# HOMOGENIZATION OF HIGH-CONTRAST AND NON SYMMETRIC CONDUCTIVITIES FOR NON PERIODIC COLUMNAR STRUCTURES

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Abstract. In this paper we determine, in dimension three, the effective conductivities of non periodic and high-contrast two-phase cylindrical composites, placed in a constant magnetic field, without any assumption on the geometry of their cross sections. Our method, in the spirit of the H-convergence of Murat-Tartar, is based on a compactness result and the cylindrical nature of the microstructure. The homogenized laws we obtain extend those of the periodic fibre-reinforcing case of [17] to the case of periodic and non periodic composites with more general transversal geometries.

1. **Introduction.** At the end of the 19th century, it was discovered [24] that a constant magnetic field $h$ modifies the symmetric conductivity matrix $\sigma$ of a conductor into a non symmetric matrix $\sigma(h)$. This is known as the Hall effect. In the Maclaurin series of the perturbed resistivity $(\sigma(h))^{-1}$ the zeroth-order term coincides with the resistivity $\sigma^{-1}$ in the absence of a magnetic field [27]. In dimension two, $h$ is a scalar and the first-order term is an antisymmetric matrix proportional to $hJ$; the coefficient of proportionality is called the Hall coefficient. In dimension three, $h \in \mathbb{R}^3$ and the first-order term, in the Maclaurin series of $(\sigma(h))^{-1}$, is of the form $\mathscr{E}(Rh)$ where $\mathscr{E}(\xi)j := \xi \times j$ and $R$ is a $3 \times 3$ matrix called the Hall matrix [16]. In this work, we consider the idealized situation when the induced non symmetric part is proportional to the applied magnetic field: $\sigma(h) = \alpha I_3 + \beta\mathscr{E}(h)$, where $\alpha$ and $\beta$ are two constant real numbers. For a given sequence of perturbed conductivities $\sigma_n(h)$, it is of great interest, in electrodynamics [27, 32], to understand the influence of the magnetic field $h$ on the effective Hall coefficient or the effective Hall matrix through the homogenization of $\sigma_n(h)$.

Let us first review a few of the mathematical theory of homogenization of elliptic partial differential equations of the form

$$\begin{cases} -\operatorname{div}\left(\sigma_n \nabla u_n\right) & = & f & \text{in } \Omega, \\ u_n & = & 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^3$, $\sigma_n$ is a sequence of matrix-valued functions in $L^\infty(\Omega)^{3\times 3}$ and $f$ is an element of $H^{-1}(\Omega)$. This topic has been intensively studied for the last four decades providing a wide literature [34, 30, 31, 2]. In the context of conduction, when the conductivity matrices $\sigma_n$ are uniformly bounded, Spagnolo [34] with the $G$-convergence theory, Murat and Tartar [30, 31] with the $H$-convergence theory showed that the solution $u_n \in H_0^1(\Omega)$ of the conductivity problem (1) strongly converges in $L^2(\Omega)$, up to a subsequence of $n$, to the solution of a limit conductivity problem of the same nature. The case of high-contrast conductivities is very different since non classical phenomena, such as nonlocal terms, may appear in the limit problem as shown, for instance, in [19, 25, 1, 18, 11, 26]. This does not happen in dimension two if the sequence $\sigma_n$ is uniformly bounded from below. Briane [10] and Casado-Días & Briane [13] proved that in that case the class of equations (1) is always compact in the sense that the limit equation of (1) is always of the same type. In [13] they proved some extensions of the well-known div-curl lemma of Murat-Tartar [31] and deduce several compactness results under the assumption of equicoerciveness coupled with the $L^1$-boundedness of the sequence of conductivities.

In this paper we are interested in the homogenization of a class of three-dimensional conductivity problems of the type

$$\begin{cases} -\operatorname{div}\left(\sigma_n(h)\nabla u_n\right) & = & f & \text{in } \Omega, \\ u_n & = & 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\sigma_n(h)$ is an equi-coercive sequence of high-contrast two-phase conductivities perturbed by a constant magnetic field $h \in \mathbb{R}^3$ of the form

$$\sigma_n(h) := (1 - \mathbb{1}_{\Omega_n})\sigma_1(h) + \mathbb{1}_{\Omega_n}\sigma_{2,n}(h)$$

where $\sigma_{2,n}(h)$ is the unbounded perturbed conductivity of the highly conducting phase $\Omega_n$ and $\sigma_1(h)$ is the perturbed conductivity of the phase surrounding $\Omega_n$.

In dimension two, for the case of low magnetic field, Bergman [3] was the first author who came up with a general formula for the effective Hall coefficient of a periodic composite material in terms of the local Hall coefficients and some local currents solving the conductivity equations in the absence of a magnetic field. We refer also to the works [28, 4, 15] for other two-dimensional composites, to [5, 7, 8, 21, 22] for composites with microstructure independent of one coordinate (the so-called columnar composites) and to [6, 9] for the case of strong magnetic field.

Recently, in dimension two, M. Briane and the second author [17] obtained the effective perturbed conductivity $\sigma_*(h)$ of a sequence of isotropic high-contrast two-phase conductivities $\sigma_n(h)$ in the case of strong magnetic field, *i.e.*, when the symmetric part and the antisymmetric part of the conductivity are of the same order. By extending a duality principle from [14] and using a suitable Dykhne transformation, which (following Milton [28, 29]) changes non symmetric matrices into symmetric ones, they proved that the symmetric part of the effective perturbed conductivity $\sigma_*(h)$ is given in terms of the effective conductivity in the absence of

a magnetic field. They subsequently compared their two-dimensional results to a three-dimensional periodic one and showed that the way a magnetic field perturbs the conductivity of a composite depends on the dimension. In order to compute the explicit perturbation formula in dimension three, they restricted themselves to a particular periodic fibre-reinforced structure, *i.e.,* a structure completely described by any two-dimensional cross section transversal to the fibres (first introduced by Fenchenko, Khruslov [19] to derive a non local effect in homogenization). To our knowledge, only few results are known on the homogenization of both high-contrast and non symmetric conductivities in dimension three.

The aim of this paper is to determine the effective perturbed conductivity of (2) for non periodic high-contrast two-phase cylindrical composites without any assumption on the geometry of the transversal microstructure.

We first investigate the periodic case, *i.e.,* $\sigma_n(h)(\cdot) = \Sigma_n(h)(\cdot/\varepsilon_n)$ where $\Sigma_n(h)(\cdot)$ is a $Y$-periodic matrix-valued function and $\varepsilon_n \to 0$ represents the size of the heterogeneities in the composite. In order to avoid non local effects in the limit problem, following Briane [12], we assume the existence of a sequence of positive numbers $c_n$ such that $\varepsilon_n^2 c_n$ tends to zero, as $n$ goes to infinity, and satisfying the weighted Poincaré-Wirtinger inequality

$$\forall\, V \in H^1(Y), \quad \int_Y |\Sigma_n(h)(y)| \left| V - \int_Y V \,\mathrm{d}y \right|^2 \mathrm{d}y \le c_n \int_Y \Sigma_n(y)\nabla V \cdot \nabla V \,\mathrm{d}y.$$

For a fixed $n \in \mathbb{N}^*$, using the theory of exact relations of Grabovsky, Milton and Sage [21, 20] (thanks to the independence of the microstructure of the variable $x_3$), we obtain the H-limit $(\sigma_n)_*$ associated with the periodic homogenization [2] of the oscillating sequence $\Sigma_n(\cdot/\varepsilon)$ as $\varepsilon \to 0$. Then, we show that the sequence of constant conductivities $(\sigma_n)_*$ converges to some $\sigma_*(h)$ which, according to [12], coincides with the homogenized conductivity associated with the limit problem of (2). The obtained effective conductivity $\sigma_*(h)$ is explicitly computed in terms of the homogenized conductivity $\widetilde{\sigma}_*(h)$ of the conduction problem posed in the $(x_1, x_2)$-plane transversal to the columnar composite (see Proposition 1).

Most of the arguments and tools used in the periodic case crucially lie on the periodic nature of the microstructure. Therefore, a fundamentally different approach is necessary for the analysis of (2) when $\sigma_n(h)$ is not periodic.

In order to study the asymptotic behavior of the problem (2) in the non periodic case, using a method, in the spirit of the H-convergence of Murat-Tartar, we determine the limit, in an appropriate sense, of the current $\sigma_n(h)\nabla u_n$. The key ingredient of this approach is a fundamental compactness result (see Lemma 3.2) based on a control of high conductivities in thin structures through weighted Poincar-Wirtinger type inequalities. This compactness lemma, combined with the two-dimensional results of [17] and the cylindrical structure of the composite allows us to obtain an explicit formula of $\sigma_*(h)$, once again, in terms of the transversal homogenized conductivity $\widetilde{\sigma}_*(h)$ and of some bounded function $\theta$ which, in some sense, takes account of the distribution of the highly conducting phase $\Omega_n$ in $\Omega$ (see Theorem 3.1).

The structure of the paper is the following: In Section 1.1 we set up some general notations. Section 2 deals with the periodic case. In Section 3 we extend the periodic result of Section 2 to a non periodic framework. Section 4 is devoted to some examples illustrating both the periodic and non periodic perturbation formulas.

Here, we give some general notations and definitions.

## 1.1. General notations and definitions.

- $\Omega$ is a bounded open subset of $\mathbb{R}^3$ with a Lipschitz boundary. The unit cube $(-\frac{1}{2}, \frac{1}{2})^3$ of $\mathbb{R}^3$ is denoted by $Y$.
- For any subset $\omega$ of $\Omega$, we denote by $\overline{\omega}$ the closure of $\omega$ in $\mathbb{R}^3$.
- $\varepsilon_n$ is a sequence of positive real numbers converging to zero as $n$ goes to infinity.
- For any matrix $\sigma$ in $\mathbb{R}^{d\times d}$, $\sigma^{\mathrm{T}}$ denotes the transpose of the matrix $\sigma$ while $\sigma^s$ denotes its symmetric part. For any invertible matrix $\sigma$ in $\mathbb{R}^{d\times d}$, $\sigma^{-\mathrm{T}}$ denotes the tranpose matrix of $\sigma^{-1}$.
- $I_d$ denotes the unit matrix in $\mathbb{R}^{d\times d}$ and $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.
- For any $h \in \mathbb{R}^3$, we denote by $\mathscr{E}(h)$ the $3 \times 3$ antisymmetric matrix defined by $\mathscr{E}(h)\,x := h \times x$, for any $x \in \mathbb{R}^3$.
- For any $\sigma, \eta \in \mathbb{R}^{d\times d}$, $\sigma \leq \eta$ means that for any $\xi \in \mathbb{R}^d$, $\sigma\xi \cdot \xi \leq \eta\xi \cdot \xi$.
- For any vector $\xi \in \mathbb{R}^3$, $\widetilde{\xi} \in \mathbb{R}^2$ denotes the vector of its first two components

$$\widetilde{\xi} := (\xi_1, \xi_2)^{\mathrm{T}}.$$

- $\nabla$ denotes the gradient operator in $\mathbb{R}^3$ with respect to the variables $x_1, x_2, x_3$ while $\widetilde{\nabla}$ is the gradient operator in $\mathbb{R}^2$ with respect to the first two variables $x_1, x_2$: for any $u \in H^1(\Omega)$, the function $\widetilde{\nabla}u$ is defined on $\widetilde{\Omega}$ by

$$\widetilde{\nabla}u := \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right)^{\mathrm{T}},$$

where $\widetilde{\Omega}$ is the projection of $\Omega$ on the $(x_1, x_2)$-plane.

- For any $3 \times 3$ matrix $\sigma$, we denote by $\widetilde{\sigma}$ the $2 \times 2$ matrix defined by

$$\widetilde{\sigma} := \left(\begin{smallmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{smallmatrix}\right).$$

- The scalar product of two vectors $u$ and $v$ of $\mathbb{R}^d$ is denoted by $u \cdot v$.
- $|\cdot|$ denotes, the euclidean norm in $\mathbb{R}^d$, the subordinate norm in $\mathbb{R}^{d\times d}$ and the Lebesgue measure.
- For a Borel subset $\omega \in \mathbb{R}^d$ and a function $u \in L^1(\omega)$ the average value of $u$ over $\omega$ is denoted by

$$\fint_\omega u \, \mathrm{d}x := \frac{1}{|\omega|} \int_\omega u \, \mathrm{d}x.$$

When $\omega = Y$, we simply denote this average value by $\langle \cdot \rangle$.

- We denote by $\mathbb{1}_\omega$ the characteristic function of the set $\omega$.
- We denote by $\mathscr{C}_c(\Omega)$ the set of continuous functions with compact support in $\Omega$. The subspace of $\mathscr{C}_c(\Omega)$ of infinitely differentiable functions with compact support in $\Omega$ is denoted by $\mathscr{D}(\Omega)$.
- We denote by $\mathscr{C}_0(\Omega)$ the space of continuous functions on $\overline{\Omega}$ vanishing on the boundary $\partial\Omega$ of $\Omega$ endowed with the usual norm.
- For any locally compact subset $X$ of $\mathbb{R}^d$, $\mathcal{M}(X)$ denotes the set of Radon measures defined on $X$.
- A sequence $(\mu_n)$ in $\mathcal{M}(\Omega)$ is said to weakly-$*$ converge to a measure $\mu$ if

$$\int_\Omega \varphi\mu_n(\mathrm{d}x) \xrightarrow[n\to\infty]{} \int_\Omega \varphi\mu(\mathrm{d}x), \quad \text{for any } \varphi \in \mathscr{C}_0(\Omega).$$

- The space of $Y$-periodic functions which belong to $L^p_{\text{loc}}(\mathbb{R}^d)$ (resp. $H^1_{\text{loc}}(\mathbb{R}^d)$) is denoted by $L^p_\#(Y)$ (resp. $H^1_\#(Y)$).
- $o(\delta)$ denotes a term of the form $\delta\zeta(\delta)$ where the limit of $\zeta(\delta)$ is zero, as $\delta$ goes to zero. For any sequences $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}^*}$, $a_n \underset{n\to\infty}{\sim} b_n$ means that $a_n = b_n + o(b_n)$.
- Throughout the paper, the letter $c$ denotes a positive constant the value of which is not given explicitly and may vary from line to line.

