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SINGULAR PERTURBATION AND BIFURCATION OF DIFFUSE TRANSITION LAYERS IN INHOMOGENEOUS MEDIA, PART I

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Abstract. We consider a singularly perturbed bistable reaction diffusion equation in a one-dimensional spatially degenerate inhomogeneous media. Degeneracy arises due to the choice of spatial inhomogeneity from some well-known class of normal forms or universal unfoldings. By means of a bilinear double well potential, we explicitly demonstrate the similarities and discrepancies between the bifurcation phenomena of the reaction diffusion equation and the limiting problem. The former is described by the location of the transition layer while the latter by the zeros of the spatial inhomogeneity function. Our result is the first which considers simultaneously the effects of *singular perturbation*, *spatial inhomogeneity* and *bifurcation phenomena*. (Part II [9] of this series analyzes the pitch-fork bifurcation for a general smooth double well potential where precise asymptotics and spectral analysis are needed.)

1. Introduction and motivation.

1. Introduction and motivation. Singularly perturbed reaction diffusion equation plays an important role in many applications ranging from phase transitions, combustions theory, pattern formation, morphogenesis, population biology to chemical reactions. There have been fruitful results on issues related to the existence and stability of stationary transition layers in the literature. The following elliptic equation is often considered:

$$\begin{aligned} \epsilon^2 \Delta u + f(u,x) \;&=\; 0, \qquad x \in \Omega \\ \frac{\partial u}{\partial n} &= 0, \qquad x \in \partial\Omega \end{aligned} \tag{1.1}$$

where $\Omega \in R^n$ is some bounded domain, and n is the unit outward normal to $\partial\Omega$. In this paper and its sequel [9], we concentrate on *bistable* nonlinearity function f of the following form:

$$f(u,x) = (1-u^2)(u-a(x)) \quad \text{where } -1 < a < 1 \tag{1.2}$$

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so that 1 and -1 are stable zeros of f while a is an unstable zero.

Equation (1.1) is in fact the Euler-Lagrange equation for stationary points of the following functional:

$$\mathcal{F}_\epsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{\epsilon^2}G(u,x)\,dx, \quad \text{where} \quad G(u,x) = \int_0^u -(1-\eta^2)(\eta - a(x))\,d\eta.$$

(The spatially homogeneous version of the above functional with $G(u,\cdot) = \frac{1}{4}(1-u^2)^2$ is the well-known *Allen-Cahn functional*. Its Euler-Lagrange equation:

$$\epsilon^2\Delta u + u(1-u^2) = 0 \tag{1.3}$$

is then called the *Allen-Cahn equation*.) One essential qualitative description of the solution u of (1.1) for $\epsilon \ll 1$ is that the underlying domain Ω is partitioned into sub-domains such that u takes on values close to 1 or -1 in each sub-domain and makes rapid but smooth transitions between them. Each transition has finite thickness of order $O(\epsilon)$ and hence is often called the *diffuse interface* or *transition layer*. We are particularly interested in information leading to the locations of these layers with respect to the underlying spatial inhomogeneity function $a(\cdot)$. For the *unbalanced* case, $G(1) \neq G(-1)$, it can be seen heuristically that the diffuse layers are roughly located near the *zeros* of a. For the *balanced* case, $G(1) = G(-1)$, the location is determined by balancing the *mean curvature* of the interface and the spatial inhomogeneity, leading to some *prescribed mean curvature problem*. In the current paper (and its sequel Part II [9]), we concentrate on the former unbalanced case.

There are several methods to study equation (1.1). In the variational setting, the technique of Γ-convergence, initiated by De Giorgi, is often used to understand rigorously the behavior of the solution in the regime $\epsilon \ll 1$. This is related to study the functional convergence of $\mathcal{F}_\epsilon$. For the Allen-Cahn Functional, Modica-Mortola [18] and Sternberg [29] proved that $\mathcal{F}_\epsilon \longrightarrow_\Gamma c_0\mathcal{F}_*$ where c_0 is some explicit constant and $\mathcal{F}_*$ is the area functional of the boundary of subsets of Ω. Or in mathematical terms,

$$\mathcal{F}_* : L^1(\Omega) \longrightarrow \overline{R}_+, \quad \mathcal{F}_*(u) = \int_\Omega |\nabla u|, \qquad \text{for } u : \Omega \longrightarrow \{-1, 1\}.$$

(Another representation of $\mathcal{F}_*$ is $\mathcal{F}_*(u) = \mathcal{H}^{n-1}\big(\partial\{u=1\}\big)$.) Kohn-Sternberg [14] proved that if u_0 is an isolated L^1-local-minimizer of $\mathcal{F}_*$, then there exists L^1-local-minimizer u^ϵ of $\mathcal{F}_\epsilon$ such that $u_\epsilon \longrightarrow_{L^1(\Omega)} u_0$. As for the convergence of critical points, under fairly general conditions, Padilla-Tonegawa [27], Hutchinson-Tonegawa [10] and Alberti [1] (for higher co-dimensions) have proved that critical points of $\mathcal{F}_\epsilon$ converge to a critical point of $\mathcal{F}_*$ in some appropriate sense.

It is also interesting to ask the question in the reverse direction, that is, given a critical point u_* of the limiting functional $\mathcal{F}_*$, can one construct critical points u^ϵ of $\mathcal{F}_\epsilon$ such that u^ϵ converges to u_* appropriately? There have been many studies in this direction for both homogeneous or inhomogeneous media. For example, in the unbalanced case, one of the earliest results include Fife-Greenlee [5] which considers an equation similar to (1.1). Solution u is constructed so that the transition layers are located near the zero level set of a. Some *non-degeneracy condition* – non-vanishing of some flux quantity across the zero level set – is imposed.

In the balanced setting, the transition layer roughly follows a minimal surface. Pacard-Rotoré [26] proved a result for the functional $\mathcal{F}_\epsilon(u) = \int_M \epsilon^2 |\nabla u|_g^2 + \frac{1}{2\epsilon}(1-u^2)^2\, d\mathrm{vol}_g$ on a Riemannian manifold (M, g) with or without the volume constraint, $\int_M u\, d\mathrm{vol}_g = m$. They show that for any *regular, non-degenerate* minimal surface or constant mean curvature surface S, there exists critical points, u^ϵ of $\mathcal{F}_\epsilon$, such that $\partial\{u^\epsilon > 0\} \longrightarrow S$. Here a surface S is regular means that S is the 0-level set of a smooth function ψ and 0 is a regular value of ψ. For the non-degeneracy condition, it is required that the associated Jacobi operator $\mathcal{L}_S := \Delta_S + |A_S|^2 + \mathrm{Ric}_M(\nu_S, \nu_S)$ has no zero eigenvalue. See also Nefedov-Sakamoto [21] for similar results with appropriate non-degenerate conditions.

In two dimensions, in Kowalczyk [15], transition layers to the Allen-Cahn equation in a two dimensional domain is studied. It is proved that for any non-degenerate straight line segment which intersects the boundary of the domain orthogonally, there exists a solution to the Allen-Cahn equation whose transition layer is located near this segment. The stability of such solutions is also analyzed and it is determined by a geometric eigenvalue problem related to the curvatures of the domain boundary. In the case of stable solutions, a result of Kohn-Sternberg [14] is recovered. As for the unstable solutions, it is proved that their Morse index is either 1 or 2.

Concerning transition layers in *spatially inhomogeneous* media, pure PDE techniques can be used to analyze the existence and stability of stationary solutions. In Angenent, Mallet-Paret and Peletier [2], the existence of stable one-dimensional transition layer solutions with the interface near the stable non-degenerate zeros of the spatial inhomogeneous term $a(x)$ is constructed via comparison principle. The stability was obtained by careful spectral analysis of the linearized operator at the stationary solution. In fact, they have classified all stable solutions. In Hale-Sakamoto [8], Liapunov-Schmidt reduction technique is used to construct both stable and unstable transition layers. Moreover, the stability property of the solutions was obtained as a byproduct of their technique. This is one example using the machinery commonly known as SLEP (singular limit eigenvalue problem [13]). There are quite a few higher dimensional analogue of the above results, see for example, do Nascimento [19, 20], and Li-Nakashima [17]

In all of the above works, various types of non-degeneracy conditions on the limiting interfacial problem are assumed. It is interesting and important to investigate the behavior and existence of layers when *degeneracy* occurs. In many cases, degeneracy is connected with bifurcation (see [3, 4, 6]). We are also motivated by the fact that bifurcation plays an important role in understanding the dynamics in models from science and engineering. In [12, 16], for a system of reaction diffusion equations with spatially homogeneous nonlinearity, it is proved that there exists pitchfork bifurcation of stationary solutions in relation to some bifurcation parameter in the system. Inspired by this, we investigate the occurrence of bifurcation of transition layers in a *spatially degenerate inhomogeneous media*. To account for degeneracy, an extra bifurcation parameter β is introduced into the spatial inhomogeneity. Precisely, we choose $a(x;\beta)$ such that the solutions of $a(x;\beta) = 0$ bifurcate at ($x = 0,\ \beta = 0$). The associated reaction diffusion problem (1.1) is degenerate because $a_x(x = 0, \beta = 0) = 0$. Up to now, all existing results require $a_x \neq 0$ at the zeros of a.

With the presence of ϵ, the above question is related to *perturbed bifurcation problems* [3, 4]. However, it is not clear if such approach is applicable due to the singular nature of ϵ. To the best of our knowledge, our work is the first comprehensive result that considers simultaneously the effects of *singular perturbation*, *spatial inhomogeneity* and *bifurcation phenomena*.

On the other hand, there are quite a few works tackling similar problems, in particular for systems of reaction diffusion equations in one-dimension. A common technique involves asymptotic expansions, with careful analysis of inner- and outer-solutions. Then the exact solution is constructed by means of some matching-conditions. These conditions are described in terms of the intersection of some manifolds corresponding to boundary conditions related to the outer- and inner-solutions. (Non-)degeneracy refers to that the manifolds intersect (tangentially) transversally. Some of the works along these lines include Hale-Lin [7], Ikeda-Mimura-Nishiura [12] which also considers the bifurcation of solutions when some underlying parameter changes. See also [16, 23]. With the presence of the singular perturation, the technique of SLEP is often used in constructing and also analyzing the stability property of the solutions. See for example [11, 22, 24, 25].

Even though in the hindsight our approach in particular of this paper, bears some similarities to the above works, we aim at characterizing the solution structure *exactly at the bifurcation point*. Our technique allows the bifurcation (and other auxiliary) parameter β to *depend on ϵ, the singular parameter*. In this way, the modification of bifurcation picture due to ϵ is described accurately. Specifically, we thrive to investigate the connection of the bifurcation of diffuse interface and that of the limiting interfacial problem. In other words, if the limiting interfacial problem bifurcates, will the diffuse interface bifurcate as well? In the unbalanced setting for the cubic nonlinearity $f(u, a)$ in (1.2), it is equivalent to asking if the number of zeros of the spatial inhomogeneity term $a(x; \beta)$ bifurcates, will the number of transition layer solutions bifurcate? If so, can we find out the bifurcation point(s) (denoted by β_*^ϵ) at which the diffuse layers start to bifurcate? What are the factors affecting the bifurcation point(s)? Will there be any differences between the bifurcations of the two problems, and in what sense? This paper is devoted to answer the above questions with explicit and yet illustrative spatial inhomogeneities. With the use of a bilinear nonlinearity function and the associated explicit solution formula, we can give a fairly complete description. In the sequel [9], we will extend some of the results here to more general nonlinear functions. The technique is based on more PDE techniques and Liaponov-Schmidt reduction.

The outline of the paper is as follows. In Section 2.1, we first give a brief description of the main concepts of bifurcation theory, in particular, the notion of equivalence and universal unfolding. This is useful in understanding the connection between the interfacial location and the background inhomogeneity. Then we introduce our problem formulation in Section 2.2. In this paper, we exclusively work on the bilinear double well potential so that we can take advantage of explicit solution formula. In Section 3, we derive the equation for the transition layer location and study some of the properties of the layer solution. Then in Section 4, we tabulate the transition varieties for all the eleven universal unfoldings with codimension no more than three for the layer locations. In the last Section 5, we give detail information for the trans-critical and pitchfork bifurcations, in particular, the appearance of imperfection in the bifurcation process.

