions in (1.9).

On the functions $k_{0,\nu}$, $\ell_{0,\nu}$, $m_{0,\nu}$, $u_{0,\nu} : \mathbb{R} \to \mathbb{R}$ we assume that

(A.3) $k_{0,\nu}$, $\ell_{0,\nu}$, $m_{0,\nu} \in C^\infty(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$, $u_{0,\nu} \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $0 \leq u_{0,\nu} \leq 1$;

(A.4) there exist

$$k,\, \ell,\, m \in BV(\mathbb{R}) \cap L^1(\mathbb{R}), \quad u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 0 \leq u_0 \leq 1,$$

such that

$$\begin{aligned} & u_{0,\nu} \to u_0, \quad k_{0,\nu} \to k, \quad \text{a.e. and in } L^p(\mathbb{R}),\, 1 \leq p < \infty, \\ & \ell_{0,\nu} \to \ell, \quad m_{0,\nu} \to m, \quad \text{a.e. and in } L^p(\mathbb{R}),\, 1 \leq p < \infty, \\ & \|u_{\nu,0}\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \quad \forall \nu, \\ & \|k_{0,\nu}\|_{L^\infty(\mathbb{R})} \leq \|k\|_{L^\infty(\mathbb{R})}, \quad \|k_{0,\nu}\|_{L^2(\mathbb{R})} \leq \|k\|_{L^2(\mathbb{R})}, \quad \forall \nu, \\ & \|\partial_x k_{0,\nu}\|_{L^1(\mathbb{R})} \leq TV(k), \quad \|\ell_{0,\nu}\|_{L^\infty(\mathbb{R})} \leq \|\ell\|_{L^\infty(\mathbb{R})}, \quad \forall \nu, \\ & \|\ell_{0,\nu}\|_{L^2(\mathbb{R})} \leq \|\ell\|_{L^2(\mathbb{R})}, \quad \|\partial_x \ell_{0,\nu}\|_{L^1(\mathbb{R})} \leq TV(\ell), \quad \forall \nu, \\ & \|m_{0,\nu}\|_{L^\infty(\mathbb{R})} \leq \|m\|_{L^\infty(\mathbb{R})}, \|m_{0,\nu}\|_{L^2(\mathbb{R})} \leq \|m\|_{L^2(\mathbb{R})}, \forall \nu, \\ & \|\partial_x m_{0,\nu}\|_{L^1(\mathbb{R})} \leq TV(m), \forall \nu. \end{aligned} \tag{1.12}$$

Our first aim in this paper is to study the behavior of solutions of (1.10) as $\nu \to 0$ and in particular, whether these solutions converge to a weak solution of the conservation law (1.1). We answer this fundamental question by stating the main result of this paper.

Theorem 1.1. *Assume* **(A.1), (A.2), (A.3), (A.4)**. *There exist a sequence* $\{\nu_n\}_n \subset (0,1]$, $\nu_n \to 0$, *a sequence* $\{u_{\nu_n}\}_{n\in\mathbb{N}}$ *of solutions of* (1.10), *and a distributional solution* u *of* (1.1) *such that*

$$u_{\nu_n} \to u, \quad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty, \text{ and a.e. in } (0,\infty)\times\mathbb{R}. \tag{1.13}$$

The second aim of this paper is to study the limit solutions of (1.10) as $\nu \to 0$ numerically. We design an efficient numerical scheme and compute the limit solutions that will approximate the physically relevant (vanishing dynamic capillary pressure limit) solutions of the scalar conservation law with discontinuous coefficient, arising in a model of two phase flow in a heterogeneous porous medium.

The rest of the paper is organized as follows. In Section 2, we present the proof of Theorem 1.1. The numerical scheme and numerical results are presented in Section 3 and a summary of the results is presented in Section 4.

2. Proof of Theorem 1.1.

Let us assume that **(A.1), (A.2), (A.3), (A.4)** hold and (1.10) admits a smooth solution $(u_\nu,\, k_\nu,\, \ell_\nu,\, m_\nu)$ such that

$$0 \le u_\nu \le 1.$$

Lemma 2.1. *The following estimates hold*

$$\begin{aligned}
&\|k_\nu\|_{L^\infty((0,\infty)\times\mathbb{R})} \le \|k\|_{L^\infty(\mathbb{R})}, \quad \|\partial_x k_\nu(t,\cdot)\|_{L^1(\mathbb{R})} \le TV(k),\\
&\|\ell_\nu\|_{L^\infty((0,\infty)\times\mathbb{R})} \le \|\ell\|_{L^\infty(\mathbb{R})}, \quad \|\partial_x \ell_\nu(t,\cdot)\|_{L^1(\mathbb{R})} \le TV(\ell),\\
&\|m_\nu\|_{L^\infty((0,\infty)\times\mathbb{R})} \le \|m\|_{L^\infty(\mathbb{R})}, \quad \|\partial_x m_\nu(t,\cdot)\|_{L^1(\mathbb{R})} \le TV(m),\\
&\|k_\nu(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\nu\int_0^t \|\partial_x k_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \le \|k\|^2_{L^2(\mathbb{R})},\\
&\|\ell_\nu(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\nu\int_0^t \|\partial_x \ell_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \le \|\ell\|^2_{L^2(\mathbb{R})},\\
&\|m_\nu(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\nu\int_0^t \|\partial_x m_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \le \|m\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

for every $t \ge 0$ *and* $\nu > 0$.

Since the proof of this lemma is straightforward we omit it.

Following [6, Lemma 4.2], we show the following result.