In the sequel, we will use the following extension of H-convergence for two-dimensional high-contrast conductivities introduced in [13] for the symmetric case and extended in [14] to the non symmetric case:

**Definition 1.1.** Let $\widetilde{\Omega}$ be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz boundary and let $\widetilde{\sigma}_n \in L^\infty(\Omega)^{2\times 2}$ be a sequence of equi-coercive matrix-valued functions. The sequence $\widetilde{\sigma}_n$ is said to $H(\mathcal{M}(\widetilde{\Omega})^2)$-converge to a matrix-valued function $\widetilde{\sigma}_*$ if for any distribution $g$ in $H^{-1}(\widetilde{\Omega})$, the solution $u_n$ of the problem

$$\begin{cases} \text{div}\left(\widetilde{\sigma}_n\widetilde{\nabla}u_n\right) = g & \text{in } \widetilde{\Omega}, \\ u_n = 0 & \text{on } \partial\widetilde{\Omega}, \end{cases}$$

satisfies the convergences

$$\begin{cases} u_n \quad\longrightarrow\quad u & \text{in } H^1_0(\widetilde{\Omega}), \\ \widetilde{\sigma}_n\widetilde{\nabla}u_n \quad\longrightarrow\quad \widetilde{\sigma}_*\widetilde{\nabla}u & \text{weakly-}* \text{ in } \mathcal{M}(\widetilde{\Omega})^2, \end{cases}$$

where $u$ is the solution of the problem

$$\begin{cases} \text{div}\left(\widetilde{\sigma}_*\widetilde{\nabla}u\right) = g & \text{in } \widetilde{\Omega}, \\ u = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

Let $\widetilde{\Omega}$ be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz boundary and $\widetilde{\Omega}_n$ be a sequence of open subsets of $\widetilde{\Omega}$. Let $\Omega$ be the bounded open cylinder $\Omega := \widetilde{\Omega} \times (0,1)$ and $\Omega_n$ the sequence of open cylinders $\Omega_n := \widetilde{\Omega}_n \times (0,1)$. Consider $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$ and two sequences $\alpha_{2,n} \geq \alpha_1$ and $\beta_{2,n} \in \mathbb{R}$. Define, for any $h \in \mathbb{R}^3$, the two-phase isotropic conductivity

$$\sigma_n(h) := \begin{cases} \sigma_1(h) := \alpha_1 I_3 + \beta_1\mathscr{E}(h) & \text{in } \Omega \setminus \Omega_n, \\ \sigma_{2,n}(h) := \alpha_{2,n} I_3 + \beta_{2,n}\mathscr{E}(h) & \text{in } \Omega_n, \end{cases}$$

where

$$\mathscr{E}(h) := \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}.$$

In the domain $\Omega$, the matrix-valued function $\sigma_n(h)$ does not depend on the variable $x_3$ and models the conductivity of a columnar heterogeneous medium. The phase $\Omega_n$ is the one of high conductivity: $\alpha_{2,n}$ and $\beta_{2,n}$ are unbounded. In order to ensure the $L^1(\Omega)^{3\times 3}$-boundedness of the conductivity, we assume that the volume fraction of the highly conducting phase $\theta_n := |\Omega|^{-1}|\Omega_n|$ converges to zero and that the convergences

$$\begin{cases} \theta_n\alpha_{2,n} \xrightarrow[n\to\infty]{} \alpha_2 > 0, \\ \theta_n\beta_{2,n} \xrightarrow[n\to\infty]{} \beta_2 \in \mathbb{R}, \end{cases} \tag{3}$$

hold. Assumption (3) can be rewritten

$$\theta_n\sigma_{2,n}(h) = \theta_n\alpha_{2,n}I_3 + \theta_n\beta_{2,n}\mathscr{E}(h) \xrightarrow[n\to\infty]{} \sigma_2(h) := \alpha_2 I_3 + \beta_2\mathscr{E}(h).$$

Our aim is to study the homogenization of the Dirichlet problem, for $f \in H^{-1}(\Omega)$,

$$\begin{cases} -\operatorname{div}\left(\sigma_n(h)\nabla u_n\right) &= f \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4}$$

On the one hand, we consider the case of a periodic cylindrical composite without any assumption on the geometry of its cross section. This framework extends the one of the three-dimensional result of [17] where the highly conducting zone is a set of circular fibres. On the other hand, by the means of a compactness result (see Lemma 3.2), we analyse the case of cylindrical but non periodic composites. In both cases, we impose conditions, adapting [12], that prevent from the appearance of non local terms so that the limit equation of (4) is a conductivity one.

In the sequel, we will omit the dependence on $h$ of $\sigma_1(h)$, $\sigma_{2,n}(h)$ and $\sigma_2(h)$ denoting simply $\sigma_1$, $\sigma_{2,n}$ and $\sigma_2$, respectively.

2. **The periodic case.** In this section, we study the influence of a constant magnetic field $h \in \mathbb{R}^3$ on the effective conductivity of a composite material where the highly conducting phase is periodically distributed but, contrary to [17], the cross section of which has a general geometry. Consider a sequence $\omega_n = \widetilde{\omega}_n \times (0,1)$ where $\widetilde{\omega}_n$ is a sequence of subsets of $(0,1)^2$ with $|\omega_n|$ converging to 0, as $n$ tends to infinity. Let $\Omega_n$ be the sequence of open subsets of $\Omega$ defined by

$$\Omega_n = \Omega \cap \bigcup_{k \in \mathbb{Z}^3} \varepsilon_n\left(\omega_n + k\right).$$

The conductivity of the heterogeneous medium occupying $\Omega$ is given by

$$\sigma_n(h)(x) = \Sigma_n(h)\left(\frac{x}{\varepsilon_n}\right), \quad \forall x \in \Omega, \tag{5}$$

where $\Sigma_n(h)(\cdot)$ is the $Y$-periodic function defined by

$$\Sigma_n(h) = a_n I_3 + b_n \mathscr{E}(h) \quad \text{with} \quad \begin{cases} a_n := \alpha_1 \mathbb{1}_{Y \setminus \omega_n} + \alpha_{2,n} \mathbb{1}_{\omega_n}, \\ b_n := \beta_1 \mathbb{1}_{Y \setminus \omega_n} + \beta_{2,n} \mathbb{1}_{\omega_n}. \end{cases} \tag{6}$$

For a fixed $n \in \mathbb{N}^*$, let $(\sigma_n)_*(h)$ be the constant matrix defined by

$$\forall \lambda \in \mathbb{R}^3, \quad (\sigma_n)_*(h)\lambda = \left\langle \Sigma_n(h)\nabla W_n^\lambda \right\rangle, \tag{7}$$

where, for any $\lambda \in \mathbb{R}^3$, $W_n^\lambda$ is the unique solution in $H_\sharp^1(Y)$ of the auxiliary problem

$$\operatorname{div}\left(\Sigma_n(h)\nabla W_n^\lambda\right) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^3) \quad \text{and} \quad \left\langle W_n^\lambda - \lambda \cdot y \right\rangle = 0, \tag{8}$$

which is equivalent to the variational cell problem

$$\begin{cases} \left\langle \Sigma_n(h)\nabla W_n^\lambda \cdot \nabla \Phi \right\rangle = 0, \quad \forall \Phi \in H_\sharp^1(Y), \\ \left\langle W_n^\lambda(y) - \lambda \cdot y \right\rangle = 0. \end{cases} \tag{9}$$

The matrix $(\sigma_n)_*(h)$ is the homogenized conductivity of the highly oscillating sequence $\Sigma_n(\cdot/\varepsilon)$ as $\varepsilon \to 0$ (see, for instance, [2] for more details).

The limit problem of the high-contrast three-dimensional equation (4) with $\sigma_n(h)$ given by (5) may include non local effects. In order to avoid such effects, we assume, following [12], that the weighted Poincar-Wirtinger inequality

$$\forall V \in H^1(Y), \quad \int_Y a_n \left| V - \int_Y V \right|^2 \le C_n \int_Y a_n |\nabla V|^2, \tag{10}$$

holds true with

$$\varepsilon_n^2 C_n \xrightarrow[n \to \infty]{} 0. \tag{11}$$

Under the assumptions (10) and (11), it was shown in [12] that the sequence of problems (4) converges to a conduction one with a homogenized conductivity $\sigma_*(h)$.

The main contribution of Proposition 1 below is to provide a formula for the effective conductivity $\sigma_*(h)$ of a cylindrical periodic composite the cross section of which has a general geometry.

**Proposition 1.** *Consider the sequence of problems* (4) *where* $\sigma_n(h)$ *is the conductivity defined by* (5)-(6). *Assume that* (3), (10) *and* (11) *are satisfied. Then, there exists a constant matrix* $\sigma_*(h)$ *such that, up to a subsequence, the solution* $u_n$ *of* (4) *weakly converges in* $H_0^1(\Omega)$ *to the solution* $u$ *of*

$$\begin{cases} -\operatorname{div}\big(\sigma_*(h)\nabla u\big) &=& f & in \ \Omega, \\ u &=& 0 & on \ \partial\Omega. \end{cases} \tag{12}$$

*Moreover, the homogenized matrix* $\sigma_*(h)$ *is the limit of* $\big(\sigma_n\big)_*(h)$ *(see* (7)*) and is given by*

$$\sigma_*(h) := \begin{pmatrix} \widetilde{\sigma}_* & p_* \\ q_*^T & \alpha_* \end{pmatrix}, \tag{13}$$

*where*

$$\begin{cases} p_* = -\big[\beta_1 I_2 + \beta_2\big(\widetilde{\sigma}_* - \widetilde{\sigma}_1\big)\widetilde{\sigma}_2^{-1}\big] J\widetilde{h}, \\ q_* = \big[\beta_1 I_2 + \beta_2\,\widetilde{\sigma}_2^{-1}\big(\widetilde{\sigma}_* - \widetilde{\sigma}_1\big)\big]^T J\widetilde{h}, \\ \alpha_* = \alpha_1 + \alpha_2 + \beta_2^2\,\widetilde{\sigma}_2^{-1}\big(\widetilde{\sigma}_1 + \widetilde{\sigma}_2 - \widetilde{\sigma}_*\big)\widetilde{\sigma}_2^{-1} J\widetilde{h} \cdot J\widetilde{h}, \end{cases} \tag{14}$$

*and, for any* $i = 1, 2$,

$$\widetilde{\sigma}_i := \begin{pmatrix} \alpha_i & -\beta_i\,h_3 \\ \beta_i\,h_3 & \alpha_i \end{pmatrix}.$$

**Remark 1.** For the sake of simplicity, throughout the paper, the symmetric part of $\sigma_n(h)$ is supposed to be isotropic. However, the results we obtain can be extended to composites the components of which have anisotropic conductivities.

**Remark 2.** It was shown in [12] that, due to the $L^1(Y)^{3\times 3}$-boundedness of $\Sigma_n(h)(\cdot)$, the sequence $\big(\sigma_n\big)_*(h)$ is bounded. Thanks to (10) and (11), Theorem 2.1 of [12] ensures that the limit $\sigma_*(h)$ obtained in the following way

$$\Sigma_n(h)\left(\frac{x}{\varepsilon}\right) \ \xrightarrow[\varepsilon\to 0]{H} \ \big(\sigma_n\big)_*(h) \ \xrightarrow[n\to\infty]{} \sigma_*(h),$$

satisfies the convergence

$$\sigma_n(h)\nabla u_n \ \longrightarrow \ \sigma_*(h)\nabla u \quad in \ \mathcal{M}(\Omega)^3,$$

and then, coincides with the homogenized conductivity matrix in the problem (12).

**Remark 3.** Since $\Omega_n$ has a columnar structure, the sequence $\widetilde{\sigma}_n(h)$ given by

$$\widetilde{\sigma}_n(h) := \widetilde{\sigma}_n(h_3) = \begin{cases} \widetilde{\sigma}_1(h_3) = \alpha_1 I_2 + \beta_1 h_3 J & in \ \widetilde{\Omega} \setminus \widetilde{\Omega}_n, \\ \widetilde{\sigma}_{2,n}(h_3) = \alpha_{2,n} I_2 + \beta_{2,n} h_3 J & in \ \widetilde{\Omega}_n. \end{cases}$$

depends only on the transversal variable $(x_1, x_2)$ and then is associated with the two-dimensional problems, for any $g \in H^{-1}(\widetilde{\Omega})$,

$$\begin{cases} -\operatorname{div}\big(\widetilde{\sigma}_n(h_3)\widetilde{\nabla} v_n\big) &=& g & in \ \widetilde{\Omega}, \\ v_n &=& 0 & on \ \partial\widetilde{\Omega}. \end{cases}$$

Similarly to (7), we define the constant matrix $\big(\widetilde{\sigma}_n\big)_*(h_3)$. For any $\lambda \perp e_3$, the solution $W_n^\lambda$ of (8) does not depend on the variable $y_3$ and then

$$\big\langle W_n^\lambda - \widetilde{\lambda} \cdot \widetilde{y} \big\rangle = 0 \quad \text{and} \quad \text{div}\,\big(\widetilde{\Sigma}_n(h_3)\widetilde{\nabla}W_n^\lambda\big) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^2).$$

This equation and (8) imply that, for any $\lambda, \mu \perp e_3$,

$$\big(\sigma_n\big)_*(h)\lambda \cdot \mu = \big\langle \Sigma_n(h)\nabla W_n^\lambda \big\rangle \cdot \mu = \big\langle \widetilde{\Sigma}_n(h_3)\widetilde{\nabla}W_n^\lambda \big\rangle \cdot \widetilde{\mu} = \big(\widetilde{\sigma}_n\big)_*(h_3)\widetilde{\lambda} \cdot \widetilde{\mu}.$$

Hence, by Remark 2, $\big(\widetilde{\sigma}_n\big)_*(h_3)$ converges to the $2 \times 2$ matrix $\widetilde{\sigma}_*$ involved in (13). A two-dimensional perturbation formula in [17] gives the influence of the magnetic field $h_3$ on $\widetilde{\sigma}_*$:

$$\widetilde{\sigma}_* := \widetilde{\sigma}_*(h_3) = \sigma_*^0\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h_3^2\big) + h_3\beta_1 J, \tag{15}$$

where $\sigma_*^0$ is a locally Lipschitz function defined on $(0, \infty)^2$, and for any $\alpha_1, \alpha_2 > 0$, the matrix $\sigma_*^0(\alpha_1, \alpha_2)$ is the transversal homogenized conductivity in the absence of a magnetic field. The independence of the microstructure of the variable $x_3$ allows us to obtain an explicit expression of $\sigma_*(h)$ in terms of the transversal homogenized conductivity $\sigma_*^0$ in the absence of a magnetic field.

**Remark 4.** In the case where the high conducting phase is a set of circular fibres, it was proved in [17] that $\sigma_*^0(\alpha_1, \alpha_2) = \alpha_1 I_2$ and the limit $\sigma_*(h)$ in (13) reduces to

$$\sigma_*(h) = \alpha_1 I_3 + \Big[\alpha_2 + \beta_2^2\,\widetilde{\sigma}_2^{-1}J\widetilde{h} \cdot J\widetilde{h}\Big]\,e_3 \otimes e_3 + \beta_1\mathscr{E}(h).$$

Now, let us proceed with the proof of Proposition 1.

*Proof of Proposition 1.* Thanks to Remarks 2 and 3, there exists a $3 \times 3$ matrix $\sigma_*(h)$ such that, up to a subsequence, we have the convergence of constant matrices

$$\big(\sigma_n\big)_*(h) \xrightarrow[n\to\infty]{} \sigma_*(h) := \begin{pmatrix} \widetilde{\sigma}_* & p_* \\ q_*^{\mathrm{T}} & \alpha_* \end{pmatrix}, \tag{16}$$

where $\widetilde{\sigma}_*$ is given by (15) and where the constants $q_*, p_* \in \mathbb{R}^2$, $\alpha_* \in \mathbb{R}$ have to be determined. To this end, we divide the proof into two steps. We first apply Grabovsky and Milton's method [21, 22, 23] to link $\big(\sigma_n\big)_*$ to a more simple problem. Then, we study the asymptotic behavior of the different coefficients of this new problem.