2. **The problems and settings.**

2.1. **Some concepts from bifurcation.**

Here we give a brief overview of some key concepts in bifurcation, in particular the notion of *equivalence*, *auxiliary parameters*, and *universal unfoldings*. We mainly follow the text Golubitsky-Schaeffer [6]. Bifurcation theory is the study of the change of the structure of solutions of an equation when some underlying parameter(s) varies. Specifically, *bifurcation* means a change in the number of solutions of an equation when a parameter passes through some critical value which is often referred as the *bifurcation point*. For a wide range of equations, including many partial differential equations, problems with multiple solutions can be reduced to the study of how the solutions of a single scalar equation

$$g(x;\beta) = 0 \tag{2.1}$$

behave as a function of parameter β. The derivation of the above equation is usually done with the technique commonly known as the *Liapunov-Schmidt* (**LS**) reduction.

In equation (2.1), the variable x is the *state variable* (not to be confused with the spatial variable in the PDE (1.1)) and β is the *bifurcation parameter*. We call the set of solutions $\{(x;\beta) : g(x;\beta) = 0\}$, the *bifurcation diagram* of g. For each β, denote $n(\beta)$ as the number of solutions x to (2.1). Classically, (x_0, β_0) is called a *bifurcation point* if $n(\beta)$ changes as β varies in the neighborhood of β_0. For instance, for

$$g(x;\beta) = x^3 - \beta x, \tag{2.2}$$

it has the property that as β crosses the value $\beta_0 = 0$, the number of solutions $n(\beta)$ changes from one to three. Hence $(x_0 = 0, \beta_0 = 0)$ is a bifurcation point of $g(x;\beta)$. In the literature, bifurcation which resembles qualitatively to that of (2.2) is called the *pitchfork bifurcation*. Usually, bifurcation study is local, i.e. it only considers $n(\beta)$ near the bifurcation point.

Bifurcation study emphasizes the qualitative properties of equation (2.1). The investigation is highlighted by the notion of *equivalence* which defines precisely when the solution sets of two equations are qualitatively similar to each other. Specifically, we want to know that given an equation $g(x;\beta) = 0$, when a second equation $a(x;\beta) = 0$ is *equivalent* to $g(x;\beta) = 0$. This problem is often known as the *recognition problem* in bifurcation theory. Mathematically, we have the following definition.

Definition 1 (Equivalence)**.** Let $a(\cdot,\cdot)$ and $g(\cdot,\cdot)$ be two bifurcation problems. Suppose a bifurcates at $(x_0;\beta_0)$. We say that g and a are *equivalent* (denoted by $g \sim a$) if they can be related as

$$S(x;\beta)g(X(x;\beta),\Lambda(\beta)) = a(x;\beta) \tag{2.3}$$

where S is a nonzero function of $(x;\beta)$ and $(X;\Lambda)$ is a local orientation preserving diffeomorphism of R^2 near $(x_0;\beta_0)$. (Again, the existence or construction of the map $(X;\Lambda)$ just needs to be local.)

If the two problems g and a are equivalent, then the two multiplicity functions are related as

$$n_g(\Lambda(\beta)) = n_a(\beta).$$

This is the most important consequence of equivalence. In order to check equivalence, i.e., solving the *recognition problem*, instead of finding explicit local diffeomorphism to establish the equivalence condition (2.3), singularity theory produces a *finite list of conditions* for the Taylor series of g such that the question of equivalence of g and a is determined solely by the values of the derivatives of g and a in this list, while all the other terms are irrelevant. In actual applications, this list specifies precisely the calculations which must be performed in order to recognize an equation of a given bifurcation type.

As an example, for a given bifurcation problem $g(x;\beta)$, if at a specific point $(x_0;\beta_0)$ it satisfies

$$g = g_x = g_{xx} = g_\beta = 0, \quad g_{xxx} > 0, \quad \text{and} \quad g_{x\beta} < 0, \tag{2.4}$$

then $g(x;\beta) = 0$ is equivalent to (2.2) around the point $(x_0 = 0; \beta_0 = 0)$. Expression (2.2) is one of the simplest satisfying conditions (2.4). Thus it is called a *normal form* for the pitchfork bifurcation. Equivalently, we say that condition (2.4) solves the recognition problem for the pitchfork normal form. Many more normal forms and how to solve their associated recognition problems can be found in [6, Chapter II, §9, 10].

So far, the bifurcation problem (2.1) involves only one bifurcation parameter β. However, there are often needs to incorporate additional parameters. For example, the original formulation of a physical model might involve several auxiliary parameters. In addition, mathematical equations resulting from the choice of physical models are often idealizations and hence a more complete description would almost surely lead to a perturbed set of equations. The deviations of the actual situation from the idealized description sometimes can be accounted for by adding auxiliary parameters into the equations.

Another central issue in bifurcation study is to investigate how bifurcation problems depend on the auxiliary parameters. It can happen that small variation of an auxiliary parameter might lead to dramatic changes in the bifurcation diagram. As an illustration, consider the perturbed pitchfork bifurcation,

$$G(x;\beta,\alpha) := x^3 - \beta x + \alpha = 0 \tag{2.5}$$

where α is an auxiliary parameter. It can be checked that the bifurcation diagrams of (2.5) with $\alpha = 0$, $\alpha > 0$, $\alpha < 0$, are shown in Figure 3-0, 1 or 3, respectively.

For a bifurcation problem $g(x;\beta) = 0$, there are many ways to introduce a perturbation. For example, we can consider

$$g(x;\beta,\gamma) = g(x;\beta) + \gamma p(x;\beta,\gamma), \quad \text{with } |\gamma| \ll 1.$$

However, people often want to construct certain distinguished family of perturbations of g which can cover *all (small)* perturbations. This leads to the concept of *universal unfolding*. The following definition formalizes this concept.

Definition 2 (Unfolding)**.** A parametrized family of mappings $G(x;\beta,\vec{\alpha})$ where $\vec{\alpha}$ lies in a parameter space R^k is an unfolding of $g(x;\beta)$ if it satisfies the following two conditions:

(a) $G(x;\beta,\vec{\alpha})$ is a perturbation of g, i.e., $G(x;\beta,0) = g(x;\beta)$;

(b) any small perturbation of g is equivalent to $G(\cdot\ ;\ \cdot,\ \vec{\alpha})$ for some $\vec{\alpha} \in R^k$ near the origin. In other words, given any perturbation $\gamma p(x;\beta,\gamma)$ with $|\gamma| \ll 1$, there are parameter values $\vec{\alpha}(\gamma) = (\alpha_1(\gamma), \cdots, \alpha_k(\gamma))$ such that

$$g(\cdot\,;\,\cdot) + \gamma p(\cdot\ ;\ \cdot,\ \gamma) \sim G(\cdot\ ;\ \cdot,\ \vec{\alpha}(\gamma))$$

in the sense of (2.3).

Remarkably, there are special unfoldings G of g which depends on a *minimum* number of auxiliary parameters and are equivalent to every other unfolding of g. Such an unfolding is called a *universal unfolding*, in a sense that there is no redundancy in the parameters. That minimum number of auxiliary parameters is called the *codimension* of g. The determination of whether an unfolding G of g is universal is called the recognition problem for universal unfolding. It is quite technical to solve, and it involves the concept of tangent space of g. Though an explicit formula of universal unfolding G is often unavailable, in actual applications, the recognition problem for a universal unfolding G depends only on finitely many derivatives of G at the origin. This is similar to the recognition for a normal form. For example, in [6, Chapter III], $a(x;\beta,\alpha) = x^3 - \beta x + \alpha_1 + \alpha_2 x^2$ is proved to be a universal unfolding of the pitchfork normal form $x^3 - \beta x$. See [6, Chapter III, §3] for a tabulation of more examples of universal unfoldings.

Dramatic changes in the bifurcation diagrams can happen as the auxiliary parameters vary. How to enumerate perturbed bifurcation diagrams is very important in the context of universal unfolding. In carrying out the enumeration, people are interested in knowing under what kind of perturbations that perturbed bifurcation diagrams in the universal unfolding of g remain unchanged in the sense of equivalence. If so, such diagrams are called persistent. To answer this question, we often turn to *non-persistent* diagrams. That is, for what values of the auxiliary parameters $\vec{\alpha}$, it produces a bifurcation diagram different from the one implied by the unperturbed bifurcation problem $g(x;\beta) = 0$?

Transition variety is thus introduced to enumerate *non-persistent* bifurcation diagrams of a perturbed bifurcation problem $G(x,\beta;\vec{\alpha}) = 0$. In other words, transition variety is the set of parameters $\vec{\alpha} \in R^k$ at which *non-persistent* bifurcation diagrams are obtained. It turns out that there are three sources of non-persistence, namely *bifurcation*, *hysteresis* and *double limit points*. See [6, Chapter III, Fig. 5.1] for these three phenomena where both unperturbed and non-persistent perturbed diagrams are sketched. The exact definition of transition variety is given as follows.

Definition 3 (Transition Variety Σ)**.** The transition variety Σ is defined as $\Sigma := \mathcal{B} \cup \mathcal{H} \cup \mathcal{D}$ where $\mathcal{B}, \mathcal{H}, \mathcal{D}$ stand for the bifurcation, hysteresis, double limits sources which are given by:

$$\mathcal{B} = \left\{\vec{\alpha} \in R^k\, |\exists(x,\beta) \text{ s.t } G(x,\beta;\vec{\alpha}) = G_x(x,\beta;\vec{\alpha}) = G_\beta(x,\beta;\vec{\alpha}) = 0 \text{ at } (x,\beta;\vec{\alpha})\right\},$$
$$\mathcal{H} = \left\{\vec{\alpha} \in R^k\, |\exists(x,\beta) \text{ s.t } G(x,\beta;\vec{\alpha}) = G_x(x,\beta;\vec{\alpha}) = G_{xx}(x,\beta;\vec{\alpha}) = 0 \text{ at } (x,\beta;\vec{\alpha})\right\},$$
$$\mathcal{D} = \left\{\vec{\alpha} \in R^k\, |\exists(x_1 \neq x_2) \text{ s.t } G(x,\beta;\vec{\alpha}) = G_x(x,\beta;\vec{\alpha}) = 0 \text{ at } (x_i,\beta;\alpha), i = 1,2\right\}.$$

Heuristically, each of the three sets in Definition 3 is described by a single scalar equation $\psi_{\mathcal{I}}(\alpha_1, \cdots, \alpha_k) = 0$ where the subscript $\mathcal{I}$ corresponds to $\mathcal{B}, \mathcal{H}$, or $\mathcal{D}$. For example, consider the defining equation for $\mathcal{B}$,

$$G(x,\beta;\vec{\alpha}) = G_x(x,\beta;\vec{\alpha}) = G_\beta(x,\beta;\vec{\alpha}) = 0.$$

If the first two equations are solved for x and β as functions of $\vec{\alpha}$, then substitute them into the third equation would give us the scalar equation for $\vec{\alpha}$ as claimed. Similar ideas apply to $\mathcal{H}$ and $\mathcal{D}$.

Using the concepts from algebraic geometry, we quote the result [6, Theorem 5.3, Chapter III] that the transition variety Σ in Definition 3 is a semi-algebraic variety in R^k of codimension 1. The implication is that in the auxiliary parameters space, $R^k \backslash \Sigma$ has finitely many connected components. In view of [6, Chapter III, Theorem 6.1], it implies that the persistent bifurcation diagrams in the universal unfolding G are enumerated by the connected components of $R^k \backslash \Sigma$. In loose terms, if $\vec{\alpha}_1, \vec{\alpha}_2$ belong to the same connected components, then $G(\cdot,\cdot;\vec{\alpha}_1)$ and $G(\cdot,\cdot;\vec{\alpha}_2)$ are equivalent. In other words, the bifurcation diagrams stay persistent as long as the auxiliary parameters are contained in each one of those components.

We now apply the above concepts to our transition layer problem. In particular, we want to investigate the equivalence between the bifurcation of the *transition layer location* and that of the *zeros of the spatial inhomogeneous function*.

2.2. **Three problems: [G, S, E].** We will investigate the following three problems.