Lemma 2.2. *Let* $\nu > 0$. *Fixed* $T > 0$, *we have*

i) the family $\{u_\nu\}_\nu$ *is bounded in* $L^\infty(0,T;L^2(\mathbb{R}))$;

ii) the families $\{\sqrt{\nu}\partial_x u_\nu\}_\nu$, $\{\nu^{\frac{3}{2}}\partial^2_{tx}u_\nu\}_\nu$, $\{\sqrt{\nu}\partial_t u_\nu\}_\nu$ *are bounded in* $L^2((0,T)\times\mathbb{R})$.

Proof. Let B be a positive constant that will be specified later. Multiplying the first equation in (1.10) by $u_\nu + \nu B\partial_t u_\nu$, we have

$$\begin{aligned}
(u_\nu + \nu B\partial_t u_\nu)\partial_t u_\nu =&\, \nu(u_\nu + \nu B\partial_t u_\nu)\partial_x(g(\ell_\nu,u_\nu)u_\nu)\\
&+ \nu^2(u_\nu + \nu B\partial_t u_\nu)\partial_x(h(m_\nu,u_\nu)\partial^2_{tx}u_\nu)\\
&- (u_\nu + \nu B\partial_t u_\nu)\partial_x f(k_\nu,u_\nu).
\end{aligned} \tag{2.1}$$

Since

$$\int_{\mathbb{R}}(u_\nu + \nu B\partial_t u_\nu)\partial_t u_\nu dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_\nu^2 dx + \nu B\int_{\mathbb{R}}(\partial_t u_\nu)^2 dx,$$

$$\nu\int_{\mathbb{R}}(u_\nu+\nu B\partial_t u_\nu)\partial_x(g(\ell_\nu,u_\nu)\partial_x u_\nu)dx$$
$$=-\nu\int_{\mathbb{R}}g(\ell_\nu,u_\nu)(\partial_x u_\nu)^2dx-\nu^2B\int_{\mathbb{R}}g(\ell_\nu,u_\nu)\partial_x u_\nu\partial_{tx}^2u_\nu dx,$$
$$\nu^2\int_{\mathbb{R}}(u_\nu+\nu B\partial_t u_\nu)\partial_x(h(m_\nu,u_\nu)\partial_{tx}^2u_\nu)dx$$
$$=-\nu^2\int_{\mathbb{R}}h(m_\nu,u_\nu)\partial_{tx}^2u_\nu\partial_x u_\nu dx-\nu^3B\int_{\mathbb{R}}h(m_\nu,u_\nu)(\partial_{tx}^2u_\nu)^2dx,$$

integrating (2.1) over $\mathbb{R}$, thanks to (1.11)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}u_\nu^2dx+\nu\alpha\int_{\mathbb{R}}(\partial_x u_\nu)^2dx+\nu^3\alpha B\int_{\mathbb{R}}(\partial_{tx}^2u_\nu)^2dx+\nu B\int_{\mathbb{R}}(\partial_t u_\nu)^2dx\\ \le-\nu B\int_{\mathbb{R}}\partial_t u_\nu\partial_x f(k_\nu,u_\nu)dx-\nu^2\int_{\mathbb{R}}h(m_\nu,u_\nu)\partial_{tx}^2u_\nu\partial_x u_\nu dx\\ -\int_{\mathbb{R}}u_\nu\partial_x f(k_\nu,u_\nu)dx-\nu^2B\int_{\mathbb{R}}g(\ell_\nu,u_\nu)\partial_x u_\nu\partial_{tx}^2u_\nu dx.\end{aligned}$$

Lemma 2.1 and (1.12) give

$$\begin{aligned}-\int_{\mathbb{R}}u_\nu\partial_x f(k_\nu,u_\nu)dx=&-\int_{\mathbb{R}}u_\nu\partial_k f(k_\nu,u_\nu)\partial_x k_\nu dx-\int_{\mathbb{R}}u_\nu\partial_u f(k_\nu,u_\nu)\partial_x u_\nu dx\\ =&-\int_{\mathbb{R}}u_\nu\partial_k f(k_\nu,u_\nu)\partial_x k_\nu dx-\int_{\mathbb{R}}\partial_x\Big(\int_0^{u_\nu}\xi\partial_u f(k_\nu,\xi)d\xi\Big)dx\\ &+\int_{\mathbb{R}}\Big(\int_0^{u_\nu}\xi\partial_{uk}^2f(k_\nu,\xi)\partial_x k_\nu d\xi\Big)dx\\ \le&\Big(\|\partial_k f\|_{L^\infty(I)}+\left\|\partial_{uk}^2f\right\|_{L^\infty(I)}\Big)TV(k)=\frac{C}{2},\end{aligned}$$

where

$$C=2\big(\left\|\partial_{uk}^2f\right\|_{L^\infty(I)}+\|\partial_k f\|_{L^\infty(I)}\big)TV(k),$$
$$I=\Big(-\|k\|_{L^\infty(\mathbb{R})},\|k\|_{L^\infty(\mathbb{R})}\Big)\times(0,1).$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}u_\nu^2dx+\nu\alpha\int_{\mathbb{R}}(\partial_x u_\nu)^2dx+\nu^3\alpha B\int_{\mathbb{R}}(\partial_{tx}^2u_\nu)^2dx+\nu B\int_{\mathbb{R}}(\partial_t u_\nu)^2dx\\ \le-\nu B\int_{\mathbb{R}}\partial_t u_\nu\partial_x f(k_\nu,u_\nu)dx-\nu^2\int_{\mathbb{R}}h(m_\nu,u_\nu)\partial_{tx}^2u_\nu\partial_x u_\nu dx\\ -\nu^2B\int_{\mathbb{R}}g(\ell_\nu,u_\nu)\partial_x u_\nu\partial_{tx}^2u_\nu dx+\frac{C}{2}\\ =-\nu B\int_{\mathbb{R}}\partial_u f\partial_x u_\nu\partial_t u_\nu dx-\nu B\int_{\mathbb{R}}\partial_k f\partial_x k_\nu\partial_t u_\nu dx\\ -\nu^2\int_{\mathbb{R}}(Bg(\ell_\nu,u_\nu)+h(m_\nu,u_\nu))\partial_x u_\nu\partial_{tx}^2u_\nu dx+\frac{C}{2}.\end{aligned}$$