**First step.** A transformation stable under homogenization. For a fixed $n \in \mathbb{N}^*$, following Grabovsky and Milton [23, 21], we consider two vectors $p_{0,n}, q_{0,n} \in \mathbb{R}^2$ and the transformation

$$\sigma_n' := \Pi_n\,\sigma_n(h)\,\widehat{\Pi}_n = \begin{pmatrix} \widetilde{\sigma}_n & p_n' \\ q_n'^{\mathrm{T}} & \alpha_n' \end{pmatrix}, \tag{17}$$

where

$$\Pi_n := \begin{pmatrix} I_2 & 0 \\ q_{0,n}^{\mathrm{T}} & 1 \end{pmatrix}, \quad \widehat{\Pi}_n := \begin{pmatrix} I_2 & p_{0,n} \\ 0 & 1 \end{pmatrix}, \tag{18}$$

and

$$p_n' = \begin{cases} \widetilde{\sigma}_1 p_{0,n} - \beta_1 J\widetilde{h} & \text{in } \Omega \setminus \Omega_n, \\ \widetilde{\sigma}_{2,n}p_{0,n} - \beta_{2,n}J\widetilde{h} & \text{in } \Omega_n, \end{cases} \qquad q_n' = \begin{cases} \widetilde{\sigma}_1^{\mathrm{T}}q_{0,n} + \beta_1 J\widetilde{h} & \text{in } \Omega \setminus \Omega_n, \\ \widetilde{\sigma}_{2,n}^{\mathrm{T}}q_{0,n} + \beta_{2,n}J\widetilde{h} & \text{in } \Omega_n. \end{cases} \tag{19}$$

Let us choose the parameters $p_{0,n}$ and $q_{0,n}$ in such a way that $p'_n$ and $q'_n$ are constant. To that aim, $p_{0,n}$ and $q_{0,n}$ have to satisfy the identities

$$\begin{cases} \widetilde{\sigma}_1 p_{0,n} - \beta_1 J\widetilde{h} & = & \widetilde{\sigma}_{2,n} p_{0,n} - \beta_{2,n} J\widetilde{h}, \\ \widetilde{\sigma}_1^{\mathrm{T}} q_{0,n} + \beta_1 J\widetilde{h} & = & \widetilde{\sigma}_{2,n}^{\mathrm{T}} q_{0,n} + \beta_{2,n} J\widetilde{h}, \end{cases}$$

which implies that

$$p_{0,n} = (\beta_{2,n} - \beta_1)(\widetilde{\sigma}_{2,n} - \widetilde{\sigma}_1)^{-1} J\widetilde{h} \quad \text{and} \quad q_{0,n} = (\beta_{2,n} - \beta_1)(\widetilde{\sigma}_1 - \widetilde{\sigma}_{2,n})^{-\mathrm{T}} J\widetilde{h}.$$

The new matrix-valued function $\sigma'_n$ defined by (17) is periodic and can be rewritten

$$\forall x \in \Omega, \quad \sigma'_n(x) = \Sigma'_n\left(\frac{x}{\varepsilon_n}\right) \quad \text{where} \quad \Sigma'_n := \begin{pmatrix} \widetilde{\Sigma}_n & p'_n \\ q'^{\mathrm{T}}_n & a'_n \end{pmatrix}. \tag{20}$$

Moreover, by (17), the coefficient $a'_n$ in (20) has the following explicit expression:

$$a'_n = \alpha'_{1,n} \mathbb{1}_{Y \setminus \omega_n} + \alpha'_{2,n} \mathbb{1}_{\omega_n} \tag{21}$$

where

$$\begin{cases} \alpha'_{1,n} = \alpha_1 + \widetilde{\sigma}_1 p_{0,n} \cdot q_{0,n} + \beta_1 (p_{0,n} - q_{0,n}) \cdot J\widetilde{h}, \\ \alpha'_{2,n} = \alpha_{2,n} + \widetilde{\sigma}_{2,n} p_{0,n} \cdot q_{0,n} + \beta_{2,n} (p_{0,n} - q_{0,n}) \cdot J\widetilde{h}. \end{cases}$$

Let us now study the homogenization of $\sigma'_n$. Define $(\sigma'_n)_*$ as in the formula (7). The conductivity $\Sigma'_n$ does not depend on the variable $y_3$. On the one hand, as in Remark 3, if $\lambda \perp e_3$, the solution $W_n^\lambda$ of the problem (9), with the conductivity $\Sigma'_n$, does not depend on the variable $y_3$ and $\nabla W_n^\lambda = (\widetilde{\nabla} W_n^\lambda, 0)^{\mathrm{T}}$. Hence, since $q'_n$ is a constant, and by Remark 3,

$$\left(\sigma'_n\right)_* \lambda = \left(\langle \widetilde{\Sigma}_n \widetilde{\nabla} W_n^\lambda \rangle, \langle q'_n \cdot \widetilde{\nabla} W_n^\lambda \rangle\right)^{\mathrm{T}} = \left(\left(\widetilde{\sigma}_n\right)_* \widetilde{\lambda}, q'_n \cdot \widetilde{\lambda}\right)^{\mathrm{T}}. \tag{22}$$

On the other hand, it is clear that, for $\lambda = e_3$, $W_n^{e_3}(y) = y_3$ satisfies (9) with the conductivity $\Sigma'_n$. Hence, since $p'_n$ is a constant, we have

$$\left(\sigma'_n\right)_* e_3 = \langle (p'_n, a'_n)^{\mathrm{T}} \rangle = \left(p'_n, \langle a'_n \rangle\right)^{\mathrm{T}}. \tag{23}$$

Then, by (21), (22), (23) and since $|\omega_n| \underset{n\to\infty}{\sim} \theta_n$, the matrix $\left(\sigma'_n\right)_*$ has the form

$$\left(\sigma'_n\right)_* = \begin{pmatrix} \left(\widetilde{\sigma}_n\right)_* & p'_n \\ q'^{\mathrm{T}}_n & \langle a'_n \rangle \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned} \langle a'_n \rangle & = & \left[\alpha_1 + \widetilde{\sigma}_1 p_{0,n} \cdot q_{0,n} + \beta_1 (p_{0,n} - q_{0,n}) \cdot J\widetilde{h}\right] \\ & + & \theta_n \left[\alpha_{2,n} + \widetilde{\sigma}_{2,n} p_{0,n} \cdot q_{0,n} + \beta_{2,n} (p_{0,n} - q_{0,n}) \cdot J\widetilde{h}\right] + o(1). \end{aligned} \tag{25}$$

**Second step.** Application of the theory of exact relations and asymptotic behavior of $\left(\sigma'_n\right)_*$. By (3) and since the volume fraction $\theta_n$ converges to 0, we have

$$\begin{cases} p_{0,n} = \theta_n(\beta_{2,n} - \beta_1)\left(\theta_n(\widetilde{\sigma}_{2,n} - \widetilde{\sigma}_1)\right)^{-1} J\widetilde{h} \quad \xrightarrow[n\to\infty]{} \quad \beta_2 \widetilde{\sigma}_2^{-1} J\widetilde{h} \\ q_{0,n} = \theta_n(\beta_{2,n} - \beta_1)\left(\theta_n(\widetilde{\sigma}_1 - \widetilde{\sigma}_{2,n})\right)^{-\mathrm{T}} J\widetilde{h} \quad \xrightarrow[n\to\infty]{} \quad -\beta_2 \widetilde{\sigma}_2^{-\mathrm{T}} J\widetilde{h}. \end{cases} \tag{26}$$

Then, by (3), (19), (18), (25) and (26) we obtain the convergences

$$
\begin{cases}
p_n' & \xrightarrow[n\to\infty]{} \quad p_*' := \left[-\beta_1 I_2 + \beta_2 \widetilde{\sigma}_1 \widetilde{\sigma}_2^{-1}\right] J\widetilde{h}, \\[4pt]
q_n' & \xrightarrow[n\to\infty]{} \quad q_*' := \left[\beta_1 I_2 - \beta_2 \widetilde{\sigma}_1^{\mathrm{T}} \widetilde{\sigma}_2^{-\mathrm{T}}\right] J\widetilde{h}, \\[4pt]
\langle a_n' \rangle & \xrightarrow[n\to\infty]{} \quad \alpha_*' := \displaystyle\sum_{i=1}^{2}\left[\alpha_i - \beta_2^2\,\widetilde{\sigma}_2^{-1}\widetilde{\sigma}_i \widetilde{\sigma}_2^{-1} J\widetilde{h}\cdot J\widetilde{h} + 2\,\beta_2\,\beta_i\,\widetilde{\sigma}_2^{-1} J\widetilde{h}\cdot J\widetilde{h}\right],
\end{cases}
\tag{27}
$$

$$
\Pi_n \xrightarrow[n\to\infty]{} \Pi := \begin{pmatrix} I_2 & 0 \\ \beta_2\widetilde{h}^{\mathrm{T}} J\widetilde{\sigma}_2^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \widehat{\Pi}_n \xrightarrow[n\to\infty]{} \widehat{\Pi} := \begin{pmatrix} I_2 & \beta_2\widetilde{\sigma}_2^{-1} J\widetilde{h} \\ 0 & 1 \end{pmatrix}. \tag{28}
$$

Since the matrix transformation (17) preserves the H-limit in the periodic case (see, for instance, [21, 22, 29]), we have

$$
\left(\sigma_n'\right)_* = \Pi_n \left(\sigma_n\right)_*(h)\, \widehat{\Pi}_n. \tag{29}
$$

Passing to the limit, as $n$ goes to infinity, in relation (29), using (16), (24), (27)-(28), we obtain

$$
\begin{pmatrix} \widetilde{\sigma}_* & p_*' \\ q_*'^{\mathrm{T}} & \alpha_*' \end{pmatrix} = \Pi\, \sigma_*\, \widehat{\Pi}. \tag{30}
$$

Inverting the identity (30) and taking into account (27) and (28) leads to (14). The proof of Proposition 1 is completed. $\qquad\square$

Now let us turn to the non periodic case.

3. **The non periodic case.** In this section, we study the homogenization of the problem (4) without any periodicity assumption. The conductivity $\sigma_n(h)$ is defined by

$$
\sigma_n(h) := \alpha_n I_3 + \beta_n \mathscr{E}(h) \quad \text{where} \quad \begin{cases} \alpha_n := \mathbb{1}_{\Omega\setminus\Omega_n}\alpha_1 + \mathbb{1}_{\Omega_n}\alpha_{2,n}, \\ \beta_n := \mathbb{1}_{\Omega\setminus\Omega_n}\beta_1 + \mathbb{1}_{\Omega_n}\beta_{2,n}. \end{cases} \tag{31}
$$

Consider the covering of $\mathbb{R}^3$ by the squares $Q_n^k$ defined by

$$
\forall k \in \mathbb{Z}^3, \quad Q_n^k = \varepsilon_n(Y + k). \tag{32}
$$

We assume that, for any $k \in \mathbb{Z}^3$, $n \in \mathbb{N}^*$, the conductivity coefficient $\alpha_n$ defined by (31) satisfies the following conditions:

$(i)$ the weighted Poincar-Wirtinger inequality

$$
\forall v \in H^1(Q_n^k), \quad \int_{Q_n^k} \alpha_n \left| v - \fint_{Q_n^k} v \right|^2 \mathrm{d}x \le c_n \int_{Q_n^k} \alpha_n |\nabla v|^2 \, \mathrm{d}x, \tag{33}
$$

where $c_n$ is a sequence of positive constants satisfying

$$
c_n \xrightarrow[n\to\infty]{} 0; \tag{34}
$$

$(ii)$ there exists a positive constant $c$ such that, for any $k \in \mathbb{Z}^3$ and $n \in \mathbb{N}^*$,

$$
\fint_{Q_n^k} \alpha_n \le c. \tag{35}
$$

**Remark 5.** Note that, in the periodic case, the hypothesis (10)-(11) is a rescaling of (33)-(34) which, similarly to the periodic case, prevents from the appearance of non local effects in the limit problem. Assumption (35) ensures that the microstructure does not concentrate on a lower dimension subset through the homogenization process since it implies that (see in the proof of Lemma 3.2)

$$
\theta_n^{-1}\mathbb{1}_{\Omega_n} \longrightarrow \theta \in L^\infty(\Omega) \quad \text{weakly-$*$ in } \mathcal{M}(\Omega). \tag{36}
$$

In the periodic case, (35) is clearly satisfied since

$$\fint_{Q_n^k} \alpha_n \ dx = \|a_n\|_{L^1(Y)} \le c,$$

where $a_n$ is defined by (6) and $\theta \equiv 1$.

We have the following result:

**Theorem 3.1.** *Assume that* (3), (32)-(35) *are satisfied. Then, there exist a matrix-valued function* $\sigma_*(h)$ *and a subsequence of* $n$, *still denoted by* $n$, *such that the solution* $u_n$ *of the problem* (4) *converges weakly in* $H_0^1(\Omega)$ *to the solution* $u$ *of the conductivity problem*

$$\begin{cases} -\operatorname{div}\big(\sigma_*(h)\nabla u\big) & = & f & in \ \Omega, \\ u & = & 0 & on \ \partial\Omega. \end{cases} \tag{37}$$

*Moreover, the effective conductivity* $\sigma_*(h)$ *in* (37) *is given by*

$$\sigma_*(h) := \begin{pmatrix} \widetilde{\sigma}_* & p_* \\ q_*^T & \alpha_* \end{pmatrix}, \tag{38}$$

*where* $\widetilde{\sigma}_*$ *is the* $H(\mathcal{M}(\widetilde{\Omega})^2)$-*limit of* $\widetilde{\sigma}_n(h)$ *in the sense of Definition* 1.1, $\theta \in L^\infty(\Omega)$ *is the weak-$*$ limit of* $\theta_n^{-1} \mathbb{1}_{\Omega_n}$ *and*

$$\begin{cases} p_* = -\big[\beta_1 I_2 + \beta_2\big(\widetilde{\sigma}_* - \widetilde{\sigma}_1\big)\widetilde{\sigma}_2^{-1}\big] J\widetilde{h}, \\ q_* = \big[\beta_1 I_2 + \beta_2\, \widetilde{\sigma}_2^{-1}\big(\widetilde{\sigma}_* - \widetilde{\sigma}_1\big)\big]^T J\widetilde{h}, \\ \alpha_* = \alpha_1 + \theta\alpha_2 + \beta_2^2\, \widetilde{\sigma}_2^{-1}\big(\widetilde{\sigma}_1 + \theta\widetilde{\sigma}_2 - \widetilde{\sigma}_*\big)\widetilde{\sigma}_2^{-1} J\widetilde{h} \cdot J\widetilde{h}, \end{cases} \tag{39}$$

*and, for any* $i = 1, 2$,

$$\widetilde{\sigma}_i := \begin{pmatrix} \alpha_i & -\beta_i\, h_3 \\ \beta_i\, h_3 & \alpha_i \end{pmatrix}.$$

**Remark 6.** The shape (32) of $Q_n^k$ is purely technical and can be generalized into any subset the diameter of which is of order $\varepsilon_n$.

**Remark 7.** Since $\Omega_n$ has a columnar structure, $\mathbb{1}_{\Omega_n}$ does not depend on the variable $x_3$. Therefore,

$$\theta_n^{-1}\mathbb{1}_{\widetilde{\Omega}_n} \ \longrightarrow \ \theta \in L^\infty(\widetilde{\Omega}) \quad \text{weakly-$*$ in } \mathcal{M}(\widetilde{\Omega}). \tag{40}$$

Hence, as in Remark 3, it was proved in [17] that there exists a function $\sigma_*^0$ defined on $(0,\infty)^2$ and a subsequence of $n$, such that, for any $\alpha_1, \alpha_2 > 0$, $\beta_1, \beta_2 \in \mathbb{R}$,

$$\widetilde{\sigma}_n(h) = \widetilde{\sigma}_n(h_3) \ \overset{H(\mathcal{M}(\widetilde{\Omega})^2)}{\longrightarrow} \ \widetilde{\sigma}_*(h) = \sigma_*^0\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h_3^2\big) + h_3\beta_1 J.$$

We obtain, once again, an explicit expression of $\sigma_*(h)$ in terms of the homogenized perturbed conductivity in the $(x_1, x_2)$-plane, in the absence of a magnetic field.