Problem [G]: *Bifurcation of transition layers for the general reaction diffusion problem (**RD**) with spatial inhomogeneity $a(x;\beta)$, i.e., solving for u satisfying,*

$$\epsilon^2 u_{xx} + (1-u^2)(u - a(x;\beta,\vec{\alpha})) = 0, \quad x \in (-1,1), \quad u_x(\pm 1) = 0. \tag{2.6}$$

This problem is exactly the Euler-Lagrange equation of the functional $\mathcal{F}_\epsilon^G$.

$$\mathcal{F}_\epsilon^G(u) := \int_\Omega \left[\frac{\epsilon}{2}u_x^2 + \frac{1}{4\epsilon}\left(1-u^2\right)^2 + \frac{a(x;\beta,\vec{\alpha})}{\epsilon}\int_{-1}^{u}\left(1-s^2\right)ds \right] dx. \tag{2.7}$$

As explained in the introduction, in the above unbalanced setting, the location of the interface for the solution of (2.6) can be connected to the underlying limiting interfacial problem which is introduced next (in the form that incorporates auxiliary parameters $\vec{\alpha}$).

Problem [S]: *Bifurcation of the zeros of the spatial inhomogeneity, i.e., finding x satisfying*

$$a(x,\beta;\vec{\alpha}) = 0, \tag{2.8}$$

where β is the scalar bifurcation parameter and $\vec{\alpha}$ is the list of auxiliary parameters.

The next problem serves as an approximation of Problem **[G]**. It is proposed as an explicit example to give indications on the bifurcation behaviors of the transition layers to Problem **[G]**.

Problem [E]: *Bifurcation of transition layers with bilinear nonlinearity where (2.8) is the spatial inhomogeneity, i.e., finding $(u^\epsilon(x;\beta,\vec{\alpha}), x_*)$ such that*

$$\begin{cases} \epsilon^2 u_{xx} = u - 1 - a(x;\beta,\vec{\alpha}), & \textit{if } x_* < x \le 1; \\ \epsilon^2 u_{xx} = u + 1 - a(x;\beta,\vec{\alpha}), & \textit{if } -1 < x \le x_*; \\ u_x(\pm 1) = 0, & \textit{Neumann boundary condition}; \\ u(x_*^-) = u(x_*^+) = 0, & C^0\textit{-matching}; \\ u_x(x_*^+) - u_x(x_*^-) = 0. & C^1\textit{-matching}. \end{cases} \tag{2.9}$$

The system (2.9) is exactly the Euler-Lagrange equation of the functional $\mathcal{F}_\epsilon^E$:

$$\mathcal{F}_\epsilon^E(u) := \int_{-1}^{1} \left[\frac{\epsilon}{2}u_x^2 + \frac{1}{2\epsilon}(1-|u|)^2 - \frac{a(x;\beta,\vec{\alpha})u}{\epsilon} \right] dx. \tag{2.10}$$

Here the quantity x_* is regarded as the location of a single transition layer. (Note that formulation (2.9) already assumes that the solution u is positive for $x > x_*$ and negative for $x < x_*$.) As we will see later, Problem **[E]** can be reduced to an explicit equation for x_* where there is one-to-one correspondence between x_* and the transition layer solution.

Note that both the functions $(1-|u|)^2$ and $(1-u^2)^2$ have a similar feature: a *double well function*. Hence we expect that the solution structure of Problems **[G]** and **[E]** are similar. Equation (2.9) is simpler to solve than (2.6) as it involves only ODEs with constant coefficients and hence we can make use of explicit solution formula. Because of this, we can obtain many precise quantitative statements. These can give us good indications for the more general Problem **[G]** which will be analyzed in the sequel [9] where detail spectral analysis and asymptotic expansion will be used.

In this paper, we consider Problems **[S]** and **[E]** in which the spatially inhomogeneous function a is *degenerate*. This is achieved by choosing a from a class of normal forms or universal unfoldings in general bifurcation study. We thrive to answer the following questions:

1. If the zeros of the spatial inhomogeneity term $a(x;\beta,\vec{\alpha})$ bifurcate, would the transition layers solutions to Problem **[E]** bifurcate also?
2. Which factors affect the bifurcation point, denoted as β_*^ϵ, for Problem **[E]**?
3. Are there any differences between the bifurcations of Problems **[S]** and **[E]**, and in what sense?

The answers to the above questions will be gradually revealed in the following sections. In Section 3, we derive the location equation for x_*. Both the spatial inhomogeneity and the singular parameter enter into the equation (3.7). In Section 4, we study the connections between the Problems **[S]** and **[E]**. In particular, We show the equivalence of the two bifurcation problems under the umbrella of universal unfolding. The well-known concepts in bifurcation study such as universal unfolding and transition varieties are also discussed. In Section 5, by adding some specific perturbations to spatial inhomogeneity, examples are provided where the two bifurcation Problems **[S]** and **[E]** are different in terms of *(im)perfection*. The rigorous justification relies on the idea of *finite determinacy*, which is a powerful machinery to recognize equivalent normal form of a bifurcation problem.

A word about notation. When we write $a \ll b$, we mean the regime that a is much smaller than b i.e. the ratio $\frac{a}{b}$ is chosen or can be made as small as possible. (Almost) all the results in this paper requires sufficiently small ϵ, i.e. $\epsilon \ll 1$. Other small parameters will also be considered. Unless mentioned explicitly, the small parameters will be taken independently of each other.

3. Deriving the location equation.

For Problem **[E]**, the solution of (2.9) is given by:

$$u^\epsilon(x) = \begin{cases} u_+ = Ae^{\frac{x}{\epsilon}} + Be^{-\frac{x}{\epsilon}} + \left(u_p^\epsilon(x,\beta;\vec{\alpha}) + 1\right) & \text{if } x_* < x \le 1; \\ u_- = Ce^{\frac{x}{\epsilon}} + De^{-\frac{x}{\epsilon}} + \left(u_p^\epsilon(x,\beta;\vec{\alpha}) - 1\right) & \text{if } -1 < x \le x_* \end{cases} \tag{3.1}$$

where x_* is the zero of u^ϵ: $u_+(x_*) = u_-(x_*) = 0$ and $u_p^\epsilon(x,\beta;\vec{\alpha})$ is a particular solution to the problem

$$\epsilon^2 u_{xx} = u - a(x,\beta;\vec{\alpha}). \tag{3.2}$$

We first obtain a system of equations for the unknown constants A, B by eliminating C, D. It follows from the C^0- and C^1-matching conditions that

$$\begin{aligned} Ae^{\frac{x_*}{\epsilon}} + Be^{-\frac{x_*}{\epsilon}} + 2 &= Ce^{\frac{x_*}{\epsilon}} + De^{-\frac{x_*}{\epsilon}}, \\ \text{and} \quad Ae^{\frac{x_*}{\epsilon}} - Be^{-\frac{x_*}{\epsilon}} &= Ce^{\frac{x_*}{\epsilon}} - De^{-\frac{x_*}{\epsilon}} \end{aligned}$$

leading to the identities

$$C = A + e^{-\frac{x_*}{\epsilon}}, \quad D = B + e^{\frac{x_*}{\epsilon}}. \tag{3.3}$$

Define quantities M and N as

$$M(u_p^\epsilon) := u_{p,x}^\epsilon(1, \beta; \vec{\alpha}), \quad N(u_p^\epsilon) := \frac{u_{p,x}^\epsilon(1, \beta; \vec{\alpha}) - u_{p,x}^\epsilon(-1, \beta; \vec{\alpha})}{2} \tag{3.4}$$

where the subscript $,x$ refers to differentiating with respect to the spatial variable x. Then the Neumann boundary condition gives us the system of equations for A, B as follows.

$$\begin{aligned} Ae^{\frac{1}{\epsilon}} - Be^{-\frac{1}{\epsilon}} &= -\epsilon M, \\ Ae^{\frac{1}{\epsilon}} - Be^{-\frac{1}{\epsilon}} &= Ce^{-\frac{1}{\epsilon}} - De^{\frac{1}{\epsilon}} - 2\epsilon N. \end{aligned}$$

Substituting the two identities (3.3) into the above, we arrive at

$$\begin{aligned} Ae^{\frac{1}{\epsilon}} - Be^{-\frac{1}{\epsilon}} &= -\epsilon M, \\ \left(e^{\frac{1}{\epsilon}} - e^{-\frac{1}{\epsilon}}\right)(A + B) &= -\left[e^{\frac{1+x_*}{\epsilon}} - e^{-\frac{1+x_*}{\epsilon}}\right] - 2\epsilon N. \end{aligned}$$

which can be re-written as

$$A = \frac{-\epsilon M\left[1 - e^{-\frac{2}{\epsilon}}\right] - \left[e^{\frac{x_*-1}{\epsilon}} - e^{-\frac{x_*+3}{\epsilon}}\right] - 2\epsilon e^{-\frac{2}{\epsilon}} N}{e^{\frac{1}{\epsilon}} - e^{-\frac{3}{\epsilon}}}, \tag{3.5}$$

$$B = \frac{\epsilon M\left[1 - e^{-\frac{2}{\epsilon}}\right] - \left[e^{\frac{x_*+1}{\epsilon}} - e^{-\frac{x_*+1}{\epsilon}}\right] - 2\epsilon N}{e^{\frac{1}{\epsilon}} - e^{-\frac{3}{\epsilon}}}. \tag{3.6}$$

We are now in a position to write down the equation for x_* which is stated in the following.

Proposition 3.1. *The location x_* to Problem **[E]** satisfies:*

$$u_p^\epsilon(x_*, \beta; \vec{\alpha}) = R^\epsilon(x_*, \beta; \vec{\alpha}) \tag{3.7}$$

where

$$\begin{aligned} R^\epsilon(x, \beta; \vec{\alpha}) := \frac{2\epsilon e^{-\frac{1}{\epsilon}}}{1 + e^{-\frac{2}{\epsilon}}} \sinh\left(\frac{x}{\epsilon}\right) M(u_p^\epsilon) + \frac{4\epsilon e^{-\frac{2}{\epsilon}}}{1 - e^{-\frac{4}{\epsilon}}} \cosh\left(\frac{x-1}{\epsilon}\right) N(u_p^\epsilon) \\ + \frac{2e^{-\frac{2}{\epsilon}}}{1 - e^{-\frac{4}{\epsilon}}} \sinh\left(\frac{2x}{\epsilon}\right), \end{aligned} \tag{3.8}$$

u_p^ϵ is a solution of (3.2), and the quantities M and N are from (3.4).

Proof. The proof is a direct consequence of the definition of x_*: $u^\epsilon(x_*) = 0$, i.e.

$$Ae^{\frac{x_*}{\epsilon}} + Be^{-\frac{x_*}{\epsilon}} + \left(u_p^\epsilon(x_*, \beta; \vec{\alpha}) + 1\right) = 0$$

and making use of the formula for A and B (3.5, 3.6). Note that the way (3.7) is written, u_p^ϵ can be any solution of (3.2). □

With the above, we re-write the location equation (3.7) which is equivalent to solving the system (2.9) in Problem **[E]**.

Definition 4 (Location Equation)**.** The zero of the transition layer solution x_*, is given by

$$G^\epsilon(x_*, \beta; \vec{\alpha}) = 0, \quad \text{where} \quad G^\epsilon(x_*, \beta; \vec{\alpha}) := u_p^\epsilon(x_*, \beta; \vec{\alpha}) - R^\epsilon(x_*, \beta; \vec{\alpha}). \qquad (3.7\text{'})$$

We now make some assumptions on $a(x; \beta, \vec{\alpha})$ throughout the remaining of the paper to make sure that the obtained solution u^ϵ (3.1) is in fact a *transition layer solution*:

(A1) $a(x; \beta, \vec{\alpha})$ is a smooth function of x, and there exists $T_1 > 0$ such that

$$\|a, a_x, a_{xx}\|_\infty \leq T_1.$$

(A2) There exist $r_0,\ \delta_0 \ll 1$ such that for $r_0 \leq |x| \leq 1$,

$$\frac{\delta_0}{2} < |a(x; \beta, \vec{\alpha})| < \delta_0,$$

in particular, the zeros of a are contained in $[-r_0, r_0]$.

From Proposition 3.1, the form of (3.7) does not depend on the actual choice of u_p^ϵ. Here we make it explicit.