Using the Young's inequality,

$$\begin{aligned}-B\nu\int_{\mathbb{R}}\partial_u f(k_\nu,u_\nu)\partial_x u_\nu\partial_t u_\nu dx-B\nu\int_{\mathbb{R}}\partial_k f(k_\nu,u_\nu)\partial_x k_\nu\partial_t u_\nu dx\\ \le\nu\left|\int_{\mathbb{R}}B\partial_u f(k_\nu,u_\nu)\partial_x u_\nu\partial_t u_\nu dx\right|+\nu\left|\int_{\mathbb{R}}B\partial_u f(k_\nu,u_\nu)\partial_x u_\nu\partial_t u_\nu dx\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \nu \int_{\mathbb{R}} |B\partial_u f(k_\nu, u_\nu)\partial_x u_\nu||\partial_t u_\nu| dx + \nu \int_{\mathbb{R}} |B\partial_k f(k_\nu, u_\nu)\partial_x k_\nu||\partial_t u_\nu| dx \\
&= \nu \int_{\mathbb{R}} \left| \frac{B\partial_u f(k_\nu, u_\nu)\partial_x u_\nu}{D_1} \right| |\partial_t u_\nu D_1| dx \\
&\quad + \nu \int_{\mathbb{R}} \left| \frac{B\partial_u f(k_\nu, u_\nu)\partial_x k_\nu}{D_2} \right| |\partial_t u_\nu D_2| dx \\
&\leq \frac{\nu B^2 \left\| \partial_u f \right\|^2_{L^\infty(I)}}{2D_1^2} \int_{\mathbb{R}} (\partial_x u_\nu)^2 dx \\
&\quad + \frac{\nu B^2 \left\| \partial_k f \right\|^2_{L^\infty(I)}}{2D_2^2} \int_{\mathbb{R}} (\partial_x k_\nu)^2 dx + \nu \frac{D_1^2 + D_1^2}{2} \int_{\mathbb{R}} (\partial_t u_\nu)^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
-\nu^2 &\int_{\mathbb{R}} (Bg(\ell_\nu, u_\nu) + h(m_\nu, u_\nu))\partial_x u_\nu \partial^2_{tx} u_\nu dx \\
&\leq \nu^2 \Big| \int_{\mathbb{R}} (Bg(\ell_\nu, u_\nu) + h(m_\nu, u_\nu))\partial_x u_\nu \partial^2_{tx} u_\nu dx \Big| \\
&= \int_{\mathbb{R}} \nu^{\frac{1}{2}} |D_3 \partial_x u_\nu| \nu^{\frac{3}{2}} \left| \frac{(Bg(\ell_\nu, u_\nu) + h(m_\nu, u_\nu))\partial^2_{tx} u_\nu}{D_3} \right| dx \\
&\leq \frac{\nu D_3^2}{2} \int_{\mathbb{R}} (\partial_x u_\nu)^2 dx + \frac{\nu^3}{2D_3^2} \int_{\mathbb{R}} (Bg(\ell_\nu, u_\nu) + h(m_\nu, u_\nu))^2 (\partial^2_{tx} u_\nu)^2 dx \\
&\leq \frac{\nu D_3^2}{2} \int_{\mathbb{R}} (\partial_x u_\nu)^2 dx + \frac{\nu^3 \left\| Bg + h \right\|^2_{L^\infty(J_1)}}{2D_3^2} \int_{\mathbb{R}} (\partial^2_{tx} u_\nu)^2 dx,
\end{aligned}
$$

where D_1, D_2, D_3 are three positive constants that will be specified later and

$$
\begin{aligned}
J_1 &= I_1 \cup J, \\
I_1 &= \Big(- \left\| \ell \right\|_{L^\infty(\mathbb{R})}, \left\| \ell \right\|_{L^\infty(\mathbb{R})} \Big) \times (0, 1), \\
J &= \Big(- \left\| m \right\|_{L^\infty(\mathbb{R})}, \left\| m \right\|_{L^\infty(\mathbb{R})} \Big) \times (0, 1).
\end{aligned}
\tag{2.2}
$$

Thus,

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} u_\nu^2 dx &+ \nu \Big(2\alpha - \frac{B^2 \left\| \partial_u f \right\|^2_{L^\infty(I)}}{D_1^2} - D_3^2 \Big) \int_{\mathbb{R}} (\partial_x u_\nu)^2 dx \\
&+ \nu^3 \Big(2\alpha B - \frac{\left\| Bg + h \right\|^2_{L^\infty(J_1)}}{D_3^2} \Big) \int_{\mathbb{R}} (\partial^2_{tx} u_\nu)^2 dx \\
&+ \nu \Big(2B - D_1^2 - D_2^2 \Big) \int_{\mathbb{R}} (\partial_t u_\nu)^2 dx \\
&\leq \nu \frac{B^2 \left\| \partial_k f \right\|^2_{L^\infty(I)}}{D_2^2} \int_{\mathbb{R}} (\partial_x k_\nu)^2 dx + C.
\end{aligned}
$$