A crucial ingredient of the proof of Theorem 3.1 is the following compactness result:

**Lemma 3.2.** *Let* $\alpha_n$ *be the sequence defined by* (31) *such that* (3) *and* (33)-(35) *hold true. Consider two sequences* $\xi_n \in L^1(\Omega)$ *and* $v_n \in H^1(\Omega)$ *satisfying*

$$\xi_n \ \longrightarrow \ \xi \ weakly\text{-}* \ in \ \mathcal{M}(\Omega) \quad and \quad v_n \ \longrightarrow \ v \ weakly \ in \ H^1(\Omega). \tag{41}$$

*We assume that*

$$\int_\Omega \alpha_n^{-1}|\xi_n|^2 \ dx + \int_\Omega \alpha_n|\nabla v_n|^2 \ dx \le c. \tag{42}$$

*Then, $\xi \in L^2(\Omega)$ and we have the convergence, in the sense of distributions*

$$\xi_n v_n \longrightarrow \xi v \quad in \ \mathscr{D}'(\Omega). \tag{43}$$

**Remark 8.** Note that Lemma 3.2 is false when the conditions (33) and (34) do not hold. This can be seen by considering the classical model example of non local effects in conduction due to Fenchenko-Khruslov [19] and presented, for instance, in [13, 10]. For the reader's convenience, we give the main steps of the counterexample. Let $\Omega := (-\frac{1}{2}, \frac{1}{2})^3$ and $\Omega_n$ be the $\frac{1}{n}$-periodic lattice of thin vertical cylinders of radius $n^{-1}e^{-n^2}$. Let $\alpha_n$ be the conductivity defined by (31) with $\alpha_1 := 1$ and $\alpha_{2,n} := \pi^{-1} e^{2n^2}$ which satisfies (3) and (35). For a fixed $f$ in $L^2(\Omega)$, let $u_n$ be the solution, in $H_0^1(\Omega)$, of the equation

$$- \operatorname{div}(\alpha_n \nabla u_n) = f \quad in \ \mathscr{D}'(\Omega).$$

For $R \in (0, \frac{1}{2})$, let $V_n$ be the $Y$-periodic function defined on $\mathbb{R}^3$ by

$$V_n(y) := \begin{cases} \dfrac{\ln r + n^2}{\ln R + n^2} & \text{if} \quad r := \sqrt{y_1^2 + y_2^2} \in (e^{-n^2}, R), \\[2mm] 0 & \text{if} \quad r \le e^{-n^2} \text{ (region of high conductivity)}, \\[2mm] 1 & \text{if} \quad r \ge R. \end{cases}$$

An easy computation shows that $\xi_n := \alpha_n \nabla u_n \cdot e_3$ and $v_n(x) := V_n(nx)$ satisfy the assumption (42) and that $v_n$ weakly converges to the constant function 1 in $H^1(\Omega)$. Moreover, Briane and Tchou [18] proved that

$$\xi_n = \alpha_n \frac{\partial u_n}{\partial x_3} \longrightarrow \xi := \frac{\partial u}{\partial x_3} + \frac{\partial v}{\partial x_3} \quad \text{weakly-} * \text{ in } \mathcal{M}(\Omega),$$

where the weak limit $u$ of $u_n$ in $H_0^1(\Omega)$ and the weak-$*$ limit $v$ of $\frac{\mathbb{1}_{\Omega_n}}{\pi e^{-2n^2}} u_n$ in the sense of Radon measures satisfy the coupled system

$$\begin{cases} -\Delta u + 2\pi (u - v) & = & f & \text{in} & \Omega, \\[2mm] -\dfrac{\partial^2 v}{\partial x_3^2} + v - u & = & 0 & \text{in} & \Omega, \\[2mm] u & = & 0 & \text{on} & \partial\Omega, \\[2mm] v(x', 0) = v(x', 1) & = & 0 & \text{if} & x' \in (-\frac{1}{2}, \frac{1}{2})^2. \end{cases} \tag{44}$$

Then, if $f$ is non zero, $u$ and $v$ are two different functions. Therefore, the convergence (43) does not hold true since, by the strong convergence, up to a subsequence, of $v_n$ to 1 in $L^2(\Omega)$ and the weak convergence of $\mathbb{1}_{\Omega \setminus \Omega_n} \nabla u_n$ to $\nabla u$ in $L^2(\Omega)$, we have

$$\xi_n v_n = \mathbb{1}_{\Omega \setminus \Omega_n} \frac{\partial u_n}{\partial x_3} v_n \longrightarrow \frac{\partial u}{\partial x_3} \ne \xi \times 1 = \frac{\partial u}{\partial x_3} + \frac{\partial v}{\partial x_3} \quad in \ \mathscr{D}'(\Omega).$$

Substituting the expression of $v$, in terms of $u$, in the first equation of (44) leads to a non local term in the equation satisfied by $u$. The Poincar-Wirtinger control (33)-(34) is fundamental to avoid such effects. In this example, (34) is false since (see [12] for more details) the optimal constant $c_n$ in (33) satisfies $c_n \ge c > 0$.

*Proof of Lemma 3.2.* On the one hand, by (3), the sequence $\alpha_n$ is bounded in $L^1(\Omega)$ and then, up to a subsequence, weakly-$*$ converges to some $a \in \mathcal{M}(\Omega)$. Moreover, the Radon measure $a$ belongs to $L^\infty(\Omega)$. Indeed, let $\varphi \in \mathscr{C}_0(\Omega)$ and denote again

by $\varphi$ its extension to $\mathbb{R}^3$ by setting $\varphi \equiv 0$ on $\mathbb{R}^3 \setminus \Omega$. There exists a finite subset $I_n$ of $\mathbb{Z}^3$ such that

$$\Omega \subset \bigcup_{k \in I_n} Q_n^k,$$

where $Q_n^k$ is defined by (32). As $\varphi$ is a uniformly continuous function, we have

$$\int_\Omega \alpha_n \varphi \, \mathrm{d}x = \sum_{k \in I_n} \int_{Q_n^k} \alpha_n \varphi \, \mathrm{d}x = \sum_{k \in I_n} \varphi(\varepsilon_n k) \int_{Q_n^k} \alpha_n \, \mathrm{d}x + o(1). \qquad (45)$$

By (35), we have

$$\sum_{k \in I_n} |\varphi(\varepsilon_n k)| \int_{Q_n^k} \alpha_n \, \mathrm{d}x \leq c \sum_{k \in I_n} |Q_n^k| |\varphi(\varepsilon_n k)| = c \|\varphi\|_{L^1(\Omega)} + o(1). \qquad (46)$$

The weak-$*$ convergence of $\alpha_n$ to $a$, combined with (45) and (46) yields

$$\left| \int_\Omega \varphi \, a(\mathrm{d}x) \right| \leq c \|\varphi\|_{L^1(\Omega)},$$

which implies that the measure $a$ is absolutely continuous with respect to the Lebesgue measure and $a \in L^\infty(\Omega)$. From (3) and the convergence of $\alpha_n$ to $a$, we have

$$\theta_n^{-1} \mathbb{1}_{\Omega_n} = (\theta_n \alpha_{2,n})^{-1} (\alpha_n - \alpha_1 \mathbb{1}_{\Omega \setminus \Omega_n}) \longrightarrow \theta := \alpha_2^{-1}(a - \alpha_1) \in L^\infty(\Omega),$$

in the sense of Radon measures, and then $a = \alpha_1 + \theta \alpha_2$.

On the other hand, by the Cauchy-Schwarz inequality combined with (41), (35) and the convergence of $\alpha_n$ to $\alpha_1 + \alpha_2 \theta$, we have, for any $\varphi \in \mathscr{C}_0(\Omega)$,

$$\left| \int_\Omega \varphi \, \xi(\mathrm{d}x) \right|^2 = \lim_{n \to \infty} \left| \int_\Omega \xi_n \varphi \, \mathrm{d}x \right|^2 \leq \limsup_{n \to \infty} \int_\Omega \alpha_n^{-1} \xi_n^2 \, \mathrm{d}x \int_\Omega \alpha_n \varphi^2 \, \mathrm{d}x$$

$$\leq c \int_\Omega (\alpha_1 + \theta \alpha_2) \varphi^2 \, \mathrm{d}x \leq c \|\alpha_1 + \theta \alpha_2\|_\infty \|\varphi\|_{L^2(\Omega)}^2,$$

which implies that the limit measure $\xi$ of $\xi_n$ in (41) is actually an element of $L^2(\Omega)$.

We now prove the convergence (43). Let $\varphi \in \mathscr{D}(\Omega)$ and let $I_n$ be a finite subset of $\mathbb{Z}^3$ such that

$$\mathrm{supp}\, \varphi \subset \bigcup_{k \in I_n} \overline{Q}_n^k \subset \Omega,$$

where $\mathrm{supp}\, \varphi$ is the support of $\varphi$. For any $w \in H^1(\Omega)$, define $\overline{w}^{\varepsilon_n}$ the piecewise constant function associated with the partition $\left(Q_n^k\right)_{k \in I_n}$ as follows:

$$\overline{w}^{\varepsilon_n} = \sum_{k \in I_n} \left( \fint_{Q_n^k} w \right) \mathbb{1}_{Q_n^k}.$$

In order to study the convergence, in the sense of distributions, of $(\xi_n v_n - \xi v)$ to $0$, we rewrite

$$\xi_n v_n - \xi v = \underbrace{\xi_n \left( v_n - \overline{v_n}^{\varepsilon_n} \right)}_{:=p_n} + \underbrace{\xi_n \left( \overline{v_n}^{\varepsilon_n} - \overline{v}^{\varepsilon_n} \right)}_{:=q_n} + \underbrace{\xi_n \overline{v}^{\varepsilon_n} - \xi v}_{:=r_n} \qquad (47)$$

and estimate each term of the identity (47) separately.

*Convergence of the term $p_n$ in* (47). Thanks to the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\left| \int_\Omega \xi_n \big( v_n - \overline{v_n}^{\varepsilon_n} \big) \varphi \ \mathrm{d}x \right|^2 &\le \left( \sum_{k \in I_n} \left| \int_{Q_n^k} \xi_n \left( v_n - \fint_{Q_n^k} v_n \right) \varphi \ \mathrm{d}x \right| \right)^2 \\
&\le \|\varphi\|_\infty^2 \left( \sum_{k \in I_n} \sqrt{\int_{Q_n^k} \alpha_n^{-1} |\xi_n|^2 \ \mathrm{d}x} \sqrt{\int_{Q_n^k} \alpha_n \left| v_n - \fint_{Q_n^k} v_n \right|^2 \ \mathrm{d}x} \right)^2 \\
&\le \|\varphi\|_\infty^2 \sum_{k \in I_n} \int_{Q_n^k} \alpha_n^{-1} |\xi_n|^2 \ \mathrm{d}x \sum_{k \in I_n} \int_{Q_n^k} \alpha_n \left| v_n - \fint_{Q_n^k} v_n \right|^2 \ \mathrm{d}x \\
&\le c_n \|\varphi\|_\infty^2 \int_\Omega \alpha_n^{-1} |\xi_n|^2 \ \mathrm{d}x \int_\Omega \alpha_n |\nabla v_n|^2 \ \mathrm{d}x,
\end{aligned}
\tag{48}
$$

where the last inequality is a consequence of (33). Finally, the inequality (48) combined with (42) and the convergence (34) yield

$$
\left| \int_\Omega \xi_n \big( v_n - \overline{v_n}^{\varepsilon_n} \big) \varphi \ \mathrm{d}x \right| \le c \sqrt{c_n} \xrightarrow[n \to \infty]{} 0.
\tag{49}
$$

*Convergence of the term $q_n$ in* (47). By the Cauchy-Schwarz inequality and (35), we have

$$
\begin{aligned}
\left| \int_\Omega \xi_n \big( \overline{v_n}^{\varepsilon_n} - \overline{v}^{\varepsilon_n} \big) \varphi \ \mathrm{d}x \right|^2 &\le \left( \sum_{k \in I_n} \left| \int_{Q_n^k} \xi_n \ \varphi \ \mathrm{d}x \right| \left| \fint_{Q_n^k} (v_n - v) \ \mathrm{d}x \right| \right)^2 \\
&\le \|\varphi\|_\infty^2 \left( \sum_{k \in I_n} |Q_n^k|^{-1} \sqrt{\int_{Q_n^k} \alpha_n \ \mathrm{d}x} \sqrt{\int_{Q_n^k} \alpha_n^{-1} |\xi_n|^2 \ \mathrm{d}x} \int_{Q_n^k} |v_n - v| \ \mathrm{d}x \right)^2 \\
&\le \|\varphi\|_\infty^2 \left( \sum_{k \in I_n} \sqrt{\fint_{Q_n^k} \alpha_n \ \mathrm{d}x} \sqrt{\int_{Q_n^k} \alpha_n^{-1} |\xi_n|^2 \ \mathrm{d}x} \sqrt{\int_{Q_n^k} (v_n - v)^2 \ \mathrm{d}x} \right)^2 \\
&\le c \|\varphi\|_\infty^2 \sum_{k \in I_n} \int_{Q_n^k} \alpha_n^{-1} |\xi_n|^2 \ \mathrm{d}x \sum_{k \in I_n} \int_{Q_n^k} (v_n - v)^2 \ \mathrm{d}x \\
&\le c \int_\Omega \alpha_n^{-1} |\xi_n|^2 \ \mathrm{d}x \int_\Omega (v_n - v)^2 \ \mathrm{d}x
\end{aligned}
$$

which yields, by (42),

$$
\left| \int_\Omega \xi_n \big( \overline{v_n}^{\varepsilon_n} - \overline{v}^{\varepsilon_n} \big) \varphi \ \mathrm{d}x \right|^2 \le c \, \|v_n - v\|_{L^2(\Omega)}^2.
\tag{50}
$$

Since $v_n$ converges weakly to $v$ in $H_0^1(\Omega)$, by Rellich's theorem, up to a subsequence, $v_n$ converges strongly to $v$ in $L^2(\Omega)$. Hence, (50) implies that

$$
\int_\Omega \xi_n \big( \overline{v_n}^{\varepsilon_n} - \overline{v}^{\varepsilon_n} \big) \varphi \ \mathrm{d}x \xrightarrow[n \to \infty]{} 0.
\tag{51}
$$

*Convergence of the term $r_n$ in* (47). Consider, for any $\delta > 0$, an approximation $\psi_\delta \in \mathscr{C}_c(\Omega)$ of $v$ for the $L^2(\Omega)$ norm, *i.e.*,

$$
\|v - \psi_\delta\|_{L^2(\Omega)} = o(\delta).
\tag{52}
$$

The term $r_n$ in (47) writes

$$
\xi_n \overline{v}^{\varepsilon_n} - \xi v = \xi_n \big( \overline{v}^{\varepsilon_n} - \overline{\psi_\delta}^{\varepsilon_n} \big) + \xi_n \big( \overline{\psi_\delta}^{\varepsilon_n} - \psi_\delta \big) + (\xi_n - \xi) \psi_\delta + \xi(\psi_\delta - v).
\tag{53}
$$

On the one hand, since $\overline{\psi_\delta}^{\,\varepsilon_n}$ converges uniformly, as $n$ goes to infinity, to $\psi_\delta \in \mathscr{C}_c(\Omega)$, the convergence (41) of $\xi_n$ to $\xi$ implies that the second term and the third term in the right hand side of the equality (53) converge to 0 in $\mathscr{D}'(\Omega)$. Moreover, by the Cauchy-Schwarz inequality and the fact that $\xi \in L^2(\Omega)$, we have

$$\left| \int_\Omega \xi(\psi_\delta - v)\varphi \ \mathrm{d}x \right| \leq \|\varphi\|_\infty \|\xi\|_{L^2(\Omega)} \|\psi_\delta - v\|_{L^2(\Omega)}.$$

On the other hand, following (50), we have

$$\left| \int_\Omega \xi_n \big(\overline{v}^{\varepsilon_n} - \overline{\psi_\delta}^{\,\varepsilon_n}\big)\varphi \ \mathrm{d}x \right| \leq c\,\|\varphi\|_\infty \|v - \psi_\delta\|_{L^2(\Omega)}. \tag{54}$$

Hence, by (52)-(54), we have

$$\limsup_{n\to\infty} \left| \int_\Omega \big(\xi_n \overline{v_n}^{\varepsilon_n} - \xi v\big)\varphi \ \mathrm{d}x \right| \leq c\,\|\psi_\delta - v\|_{L^2(\Omega)} = o(\delta), \tag{55}$$

for arbitrary $\delta > 0$.