Theorem 3.2. *For $a(x; \beta, \vec{\alpha})$ satisfying **(A1)**, we can choose*

$$u_p^\epsilon(x; \beta, \vec{\alpha}) = a(x; \beta, \vec{\alpha}) \ - \ \epsilon^2 T_a(x; \beta, \vec{\alpha}) \qquad (3.9)$$

where

$$T_a(x) = \frac{1}{2\epsilon} e^{\frac{x}{\epsilon}} \int_1^x a_{xx}(s) e^{-\frac{s}{\epsilon}}\, ds - \frac{1}{2\epsilon} e^{-\frac{x}{\epsilon}} \int_{-1}^x a_{xx}(s) e^{\frac{s}{\epsilon}}\, ds \qquad (3.10)$$

It follows that there exists a constant $C = C(a)$ (which depends on the function a but independently of ϵ) such that

$$\|T_a(x),\ \epsilon T_{a,x}(x),\ \epsilon^2 T_{a,xx}(x)\|_\infty \leq C. \qquad (3.11)$$

As a consequence,

$$\|u_p^\epsilon,\ u_{p,x}^\epsilon,\ u_{p,xx}^\epsilon\|_\infty \leq C \qquad (3.12)$$

and

$$\left|M(u_p^\epsilon)\right|, \ \left|N(u_p^\epsilon)\right| \leq C. \qquad (3.13)$$

Proof. Recall that u_p^ϵ is a particular solution to (3.2), i.e. it solves $\epsilon^2 u_{p,xx}^\epsilon - u_p^\epsilon = -a(x; \beta, \vec{\alpha})$. Consider the decomposition $u_p^\epsilon(x) = a(x; \beta, \vec{\alpha}) - \epsilon^2 T_a(x)$. Then $T_a(x)$ solves

$$\epsilon^2 T_{a,xx} - T_a(x) = a_{xx}. \qquad (3.14)$$

Using the two linearly independent solutions $u_1(x) = e^{\frac{x}{\epsilon}}$ and $u_2(x) = e^{-\frac{x}{\epsilon}}$ and their associated Wronskian:

$$W(x) = \det \begin{pmatrix} u_1 & u_2 \\ u_{1,x} & u_{2,x} \end{pmatrix} = -\frac{2}{\epsilon},$$

a solution of $\epsilon^2 v_{xx} - v = p(x)$ is given by

$$\begin{aligned} v(x) \ &= \ -u_1(x) \int_1^x \frac{p(s) u_2(s)}{\epsilon^2 W(s)}\, ds + u_2(x) \int_{-1}^x \frac{p(s) u_1(s)}{\epsilon^2 W(s)}\, ds \\ &= \ \frac{1}{2\epsilon} e^{\frac{x}{\epsilon}} \int_1^x p(s) e^{-\frac{s}{\epsilon}}\, ds - \frac{1}{2\epsilon} e^{-\frac{x}{\epsilon}} \int_{-1}^x p(s) e^{\frac{s}{\epsilon}}\, ds. \end{aligned}$$

Setting $p(x) = a_{xx}$ gives the formula (3.10) for T_a stated above.

To show that $\|T_a(x)\|_\infty \le C$, we only show $\left\| \frac{1}{2\epsilon} e^{\frac{x}{\epsilon}} \int_1^x a_{xx}(s) e^{-\frac{s}{\epsilon}}\, ds \right\|_\infty \le C$. This is obvious as

$$\left\| \frac{1}{2\epsilon} e^{\frac{x}{\epsilon}} \int_1^x a_{xx}(s) e^{-\frac{s}{\epsilon}}\, ds \right\|_\infty \le \frac{\|a_{xx}\|_\infty}{2} \left(e^{\frac{x-s}{\epsilon}} \right)\Big|_x^1 \le C \left(\|a_{xx}\| \right).$$

A similar proof establishes that $\|\epsilon T_{a,x}(x)\|_\infty \le C$. The estimate $\|\epsilon^2 T_{a,xx}(x)\|_\infty \le C$ holds because $\epsilon^2 T_{a,xx}(x) = T_a(x) + \epsilon^2 a_{xx}$.

The estimates for u_p^ϵ, M and N follow from $u_p^\epsilon = a_{xx} - \epsilon^2 T_a$ and (3.4). □

Remark 3.3. For general spatial inhomogeneity satisfying assumption **(A1)**, we will choose T_a as given by (3.10). However, in practice, all we need is a solution T_a of (3.14) satisfying (3.11) from which the estimates (3.12) for u_p^ϵ follow. (The boundary conditions will be taken care of by the quantity $M(u_p^\epsilon)$ and $N(u_p^\epsilon)$.) For the rest of the paper, we are particularly interested in those a which are explicit normal forms or universal unfolding. These are explicitly given by some polynomial functions. In these cases, T_a can be easily found, for example, by the method of undetermined coefficients.

For the next three Lemmas, we will impose assumptions **(A2)** (in addition to **(A1)**). We will show that the solution u of Problem **[E]** resembles a transition layer solution and the error term R^ϵ is exponentially small so that equation (3.7') is dominated by the term u_p^ϵ.

Lemma 3.4. *For $\epsilon \ll 1$, any solution(s) x_* to the location equation (3.7') satisfies either $|x_*| < r_0$ or $1 - \sigma_0 < |x_*| < 1$ for some small number σ_0, i.e. x_* must lie either near the origin or the boundary of the interval.*

Proof. Recalling the definition of $R^\epsilon(x, \beta; \vec{\alpha})$ in Proposition 3.1(3.8), we write

$$R^\epsilon(x, \beta; \vec{\alpha}) := \frac{2e^{-\frac{2}{\epsilon}}}{1 - e^{-\frac{4}{\epsilon}}} \sinh\left(\frac{2x}{\epsilon}\right) + R_1^\epsilon(x) + R_2^\epsilon(x) \tag{3.15}$$

where

$$R_1^\epsilon(x) := \frac{2\epsilon e^{-\frac{1}{\epsilon}}}{1 + e^{-\frac{2}{\epsilon}}} \sinh\left(\frac{x}{\epsilon}\right) M(u_p^\epsilon), \;\text{ and }\; R_2^\epsilon(x) := \frac{4\epsilon e^{-\frac{2}{\epsilon}}}{1 - e^{-\frac{4}{\epsilon}}} \cosh\left(\frac{x-1}{\epsilon}\right) N(u_p^\epsilon).$$

As $\left\|u_{p,x}^\epsilon\right\|_\infty \le C$, we have $|M|,\ |N| \le C$. Hence there exists a constant (denoted again by C) such that $\|R_1^\epsilon,\ R_2^\epsilon\|_\infty \le C\epsilon$.

From Theorem 3.2, we then write the location equation (3.7) as

$$a(x; \beta, \vec{\alpha}) = \epsilon^2 T_a(x) + \frac{2e^{-\frac{2}{\epsilon}}}{1 - e^{-\frac{4}{\epsilon}}} \sinh\left(\frac{2x}{\epsilon}\right) + R_1^\epsilon(x) + R_2^\epsilon(x), \tag{3.16}$$

Note that for $r_0 < |x| \le 1 - \sigma_0$, by **(A2)**, $|a|$ is bounded from below by $\frac{\delta_0}{2}$ and all the remaining terms in the above are bounded by $C\epsilon$. Hence there cannot be any solution in this range. □

Remark 3.5. We elaborate upon the above result. From (3.16), as the right hand side of the equation varies from $O(\epsilon)$ to the value ± 2 while the range of the function a lies in $(-\delta_0, -\frac{\delta_0}{2}) \cup (\frac{\delta_0}{2}, \delta_0)$, there must be a solution in $(-1, -1 + \sigma_0) \cup (1 - \sigma_0, 1)$. Such a solution corresponds to "nucleation" of interfaces (or mountain-pass solution) when the function u changes from the state $u \sim 1$ (-1) to the state $u \sim -1$ (1). Whether there is actually a solution in the interval $(-r_0, r_0)$ depends on the property of a. For the rest of the paper, we will concentrate on the situations such that there are solutions, for example, when a varies from negative to positive values as in the pitch-fork bifurcation case (2.2).

Lemma 3.6 (Exponential Smallness of R^ϵ)**.** *Under the assumptions (**A1,A2**), the function R^ϵ in (3.8) is exponentially small for $\epsilon \ll 1$ and $|x| < r_0$. Precisely, there exists a constant $C > 0$ such that for all $|x| < r_0$*

$$|R^\epsilon(x, \beta; \vec{\alpha})| \leq C \exp\left(-\frac{1 - r_0}{\epsilon}\right)$$

where r_0 is the same as in in Lemma 3.4.

Proof. This clearly follows from the formula of R^ϵ (3.15) and the fact that $|M|, |N| \leq C$ (Theorem 3.2). □

Lemma 3.7. *Under the assumptions (**A1,A2**), suppose there is a solution x_* of (3.7') in $[-r_0, r_0]$, then for sufficiently small ϵ, we have*

(1) *for $x \in (x_*, 1]$, then $u^\epsilon(x) = u_+(x) > 0$; for $x \in [-1, x_*)$, then $u^\epsilon(x) = u_-(x) < 0$.*

(2) *for $r_0 \leq x \leq 1$, then $|u^\epsilon(x) - 1| \leq 2\delta_0$, and for $-1 \leq x \leq r_0$, then $|u^\epsilon(x) + 1| \leq 2\delta_0$.*

Proof. We prove the result for $x \in (x_*, 1]$ only. From (3.1), (3.5), (3.6), we have,

$$\begin{aligned} u_+(x) &= \left(\frac{-\epsilon M\left[1 - e^{-\frac{2}{\epsilon}}\right] - \left[e^{\frac{x_*-1}{\epsilon}} - e^{-\frac{x_*+3}{\epsilon}}\right] - 2\epsilon e^{-\frac{2}{\epsilon}} N}{1 - e^{-\frac{4}{\epsilon}}}\right) e^{\frac{x-1}{\epsilon}} \\ &\quad + \left(\frac{\epsilon M\left[1 - e^{-\frac{2}{\epsilon}}\right] - \left[e^{\frac{x_*+1}{\epsilon}} - e^{-\frac{x_*+1}{\epsilon}}\right] - 2\epsilon N}{1 - e^{-\frac{4}{\epsilon}}}\right) e^{-\frac{x+1}{\epsilon}} + u_p^\epsilon(x) + 1. \end{aligned} \tag{3.17}$$

We consider two cases: (a) $x = [x_* + L\epsilon, 1]$ where L is some large but fixed number; (b) $x \in \left(x_*, x_* + \frac{1}{2}\epsilon \left|\log \epsilon\right|\right]$.

In case (a), if $\epsilon \ll 1$, from (3.17), we have

$$u_+(x) = u_p^\epsilon(x) + 1 + e^{\frac{x_*-x}{\epsilon}} + O(\epsilon)$$

so that

$$|u_+(x) - 1| \leq \left|u_p^\epsilon(x)\right| + e^{-L} + O(\epsilon).$$

By (3.9) and assumption **(A2)**, conclusion **(2)** follows (for large enough L).

For case (b), we consider the derivative of u_+:

$$\begin{aligned} u_{+,x}(x) &= \frac{1}{\epsilon}\left(\frac{-\epsilon M\left[1 - e^{-\frac{2}{\epsilon}}\right] - \left[e^{\frac{x_*-1}{\epsilon}} - e^{-\frac{x_*+3}{\epsilon}}\right] - 2\epsilon e^{-\frac{2}{\epsilon}} N}{1 - e^{-\frac{4}{\epsilon}}}\right) e^{\frac{x-1}{\epsilon}} \\ &\quad -\frac{1}{\epsilon}\left(\frac{\epsilon M\left[1 - e^{-\frac{2}{\epsilon}}\right] - \left[e^{\frac{x_*+1}{\epsilon}} - e^{-\frac{x_*+1}{\epsilon}}\right] - 2\epsilon N}{1 - e^{-\frac{4}{\epsilon}}}\right) e^{-\frac{x+1}{\epsilon}} + u_p^\epsilon(x) + 1 \\ &= \frac{1}{\epsilon} e^{\frac{x_*-x}{\epsilon}} + u_{p,x}^\epsilon(x) + O(1) \geq \frac{e^{-\frac{1}{2}|\log \epsilon|}}{\epsilon} + O(1) \geq \frac{C}{\epsilon^{\frac{1}{2}}} > 0. \end{aligned}$$

As $u_+(x_*) = 0$, the conclusion follows. □

The above statements show that solving the location equation is essentially the same as finding the zeros of the spatially inhomogeneity function a. The error is of algebraic order in ϵ and is described by the function T_a. The solution constructed has the form of a single interface. (It is conceivable that there can be solutions consisting of clusters of microscopic layers. The structures of these solutions are beyond the scope of the current method. However, see for example [28] for some results regarding such solutions.)