An integration over $(0, t)$ and Lemma 2.1 give

$$
\begin{aligned}
\left\| u_\nu(t, \cdot) \right\|^2_{L^2(\mathbb{R})} &+ \nu \Big(2\alpha - \frac{B^2 \left\| \partial_u f \right\|^2_{L^\infty(I)}}{D_1^2} - D_3^2 \Big) \int_0^t \left\| \partial_x u_\nu(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \\
&+ \nu^3 B \Big(2\alpha B - \frac{\left\| Bg + h \right\|^2_{L^\infty(J_1)}}{D_3^2} \Big) \int_0^t \left\| \partial^2_{tx} u_\nu(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds
\end{aligned}
$$

$$+\nu\Big(2B-D_1^2-D_2^2\Big)\int_0^t \|\partial_t u_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}ds$$
$$\le Ct+\frac{B^2\|\partial_k f\|^2_{L^\infty(I)}}{D_2^2}\|k\|^2_{L^2(\mathbb{R})}+\|u_0\|^2_{L^2(\mathbb{R})}.$$

We can find three constants D_1, D_2, D_3 such that

$$\begin{cases} 2\alpha-\dfrac{B^2\|\partial_u f\|^2_{L^\infty(I)}}{D_1^2}-D_3^2>0,\\ 2\alpha B-\dfrac{\|Bg+h\|^2_{L^\infty(J_1)}}{D_3^2}>0,\\ 2B-D_1^2-D_2^2>0.\end{cases}\tag{2.3}$$

Indeed B can be taken very big, and up to rescaling we can have $\alpha=B^3$. Therefore, the proof is done. □

To prove Theorem 1.1, the following technical lemma is needed [20].

Lemma 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Suppose the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n=\mathcal{L}_{1,n}+\mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Proof of Theorem 1.1. Let $c\in\mathbb{R}$ be fixed, we claim that the family

$$\{\partial_t|u_\nu-c|+\partial_x(\mathrm{sign}\,(u_\nu-c)\,(f(k,u_\nu)-f(k,c)))\}_{\nu>0}$$

is compact in $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$. For the sake of notational simplicity we write

$$\eta_0(u)=|u-c|-|c|,$$
$$Q_0(k,u)=\mathrm{sign}\,(u-c)\,(f(k,u)-f(k,c))-\mathrm{sign}\,(-c)\,(f(k,0)-f(k,c)).$$

Observe that

$$\eta_0(0)=Q_0(k,0)=0,$$

and

$$\begin{aligned}\partial_t|u_\nu-c|&+\partial_x(\mathrm{sign}\,(u_\nu-c)\,(f(k,u_\nu)-f(k,c)))\\ &=\partial_t\eta_0(u_\nu)+\partial_x Q_0(k,u_\nu)\\ &\quad+\mathrm{sign}\,(-c)\,\partial_x(f(k,0)-f(k,c)).\end{aligned}\tag{2.4}$$

Let $\{(\eta_\nu,Q_\nu)\}_{\nu>0}$ be a family of maps such that

$$\begin{aligned}&\eta_\nu\in C^\infty(\mathbb{R}),\quad Q_\nu\in C^\infty(\mathbb{R}^2),\\ &\partial_u Q_\nu(k,u)=\partial_u f(k,u)\eta_\nu'(u),\quad \eta_\nu''\ge 0,\\ &\|\eta_\nu-\eta_0\|_{L^\infty(0,1)}\le\nu,\quad \|\eta_\nu'-\eta_0'\|_{L^1(0,1)}\le\nu,\\ &\|\eta_\nu'\|_{L^\infty(0,1)}\le 1,\quad \eta_\nu(0)=Q_\nu(k,0)=0,\end{aligned}\tag{2.5}$$

for each $\nu\ge 0$. Since

$$Q_0(k,u)=\int_0^u\partial_u f(k,\xi)\eta_0'(\xi)d\xi,\quad Q_\nu(k,u)=\int_0^u\partial_u f(k,\xi)\eta_\nu'(\xi)d\xi,$$

we also have

$$\|\partial_k Q_\nu\|_{L^\infty(I)}\le\left\|\partial^2_{uk}f\right\|_{L^\infty(I)},\quad \|Q_\nu-Q_0\|_{L^\infty(I)}\le\|\partial_u f\|_{L^\infty(I)}\,\nu.$$

By (1.10),

$$\begin{aligned}
&\partial_t\eta_0(u_\nu)+\partial_x Q_0(k,u_\nu)\\
&\quad=\partial_t\eta_\nu(u_\nu)+\partial_x Q_\nu(k_\nu,u_\nu)+\partial_t(\eta_0(u_\nu)-\eta_\nu(u_\nu))\\
&\qquad+\partial_x(Q_0(k,u_\nu)-Q_\nu(k,u_\nu))+\partial_x(Q_\nu(k,u_\nu)-Q_\nu(k_\nu,u_\nu))\\
&\quad=\nu\partial_x\big(\eta_\nu'(u_\nu)g(\ell_\nu,u_\nu)\partial_x u_\nu\big)-\nu\eta_\nu''(u_\nu)g(\ell_\nu,u_\nu)(\partial_x u_\nu)^2\\
&\qquad+\nu^2\partial_x\big(\eta_\nu'(u_\nu)h(m_\nu,u_\nu)\partial_{tx}^2 u_\nu\big)-\nu^2\eta_\nu''(u_\nu)h(m_\nu,u_\nu)\partial_x u_\nu\partial_{tx}^2 u_\nu\\
&\qquad-[\eta_\nu'(u_\nu)\partial_u f(k_\nu,u_\nu)-\partial_k Q(k_\nu,u_\nu)]\partial_x k_\nu\\
&\qquad+\partial_t(\eta_0(u_\nu)-\eta_\nu(u_\nu))+\partial_x(Q_0(k,u_\nu)-Q_\nu(k_\nu,u_\nu))\\
&\qquad+\partial_x(Q_\nu(k,u_\nu)-Q_\nu(k_\nu,u_\nu)).
\end{aligned}$$