Finally, putting together (47), (49), (51) and (55), we obtain that

$$\limsup_{n\to\infty} \left| \int_\Omega (\xi_n v_n - \xi v)\varphi \ \mathrm{d}x \right| = o(\delta),$$

which concludes the proof of Lemma 3.2.                                        □

In the sequel we apply Lemma 3.2 to sequences $\xi_n$ of vector-valued functions in $L^1(\Omega)^2$ or $L^1(\Omega)^3$.

*Proof of Theorem 3.1.* Thanks to the equi-coerciveness $\sigma_n \geq \alpha_1 I_3$, the solution $u_n$ of the problem (4) satisfies the convergence, up to a subsequence,

$$u_n \longrightarrow u \quad \text{weakly in } H_0^1(\Omega), \tag{56}$$

for some $u$ in $H_0^1(\Omega)$. Moreover, putting $u_n$ as a test function in the equation (4), we obtain that

$$\int_\Omega \alpha_n |\nabla u_n|^2 \ \mathrm{d}x = \int_\Omega \sigma_n \nabla u_n \cdot \nabla u_n \ \mathrm{d}x = \langle f, u_n \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leq c. \tag{57}$$

Since $\alpha_{2,n}^{-1} \sigma_{2,n} = I_3 + \alpha_{2,n}^{-1} \beta_{2,n} \mathscr{E}(h)$, by (3) the sequence $|\alpha_{2,n}^{-1} \sigma_{2,n}|$ is bounded. Then, as the sequence $\alpha_n$ is bounded in $L^1(\Omega)$, the Cauchy-Schwarz inequality and (57) give

$$\left( \int_\Omega |\sigma_n \nabla u_n| \ \mathrm{d}x \right)^2 \leq c \int_\Omega \alpha_n \ \mathrm{d}x \int_\Omega \alpha_n |\nabla u_n|^2 \ \mathrm{d}x \leq c.$$

Hence, we have the convergence of the current $\sigma_n \nabla u_n$, up to a subsequence,

$$\sigma_n \nabla u_n \longrightarrow \xi_0 \quad \text{weakly-$*$ in } \mathcal{M}(\Omega)^3, \tag{58}$$

for some $\xi_0 \in \mathcal{M}(\Omega)^3$. Moreover, by the boundedness of $|\alpha_{2,n}^{-1} \sigma_{2,n}|$ and (57), we have

$$\int_\Omega \alpha_n^{-1} |\sigma_n \nabla u_n|^2 \ \mathrm{d}x \leq c \int_\Omega \alpha_n |\nabla u_n|^2 \ \mathrm{d}x \leq c.$$

Then, by Lemma 3.2 applied to $\xi_n := \sigma_n \nabla u_n$, the measure $\xi_0$ is actually an element of $L^2(\Omega)^3$.

The rest of the proof, which is divided into three steps, is devoted to the determination of the form of the limit current $\xi_0$. To that end, we use a method in the spirit of H-convergence of Murat-Tartar which is based on the cylindrical nature of the microstructure and the compactness result of Lemma 3.2 for sequences

only bounded in $L^2(\Omega; \sigma_n^{-1/2} \mathrm{d}x)$. In the first two steps, we compute the components $\xi_0 \cdot e_1$ and $\xi_0 \cdot e_2$ by combining Lemma 3.2 with a corrector function associated with the transversal conductivity $\widetilde{\sigma}_n$, the existence of which is ensured by the two-dimensional results of [14, 17]. Since the corrector function considered in the previous steps is independent of the variable $x_3$, the component $\xi_0 \cdot e_3$ needs a different approach. This is the object of the last step.

**First step.** Building a corrector. Thanks to Remark 7, there exists a subsequence of $\widetilde{\sigma}_n$ still denoted by $\widetilde{\sigma}_n$ which $H(\mathcal{M}(\widetilde{\Omega})^2)$-converges to some coercive matrix-valued function $\widetilde{\sigma}_*$. Then, the sequence $\widetilde{\sigma}_n^{\mathrm{T}}$ $H(\mathcal{M}(\widetilde{\Omega})^2)$-converges to $\widetilde{\sigma}_*^{\mathrm{T}}$ (see Theorem 2.1 of [14]). Let $\lambda \in \mathbb{R}^3$ with $\lambda \perp e_3$. For $\widetilde{\lambda} = (\lambda_1, \lambda_2)^{\mathrm{T}} \in \mathbb{R}^2$, let $v_n^{\widetilde{\lambda}}$ be the solution of

$$\begin{cases} \mathrm{div}\left(\widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\widetilde{\lambda}}\right) & = & \mathrm{div}\left(\widetilde{\sigma}_*^{\mathrm{T}} \widetilde{\lambda}\right) & \text{in } \widetilde{\Omega}, \\ v_n^{\widetilde{\lambda}} & = & \widetilde{\lambda} \cdot \widetilde{x} & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{59}$$

By Definition 1.1, we have the convergences

$$\begin{cases} v_n^{\widetilde{\lambda}} & \longrightarrow & \widetilde{\lambda} \cdot \widetilde{x} = \lambda \cdot x & \text{weakly in } H^1(\widetilde{\Omega}), \\ \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\widetilde{\lambda}} & \longrightarrow & \widetilde{\sigma}_*^{\mathrm{T}} \widetilde{\lambda} & \text{weakly-}* \text{ in } \mathcal{M}(\widetilde{\Omega})^2. \end{cases} \tag{60}$$

Setting, for $x \in \Omega$, $v_n^{\lambda}(x_1, x_2, x_3) = v_n^{\widetilde{\lambda}}(x_1, x_2)$, we have the convergences

$$\begin{cases} v_n^{\lambda} & \longrightarrow & \lambda \cdot x & \text{weakly in } H^1(\Omega), \\ \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\lambda} & \longrightarrow & \widetilde{\sigma}_*^{\mathrm{T}} \widetilde{\lambda} & \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2, \end{cases} \tag{61}$$

and the energy inequality, as in (57),

$$\int_{\Omega} \alpha_n \left|\nabla v_n^{\lambda}\right|^2 \mathrm{d}x = \int_{\widetilde{\Omega}} \widetilde{\sigma}_n \widetilde{\nabla} v_n^{\widetilde{\lambda}} \cdot \widetilde{\nabla} v_n^{\widetilde{\lambda}} \mathrm{d}\widetilde{x} \leq c. \tag{62}$$

Let $\varphi \in \mathscr{D}(\Omega)$. By the convergence (58) and since $v_n^{\lambda}$ converges weakly to $\lambda \cdot x$ in $H^1(\Omega)$, putting $v_n^{\lambda} \varphi$ as a test function in (4) yields

$$\int_{\Omega} \sigma_n \nabla u_n \cdot \nabla\left(v_n^{\lambda} \varphi\right) \mathrm{d}x \xrightarrow[n\to\infty]{} \langle f, \varphi \, \lambda \cdot x \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_{\Omega} \xi_0 \cdot \nabla(\varphi \lambda \cdot x) \mathrm{d}x. \tag{63}$$

Since $\sigma_n$ and $v_n^{\lambda}$ do not depend on the variable $x_3$, we have the identity

$$\sigma_n \nabla u_n \cdot \nabla v_n^{\lambda} = \widetilde{\sigma}_n \widetilde{\nabla} u_n \cdot \widetilde{\nabla} v_n^{\lambda} - \partial_3\left(\beta_n \widetilde{\nabla} v_n^{\lambda} \cdot J\widetilde{h} \, u_n\right). \tag{64}$$

Then, by (64), an integration by parts gives

$$\int_{\Omega} \sigma_n \nabla u_n \cdot \nabla\left(v_n^{\lambda} \varphi\right) \mathrm{d}x = \int_{\Omega} \sigma_n \nabla u_n \cdot \nabla\varphi \, v_n^{\lambda} \, \mathrm{d}x \tag{65}$$

$$+ \int_{\Omega} \beta_n \widetilde{\nabla} v_n^{\lambda} \cdot J\widetilde{h} \, u_n \, \frac{\partial\varphi}{\partial x_3} \, \mathrm{d}x \tag{66}$$

$$+ \int_{\Omega} \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\lambda} \cdot \widetilde{\nabla} u_n \, \varphi \, \mathrm{d}x. \tag{67}$$

**Step 2.** Estimates of the terms in (65)-(67). The convergence of these terms are consequences of Lemma 3.2 and the generalized two-dimensional div-curl lemma in a high-contrast context of [13].

*Convergence of the term on the right hand side of* (65). On the one hand, by the boundedness of $\alpha_n^{-1}\sigma_n$ and (57), we have the inequality

$$\int_\Omega \alpha_n^{-1} |\sigma_n \nabla u_n \cdot \nabla \varphi|^2 \, \mathrm{d}x \le c \, \|\nabla \varphi\|_\infty^2 \int_\Omega \alpha_n |\nabla u_n|^2 \le c.$$

On the other hand, the convergence (58), the inequality (62), and the convergence (61) of $v_n^\lambda$ to $\lambda \cdot x$, show that the sequences $\xi_n := \sigma_n \nabla u_n$ and $v_n := v_n^\lambda$ satisfy the assumptions of Lemma 3.2. Hence, we obtain

$$\int_\Omega \sigma_n \nabla u_n \cdot \nabla \varphi \, v_n^\lambda \, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_\Omega \xi_0 \cdot \nabla \varphi \, \lambda \cdot x \, \mathrm{d}x. \tag{68}$$

*Convergence of the term in* (66). We first compute the limit of $\beta_n \widetilde{\nabla} v_n^\lambda$ in the sense of Radon measures. We have the identity

$$
\begin{aligned}
\beta_n \widetilde{\nabla} v_n^\lambda &= \beta_n \left( \frac{\partial v_n^\lambda}{\partial x_1}, \frac{\partial v_n^\lambda}{\partial x_2} \right)^{\mathrm{T}} \\
&= \mathbb{1}_{\Omega \setminus \Omega_n} \beta_1 \widetilde{\nabla} v_n^\lambda + \beta_{2,n} \widetilde{\sigma}_{2,n}^{-T} \left[ \mathbb{1}_{\Omega_n} \widetilde{\sigma}_{2,n}^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \right] \\
&= \mathbb{1}_{\Omega \setminus \Omega_n} \beta_1 \widetilde{\nabla} v_n^\lambda + \beta_{2,n} \widetilde{\sigma}_{2,n}^{-T} \left[ \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda - \mathbb{1}_{\Omega \setminus \Omega_n} \widetilde{\sigma}_1^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \right],
\end{aligned}
\tag{69}
$$

where

$$\widetilde{\sigma}_1 := \begin{pmatrix} \alpha_1 & -\beta_1 \, h_3 \\ \beta_1 \, h_3 & \alpha_1 \end{pmatrix} \quad \text{and} \quad \widetilde{\sigma}_{2,n} := \begin{pmatrix} \alpha_{2,n} & -\beta_{2,n} \, h_3 \\ \beta_{2,n} \, h_3 & \alpha_{2,n} \end{pmatrix}. \tag{70}$$

By (3), we have

$$\beta_{2,n} \widetilde{\sigma}_{2,n}^{-T} \xrightarrow[n\to\infty]{} \beta_2 \widetilde{\sigma}_2^{-\mathrm{T}}, \quad \text{where} \quad \widetilde{\sigma}_2 := \begin{pmatrix} \alpha_2 & -\beta_2 \, h_3 \\ \beta_2 \, h_3 & \alpha_2 \end{pmatrix}. \tag{71}$$

Combining this convergence with the ones in (60), we obtain that

$$\beta_n \widetilde{\nabla} v_n^\lambda \longrightarrow \beta_1 \widetilde{\lambda} + \beta_2 \widetilde{\sigma}_2^{-\mathrm{T}} \left[ \widetilde{\sigma}_*^{\mathrm{T}} - \widetilde{\sigma}_1^{\mathrm{T}} \right] \widetilde{\lambda} \quad \text{weakly-} * \text{ in } \mathcal{M}(\Omega)^2. \tag{72}$$

By the boundedness of $\alpha_n^{-1} \sigma_n$, (3) and (62), we have

$$\int_\Omega \alpha_n^{-1} |\beta_n \widetilde{\nabla} v_n^\lambda|^2 \, \mathrm{d}x \le c \int_\Omega \alpha_n |\nabla v_n^\lambda|^2 \le c.$$

This inequality together with (57), (72) and the weak convergence (56) of $u_n$ to $u$ in $H_0^1(\Omega)$ show that the sequences $\xi_n := \beta_n \widetilde{\nabla} v_n^\lambda$ and $v_n := u_n$ satisfy the assumptions of Lemma 3.2. Then,

$$\int_\Omega \beta_n \widetilde{\nabla} v_n^\lambda \cdot J\widetilde{h} \, u_n \, \frac{\partial \varphi}{\partial x_3} \, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_\Omega \left( \beta_1 I_2 + \beta_2 \left[ \widetilde{\sigma}_* - \widetilde{\sigma}_1 \right] \widetilde{\sigma}_2^{-1} \right) J\widetilde{h} \cdot \widetilde{\lambda} \, u \, \frac{\partial \varphi}{\partial x_3} \, \mathrm{d}x. \tag{73}$$

*Convergence of the term in* (67). Integrating by parts in (67), we obtain that

$$\int_\Omega \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \cdot \widetilde{\nabla} u_n \, \varphi \, \mathrm{d}x = - \int_\Omega \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \cdot \widetilde{\nabla} \varphi \, u_n \, \mathrm{d}x + \int_\Omega \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \cdot \widetilde{\nabla} (u_n \varphi) \, \mathrm{d}x. \tag{74}$$

On the one hand, the boundedness of $\alpha_n^{-1} \sigma_n$ and (62) yields

$$\int_\Omega \alpha_n^{-1} \left| \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \right|^2 \le c \int_\Omega \alpha_n \left| \widetilde{\nabla} v_n^\lambda \right|^2 \le c.$$

By the second convergence of (61), the weak convergence (56) of $u_n$ to $u$ in $H_0^1(\Omega)$ and (57), the sequences $\xi_n := \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda$ and $v_n := u_n$ satisfy the assumptions of Lemma 3.2. Hence,

$$\int_\Omega \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \cdot \widetilde{\nabla} \varphi \, u_n \, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_\Omega \widetilde{\sigma}_*^{\mathrm{T}} \widetilde{\lambda} \cdot \widetilde{\nabla} \varphi \, u \, \mathrm{d}x. \tag{75}$$

On the other hand, since $\widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\widetilde{\lambda}}$ does not depend on the variable $x_3$, the second term on the right hand side of (74) can be rewritten under the form

$$\int_\Omega \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \cdot \widetilde{\nabla} \left( u_n \varphi \right) \, \mathrm{d}x = \int_{\widetilde{\Omega}} \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\widetilde{\lambda}}(x') \cdot \widetilde{\nabla} \left[ \int_0^1 \left( u_n \varphi \right)(x', x_3) \, \mathrm{d}x_3 \right] \, \mathrm{d}x' \tag{76}$$

where $x' = (x_1, x_2)$.