In order to concentrate on the key issues, for the rest of the paper, we will completely ignore the term $R^\epsilon(x, \beta; \vec{\alpha})$ which only contributes exponentially small terms. Then the bifurcation of the location x_* is completely captured by the interaction between a and T_a.

4. **Normal forms and universal unfolding.** Here we will tie the statement of Theorem 3.2 with the concept of universal unfolding (Definition 2). From the representation (3.9) together with the estimates (3.11), for $\epsilon \ll 1$, the location equation is always a small perturbation of $a = 0$ (under the assumptions **(A1)** and **(A2)**). And hence it can be embedded into its universal unfolding. Abstractly,

$$a(x; \beta, \vec{\alpha}) - \epsilon^2 T_a(x; \beta, \vec{\alpha}) \;\sim\; a(x; \widetilde{\beta}, \widetilde{\vec{\alpha}}). \tag{4.1}$$

It is in this sense we say that Problems **[S]** and **[E]** are equivalent. In actual applications, it is certainly useful to know the transformation $(\beta, \vec{\alpha}) \longrightarrow (\widetilde{\beta}, \widetilde{\vec{\alpha}})$ as explicit as possible. Furthermore, from Definition 3, the transition variety Σ holds crucial information as of how the equivalence of the bifurcation diagram changes as the auxiliary parameters vary. Hence we will be interested in the modification of Σ due to the presence of the singular perturbation. On the other hand, it is also instructive to know just how the bifurcation diagram of $u_p^\epsilon(x; \beta, \vec{\alpha}) = 0$ differs from that of $a(x; \beta, \vec{\alpha}) = 0$ when all the parameters remain the same. We call this the *point-wise comparison*. In this section, we emphasize the framework of unfolding while the next will deal with the latter.

In order to have concrete examples, we concentrate here on universal unfoldings with co-dimension no more than three. These are listed in [6, p. 205-211]. For convenience, we switch the sign of the leading term. We denote the necessary translations as $\widetilde{\beta}_s(\epsilon, \vec{\alpha}), \widetilde{\vec{\alpha}}_s(\epsilon, \vec{\alpha})$ so that Problems **[S]** and **[E]** are equivalent, that is,

$$\widetilde{\beta} = \beta + \widetilde{\beta}_s(\epsilon, \vec{\alpha}) \quad \text{ and } \quad \widetilde{\vec{\alpha}} = \vec{\alpha} + \widetilde{\vec{\alpha}}_s(\epsilon, \vec{\alpha}) \tag{4.2}$$

which are defined such that (4.1) is true. (We recall again that the exponentially small term R^ϵ is always neglected.) It turns out that for the explicit examples we will consider, which consists of polynomials, (4.1) can be strengthened to be

$$a(x; \beta, \vec{\alpha}) - \epsilon^2 T_a(x; \beta, \vec{\alpha}) = a(x; \widetilde{\beta}, \widetilde{\vec{\alpha}}). \tag{4.3}$$

Recall the equation (3.2) for u_p^ϵ: $\epsilon^2 u_{p,xx}^\epsilon - u_p^\epsilon = -a$. Now let $a(x; \beta, \vec{\alpha})$ be some polynomial, for instance, x^n. Then we have $L^\epsilon(u_p^\epsilon) := \epsilon^2 u_{p,xx}^\epsilon - u_p^\epsilon = -x^n$. As explained in Remark 3.3 (1), we can take

$$u = x^n + \epsilon^2 n(n-1)x^{n-2} + \epsilon^4 n(n-1)(n-2)(n-3)x^{n-4} + \cdots \tag{4.4}$$

To be explicit, we have:

$$n = 1, \qquad u = x; \tag{4.5}$$
$$n = 2, \qquad u = x^2 + 2\epsilon^2; \tag{4.6}$$
$$n = 3, \qquad u = x^3 + 6\epsilon^2 x; \tag{4.7}$$
$$n = 4, \qquad u = x^4 + 12\epsilon^2 x^2 + 24\epsilon^4; \tag{4.8}$$
$$n = 5, \qquad u = x^5 + 20\epsilon^2 x^3 + 120\epsilon^4 x. \tag{4.9}$$

We list here the particular solutions for each of the following universal unfolding and identify the translations as described in (4.3). Note that for *limit point* normal form, its universal unfolding is just itself.

1. **Limit point** universal unfolding: $a(x, \beta) = -x^2 \mp \beta$. Then
$$a(x; \beta) - \epsilon^2 T_a(x) = -x^2 \mp \beta - 2\epsilon^2$$
Hence $\widetilde{\beta} = \beta \pm 2\epsilon^2$.
2. **Simple** universal unfolding: $a(x, \beta; \alpha) = -x^2 + \beta^2 - \alpha$. Then
$$a(x; \beta) - \epsilon^2 T_a(x) = -x^2 + \beta^2 - (\alpha + 2\epsilon^2)$$
Hence $\widetilde{\beta} = \beta$ and $\widetilde{\alpha} = \alpha + 2\epsilon^2$.
3. **Isola Center** universal unfolding: $a(x, \beta; \alpha) = -x^2 - \beta^2 - \alpha$. Then
$$a(x; \beta) - \epsilon^2 T_a(x) = -x^2 - \beta^2 - (\alpha + 2\epsilon^2)$$
Hence $\widetilde{\beta} = \beta$ and $\widetilde{\alpha} = \alpha + 2\epsilon^2$.
4. **Hysteresis** universal unfolding: $a(x, \beta; \alpha) = -x^3 + \beta - \alpha x$. Then
$$a(x; \beta) - \epsilon^2 T_a(x) = -x^3 + \beta - (\alpha + 6\epsilon^2)x$$
Hence $\widetilde{\beta} = \beta$ and $\widetilde{\alpha} = \alpha + 6\epsilon^2$.
5. **Asymmetric cusp**: $a(x, \beta; \alpha_1, \alpha_2) = -x^2 + \beta^3 - \alpha_1 - \alpha_2\beta$. Then
$$a(x; \beta) - \epsilon^2 T_a(x) = -x^2 + \beta^3 - (\alpha_1 + 2\epsilon^2) - \alpha_2\beta$$
Hence $\widetilde{\beta} = \beta$, $\widetilde{\alpha}_1 = \alpha_1 + 2\epsilon^2$ and $\widetilde{\alpha}_2 = \alpha_2$.
6. **Pitchfork**: $a(x, \beta; \alpha_1, \alpha_2) = -x^3 + \beta x - \alpha_1 - \alpha_2 x^2$. Then
$$a(x; \beta) - \epsilon^2 T_a(x) = -x^3 + (\beta - 6\epsilon^2)x - (\alpha_1 + 2\epsilon^2\alpha_2) - \alpha_2 x^2$$
Hence $\widetilde{\beta} = \beta - 6\epsilon^2$, $\widetilde{\alpha}_1 = \alpha_1 + 2\epsilon^2\alpha_2$ and $\widetilde{\alpha}_2 = \alpha_2$.
7. **Quartic fold**: $a(x, \beta; \alpha_1, \alpha_2) = -x^4 + \beta - \alpha_1 x - \alpha_2 x^2$. Then
$$\begin{aligned} a(x, \beta; \alpha_1, \alpha_2) - \epsilon^2 T_a(x) &= -x^4 - 12\epsilon^2 x^2 - 24\epsilon^4 + \beta - \alpha_1 x - \alpha_2(x^2 + 2\epsilon^2) \\ &= -x^4 + (\beta - 2\alpha_2\epsilon^2 - 24\epsilon^4) - \alpha_1 x - (\alpha_2 + 12\epsilon^2)x^2 \end{aligned}$$
Hence $\widetilde{\beta} = \beta - 2\alpha_2\epsilon^2 - 24\epsilon^4$, $\widetilde{\alpha}_1 = \alpha_1$ and $\widetilde{\alpha}_2 = \alpha_2 + 12\epsilon^2$.
8. $a(x, \beta; \alpha_1, \alpha_2, \alpha_3) = -x^2 \mp \beta^4 - \alpha_1 - \alpha_2\beta - \alpha_3\beta^2$. Then
$$a(x, \beta; \alpha_1, \alpha_2, \alpha_3) - \epsilon^2 T_a(x) = -x^2 \mp \beta^4 - (\alpha_1 + 2\epsilon^2) - \alpha_2\beta - \alpha_3\beta^2.$$
Hence $\widetilde{\beta} = \beta$, $\widetilde{\alpha}_1 = \alpha_1 + 2\epsilon^2$, $\widetilde{\alpha}_2 = \alpha_2$ and $\widetilde{\alpha}_3 = \alpha_3$.
9. **Winged cusp**: $a(x, \beta; \alpha_1, \alpha_2, \alpha_3) = -x^3 - \beta^2 - \alpha_1 - \alpha_2 x - \alpha_3\beta x$. Then
$$\begin{aligned} a(x, \beta; \alpha_1, \alpha_2, \alpha_3) - \epsilon^2 T_a(x) &= -x^3 - \beta^2 - \alpha_1 - \alpha_2 x - \alpha_3\beta x - 6\epsilon^2 x \\ &= -x^3 - \beta^2 - \alpha_1 - (\alpha_2 + 6\epsilon^2)x - \alpha_3\beta x \end{aligned}$$
Hence $\widetilde{\beta} = \beta$, $\widetilde{\alpha}_1 = \alpha$, $\widetilde{\alpha}_2 = \alpha_2 + 6\epsilon^2$ and $\widetilde{\alpha}_3 = \alpha_3$.

10. $a(x,\beta;\alpha_1,\alpha_2,\alpha_3) = -x^4 + \beta x - \alpha_1 - \alpha_2\beta - \alpha_3 x^2$. Then

$$\begin{aligned} & a(x,\beta;\alpha_1,\alpha_2,\alpha_3) - \epsilon^2 T_a(x) \\ = \; & -x^4 - 12\epsilon^2 x^2 - 24\epsilon^4 + \beta x - \alpha_1 - \alpha_2\beta - \alpha_3(x^2 + 2\epsilon^2) \\ = \; & -x^4 + \beta x - (\alpha_1 + 2\alpha_3\epsilon^2 + 24\epsilon^4) - \alpha_2\beta - (\alpha_3 + 12\epsilon^2)x^2. \end{aligned}$$

Hence $\widetilde{\beta} = \beta$, $\widetilde{\alpha}_1 = \alpha_1 + 2\epsilon^2\alpha_3 + 24\epsilon^4$, $\widetilde{\alpha}_2 = \alpha_2$, and $\widetilde{\alpha}_3 = \alpha_3 + 12\epsilon^2$.

11. $a(x,\beta;\alpha_1,\alpha_2,\alpha_3) = -x^5 + \beta - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3$. Then

$$\begin{aligned} & a(x,\beta;\alpha_1,\alpha_2,\alpha_3) - \epsilon^2 T_a(x) \\ = \; & -x^5 - 20\epsilon^2 x^3 - 120\epsilon^4 x + \beta - \alpha_1 x - \alpha_2(x^2 + 2\epsilon^2) - \alpha_3(x^3 + 6\epsilon^2 x) \\ = \; & -x^5 + (\beta - 2\alpha_2\epsilon^2) - (\alpha_1 + 6\epsilon^2\alpha_3 + 120\epsilon^4)x - \alpha_2 x - (\alpha_3 + 20\epsilon^2)x^3 \end{aligned}$$

Hence $\widetilde{\beta} = \beta - 2\epsilon^2\alpha_2$, $\widetilde{\alpha}_1 = \alpha_1 + 6\epsilon^2\alpha_3 + 120\epsilon^4$, $\widetilde{\alpha}_2 = \alpha_2$ and $\widetilde{\alpha}_3 = \alpha_3 + 20\epsilon^2$.

Tables 1 and 2 summarize transition varieties of both Problems [**S**] and [**E**].