Then, for (2.4),

$$\begin{aligned}
&\partial_t|u_\nu-c|+\partial_x(\operatorname{sign}(u_\nu-c)\,(f(k,u_\nu)-f(k,c)))\\
&\qquad=I_{1,\nu}+I_{2,\nu}+I_{3,\nu}+I_{4,\nu}+I_{5,\nu}+I_{6,\nu}+I_{7,\nu}+I_{8,\nu}+I_9,
\end{aligned}$$

where

$$\begin{aligned}
I_{1,\nu}&=\nu\partial_x\big(\eta_\nu'(u_\nu)g(\ell_\nu,u_\nu)\partial_x u_\nu\big),\\
I_{2,\nu}&=-\nu\eta_\nu''(u_\nu)g(\ell_\nu,u_\nu)(\partial_x u_\nu)^2,\\
I_{3,\nu}&=\nu^2\partial_x\big(\eta_\nu'(u_\nu)h(m_\nu,u_\nu)\partial_{tx}^2 u_\nu\big),\\
I_{4,\nu}&=-\nu^2\eta_\nu''(u_\nu)h(m_\nu,u_\nu)\partial_x u_\nu\partial_{tx}^2 u_\nu,\\
I_{5,\nu}&=-[\eta_\nu'(u_\nu)\partial_u f(k,u_\nu)-\partial_k Q(k,u_\nu)]\partial_x k_\nu,\\
I_{6,\nu}&=\partial_t(\eta_0(u_\nu)-\eta_\nu(u_\nu)),\\
I_{7,\nu}&=\partial_x(Q_0(k,u_\nu)-Q_\nu(k,u_\nu)),\\
I_{8,\nu}&=\partial_x(Q_\nu(k,u_\nu)-Q_\nu(k_\nu,u_\nu)),\\
I_9&=\operatorname{sign}(-c)\,\partial_x(f(k,0)-f(k,c)).
\end{aligned}$$

Let us show that

$$I_{1,\nu}\to 0\quad\text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0. \tag{2.6}$$

Thanks to Lemma 2.2,

$$\begin{aligned}
&\|\nu\eta_\nu'(u_\nu)g(\ell_\nu,u_\nu)\partial_x u_\nu\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\qquad\le\|\eta_\nu'\|^2_{L^\infty(0,1)}\|g\|^2_{L^\infty(I_1)}\,\nu^2\int_0^T\|\partial_x u_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds\\
&\qquad\le\|\eta_\nu'\|^2_{L^\infty(0,1)}\|g\|^2_{L^\infty(I_1)}\,\nu C_1(T+1)\to 0,
\end{aligned}$$

where $C_1>0$ is independent on ν and T, and I_1 is defined in (2.2).

Let us show that

$$\{I_{2,\nu}\}_{\nu>0}\quad\text{is bounded in } L^1((0,T)\times\mathbb{R}).$$

Again for Lemma 2.2,

$$\begin{aligned}
&\big\|\nu\eta_\nu''(u_\nu)g(\ell_\nu,u_\nu)(\partial_x u_\nu)^2\big\|_{L^1((0,T)\times\mathbb{R})}\\
&\qquad\le\|\eta_\nu''\|_{L^\infty(0,1)}\|g\|_{L^\infty(I_1)}\,\nu\int_0^T\|\partial_x u_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds\\
&\qquad\le\|\eta_\nu''\|_{L^\infty(0,1)}\|g\|_{L^\infty(I_1)}\,C_2(T+1),
\end{aligned}$$

where $C_2 > 0$ is independent on ν and T.

We claim that

$$I_{3,\nu} \to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\, T>0. \tag{2.7}$$

Thanks to Lemma 2.2,

$$\begin{aligned}
&\left\|\nu^2\eta_\nu'(u_\nu)h(m_\nu,u_\nu)\partial_{tx}^2 u_\nu\right\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le \|\eta_\nu'\|^2_{L^\infty(0,1)}\|h\|_{L^\infty(J)}\,\nu^4\int_0^T\left\|\partial_{tx}^2 u_\nu(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\le \|\eta_\nu'\|^2_{L^\infty(0,1)}\|h\|_{L^\infty(J)}\,\nu C_3(T+1)\to 0,
\end{aligned}$$

where $C_3 > 0$ is independent on ν and T, and J is defined in (2.2). Let us show that

$$\{I_{4,\nu}\}_{\nu>0} \quad \text{is bounded in } L^1((0,T)\times\mathbb{R}).$$