In order to study the asymptotic behavior of (76), we apply a two-dimensional div-curl lemma of [17] which is an extension to the non symmetric case of [13]. Set, for any $x' \in \Omega$,

$$\eta_n(x') := \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\widetilde{\lambda}}(x') \quad \text{and} \quad v_n(x') := \int_0^1 \left( u_n \varphi \right)(x', x_3) \, \mathrm{d}x_3. \tag{77}$$

Due to the convergences (60) and (56), we have

$$\begin{cases} \eta_n & \longrightarrow \quad \widetilde{\sigma}_*^{\mathrm{T}} \widetilde{\lambda} & \text{weakly-}* \text{ in } \mathcal{M}(\widetilde{\Omega})^2, \\[2mm] v_n(x') & \longrightarrow \quad v(x') := \int_0^1 \left( u\varphi \right)(x', x_3) \, \mathrm{d}x_3 & \text{weakly in } H^1(\widetilde{\Omega}). \end{cases} \tag{78}$$

The convergences (3) and (40), the definition (59) of the corrector $v_n^{\widetilde{\lambda}}$, (60)-(62) and (78) imply that the sequences $\eta_n$ and $v_n$ defined in (77) satisfy the assumptions of Lemma 2.1 in [17]. Then,

$$\widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\widetilde{\lambda}}(x') \cdot \widetilde{\nabla} \left[ \int_0^1 \left( u_n \varphi \right)(x', x_3) \, \mathrm{d}x_3 \right] \xrightarrow{\mathscr{D}'(\widetilde{\Omega})} \widetilde{\sigma}_*^{\mathrm{T}} \widetilde{\lambda} \cdot \widetilde{\nabla} \left[ \int_0^1 \left( u\varphi \right)(x', x_3) \, \mathrm{d}x_3 \right]. \tag{79}$$

Let $\psi \in \mathscr{D}(\widetilde{\Omega})$ such that $\psi \equiv 1$ on the projection of the support of $\varphi$ on the $(x_1, x_2)$-plane. Taking $\psi$ as a test function in (79), we obtain

$$\int_{\widetilde{\Omega}} \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^{\widetilde{\lambda}}(x') \cdot \widetilde{\nabla} \left[ \int_0^1 \left( u_n \varphi \right)(x', x_3) \, \mathrm{d}x_3 \right] \, \mathrm{d}x' \xrightarrow[n\to\infty]{} \int_\Omega \sigma_*^{\mathrm{T}} \widetilde{\lambda} \cdot \widetilde{\nabla} \left( u\varphi \right).$$

Finally, this convergence combined with (74), (75) and (76) gives

$$\int_\Omega \widetilde{\sigma}_n^{\mathrm{T}} \widetilde{\nabla} v_n^\lambda \cdot \widetilde{\nabla} u_n \, \varphi \, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_\Omega \widetilde{\sigma}_*^{\mathrm{T}} \widetilde{\lambda} \cdot \widetilde{\nabla} u \, \varphi \, \mathrm{d}x. \tag{80}$$

Putting together (68), (73) and (80) with the equality (65)-(67), we obtain that

$$\int_\Omega \sigma_n \nabla u_n \cdot \nabla \left( v_n^\lambda \varphi \right) \, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_\Omega \xi_0 \cdot \nabla \varphi \, \lambda \cdot x \, \mathrm{d}x + \int_\Omega \widetilde{\sigma}_* \widetilde{\nabla} u \cdot \widetilde{\lambda} \, \varphi \, \mathrm{d}x$$
$$+ \int_\Omega \left( \beta_1 I_2 + \beta_2 \left[ \widetilde{\sigma}_* - \widetilde{\sigma}_1 \right] \widetilde{\sigma}_2^{-1} \right) J\widetilde{h} \cdot \widetilde{\lambda} \, u \, \frac{\partial \varphi}{\partial x_3} \, \mathrm{d}x.$$

Since $\widetilde{\sigma}_*$ depends only on the variable $(x_1, x_2)$, this convergence, an integration by parts and (63) give

$$\int_\Omega \xi_0 \cdot \lambda \, \varphi \, \mathrm{d}x = \int_\Omega \left[ \widetilde{\sigma}_* \widetilde{\nabla} u - \frac{\partial u}{\partial x_3} \left( \beta_1 I_2 + \beta_2 \left[ \widetilde{\sigma}_* - \widetilde{\sigma}_1 \right] \widetilde{\sigma}_2^{-1} \right) J\widetilde{h} \right] \cdot \widetilde{\lambda} \, \varphi \, \mathrm{d}x. \tag{81}$$

Finally, since the equation (81) holds for any $\varphi \in \mathscr{D}(\Omega)$ and any $\lambda \perp e_3$, we obtain the first two components of $\xi_0$

$$\widetilde{\xi}_0 = \widetilde{\sigma}_* \widetilde{\nabla} u - \frac{\partial u}{\partial x_3} \left( \beta_1 I_2 + \beta_2 \left[ \widetilde{\sigma}_* - \widetilde{\sigma}_1 \right] \widetilde{\sigma}_2^{-1} \right) J \widetilde{h}. \tag{82}$$

**Step 3.** Computation of $\xi_0 \cdot e_3$. By (58), we have the convergence

$$\alpha_n \frac{\partial u_n}{\partial x_3} + \beta_n \widetilde{\nabla} u_n \cdot J \widetilde{h} = \sigma_n \nabla u_n \cdot e_3 \longrightarrow \xi_0 \cdot e_3 \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{83}$$

We first study the asymptotic behaviour of $\alpha_n \partial_3 u_n$ (which also gives the limit of $\beta_n \partial_3 u_n$ due to the fact that, by virtue of (3), $\alpha_{2,n}$ and $\beta_{2,n}$ are of the same order). On the one hand, since $\theta_n = |\Omega|^{-1} |\Omega_n|$, by the convergence (3), we have

$$\int_\Omega \alpha_n^{-1} |\theta_n^{-1} \mathbb{1}_{\Omega_n}|^2 \, \mathrm{d}x = \frac{\theta_n^{-1} |\Omega_n|}{\theta_n \alpha_{2,n}} = \frac{|\Omega|}{\theta_n \alpha_{2,n}} \leq c.$$

On the other hand, by (57), the weak convergence (56) of $u_n$ to $u$ in $H_0^1(\Omega)$ and (36), the sequences $\xi_n := \theta_n^{-1} \mathbb{1}_{\Omega_n}$ and $v_n := u_n$ satisfy, once again, the assumptions of Lemma 3.2. Hence,

$$\theta_n^{-1} \mathbb{1}_{\Omega_n} u_n \longrightarrow \theta u \quad \text{in } \mathscr{D}'(\Omega).$$

Moreover, since $\mathbb{1}_{\Omega_n}$ does not depend on the variable $x_3$, we have

$$\theta_n^{-1} \mathbb{1}_{\Omega_n} \frac{\partial u_n}{\partial x_3} \longrightarrow \theta \frac{\partial u}{\partial x_3} \quad \text{in } \mathscr{D}'(\Omega). \tag{84}$$

Finally, thanks to (84) and (3), we obtain the convergences, in the sense of Radon measures,

$$\begin{cases} \alpha_n \dfrac{\partial u_n}{\partial x_3} = \alpha_1 \mathbb{1}_{\Omega \setminus \Omega_n} \dfrac{\partial u_n}{\partial x_3} + (\theta_n \alpha_{2,n}) \theta_n^{-1} \mathbb{1}_{\Omega_n} \dfrac{\partial u_n}{\partial x_3} \quad \rightharpoonup \quad (\alpha_1 + \theta \alpha_2) \dfrac{\partial u}{\partial x_3}, \\[2mm] \beta_n \dfrac{\partial u_n}{\partial x_3} = \beta_1 \mathbb{1}_{\Omega \setminus \Omega_n} \dfrac{\partial u_n}{\partial x_3} + (\theta_n \beta_{2,n}) \theta_n^{-1} \mathbb{1}_{\Omega_n} \dfrac{\partial u_n}{\partial x_3} \quad \rightharpoonup \quad (\beta_1 + \theta \beta_2) \dfrac{\partial u}{\partial x_3}. \end{cases} \tag{85}$$

Now, in order to obtain the limit of the term $\beta_n \widetilde{\nabla} u_n$ in (83), which similarly to (69)-(70), writes

$$\beta_n \widetilde{\nabla} u_n = \beta_1 \mathbb{1}_{\Omega \setminus \Omega_n} \widetilde{\nabla} u_n + \beta_{2,n} \widetilde{\sigma}_{2,n}^{-1} \left[ \widetilde{\sigma}_n \widetilde{\nabla} u_n - \mathbb{1}_{\Omega \setminus \Omega_n} \widetilde{\sigma}_1 \widetilde{\nabla} u_n \right], \tag{86}$$

it remains to estimate $\widetilde{\sigma}_n \widetilde{\nabla} u_n$. Since $\xi_0$ is the limit of the current $\sigma_n \nabla u_n$ (58) and since

$$\forall \lambda \perp e_3, \quad \sigma_n \nabla u_n \cdot \lambda = \widetilde{\sigma}_n \widetilde{\nabla} u_n \cdot \widetilde{\lambda} - \beta_n \frac{\partial u_n}{\partial x_3} J \widetilde{h} \cdot \widetilde{\lambda},$$

the equality (82) gives

$$\widetilde{\sigma}_n \widetilde{\nabla} u_n - \beta_n \frac{\partial u_n}{\partial x_3} J \widetilde{h} \longrightarrow \widetilde{\sigma}_* \widetilde{\nabla} u - \frac{\partial u}{\partial x_3} \left( \beta_1 I_2 + \beta_2 \left[ \widetilde{\sigma}_* - \widetilde{\sigma}_1 \right] \widetilde{\sigma}_2^{-1} \right) J \widetilde{h},$$

in the sense of Radon measures. Then, combining this convergence with (85), we have

$$\widetilde{\sigma}_n \widetilde{\nabla} u_n \longrightarrow \widetilde{\sigma}_* \widetilde{\nabla} u + \beta_2 \frac{\partial u}{\partial x_3} \left( \left[ \widetilde{\sigma}_1 + \theta \widetilde{\sigma}_2 - \widetilde{\sigma}_* \right] \widetilde{\sigma}_2^{-1} \right) J \widetilde{h} \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2. \tag{87}$$

Finally, passing to the limit in (86), taking into account (87) and (71), we obtain the convergence, in the sense of Radon measures,

$$\beta_n \widetilde{\nabla} u_n \longrightarrow \left[ \beta_1 I_2 + \beta_2 \widetilde{\sigma}_2^{-1} (\widetilde{\sigma}_* - \widetilde{\sigma}_1) \right] \widetilde{\nabla} u + \beta_2^2 \frac{\partial u}{\partial x_3} \widetilde{\sigma}_2^{-1} (\widetilde{\sigma}_1 + \theta \widetilde{\sigma}_2 - \widetilde{\sigma}_*) \widetilde{\sigma}_2^{-1} J \widetilde{h}. \tag{88}$$

Putting together $(83)$, $(85)$ and $(88)$ yields

$$
\begin{aligned}
\xi_0 \cdot e_3 = \quad & \left[\beta_1 I_2 + \beta_2 \widetilde{\sigma}_2^{-1}(\widetilde{\sigma}_* - \widetilde{\sigma}_1)\right]^{\mathrm{T}} J\widetilde{h} \cdot \widetilde{\nabla} u \\
& + \left[(\alpha_1 + \alpha_2 \theta) + \beta_2^2 \widetilde{\sigma}_2^{-1}(\widetilde{\sigma}_1 + \theta \widetilde{\sigma}_2 - \widetilde{\sigma}_*)\widetilde{\sigma}_2^{-1} J\widetilde{h} \cdot J\widetilde{h}\right] \frac{\partial u}{\partial x_3}.
\end{aligned}
\tag{89}
$$

Finally, since the current $\sigma_n \nabla u_n$ weakly-$*$ converges to $\xi_0$ in $(58)$, we have the limit equation

$$
-\operatorname{div}(\xi_0) = f,
$$

where, by $(82)$ and $(89)$,

$$
\xi_0 = (\widetilde{\xi}_0, \xi_0 \cdot e_3)^{\mathrm{T}} = \sigma_*(h)\nabla u,
$$

which yields to the expression $(38)$-$(39)$ of $\sigma_*(h)$. Theorem $3.1$ is proved. $\qquad\square$

4. **Two examples.** In this section we present two examples where the perturbation formulas for the effective conductivities of non periodic high-contrast columnar composites are fully explicitly computed.

4.1. **Circular fibres with variable radius.** Let $\rho$ be a continuous function on $\overline{\Omega}$ depending only on the variable $x' = (x_1, x_2)$ satisfying

$$
\exists c_1, c_2 > 0, \quad c_1 \leq \rho(x') \leq c_2, \quad \forall x = (x', x_3) \in \overline{\Omega},
\tag{90}
$$

and let $r_n$ be a sequence of positive numbers converging to $0$, as $n$ goes to infinity. We assume, without loss of generality, that

$$
\fint_{\Omega} \rho \, \mathrm{d}x = 1.
\tag{91}
$$

Define, for any $k \in \mathbb{Z}^3$, the sequence $(r_{n,k})_{n \in \mathbb{N}^*}$ by

$$
r_{n,k} := r_n \sqrt{\rho(\varepsilon_n k)}.
$$

We consider the case where the highly conducting zone $\Omega_n$ is the set of circular fibres $\omega_{n,k} = \{y \in Y \mid y_1^2 + y_2^2 \leq r_{n,k}^2\}$ (see Figure $1$)

$$
\Omega_n = \Omega \cap \bigcup_{k \in \mathbb{Z}^3} \varepsilon_n(\omega_{n,k} + k).
\tag{92}
$$

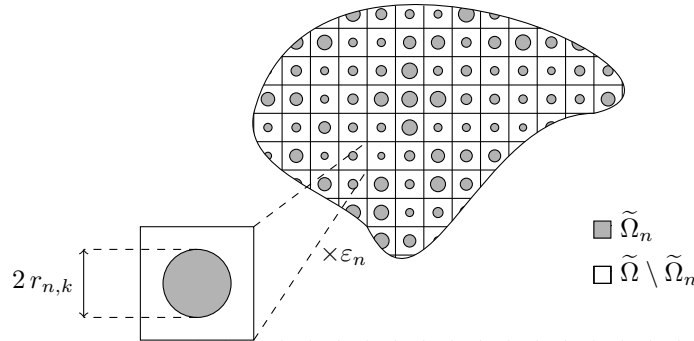Note that the fibres $\omega_{n,k}$ do not have the same radius.