5. Point-wise comparison of trans-critical and pitchfork bifurcations.

In this section, we compare bifurcations of Problems [**S**] and [**E**] in a point-wise manner. Of course, as explained in the previous section, two problems are equivalent under the framework of universal unfolding. Here we want to see, given an a, how Problem [**E**] differs from [**S**] simply due to the presence of the singular perturbation (without taking into account the effect of auxiliary parameters). We do this for two explicit spatial inhomogeneities: (i) *trans-critical* bifurcation, $a(x;\beta) = -\frac{1}{2}x^2 + \beta x$, and (ii) *pitchfork* bifurcation: $-\frac{1}{2}x^3 + \beta x$. For the latter, we also consider its perturbed version $a(x,\beta;c) = -\frac{1}{2}x^3 + \beta x + \frac{c}{n}x^n$ where n is an integer.

5.1. Transcritical bifurcation.

For Problem [**S**], the trans-critical spatial inhomogeneity is interesting in two aspects: **(a)** it has a unique bifurcation point $\beta = 0$, while the associated Problem [**E**] has two; **(b)** it always has two solutions if $\beta \neq 0$ while for Problem [**E**], there is certain range of β where transition layers do not exist.

With $a(x;\beta) = -\frac{1}{2}x^2 + \beta x$, it follows from (4.6) that

$$u_p^\epsilon(x,\beta) = -\frac{1}{2}x^2 + \beta x - \epsilon^2. \tag{5.1}$$

We make the following two remarks for the associated problem (3.7): $G^\epsilon(x,\beta) = 0$.

(1) *(Non-existence of solutions).* For $\epsilon \ll 1$, there exists critical values $\beta_{1,*}^\epsilon = \sqrt{2}\epsilon$ and $\beta_{2,*}^\epsilon = -\sqrt{2}\epsilon$ such that for $\beta \in \left(\beta_{2,*}^\epsilon, \beta_{1,*}^\epsilon\right)$, the equation $G^\epsilon(x,\beta) = 0$ does not have a solution while for $\beta < \beta_{2,*}^\epsilon$ or $\beta > \beta_{1,*}^\epsilon$, there are two solutions (see Figure 1). This statement follows easily from the quadratic formula.

(2) The above non-equivalence between Problem [**S**] and [**E**] can of course be reconciled in the framework of universal unfolding as noted previously. Note that,

$$-\frac{1}{2}x^2 + \beta x - \epsilon^2 = -\frac{1}{2}(x+\beta)^2 + \frac{1}{2}\beta^2 - \epsilon^2.$$

Hence (5.1) can be embedded into the simple bifurcation unfolding: $-\frac{1}{2}(x+\beta)^2 + \frac{1}{2}\beta^2 - \alpha$ (case (2) in page 1023). By adjusting the value of α, all the three bifurcation diagrams in Figure 2 can be realized.

Normal Form	$\mathcal{B}$	$\mathcal{H}$	$\mathcal{D}$
$-x^2 \mp \beta$	$\emptyset$	$\emptyset$	$\emptyset$
$-x^2 + \beta^2 - \alpha$	$\alpha = 0$	$\emptyset$	$\emptyset$
$-x^2 - \beta^2 - \alpha$	$\alpha = 0$	$\emptyset$	$\emptyset$
$-x^3 + \beta - \alpha x$	$\emptyset$	$\alpha = 0$	$\emptyset$
$-x^2 + \beta^3 - \alpha_1 - \alpha_2\beta$	$\frac{\alpha_1^2}{4} = \frac{\alpha_2^3}{27}$	$\emptyset$	$\emptyset$
$-x^3 + \beta x - \alpha_1 - \alpha_2 x^2$	$\alpha_1 = 0$	$\alpha_1 = \frac{1}{27}\alpha_2^3$	$\emptyset$
$-x^4 + \beta - \alpha_1 - \alpha_2 x^2$	$\emptyset$	$\left(\frac{\alpha_1}{8}\right)^2 = \left(\frac{\alpha_2}{6}\right)^3$	$\alpha_1 = 0$ $\alpha_2 \leq 0$
$-x^2 \mp \beta^4 - \alpha_1 - \alpha_2\beta - \alpha_3\beta^2$	$\alpha_1 = \pm 3\beta^4 + \alpha_3\beta^2$ $\alpha_2 = \mp 4\beta^3 - 2\alpha_3\beta$	$\emptyset$	$\emptyset$
$-x^3 - \beta^2 - \alpha_1 - \alpha_2 x - \alpha_3\beta x$	$\alpha_1 = 2x^3 - \frac{\alpha_3^2}{4}x^2$ $\alpha_2 = -3x^2 + \frac{\alpha_3^2}{2}x^2$	$\alpha_1\alpha_3^2 + \alpha_2^2 = 0$ $\alpha_1 \leq 0$	$\emptyset$
$-x^4 + \beta x - \alpha_1 - \alpha_2\beta - \alpha_3 x^2$	$\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^4 = 0$	$\alpha_1 + \frac{\alpha_3^2}{12} + \frac{8}{27}\alpha_2^3\alpha_3^3 = 0$	$4\alpha_1 = \alpha_3^2$ $\alpha_3 \leq 0$
$-x^5 + \beta - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3$	$\emptyset$	$\alpha_1 = 15x^4 + 3\alpha_3 x^2$ $\alpha_2 = -10x^3 - 3\alpha_3 x$	$*$

TABLE 1. Transition Varieties for Problem [**S**]. Explicit formulas for $*$ are not provided due to the complexity and length of the expressions. But they can all be found via Definition 3.

Normal Form	$\mathcal{B}^\epsilon$	$\mathcal{H}^\epsilon$	$\mathcal{D}^\epsilon$
$-x^2 \pm \beta$	$\emptyset$	$\emptyset$	$\emptyset$
$-x^2 + \beta^2 - \alpha$	$\alpha^\epsilon = -2\epsilon^2$	$\emptyset$	$\emptyset$
$-x^2 - \beta^2 - \alpha$	$\alpha^\epsilon = -2\epsilon^2$	$\emptyset$	$\emptyset$
$-x^3 + \beta - \alpha x$	$\emptyset$	$\alpha^\epsilon = -6\epsilon^2$	$\emptyset$
$-x^2 + \beta^3 - \alpha_1 - \alpha_2\beta$	$\dfrac{(\alpha_1^\epsilon + 2\epsilon^2)^2}{4} = \dfrac{(\alpha_2^\epsilon)^3}{27}$	$\emptyset$	$\emptyset$
$-x^3 + \beta x - \alpha_1 - \alpha_2 x^2$	$\alpha_1^\epsilon = -2\epsilon^2\alpha_2^\epsilon$	$\alpha_1^\epsilon + 2\epsilon^2\alpha_2^\epsilon = \dfrac{1}{27}(\alpha_2^\epsilon)^3$	$\emptyset$
$-x^4 + \beta - \alpha_1 - \alpha_2 x^2$	$\emptyset$	$\left(\dfrac{\alpha_1^\epsilon}{8}\right)^2 = \left(\dfrac{\alpha_2^\epsilon + 12\epsilon^2}{6}\right)^3$	$\alpha_1^\epsilon = 0$ $\alpha_2^\epsilon + 12\epsilon^2 \leq 0$
$-x^2 \mp \beta^4 - \alpha_1 - \alpha_2\beta - \alpha_3\beta^2$	$\alpha_1^\epsilon + 2\epsilon^2 = \pm 3\beta^4 + \alpha_3^\epsilon\beta^2$ $\alpha_2^\epsilon = \mp 4\beta^3 - 2\alpha_3^\epsilon\beta$	$\emptyset$	$\emptyset$
$-x^3 - \beta^2 - \alpha_1 - \alpha_2 x - \alpha_3\beta x$	$\alpha_1^\epsilon = 2x^3 - \frac{(\alpha_3^\epsilon)^2}{4}x^2$ $\alpha_2^\epsilon = -3x^2 + \frac{(\alpha_3^\epsilon)^2}{2}x^2 - 6\epsilon^2$	$\alpha_1^\epsilon(\alpha_3^\epsilon)^2 + (\alpha_2^\epsilon + 6\epsilon^2)^2 = 0$ $\alpha_1^\epsilon \leq 0$	$\emptyset$
$-x^4 + \beta x - \alpha_1 - \alpha_2\beta - \alpha_3 x^2 \diamond$	$\widetilde{\alpha}_1 + \widetilde{\alpha}_2^2\widetilde{\alpha}_3 + \widetilde{\alpha}_3^4 = 0$	$\widetilde{\alpha}_1 + \dfrac{\widetilde{\alpha}_3^2}{12} + \dfrac{8}{27}\widetilde{\alpha}_2^3\widetilde{\alpha}_3^3 = 0$	$4\widetilde{\alpha}_1 = \widetilde{\alpha}_3^2$ $\widetilde{\alpha}_3 \leq 0$
$-x^5 + \beta - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3 \star$	$\emptyset$	$\widetilde{\alpha}_1 = 15x^4 + 3\widetilde{\alpha}_3 x^2$ $\widetilde{\alpha}_2 = -10x^3 - 3\widetilde{\alpha}_3 x$	$*$

TABLE 2. Transition Varieties for Problems [**E**]. Formulas for Problem [**E**] are correct up to exponentially small error as ϵ approaches zero. Explicit formulas for $*$ are not provided due to the complexity and length of the expressions. They can all be found via definition (3). In ($\diamond$), $\widetilde{\alpha}_1 = \alpha_1^\epsilon + 24\epsilon^4 + 2\epsilon^2\alpha_3^\epsilon, \widetilde{\alpha}_2 = \alpha_2^\epsilon, \widetilde{\alpha}_3 = \alpha_3^\epsilon + 6\epsilon^2$. In ($\star$), $\widetilde{\alpha}_1 = \alpha_1^\epsilon + 120\epsilon^4 + 6\epsilon^2\alpha_3^\epsilon, \widetilde{\alpha}_2 = \alpha_2^\epsilon, \widetilde{\alpha}_3 = \alpha_3^\epsilon + 20\epsilon^2$.

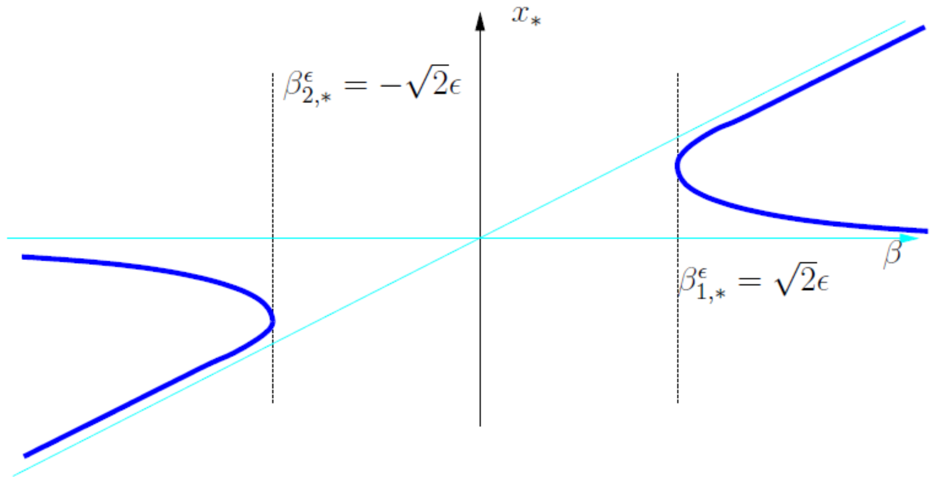


FIGURE 1. Trans-critical Spatial Inhomogeneity: The two disjointed branches are bifurcating branches of Problem **[E]**. In comparison, bifurcation of Problem **[S]** is in blue.