Again for Lemma 2.2 and Young's inequality,

$$\begin{aligned}
&\left\|\nu^2\eta_\nu''(u_\nu)h(m_\nu,u_\nu)\partial_x u_\nu\partial_{tx}^2 u_\nu\right\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\le \|\eta_\nu''\|_{L^\infty(0,1)}\|h\|_{L^\infty(J)}\,\nu^2\int_0^T\int_{\mathbb{R}}|\partial_x u_\nu||\partial_{tx}^2 u_\nu|dsdx\\
&\quad\le \|\eta_\nu''\|_{L^\infty(0,1)}\|h\|_{L^\infty(J)}\int_0^T\int_{\mathbb{R}}\nu^{\frac12}|\partial_x u_\nu|\nu^{\frac32}|\partial_{tx}^2 u_\nu|dsdx\\
&\quad\le \frac{\|\eta_\nu''\|_{L^\infty(0,1)}\|h\|_{L^\infty(J)}}{2}\nu\int_0^T\|\partial_x u_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\qquad+\frac{\|\eta_\nu''\|_{L^\infty(0,1)}\|h\|_{L^\infty(J)}}{2}\nu^3\int_0^T\left\|\partial_{tx}^2 u_\nu(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\le \|\eta_\nu''\|_{L^\infty(0,1)}\|h\|_{L^\infty(J)}\,C_4(T+1),
\end{aligned}$$

where $C_4 > 0$ is independent on ν and T.

Moreover, thanks to (2.5),

$$\begin{aligned}
&\{I_{5,\nu}\}_{\nu>0} \quad \text{is bounded in} \quad L^1((0,T)\times\mathbb{R}) \quad \text{for each} \quad T>0,\\
&I_{6,\nu}\to 0,\quad I_{7,\nu}\to 0 \quad \text{in } H^{-1}_{loc}((0,\infty)\times\mathbb{R}),\\
&\{I_{8,\nu}\}_{\nu>0} \quad \text{is compact in} \quad H^{-1}_{loc}((0,\infty)\times\mathbb{R}),\\
&I_9\in\mathcal{M}_{loc}((0,\infty)\times\mathbb{R}).
\end{aligned}$$

Therefore, Lemma 2.3, [23], and [22, Theorem 5] give (1.13).

Let us show that u is a distributional solution of (1.1). Let $\phi\in C^\infty(\mathbb{R}^2)$ be a test function with compact support, we have to prove that

$$\int_0^\infty\int_{\mathbb{R}}(u\partial_t\phi+f(k,u)\partial_x\phi)dtdx+\int_{\mathbb{R}}u_0(x)\phi(0,x)dx=0. \tag{2.8}$$

We have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}(u_{\nu_n}\partial_t\phi+f(k_{\nu_n},u_{\nu_n})\partial_x\phi)dtdx+\int_{\mathbb{R}}u_{0,\nu_n}(x)\phi(0,x)dx\\
&+\nu_n\int_0^\infty\int_{\mathbb{R}}g(\ell_{\nu_n},u_{\nu_n})\partial_x u_{\nu_n}\partial_x\phi dtdx-{\nu_n}^2\int_0^\infty\int_{\mathbb{R}}h(m_{\nu_n},u_{\nu_n})\partial_{tx}^2 u_{\nu_n}\partial_x\phi=0.
\end{aligned}$$

Therefore, (2.8) follows from (1.13), (1.12), Lemma 2.2, and the Dominated Convergence Theorem. □

3. Numerical results. Theorem 1.1 establishes that the limit of solutions of the two phase flow in a heterogeneous porous medium with vanishing dynamic capillary pressure exists and is a weak solution of the scalar conservation law (1.1). We aim to characterize this limit in this section using numerical experiments.

3.1. Numerical scheme. We will consider the regularized problem (1.10) and discretize it using a finite difference scheme. Consider a spatial domain $[X_l, X_r]$ and discretize it (for simplicity) into N equally spaced points $x_j = X_l + j * h$ with $h = (X_r - X_l)/N$ with $j = 0, 1, \cdots, N-1$. Then the approximate solution is defined as

$$u_j(t) \approx u(t, x_j).$$

Similarly the coefficients are represented on the grid as,

$$(k_j(t), \ell_j(t), m_j(t)) \approx (k(t, x_j), \ell(t, x_j), m(t, x_j)).$$

The approximate solution is computed using the following semi-discrete finite difference scheme:

$$\begin{aligned}
(u_j)_t = & -\frac{f(k_{j+1}, u_{j+1}) - f(k_{j-1}, u_{j-1})}{2\Delta x} \\
& + \frac{\nu}{(\Delta x)^2} \left\{ \frac{g(\ell_j, u_j) + g(\ell_{j+1}, u_{j+1})}{2} (u_{j+1} - u_j) \right. \\
& \qquad \left. - \frac{g(\ell_j, u_j) + g(\ell_{j-1}, u_{j-1})}{2} (u_j - u_{j-1}) \right\} \\
& + \frac{\tau\nu^2}{(\Delta x)^2} \left\{ \frac{h(m_j, u_j) + h(m_{j+1}, u_{j+1})}{2} (u_{j+1} - u_j)_t \right. \\
& \qquad \left. - \frac{h(m_j, u_j) + h(m_{j-1}, u_{j-1})}{2} (u_j - u_{j-1})_t \right\}.
\end{aligned} \tag{3.1}$$

We take the time-dependent terms to the left-hand side of the equation, apply transparent boundary conditions and then invert the matrix to obtain a semi-discrete formulation of the form

$$U_t = \mathcal{L}(U).$$

Here $U = \{u_j\}_{j=\{0,\cdots,N-1\}}$ is the vector of unknowns. In order to discretize the above ODE system, we will use a third-order Runge-Kutta time stepping of the form:

$$\begin{aligned}
U^{(1)} &= U^n + \Delta t \mathcal{L}\left(U^n\right), \\
U^{(2)} &= \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{1}{4} \Delta t \mathcal{L}\left(U^{(1)}\right), \\
U^{n+1} &= \frac{1}{3} U^n + \frac{2}{3} U^{(2)} + \frac{2}{3} \Delta t \mathcal{L}\left(U^{(2)}\right).
\end{aligned} \tag{3.2}$$