FIGURE 1. The cross section of the non periodic microstructure

We have the following result:

**Proposition 2.** *Let $\Omega_n$ be the sequence of subsets of $\Omega$ defined by (92) and $\sigma_n(h)$ be the associated conductivity in the problem (4). Assume that*

$$\varepsilon_n^2 |\ln r_n| \xrightarrow[n\to\infty]{} 0. \tag{93}$$

*Then, there exist a matrix-valued function $\sigma_*(h)$ and a subsequence of $n$, still denoted by $n$, such that the solution $u_n$ of the problem (4) converges weakly in $H_0^1(\Omega)$ to the solution $u$ of the conductivity problem*

$$\begin{cases} -\operatorname{div}\big(\sigma_*(h)\nabla u\big) &=& f & in\ \Omega, \\ u &=& 0 & on\ \partial\Omega, \end{cases}$$

*where $\sigma_*(h)$ is given, for any $x = (x', x_3) \in \Omega$, by*

$$\sigma_*(h)(x) = \alpha_1 I_3 + \rho(x')\left(\frac{\alpha_2^3 + \alpha_2\beta_2^2|h|^2}{\alpha_2^2 + \beta_2^2 h_3^2}\right) e_3 \otimes e_3 + \beta_1 \mathscr{E}(h). \tag{94}$$

**Remark 9.** We can easily check that the homogenized conductivity $\widetilde{\sigma}_*(h)$ of the two dimensional microstructure of the Figure 1 is given by

$$\widetilde{\sigma}_*(h) = \alpha_1 I_2 + \beta_1 h_3 J = \widetilde{\sigma}_1(h).$$

This leads to the simple form (94) of $\sigma_*(h)$.

*Proof of Proposition 2.* In order to apply Theorem 3.1, we need to check that the conditions (33)-(35) hold true. On the one hand, the Poincar-Wirtinger inequality combined with (3) imply the existence, for any $k \in \mathbb{Z}^3$, of a sequence of positive constants $c_{n,k}$ such that

$$\forall v \in H^1(Q_n^k), \quad \int_{Q_n^k} \alpha_n \left| v - \fint_{Q_n^k} v \right|^2 \, \mathrm{d}x \le c_{n,k} \int_{Q_n^k} \alpha_n |\nabla v|^2 \, \mathrm{d}x, \tag{95}$$

where $Q_n^k = \varepsilon_n(Y + k)$. Using estimates derived in [12], one can show that the best constant in the weighted Poincar-Wirtinger inequality (95) satisfies

$$\forall k \in \mathbb{Z}^3, \quad \forall n \in \mathbb{N}^*, \quad 0 < c_{n,k} \le c\,\varepsilon_n^2 \left| \ln\big(r_n \sqrt{\rho(\varepsilon_n k)}\big) \right|,$$

for some positive constant $c$. Therefore, by (90) and (93), we have

$$0 < c_{n,k} \le c\,\varepsilon_n^2 |\ln r_n| + o(1) \xrightarrow[n\to\infty]{} 0,$$

uniformly with respect to $k \in \mathbb{Z}^3$. Conditions (33) and (34) of Theorem 3.1 are satisfied. On the other hand, by the definition of $\Omega_n$ and (91), we have the following estimate for the volume fraction

$$\theta_n = \frac{|\Omega_n|}{|\Omega|} \underset{n\to\infty}{\sim} \frac{1}{|\Omega|} \sum_{\varepsilon_n k \in \Omega} \varepsilon_n^2 \pi r_n^2 \rho(\varepsilon_n k) \underset{n\to\infty}{\sim} \pi r_n^2 \fint_\Omega \rho \, \mathrm{d}x = \pi r_n^2,$$

which, by (90), implies that for any $n \in \mathbb{N}^*$ and $k \in \mathbb{Z}^3$,

$$\fint_{Q_n^k} \alpha_n \, \mathrm{d}x = \alpha_1\big(1 - \pi r_n^2 \rho(\varepsilon_n k)\big) + \alpha_{2,n} \pi r_n^2 \rho(\varepsilon_n k) \le c + c\,\theta_n \alpha_{2,n} \le c.$$

Then, condition (35) of Theorem 3.1 is satisfied. Theorem 3.1 and Remark 9 ensure the existence of an effective conductivity $\sigma_*(h)$ which, after an easy computation, writes, for any $x = (x', x_3) \in \widetilde{\Omega} \times (0, 1)$,

$$\sigma_*(h)(x) = \alpha_1 I_3 + \theta(x')\left(\frac{\alpha_2^3 + \alpha_2\beta_2^2|h|^2}{\alpha_2^2 + \beta_2^2 h_3^2}\right) e_3 \otimes e_3 + \beta_1 \mathscr{E}(h), \tag{96}$$

where $\theta$ is the weak limit of $\theta_n^{-1}\mathbb{1}_{\Omega_n}$. The function $\theta$ in (96) coincides with $\rho$. Indeed, since $\rho$ is continuous, we obtain, for any $\varphi \in \mathscr{C}_0(\Omega)$ extended to $\mathbb{R}^3$ by setting $\varphi \equiv 0$ on $\mathbb{R}^3 \setminus \Omega$,

$$\int_\Omega \theta_n^{-1}\mathbb{1}_{\Omega_n}\varphi \, \mathrm{d}x = \frac{1}{\pi r_n^2}\sum_{k\in\mathbb{Z}^3}\int_{\omega_{n,k}}\varphi \, \mathrm{d}x + o(1) = \frac{1}{\pi r_n^2}\sum_{\varepsilon_n k\in\Omega}\varepsilon_n^2\pi r_n^2\rho(\varepsilon_n k)\varphi(\varepsilon_n k) + o(1)$$

which implies that

$$\int_\Omega \theta_n^{-1}\mathbb{1}_{\Omega_n}\varphi \, \mathrm{d}x = \int_\Omega \rho\varphi \, \mathrm{d}x + o(1).$$

Finally $\theta_n^{-1}\mathbb{1}_{\Omega_n}$ converges weakly-$*$ to $\rho$ in $\mathcal{M}(\Omega)$ and, then, $\theta \equiv \rho$. This concludes the proof of Proposition 2.                                                                                    $\square$

4.2. **Thin squared grids.** In this section, we consider the case of a columnar composite the cross section of which is a highly conducting grid surrounded by another conducting medium (see Figure 2). Let $t_n$ be a positive sequence converging to 0 as $n$ goes to infinity. Let $\rho$ be a continuous function on $\overline{\Omega}$, depending only on the variable $x' = (x_1, x_2)$ and satisfying

$$\exists c_1, c_2 > 0, \quad c_1 \le \rho(x') \le c_2, \quad \forall x = (x', x_3) \in \overline{\Omega}. \tag{97}$$

We assume, without loss of generality, that

$$\fint_\Omega \rho \, \mathrm{d}x = 1.$$

Define, for any $k$ in $\mathbb{Z}^3$, the sequence $(t_{n,k})_{n\in\mathbb{N}^*}$ by

$$t_{n,k} := \rho(\varepsilon_n k)\, t_n. \tag{98}$$

Let $\Omega_n$ be the set of non periodically distributed squared fibres

$$\Omega_n = \Omega \cap \bigcup_{k\in\mathbb{Z}^3}\varepsilon_n(\omega_{n,k} + k), \tag{99}$$

where

$$\omega_{n,k} := \left\{y \in Y \mid \max(|y_1|, |y_2|) \ge \tfrac{1}{2} - t_{n,k}\right\}.$$

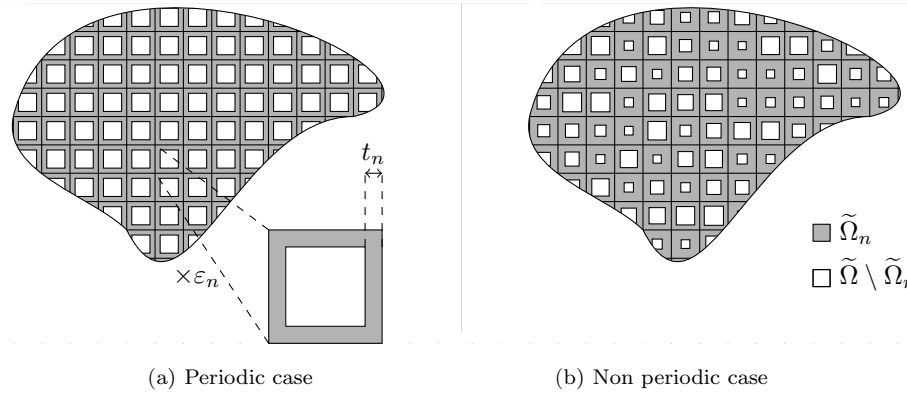Note that the case $\rho \equiv 1$ leads to a periodic distribution of the squared fibres in $\Omega$.



(a) Periodic case                          (b) Non periodic case

FIGURE 2. The cross section of the structure

We have the following result:

**Proposition 3.** *Let $\Omega_n$ be the sequence of subsets of $\Omega$ defined by* (99) *and $\sigma_n(h)$ be the associated conductivity in the problem* (4). *Assume that*

$$4\,t_n\alpha_{2,n} \xrightarrow[n\to\infty]{} \alpha_2 > 0 \quad and \quad 4\,t_n\beta_{2,n} \xrightarrow[n\to\infty]{} \beta_2 \in \mathbb{R}. \tag{100}$$

*Then, there exist a matrix-valued function $\sigma_*(h)$ and a subsequence of $n$, still denoted by $n$, such that the solution $u_n$ of the problem* (4) *converges weakly in $H_0^1(\Omega)$ to the solution $u$ of the conductivity problem*

$$\begin{cases} -\operatorname{div}\big(\sigma_*(h)\nabla u\big) &= f \quad in \ \Omega, \\ u &= 0 \quad on \ \partial\Omega, \end{cases} \tag{101}$$

*where $\sigma_*(h)$ is given by*

$$\sigma_*(h) := \begin{pmatrix} \widetilde{\sigma}_*(h) & p_* \\ q_*^T & \alpha_* \end{pmatrix} \tag{102}$$

*and, for any $(x', x_3) \in \Omega$,*

$$\begin{cases} \widetilde{\sigma}_*(h) = \left(\alpha_1 + \rho(x')\,\dfrac{\alpha_2^2 + \beta_2^2 h_3^2}{2\alpha_2}\right) I_2 + \beta_1 h_3 J, \\[2mm] p_* = -\left[\beta_1 + \rho(x')\,\dfrac{\beta_2}{2}\right] J\widetilde{h} + \rho(x')\,\dfrac{\beta_2^2\, h_3}{2\alpha_2}\widetilde{h}, \\[2mm] q_* = \left[\beta_1 + \rho(x')\,\dfrac{\beta_2}{2}\right] J\widetilde{h} + \rho(x')\,\dfrac{\beta_2^2\, h_3}{2\alpha_2}\widetilde{h}, \\[2mm] \alpha_* = \alpha_1 + \rho(x')\,\alpha_2 + \rho(x')\,\dfrac{\beta_2^2}{2\,\alpha_2}(h_1^2 + h_2^2). \end{cases} \tag{103}$$

*In formula* (103), *$\rho \equiv 1$ corresponds to the periodic case.*

*Proof of Proposition 3.* Let us first consider the periodic case. In order to apply Proposition 1, we need to check that (10) and (11) are satisfied. To this end, consider $V \in \mathscr{C}^1(\overline{Y})$ such that $\langle V \rangle = 0$. Define, for any $n \in \mathbb{N}^*$, the subsets $K_n^i$, $i = 1, 2, 3, 4$, of $\overline{Y}$ by

$K_n^1 := \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[\frac{1}{2} - t_n, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$, $K_n^2 := \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2} + t_n, -\frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$K_n^3 := \left[\frac{1}{2} - t_n, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$, $K_n^4 := \left[-\frac{1}{2} + t_n, -\frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$.

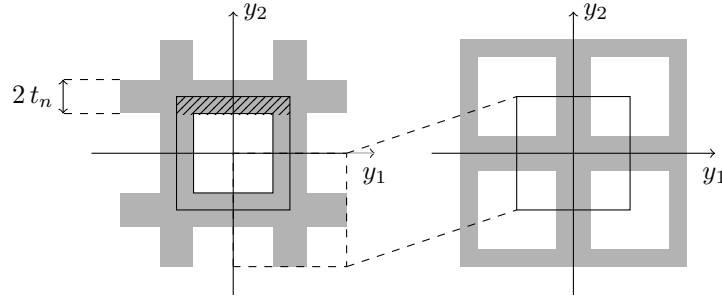For instance, the projection of $K_n^1$, in the $(y_1, y_2)$-plane, is the shaded zone in Figure 3.