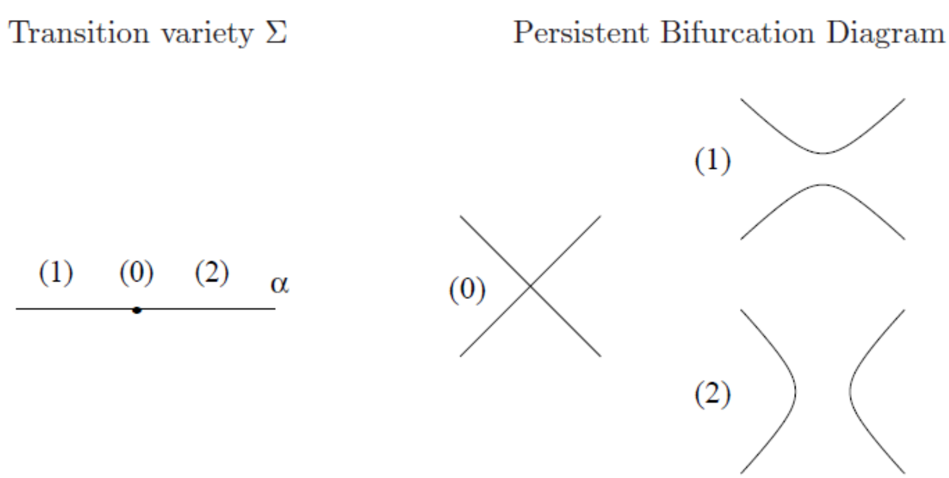


FIGURE 2. Simple Bifurcation: $-\frac{1}{2}x^2 + \beta^2 - \alpha = 0$.

5.2. **Pitchfork bifurcation.** In this subsection, we take $a(x, \beta; c) = -\frac{1}{2}x^3 + \beta x + \frac{c}{n}x^n$ as the spatial inhomogeneity with n being an integer and c a small parameter. Denote problem (3.7) for Problem **[E]** as $G^\epsilon_n(x, \beta; c) = 0$ where the subscript n indicates the order of variation. In the following, we explain the difference between Problems **[S]** and **[E]** in a point-wise manner.

[1]: Problem [S] preserves perfection for $n \geq 3$. For the algebraic problem $a(x, \beta; c) = -\frac{1}{2}x^3 + \beta x + \frac{c}{n}x^n = 0$, $n \geq 3$, with $|c| \ll 1$, it holds that

1. when $c = 0$, it gives global perfect pitchfork bifurcation, given by Figure 3(0);
2. when $c \neq 0$, it gives local perfect pitchfork bifurcation. However, the bifurcation diagram is a distorted version of Figure 3(0) and it only holds for a smaller local region of $\beta = 0$ and $\alpha = 0$.

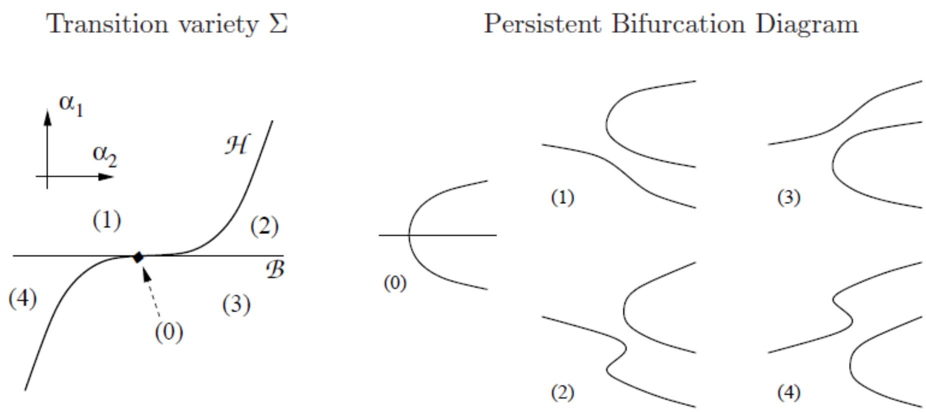


Figure 3. Pitchfork Bifurcation: $-x^3 + \beta x - \alpha_1 - \alpha_2 x^2 = 0$. Here on the left, the transition variety set is formed by two curves $\mathcal{B} = \{\alpha_1 = 0\}$ and $\mathcal{H} = \left\{\alpha_1 = \dfrac{\alpha_2^3}{27}\right\}$, which divide the auxiliary parameter plane (α_1, α_2) into four regions. On the right, bifurcation diagrams (1,2,3,4) correspond to each one of the region.

We say that, the pitchfork bifurcation is *perfect* if it is equivalent to Figure 3(0). Diagrams (1–4) of Figure 3 are all referred as *imperfect* pitchfork bifurcations. In Diagrams (1,3) of Figure 3, the primary branch (its existence is independent of the value of bifurcation parameter) is said to have no hysteresis point (or kink) while in Diagrams (2,4) of Figure 3, the primary branch has hysteresis point. We next verify that with the non-zero higher order perturbation terms, the bifurcation of Problems **[S]** and **[E]** are *non-equivalent* in the sense of *(im)perfection*.

[2]: Problem [E], perfect bifurcation for $n = 2k + 1$. In this case, the bifurcation of single transition layers to Problem **[E]** is still a perfect pitchfork bifurcation for any $|c| \ll 1$. The reason is that the resulting location equation $G^\epsilon_{2k+1}(x, \beta; c) = 0$ is an odd function in x which inherits from the oddness of the spatial inhomogeneity. Hence $x_* = 0$ is always a solution for all β. Moreover, the two other solution branches bifurcate from the trivial branch $x_* = 0$.

The bifurcation point $\beta^\epsilon_*(c)$ is computed as follows. We write

$$G^\epsilon_n(x, \beta; c) = A^\epsilon_n(\beta; c)x^3 + B^\epsilon_n(\beta; c)x + \alpha^\epsilon_{1,n}(\beta, c) + \alpha^\epsilon_{2,n}(\beta, c)x^2 + \text{ higher order terms.} \tag{5.2}$$

Consider the case $n = 2k + 1$ with $k \geq 1$. (One can discuss the case where the odd perturbation term is of the form cx. It gives a perfect pitchfork bifurcation as well. However, it changes the leading asymptotic form of the bifurcation value $\beta^\epsilon_*(c)$, for this reason, we only consider $k \geq 1$). Then $G^\epsilon_{2k+1}(x, \beta; c) := \dfrac{1}{2}x^3 - (\beta - 3\epsilon^2)x - c\,[Q_{2k+1}(x; \epsilon)]$ where $Q_{2k+1}(x; \epsilon)$ is a bounded particular solution to

$\epsilon^2 u_{xx} - u = \dfrac{1}{2k+1}x^{2k+1}$. Using (4.4), we have,

$$A^{\epsilon}_{2k+1}(\beta;c) \;=\; \frac{1}{2} - c\frac{(2k)!}{3!}\epsilon^{2k-2},\;\; B^{\epsilon}_{2k+1}(\beta;c) \;=\; -(\beta - 3\epsilon^2) - c(2k)!\epsilon^{2k},$$
$$\alpha^{\epsilon}_{1,2k+1}(\beta,c) \;\equiv\; 0,\;\; \alpha^{\epsilon}_{2,2k+1}(\beta,c) \;\equiv\; 0.$$

The above gives $x_* = 0$ and

$$\beta^{\epsilon}_*(c) = 3\epsilon^2 + \tau^{\epsilon}_{o*}(c) := 3\epsilon^2 - c(2k)!\epsilon^{2k}, \tag{5.3}$$

where $\tau^{\epsilon}_{o*}(c)$ is the change in the bifurcation point, due to the presence of *odd* perturbation term $\frac{c}{2k+1}x^{2k+1}$. The above information is depicted in Figure 4(a).

[3]: Problem [E], Imperfect Bifurcation without Hysteresis for $n \geq 4$, even. In (5.2), from Definition 3, the hysteresis transition and bifurcation sources sets are characterized as

$$\mathcal{H} = \left\{ \frac{\alpha^{\epsilon}_{1,n}(\beta,c)}{A^{\epsilon}(\beta;c)} = \left[\frac{\alpha^{\epsilon}_{2,n}(\beta,c)}{A^{\epsilon}(\beta;c)}\right]^3 \frac{1}{27} \right\} \quad \text{and} \quad \mathcal{B} = \left\{ \frac{\alpha^{\epsilon}_{1,n}(\beta,c)}{A^{\epsilon}(\beta;c)} = 0 \right\}.$$

(For pitchfork bifurcation, the double points source $\mathcal{D}$ is empty.) In order to determine which types (imperfect or perfect, with hysteresis or not) of pitchfork bifurcation it produces, we need to study each the coefficients quantitatively. We denote $n = 2k$ with $k \geq 2$. Then

$$G^{\epsilon}_{2k}(x,\beta;c) \;:=\; \frac{1}{2}x^3 - (\beta - 3\epsilon^2)x - c\left[Q_{2k}(x;\epsilon)\right]$$

where $Q_{2k}(x;\epsilon)$ is a bounded particular solution to $\epsilon^2 u_{xx} - u = \dfrac{1}{2k}x^{2k}$. For convenience, we write $a(\epsilon) \propto b(\epsilon)$ meaning that $\liminf_{\epsilon\downarrow 0} \dfrac{a(\epsilon)}{b(\epsilon)} > 0$. Using (4.4), we have,

$$A^{\epsilon}_{2k}(\beta;c) \;=\; \frac{1}{2},\;\; B^{\epsilon}_{2k}(\beta;c) \;=\; -(\beta - 3\epsilon^2),$$
$$\alpha^{\epsilon}_{1,2k}(\beta,c) \;\propto\; -c\epsilon^{2k},\;\; \alpha^{\epsilon}_{2,2k}(\beta,c) \;\propto\; -c\epsilon^{2k-2}. \tag{5.4}$$

The above lead to the following consequences:

1. If $c = 0$, then $\alpha^{\epsilon}_{1,2k}(\beta,c) = \alpha^{\epsilon}_{2,2k}(\beta,c) = 0$. This implies $G^{\epsilon}_{2k}(x,\beta;c=0) = 0$ has perfect pitchfork bifurcation. See the Figure 4(a).
2. If $c > 0$, for $\epsilon \ll 1$, then
$$A^{\epsilon}_{2k}(\beta;c) > \frac{1}{4}, \quad \alpha^{\epsilon}_{1,2k}(\beta,c) < -\frac{c}{2}\epsilon^{2k} < 0, \quad \alpha^{\epsilon}_{2,2k}(\beta,c) < -\frac{c}{2}\epsilon^{2k-2} < 0.$$
Hence $\dfrac{\alpha^{\epsilon}_{1,2k}(\beta,c)}{A^{\epsilon}_{2k}(\beta;c)} \ll \left[\dfrac{\alpha^{\epsilon}_{2,2k}(\beta,c)}{A^{\epsilon}_{2k}(\beta;c)}\right]^3 \dfrac{1}{27}$. This implies that the bifurcation diagram of $G^{\epsilon}_{2k} = 0$ falls into the region (3) in Figure 3, i.e., imperfect pitchfork bifurcation without hysteresis.
3. If $c < 0$, for $\epsilon \ll 1$, then
$$A^{\epsilon}_{2k}(\beta;c) > \frac{1}{4}, \quad \alpha^{\epsilon}_{1,2k}(\beta,c) > -\frac{c}{2}\epsilon^{2k} > 0, \quad \alpha^{\epsilon}_{2,2k}(\beta,c) > -\frac{c}{2}\epsilon^{2k-2} > 0.$$
Hence $\dfrac{\alpha^{\epsilon}_{1,2k}(\beta,c)}{A^{\epsilon}_{2k}(\beta;c)} \gg \left[\dfrac{\alpha^{\epsilon}_{2,2k}(\beta,c)}{A^{\epsilon}_{2k}(\beta;c)}\right]^3 \dfrac{1}{27}$. This implies that the bifurcation diagram of $G^{\epsilon}_{2k} = 0$ falls into the region (1) in Figure 3, i.e., imperfect pitchfork bifurcation without hysteresis.