3.2. Numerical experiments. We will focus on results when the scheme is applied to the flow of two phases in a heterogeneous porous medium with vanishing dynamic capillary pressure (1.9). The following identifications,

$$\begin{aligned}
u(x,t) &= S^w(x,t), \\
k(x) = l(x) = m(x) &\equiv K, \\
f(k, S^w) &= \frac{q\lambda^w}{\lambda^w + \lambda^o} - \frac{K\lambda^w \lambda^o (\rho^w - \rho^o) g}{\lambda^w + \lambda^o},
\end{aligned}$$

$$g(l, S^w) = \frac{K\lambda^w\lambda^o}{\lambda^w + \lambda^o}\bar{P}^{c,\prime}(S^w),$$
$$h(m, S^w) = \frac{K\lambda^w\lambda^o}{\lambda^w + \lambda^o},$$

are used to represent two phase flow in the form (1.10) and we can apply the scheme (3.1) to these equations.

For the rest of this section, we will use the following parameters,

$$\begin{aligned} q &= 1, \\ \lambda^w &= (S^w)^2, \\ \lambda^o &= (S^o)^2 = (1 - S^w)^2, \\ \rho^w &= 1,\ \rho^o = 0.9,\ g = 10\ \rightarrow\ (\rho^w - \rho^o)g = 1, \\ \bar{P}^c(S^w) &= \left((S^w)^{-4/3} - 1\right)^{1/4}. \end{aligned}$$

The above relative permeability and capillary pressure functions are taken from [11].

For the numerical experiments, we compute on a uniform grid on the interval $[0, 2]$ and use the following parameter settings:

$$\begin{aligned} \tau &= 1, \\ \nu &= 6\Delta x, \\ CFL &= 0.45, \\ (S^w)^0 &= 0.8 \cdot 1_{\{x \le 0.25\}} + 0.2 \cdot 1_{\{x > 0.25\}}, \\ T &= 0.6. \end{aligned}$$

Thus, we set the small scale parameter in terms of the mesh width and expect to capture the small scale limit as the mesh is refined.

3.2.1. *Flow in a homogeneous medium.* In order to test the numerical schemes, we start with a simple homogeneous medium (single rock type) by setting the rock permeability as

$$K \equiv 1.$$

The results of a mesh refinement study for the above Riemann problem are presented in Figure 1. In this figure, we show the water saturation at time $T = 0.6$ for a series of mesh resolutions ranging between 200 and 1600 mesh points. In the absence of an exact solution, we notice that the simulations converge to a weak solution of the scalar conservation (1.10) that consists of a classical shock wave (satisfying the Lax entropy condition) as well as a *non-classical* shock wave with an intermediate state in between. The non-classical shock violates the Lax entropy condition and is a consequence of the fact that the flux function f in (1.1) is non-convex. Non-classical shocks for two phase flow with dynamic capillary pressure have also been studied in [9] and [17]. A detailed discussion on non-classical shocks is presented in [18]. The results presented in Figure 1 clearly show that the scheme (3.1) provides a good approximation of the equation (1.1) in the case of a homogeneous medium and is able to resolve interesting small scale dependent shock waves such as non-classical shock waves. Note that a good resolution of the non-classical shock wave (in particular of the intermediate state) is provided, even on a fairly coarse resolution of 200 mesh points.

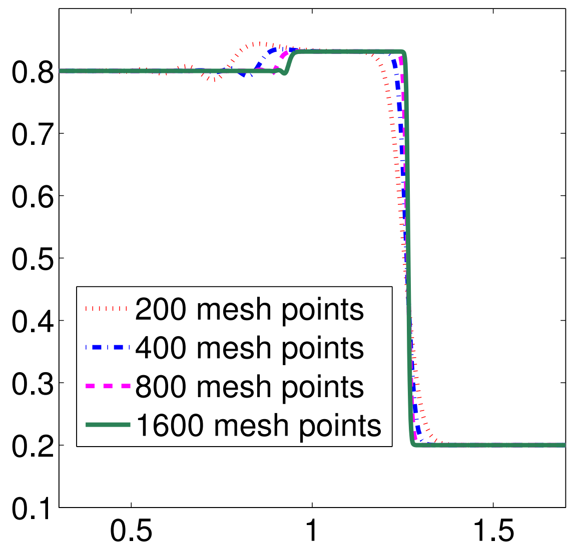

FIGURE 1. Water saturation (Y-axis) with respect to the spatial domain (X-axis) at time $T = 0.6$ for the homogeneous medium with a single rock permeability $K = 1$. Results of a mesh convergence study showing convergence to a non-classical solution of the underlying scalar conservation law.

3.2.2. *Flow in a heterogeneous medium.* Next, we consider a heterogeneous medium by using the rock permeability:

$$K = 1 \cdot 1_{\{x \leq 0.6\}} + 1.2 \cdot 1_{\{x > 0.6\}}. \tag{3.3}$$

Thus, we consider flow in a medium that has two rock types with a sharp interface at $x = 0.6$. We use the same parameters and initial data as in the previous numerical experiment and show the results of a mesh convergence study in Figure 2. The figure shows that the approximate solutions generated by the scheme (3.1) converge to a weak solution of the scalar conservation law with discontinuous coefficient (1.10) that consists of the following three waves: i) a non-classical shock wave on the right, ii) a classical shock wave in the middle, and iii) a discontinuity at the rock type interface at $x = 0.6$. It is easy to check that the standing wave at $x = 0.6$ satisfies flux continuity conditions (Rankine-Hugoniot conditions for the scalar conservation law with discontinuous flux). Again, the results show the robustness of the scheme and its ability to resolve interesting solution features, even at coarse resolutions.