FIGURE 3. The period cell of the cross section of the microstructure

By the definition (6) of $a_n$, we have

$$
\begin{aligned}
\int_Y a_n V^2 \, \mathrm{d}y \;\; &\leq \;\; \int_{Y \setminus \bigcup_{i=1}^4 K_n^i} a_n V^2 \, \mathrm{d}y + \sum_{i=1}^4 \int_{K_n^i} a_n V^2 \, \mathrm{d}y \\
&\leq \;\; \alpha_1 \int_{Y \setminus \bigcup_{i=1}^4 K_n^i} V^2 \, \mathrm{d}y + \alpha_{2,n} \sum_{i=1}^4 \int_{K_n^i} V^2 \, \mathrm{d}y \\
&\leq \;\; \alpha_1 \int_Y V^2 \, \mathrm{d}y + \alpha_{2,n} \sum_{i=1}^4 \int_{K_n^i} V^2 \, \mathrm{d}y.
\end{aligned}
$$

Since $\langle V \rangle = 0$, this inequality and the Poincar-Wirtinger inequality in $H^1(Y)$, yield

$$
\int_Y a_n V^2 \, \mathrm{d}y \leq \alpha_1 \int_Y |\nabla V|^2 \, \mathrm{d}y + \alpha_{2,n} \sum_{i=1}^4 \int_{K_n^i} V^2 \, \mathrm{d}y. \tag{104}
$$

We now estimate the second term of the right hand side of this inequality. On the one hand, since $K_n^1$ is convex, the Poincar-Wirtinger constant in $H^1(K_n^1)$ is bounded from above by the diameter of $K_n^1$ divided by $\pi$ [33] and, therefore

$$
\begin{aligned}
\int_{K_n^1} V^2 \, \mathrm{d}y \;\; &\leq \;\; 2 \int_{K_n^1} \left| V - \fint_{K_n^1} V \, \mathrm{d}y \right|^2 + 2 \, |K_n^1| \left| \fint_{K_n^1} V \, \mathrm{d}y \right|^2 \\
&\leq \;\; c \left( \int_{K_n^1} |\nabla V|^2 \, \mathrm{d}y + |K_n^1| \left| \fint_{K_n^1} V \, \mathrm{d}y \right|^2 \right).
\end{aligned} \tag{105}
$$

On the other hand, for any $-\frac{1}{2} \leq r, \, s, \, t \leq \frac{1}{2}$, we have

$$
\widehat{V}(s) - \widehat{V}(r) = \int_r^s \widehat{V}'(t) \, \mathrm{d}t, \quad \text{where} \quad \widehat{V}(t) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} V(y_1, t, y_3) \, \mathrm{d}y_1 \mathrm{d}y_3. \tag{106}
$$

Integrating the first equality in (106) with respect to $s \in [\frac{1}{2} - t_n, \frac{1}{2}]$ and $r \in [-\frac{1}{2}, \frac{1}{2}]$, we have

$$
\left| \fint_{K_n^1} V \, \mathrm{d}y - \fint_Y V \, \mathrm{d}y \right| \leq \fint_{\frac{1}{2}-t_n}^{\frac{1}{2}} \fint_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\widehat{V}'(t)| \, \mathrm{d}t \leq \int_Y \left| \frac{\partial V}{\partial y_2} \right| \, \mathrm{d}y,
$$

which, since $\langle V \rangle = 0$, implies that

$$
\left| \fint_{K_n^1} V \, \mathrm{d}y \right| \leq \int_Y |\nabla V| \, \mathrm{d}y \leq \|\nabla V\|_{L^2(Y)^3}. \tag{107}
$$

Then, combining (105) and (107) with the boundedness (100) of $|K_n^1|\alpha_{2,n} = t_n\alpha_{2,n}$, we obtain that

$$\alpha_{2,n}\int_{K_n^1} V^2 \,\mathrm{d}y \leq c\left(\alpha_{2,n}\int_{K_n^1} |\nabla V|^2 \,\mathrm{d}y + \|\nabla V\|_{L^2(Y)^3}^2\right) \leq c\int_Y a_n|\nabla V|^2 \,\mathrm{d}y. \quad (108)$$

Similarly to (108), we have, for $i = 2, 3, 4$,

$$\alpha_{2,n}\int_{K_n^i} V^2 \,\mathrm{d}y \leq c\int_Y a_n|\nabla V|^2 \,\mathrm{d}y. \quad (109)$$

Finally, (104), (108) and (109) imply

$$\int_Y a_n V^2 \,\mathrm{d}y \leq c\int_Y a_n|\nabla V|^2 \,\mathrm{d}y. \quad (110)$$

By a density argument, (110) is satisfied for any $V \in H^1(Y)$ with $\langle V \rangle = 0$. Since $\varepsilon_n$ converges to $0$, the hypotheses (10) and (11) of Proposition 1 are satisfied. Then, there exists a homogenized matrix which is given in terms of the transversal effective conductivity $\widetilde{\sigma}_*$ of the microstructure of Figure 2a. It remains to determine $\widetilde{\sigma}_*$. Since one can choose the cross-like shape of the Figure 3 as the period cell of the transversal microstructure of the heterogeneous medium occupying $\Omega$, Proposition 3.2 of [17] ensures that

$$\widetilde{\sigma}_* = \left(\alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h_3^2}{2\alpha_2}\right) I_2 + \beta_1 h_3 J, \quad (111)$$

and formula (102)-(103) of $\sigma_*(h)$ is a consequence of (13)-(14) where $\widetilde{\sigma}_*$ is given by (111). The periodic case is then proved.

The existence of $\sigma_*(h)$ in the non periodic case is a consequence of Theorem 3.1. Indeed, for any $k \in \mathbb{Z}^3$ and $n \in \mathbb{N}^*$, a rescaling of (110) gives

$$\forall v \in H^1(Q_n^k), \quad \int_{Q_n^k} \alpha_n \left|v - \fint_{Q_n^k} v\right|^2 \,\mathrm{d}x \leq c\,\varepsilon_n^2\int_{Q_n^k} \alpha_n|\nabla v|^2 \,\mathrm{d}x,$$

and, by (98),

$$\fint_{Q_n^k} \alpha_n \,\mathrm{d}x = \alpha_1\left(1 - 4\,t_{n,k}(1 - t_{n,k})\right) + 4\,\alpha_{2,n}\,t_{n,k}(1 - t_{n,k}) \leq c + c\,t_n\alpha_{2,n} \leq c.$$

The assumptions of Theorem 3.1 are satisfied. Then, there exist a matrix-valued function $\sigma_*(h)$ and a subsequence of $n$, still denoted by $n$, such that the solution $u_n$ of the problem (4) converges weakly in $H_0^1(\Omega)$ to the solution $u$ of the conductivity problem (101). In view of the formulas (38) and (39), the expression of $\sigma_*(h)$ becomes explicit as soon as $\widetilde{\sigma}_*$ and $\theta$ are identified.

On the one hand, it is easy to check, similarly to the proof of Proposition 2, that $\rho$ is the weak-$*$ limit, in the sense of Radon measures, of the sequence $\theta_n^{-1}\mathbb{1}_{\Omega_n}$. Then, the function $\theta$ in Theorem 3.1 turns out to be $\rho$. On the other hand, by Remark 7, in order to compute $\widetilde{\sigma}_*$, one has to determine $\sigma_*^0(\alpha_1, \alpha_2)$, which is the $H(\mathcal{M}(\widetilde{\Omega})^2)$-limit, in the sense of Definition 1.1, of the conductivity $\widetilde{\sigma}_n(0)$, in the absence of a magnetic field, given by, for any $x' \in \widetilde{\Omega}$,

$$\widetilde{\sigma}_n(0) := \begin{cases} \alpha_1 I_2 & \text{in } \widetilde{\Omega} \setminus \widetilde{\Omega}_n, \\ \alpha_{2,n} I_2 & \text{in } \widetilde{\Omega}_n. \end{cases}$$

Due to the local nature [13] of the $H(\mathcal{M}(\widetilde{\Omega})^2)$-convergence, it is sufficient to compute $\sigma^0_*(\alpha_1, \alpha_2)$ locally in $\widetilde{\Omega}$. To that aim, consider $x' \in \widetilde{\Omega}$ and $\varepsilon > 0$ small enough such that the closed disk $\overline{D}(x', \varepsilon) \subset \widetilde{\Omega}$. Since $\rho$ is continuous and by (97), we have

$$0 < c_{1,\varepsilon}(x') := \inf_{z \in \overline{D}(x',\varepsilon)} \rho(z) \le \rho(x') \le c_{2,\varepsilon}(x') := \sup_{z \in \overline{D}(x',\varepsilon)} \rho(z). \tag{112}$$
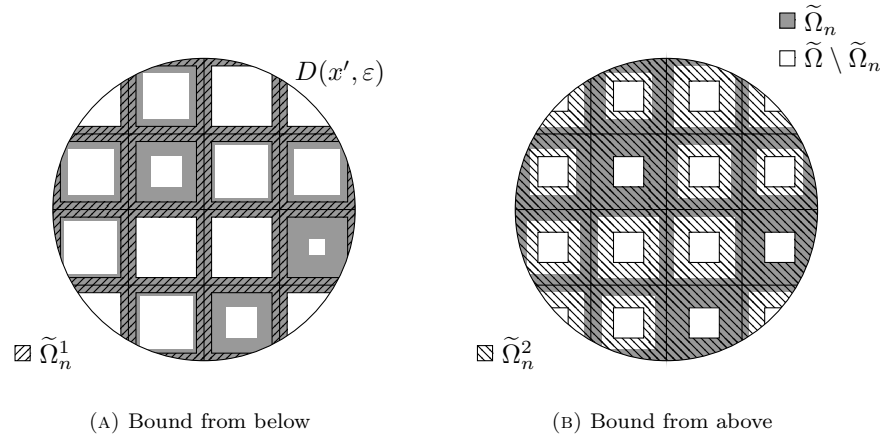


FIGURE 4. Bounds from below and above of $\widetilde{\sigma}_n(0)$

For $i = 1, 2$, let $\widetilde{\Omega}^i_n$ be the subset of $\overline{D}(x', \varepsilon)$ defined by (see Figure 4)

$$\widetilde{\Omega}^{i,\varepsilon}_n = \overline{D}(x', \varepsilon) \cap \bigcup_{k \in \mathbb{Z}^2} \varepsilon_n \big(k + \big\{ y \in (-\tfrac{1}{2}, \tfrac{1}{2})^2 \mid$$
$$\max(|y_1|, |y_2|) \ge \tfrac{1}{2} - c_{i,\varepsilon}(x')\, t_n \big\}\big), \tag{113}$$

and let $\widetilde{\sigma}^i_n$ be the periodic conductivity defined on $\overline{D}(x', \varepsilon)$ by

$$\widetilde{\sigma}^{i,\varepsilon}_n := \begin{cases} \alpha_1 I_2 & \text{in} \quad \overline{D}(x', \varepsilon) \setminus \widetilde{\Omega}^{i,\varepsilon}_n, \\ \alpha_{2,n} I_2 & \text{in} \quad \widetilde{\Omega}^{i,\varepsilon}_n. \end{cases} \tag{114}$$

By the definitions (112) and (114), we have for any $z \in \overline{D}(x', \varepsilon)$, the inequalities

$$\widetilde{\sigma}^{1,\varepsilon}_n(z) \le \widetilde{\sigma}_n(0)(z) \le \widetilde{\sigma}^{2,\varepsilon}_n(z). \tag{115}$$

For the rest of the proof, we need the following result which is a consequence of the two-dimensional div-curl lemma, in a high contrast context, of [13]:

**Lemma 4.1.** *Let $D$ be a bounded domain of $\mathbb{R}^2$ and, for $i = 1, 2$, consider an equi-coercive sequence of symmetric matrix-valued functions $A^i_n \in L^\infty(D)^{2 \times 2}$ bounded in $L^1(D)^{2 \times 2}$ which $H(\mathcal{M}(D)^2)$-converges to $A^i_*$ in the sense of Definition 1.1. We assume that*

$$\forall\, n \in \mathbb{N}^*, \quad A^1_n \le A^2_n \quad a.e. \text{ in } D. \tag{116}$$

*Then, we have the inequality*

$$A^1_* \le A^2_* \quad a.e. \text{ in } D.$$

*Proof of Lemma 4.1.* Let $\lambda \in \mathbb{R}^2$. Consider, for $i = 1, 2$, the solution $v_n^{\lambda,i}$ of

$$\begin{cases} \operatorname{div}\left(A_n^i \widetilde{\nabla} v_n^{\lambda,i}\right) & = & \operatorname{div}\left(A_*^i \lambda\right) & \text{in } D, \\ v_n^{\lambda,i} & = & \lambda \cdot x & \text{on } \partial D. \end{cases}$$

By Definition 1.1, we have the convergences, for $i = 1, 2$,

$$\begin{cases} v_n^{\lambda,i} & \longrightarrow & \lambda \cdot x & \text{weakly in } H^1(D), \\ A_n^i \widetilde{\nabla} v_n^{\lambda,i} & \longrightarrow & A_*^i \lambda & \text{weakly-$*$ in } \mathcal{M}(D)^2. \end{cases}$$

On the one hand, by (116), we have the inequality, almost everywhere in $D$

$$2 \, A_n^1 \widetilde{\nabla} v_n^{\lambda,1} \cdot \widetilde{\nabla} v_n^{\lambda,2} - A_n^1 \widetilde{\nabla} v_n^{\lambda,1} \cdot \widetilde{\nabla} v_n^{\lambda,1} \leq A_n^1 \widetilde{\nabla} v_n^{\lambda,2} \cdot \widetilde{\nabla} v_n^{\lambda,2} \leq A_n^2 \widetilde{\nabla} v_n^{\lambda,2} \cdot \widetilde{\nabla} v_n^{\lambda,2}. \quad (117)$$

On the other hand, applying, for $i, j = 1, 2$, the two-dimensional div-curl lemma of [13] (Theorem 2.1) to $\xi_n := A_n^i \widetilde{\nabla} v_n^{\lambda,i}$ and $v_n := v_n^{\lambda,j}$, we have the convergences, in the sense of distributions,

$$\forall \, i, j = 1, 2, \quad \xi_n \cdot \widetilde{\nabla} v_n = A_n^i \widetilde{\nabla} v_n^{\lambda,i} \cdot \widetilde{\nabla} v_n^{\lambda,j} \longrightarrow A_*^i \lambda \cdot \lambda \quad \text{in} \quad \mathscr{D}'(D). \quad (118)$$

Finally, combining (117) and (118), we obtain

$$2 \, A_*^1 \lambda \cdot \lambda - A_*^1 \lambda \cdot \lambda \leq A_*^2 \lambda \cdot \lambda \quad \text{in} \quad \mathscr{D}'(D),$$

which concludes the proof of Lemma 4.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since, for $i = 1, 2$, $\widetilde{\sigma}_n^{i,\varepsilon}$ is an equi-coercive sequence of periodic matrix-valued functions bounded in $L^1(D(x', \varepsilon))$, $\widetilde{\sigma}_n^{i,\varepsilon}$ $H(\mathcal{M}(D)^2)$-converges to a constant matrix $\widetilde{\sigma}_*^{i,\varepsilon}$. Then, applying Lemma 4.1 with $D = D(x', \varepsilon)$ and (115), we have

$$\widetilde{\sigma}_*^{1,\varepsilon} \leq \sigma_*^0(\alpha_1, \alpha_2)(z) \leq \widetilde{\sigma}_*^{2,\varepsilon} \quad \text{a.e. } z \in \overline{D}(x', \varepsilon). \quad (119)$$

Moreover, due to the definition (113) of $\widetilde{\Omega}_n^{i,\varepsilon}$ and the convergence (100), we have

$$|\widetilde{\Omega}_n^{i,\varepsilon}| \, |\widetilde{\Omega}|^{-1} \alpha_{2,n} = 4 \, t_n \alpha_{2,n} \, c_{i,\varepsilon}(x') + o(1) \xrightarrow[n \to \infty]{} \alpha_2 \, c_{i,\varepsilon}(x') > 0.$$

Then, substituting $\alpha_2 \, c_{i,\varepsilon}(x')$ for $\alpha_2$ in the formula (111) in the absence of a magnetic field (*i.e.*, $h_3 = 0$), we obtain, for $i = 1, 2$,

$$\widetilde{\sigma}_*^{i,\varepsilon} = \left(\alpha_1 + c_{i,\varepsilon}(x') \, \frac{\alpha_2}{2}\right) I_2. \quad (120)$$

By (112) and (120), taking the limit, as $\varepsilon$ goes to 0, in the inequalities (119), we obtain, for any Lebesgue point $x'$ of $\sigma_*^0(\alpha_1, \alpha_2)$ in $\widetilde{\Omega}$,

$$\sigma_*^0(\alpha_1, \alpha_2) = \left(\alpha_1 + \rho(x') \, \frac{\alpha_2}{2}\right) I_2.$$

Therefore, by Remark 7, we have

$$\widetilde{\sigma}_* = \left(\alpha_1 + \rho(x') \, \frac{\alpha_2^2 + \beta_2^2 h_3^2}{2 \, \alpha_2}\right) I_2 + \beta_1 h_3 J. \quad (121)$$

Finally, we apply the formula (38)-(39) for $\sigma_*(h)$ in Theorem 3.1, with $\widetilde{\sigma}_*$ given by (121), to obtain (102)-(103). This concludes the proof of Proposition 3. $\qquad\square$

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