The bifurcation point $(x_*^\epsilon(c), \beta_*^\epsilon(c))$ must solve $G_{2k}^\epsilon(x,\beta;c) = 0$, and $\dfrac{dG_{2k}^\epsilon}{dx}(x,\beta;c) = 0$. Neglecting higher order terms, one has

$$A_{2k}^\epsilon x^3 - B_{2k}^\epsilon x + \alpha_{1,2k}^\epsilon + \alpha_{2,2k}^\epsilon x^2 = 0, \quad 3A_{2k}^\epsilon x^2 - B_{2k}^\epsilon + 2\alpha_{2,2k}^\epsilon x = 0.$$

Substituting

$$B_{2k}^\epsilon = 3A_{2k}^\epsilon x^2 + 2\alpha_{2,2k}^\epsilon x \tag{5.5}$$

into the first equation yields

$$-2A_{2k}^\epsilon x^3 - \alpha_{2,2k}^\epsilon x^2 + \alpha_{1,2k}^\epsilon = 0. \tag{5.6}$$

Note that for the cubic equation

$$f(x) = x^3 - Bx^2 + C = 0,$$

it has two critical points 0 and $\dfrac{2B}{3}$. In addition $f(0) = C, f(\dfrac{2B}{3}) = C - \dfrac{4B^3}{27}$. It is simple fact that $f(x) = 0$ has unique real solution if and only if $C\left(C - \dfrac{4B^3}{27}\right) \geq 0$. Otherwise, it has three real roots. In the case that $k \geq 2$, (5.6) has unique zero x_*^ϵ due to asymptotic behavior of $\alpha_{i,2k}^\epsilon$ in (5.4). Furthermore, $x_*^\epsilon(c) \propto (-c)^{\frac{1}{3}}\epsilon^{\frac{2k}{3}}$. Since $B_{2k}^\epsilon = -(\beta - 3\epsilon^2)$, from (5.5), we obtain

$$\beta_*^\epsilon(c) \;=\; 3\epsilon^2 + \tau_{e*}^\epsilon(c) \;:=\; 3\epsilon^2 - 3A_{2k}^\epsilon(x_*^\epsilon(c))^2 - 2\alpha_{2,2k}^\epsilon x_*^\epsilon(c), \tag{5.7}$$

where $\tau_{e*}^\epsilon(c)$ is the change in the bifurcation point, due to the presence of *even* perturbation term $\frac{c}{2k}x^{2k}$. The above information is depicted in Figure 4(b) for the case $c < 0$.

We remark that, all four regions in Figure 3 are realized if we add lower order terms c and cx^2 to $a(x,\beta) = -x^3 + \beta x$ for Problem [**S**]. More explicitly, $a(x,\beta;c) = -x^3 + \beta x + c = 0$ produces imperfect pitchfork bifurcation without hysteresis point, while $a(x,\beta;c) = -x^3 + \beta x + cx^2 = 0$ produces imperfect pitchfork bifurcation with hysteresis point. It turns out that in these cases the bifurcation of Problem [**E**] is *consistent* with that of Problem [**S**]. For completeness, we present the results here though the analytical computations are similar to the previous discussions.

[4]: Problem [E], Imperfection with Hysteresis for $n = 2$. Here we consider

$$a(x,\beta;c) = -\frac{1}{2}x^3 + \beta x + cx^2.$$

Then the coefficients in (5.2) are given as:

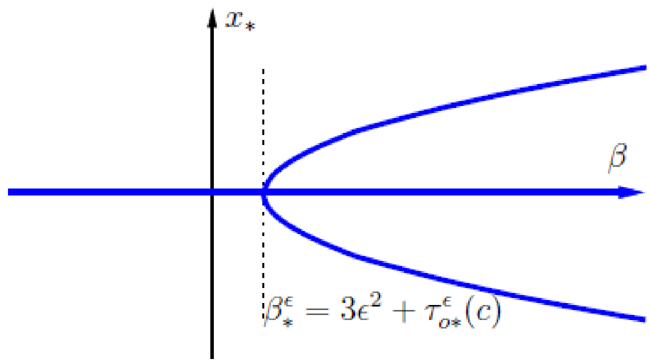
$$A_2^\epsilon(\beta;c) \;=\; \frac{1}{2},\; B_2^\epsilon(\beta;c) \;=\; -(\beta - 3\epsilon^2),\; \alpha_{1,2}^\epsilon(\beta,c) \;=\; -2\epsilon^2\, c,\; \alpha_{2,2}^\epsilon(\beta,c) \;=\; -c.$$

The above lead to that for $\epsilon \ll 1$ and $|c| \ll 1$:

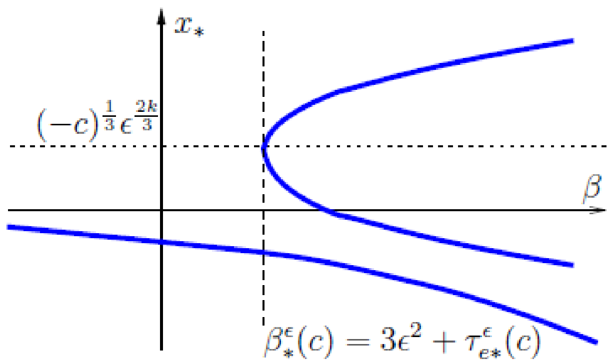
1. If $c > 0$ fixed, then $\alpha_{1,2}^\epsilon(\beta,c) < 0$, $\alpha_{2,2}^\epsilon(\beta,c) < 0$. Hence

$$\frac{\alpha_{1,2}^\epsilon(\beta,c)}{A_2^\epsilon(\beta;c)} = \frac{-c\epsilon^2}{A_2^\epsilon(\beta;c)} \gg \left[\frac{-c}{A_2^\epsilon(\beta;c)}\right]^3 \frac{1}{27} = \left[\frac{\alpha_{2,2}^\epsilon(\beta,c)}{A_2^\epsilon(\beta;c)}\right]^3 \frac{1}{27}.$$

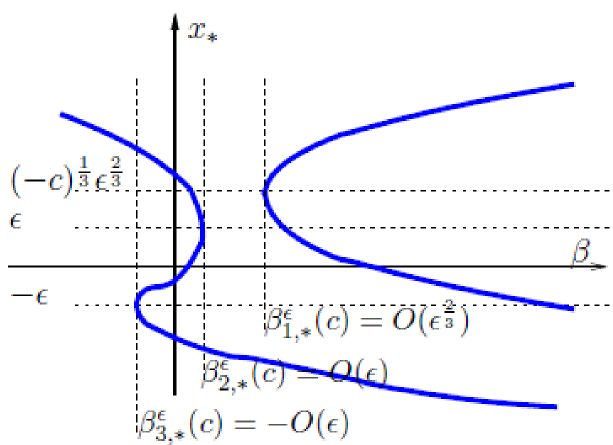
This implies that the bifurcation diagram of $G_2^\epsilon = 0$ falls into the region (4) in Figure 3, i.e., imperfect pitchfork bifurcation with hysteresis.



(a) Perfect pitchfork bifurcation, $c = 0$ or n is odd



(b) Imperfect pitchfork bifurcation without hysteresis, $c < 0$ and $n = 2k, k \geq 2$



(c) Imperfect pitchfork bifurcation with hysteresis, $c < 0$ and $n = 2$

FIGURE 4. Pitchfork Bifurcations of Problem **[E]**. (**a**): perfect pitchfork bifurcation where the spatial inhomogeneity has odd symmetry. (**b**): imperfect pitchfork bifurcation without hysteresis which results from adding higher-order even perturbation and $c < 0$. (**c**): imperfect pitchfork bifurcation with hysteresis which results from adding cx^2 and $c < 0$. The terms $\tau_{o*}^\epsilon, \tau_{e*}^\epsilon$ are lower order corrections to the bifurcation points. They are defined in (5.3), (5.7) respectively. $\beta_{i,*}^\epsilon(c)$ are given in (5.8).

2. If $c < 0$ fixed, then $\alpha^\epsilon_{1,2}(\beta, c) > 0$, $\alpha^\epsilon_{2,2}(\beta, c) > 0$. Hence

$$\frac{\alpha^\epsilon_{1,2}(\beta, c)}{A^\epsilon_2(\beta; c)} = \frac{-c\epsilon^2}{A^\epsilon_2(\beta; c)} \ll \left[\frac{-c}{A^\epsilon_0(\beta; c)}\right]^3 \frac{1}{27} = \left[\frac{\alpha^\epsilon_{2,2}(\beta, c)}{A^\epsilon_0(\beta; c)}\right]^3 \frac{1}{27}.$$

   This implies that the bifurcation diagram of $G^\epsilon_2 = 0$ falls into the region (2) in Figure 3, i.e., imperfect pitchfork bifurcation with hysteresis.
3. Applying perturbation analysis to the equation (5.6) with $k = 1$, i.e., $-x^3 + 2cx^2 - 4c\epsilon^2 = 0$ leads to that there are three bifurcation points $\left(x^\epsilon_{*,i}(c), \beta^\epsilon_{*,i}(c)\right)$, $i = 1, 2, 3$:

$$x^\epsilon_{*,1}(c) \propto (-c)^{\frac{1}{3}}\epsilon^{\frac{2}{3}} \text{ and } x^\epsilon_{*,i}(c) \propto \pm\epsilon \text{ for } i = 2 \text{ or } 3$$

$$\text{and} \qquad \beta^\epsilon_{*,i}(c) := 3\epsilon^2 + 2cx^\epsilon_{*,i} - 3A^\epsilon_2(x^\epsilon_{*,i})^2 \text{ for } i = 1, 2, 3. \tag{5.8}$$

See Figure 4(c) for the details when $c < 0$.

[5]: Problem [E], Imperfection without Hysteresis for $n = 0$. For this, we consider

$$a(x, \beta; c) = -\frac{1}{2}x^3 + \beta x + c.$$

In this case, we have for (5.2), i.e., $G^\epsilon_0 = 0$ that

$$A^\epsilon_0(\beta; c) = \frac{1}{2},\ B^\epsilon_0(\beta; c) = -(\beta - 3\epsilon^2),\ \alpha^\epsilon_{1,0}(\beta, c) = -c,\ \alpha^\epsilon_{2,0}(\beta, c) \equiv 0.$$

Similarly, for all $\epsilon \ll 1$ and $|c| \ll 1$, we have the following.

1. If $c > 0$, then $\alpha^\epsilon_{1,0}(\beta, c) = -c < 0$, $\alpha^\epsilon_{2,0}(\beta, c) = 0$. Hence

$$\frac{\alpha^\epsilon_{1,0}(\beta, c)}{A^\epsilon_0(\beta; c)} < 0 = \left[\frac{\alpha^\epsilon_{2,0}(\beta, c)}{A^\epsilon_0(\beta; c)}\right]^3 \frac{1}{27}.$$

   This implies that it is imperfect without hysteresis.
2. If $c < 0$, then $\alpha^\epsilon_{1,0}(\beta, c) > -c > 0$, $\alpha^\epsilon_{2,0}(\beta, c) = 0$. Hence

$$\frac{\alpha^\epsilon_{1,0}(\beta, c)}{A^\epsilon_0(\beta; c)} > 0 = \left[\frac{\alpha^\epsilon_{2,0}(\beta, c)}{A^\epsilon_0(\beta; c)}\right]^3 \frac{1}{27}.$$

   This again implies that it is imperfect without hysteresis.

We end our discussion with an important observation.

Remark 5.1 (Bifurcation point vs. the form of spatial inhomogeneity)**.** The bifurcation point to the Problem [**E**] depends on the form of spatial inhomogeneity in an interesting way. To be more precise, in case of trans-critical spatial inhomogeneity, bifurcation point is at the order of ϵ. While for the pitchfork spatial inhomogeneity, the bifurcation is at the order of ϵ^2. But for both the underlying interfacial Problems [**S**], their bifurcation points are exactly $\beta = 0$. These delicate dependence on the form of spatial inhomogeneity is recovered in the subsequent work [9] on Problem [**G**] for a smooth nonlinear double well potential function. It would be possible that the bifurcation points for Problem [**G**] become much more delicate as the domain dimension increases.

6. **Summary and comment on Problem [G].** In this work, we have studied the similarities and differences between Problems [**S**] and [**E**] from the perspective of bifurcation. We found that bifurcation of transition layers to Problem [**E**] is *equivalent* to that of Problem [**S**] under the umbrella of *universal unfolding*. In other words, the two problems are equivalent with suitable translations of both bifurcation point and transition varieties. These translations are necessary due to physical presence of the singular parameter ϵ. However, there are explicit examples that bifurcation of Problem [**S**] and that of Problem [**E**] are *different* in terms of *imperfection*. We provide such explicit examples by introducing higher-order perturbation terms to pitchfork normal form.

This is part one of our study of bifurcation of transition layers to reaction diffusion equations in degenerate spatially inhomogeneous media. Here we have studied the explicit transition layer Problem [**E**] to shed light on study of the Problem [**G**]. In the sequel Huang-Yip [9], similar results are recovered in Problem [**G**], where bifurcation of transition layers are obtained through careful asymptotic expansions and reduced bifurcation equation.

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