3.2.3. *Comparison.* We conclude this section by providing a comparison between a homogeneous and a heterogeneous medium. The parameters are from the two previous numerical experiments and the water saturation, computed with scheme (3.1), is plotted in Figure 3. The figure brings out the role of the discontinuity quite well. The discontinuity creates the standing wave at the interface and changes the speeds of both the classical as well as the non-classical shock waves. Thus, it is essential to consider the effect of the discontinuity in rock permeability as it has a

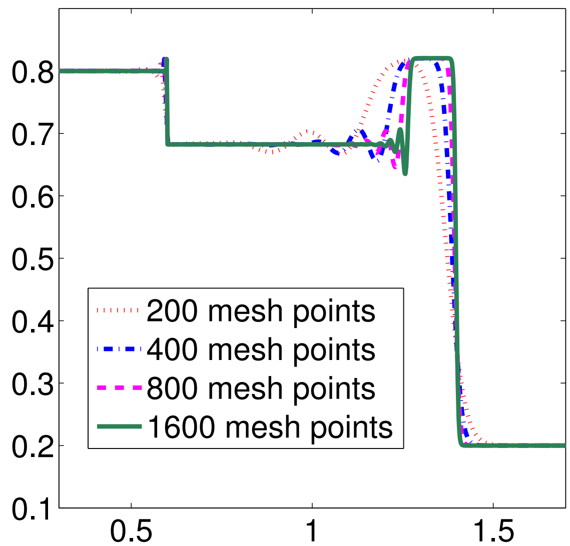

FIGURE 2. Water saturation (Y-axis) with respect to the spatial domain (X-axis) in a heterogeneous porous medium with discontinuous rock permeability (3.3), computed with the scheme (3.1) at time $T = 0.6$. Note the resolution of both the non-classical shock wave as well as the standing wave at the interface $x = 0.6$.

major effect on the wave structure of the flow. Also, the effect of dynamic capillary pressure is crucial as it generates non-classical shock waves in the vanishing capillary pressure limit. This should be contrasted with the solutions obtained with a static capillary pressure, as were considered in [1, 19], where only classical shock waves were present away from the interface.

4. **Conclusion.** We considered a scalar conservation law with a spatially dependent and possibly discontinuous coefficient. It is well known that these equations contain infinitely many L^1 contractive semigroups of solutions and the standard Kruzkhov entropy conditions do not suffice to single out a physically relevant solution. It has also been established in [1, 2] and references therein that the solutions to such equations depend *explicitly* on the underlying small scale effects. Here, we consider a concrete model of two phase flow in a heterogeneous porous medium. The relevant small scale effect is a dynamic capillary pressure term, that was introduced in [12]. Compared to the standard capillary pressure models [4], the addition of the new term resulted in a model that contain higher-order mixed spatio-temporal derivatives.

In the current paper, we considered the regularized problem corresponding to the dynamic capillary pressure as well as a discontinuous spatial coefficient in the flux, diffusion and dispersion terms. We showed rigorously that the limit under vanishing capillary pressure exists and is a weak solution of the corresponding scalar conservation law with discontinuous coefficient. Thus, it makes sense to talk about the

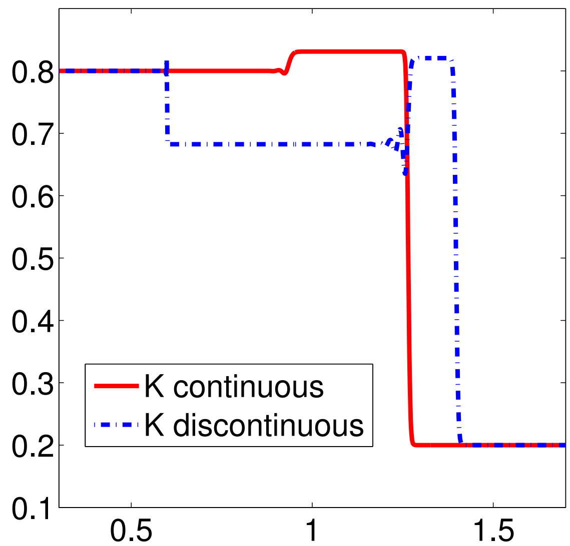

FIGURE 3. Comparison of water saturation (Y-axis) with respect to the spatial domain (X-axis), for a homogeneous medium with constant rock permeability and a heterogeneous medium with discontinuous rock.

semigroup of solutions for the scalar conservation law with discontinuous coefficients that arises as a limit of vanishing dynamic capillary pressure.

We introduced a finite difference scheme in order to approximate the regularized problem. The numerical scheme was tested on a suite of numerical experiments and was shown to be robust. It enabled us to characterize the vanishing dynamic capillary pressure limit, in some concrete examples. In particular, we found that the limit solution can contain non-classical shock waves as well as standing waves at the interface between different rock types. This numerical scheme can be employed to identify the vanishing dynamic capillary pressure limit in a large number of cases.

In the future, we would like to extend the methods of this paper to models of multi-dimensional porous medium flows. The numerical scheme will also be extended in this setting to characterize the vanishing dynamic capillary pressure limit in several space dimensions.

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