

QUALITATIVE ANALYSIS OF SOME PDE MODELS OF TRAFFIC FLOW

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ABSTRACT. We review our previous results on partial differential equation(PDE) models of traffic flow. These models include the first order PDE models, a nonlocal PDE traffic flow model with Arrhenius look-ahead dynamics, and the second order PDE models, a discrete model which captures the essential features of traffic jams and chaotic behavior. We study the well-posedness of such PDE problems, finite time blow-up, front propagation, pattern formation and asymptotic behavior of solutions including the stability of the traveling fronts. Traveling wave solutions are wave front solutions propagating with a constant speed and propagating against traffic.

1. Introduction. Traffic congestion has a significant impact on economic activity throughout much of the world. An essential step towards active congestion control is the creation of accurate, reliable traffic monitoring and control systems. These systems usually run algorithms which rely on mathematical models of traffic used to power estimation and control schemes. Tremendous efforts have been devoted to model traffic congestion [1]-[12] [14]-[50]. There are many important approaches to the modeling of traffic phenomena: microscopic models, mesoscopic models and macroscopic models. Macroscopic models describe traffic phenomena through parameters which characterize collective traffic properties. Different mathematical approaches correspond to the three different observations and modeling scales.

The first continuum model of traffic flow is the LWR theory developed independently by Lighthill and Whitham [37] and Richards [43]. The LWR theory assumes that there exists an *equilibrium* speed-concentration relationship $v = v_e(\rho)$. The LWR model is a scalar nonlinear conservation law

$$\rho_t + (\rho v_e(\rho))_x = 0 \tag{1.1}$$

where $v_e(\rho)$ is the equilibrium speed satisfying $v'_e(\rho) < 0$, $v_e(0) = v_f$ and $v_e(\rho_j) = 0$, where v_f is the free flow speed and ρ_j is the jam concentration. $q(\rho) = \rho v_e(\rho)$ is called a fundamental diagram in traffic flow. The LWR model can describe

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the formation of shock waves but fails in describing more complicated traffic flow patterns.

Higher order models were developed in the literature [1] [10] [41] [46] [49] [50]. Payne [41] and Whitham [46] proposed the second order PW model

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + vv_x + \frac{c_0^2}{\rho} \rho_x = \frac{v_e(\rho) - v}{\tau} \end{cases} \quad (1.2)$$

where $\tau > 0$ is the relaxation time, about 5 seconds, c_0 is the traffic sound speed, about 15m/s or 54km/h, and $v_e(\rho)$ is the desired speed. The second equation describes drivers' acceleration behavior: a relaxation to the equilibrium speed and an anticipation which expresses the effect of drivers reacting to conditions downstream.

The relaxation model (1.2) is stable if

$$\lambda_1 = v - c_0 < \lambda_* < \lambda_2 = v + c_0 \quad (1.3)$$

on the equilibrium curve $v = v_e(\rho)$, where λ_* is the characteristic speed of the LWR model (1.1), see Whitham [46]. Under the strict subcharacteristic condition (1.3), Li and Liu [31] and Li and Wu [35] established the nonlinear stability of traveling wave solutions for PW model (1.2) with general equilibrium flux. Traveling wave solutions are wave front solutions propagating with a constant speed and propagating against traffic. Li also obtained [25] the well-posedness and zero relaxation limit for an anisotropic second order model of traffic flow [1] [8] [49], ARZ model, with general fundamental diagrams

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + vv_x + \rho v'_e(\rho) v_x = \frac{v_e(\rho) - v}{\tau}. \end{cases} \quad (1.4)$$

We also established the existence and stability of traveling wave solutions of a quasi-linear hyperbolic system with both relaxation and diffusion in [30].

We derived [28] a class of dynamic traffic flow models from the PW model (1.2) that captured the essential features of traffic jams in *the unstable regions*. There is a qualitative agreement when the analytical results are compared with previous empirical findings for freeway traffic and numerical simulations: [14] [15] [17] where spontaneous appearance of a lot of interacting clusters of vehicles was observed in *the unstable regions* with a nonconcave fundamental diagram.

We considered the following macroscopic traffic flow model with a nonlocal flux in [21]

$$\partial_t \rho + \partial_x (\rho(1 - \rho)e^{-J \circ \rho}) = 0 \quad (1.5)$$

where the function $\rho(t, x)$ represents the density of traffic flow, the kernel J acts only on the spatial variable x , see (2.2) (2.3).

The outline of the paper is the following. In Section 2, we state the results on the nonlocal model (1.5). We also establish the blowup alternative which quantifies the nature of blowup, and prove an interesting maximum principle which shows that the L^∞ norm of the solution cannot increase in time. We study the finite time singularities or shock formation in solutions to (2.1). Finite time singularity scenarios are analyzed by using method of characteristics. In Section 3, we survey well-posedness results and stability of traveling fronts of the second order models. A discrete model which captures the essential features of traffic jams is presented in Section 4. In particular, the model can describe chaos which can be explained as the appearance of a phantom traffic jam. We present results on a model with

both relaxation and diffusion in Section 5. In Section 6, we give some concluding remarks.

2. A nonlocal traffic flow model. In this section, we study the nonlocal model (2.1) which was derived in [44] based on stochastic microscopic dynamics with Arrhenius look-ahead dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho(1 - \rho)e^{-J \circ \rho}) = 0, & \text{in } (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R} \end{cases} \tag{2.1}$$

where the function $\rho(t, x)$ represents the density of traffic flow, the kernel J acts only on the spatial variable x

$$(J \circ \rho)(t, x) = \int_x^\infty J(y - x)\rho(t, y)dy \tag{2.2}$$

and

$$J(r) = \begin{cases} \frac{J_0}{\gamma}, & \text{if } 0 \leq r \leq \gamma \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

is an anisotropic short range inter-vehicle interaction potential, $\gamma > 0$ is proportional to the look-ahead distance and $J_0 > 0$ is the interaction strength. We suppress the dependence of J on γ and J_0 for simplicity of notation. It takes into account interactions of every vehicle with other vehicles ahead. Numerical simulations in [18] indicated that, when $\gamma > 0$, there are shock formations in finite time in the solutions to (2.1) which corresponds to congestion formation in traffic flow. Other non-local models were derived in [4, 7, 12].

When the look-ahead distance $\gamma \rightarrow +\infty$, the global flux in (2.1) becomes a non-global one $\rho(1 - \rho)$. The model (2.1) is then reduced to the classical Lightwill-Whitham-Richards(LWR) model [37, 43]

$$\partial_t \rho + \partial_x(\rho(1 - \rho)) = 0, \quad \text{in } (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{2.4}$$

If, on the other hand, $\gamma \rightarrow 0$, then the global flux in (2.1) is again reduced to a non-global one $\rho(1 - \rho)\exp(-J_0\rho)$ where $\exp(-J_0\rho)$ is a slow down factor in the limiting low visibility. This flux is concave if $J_0 < 3$ and changes concavity when $J_0 \geq 3$. It is well-known that the LWR model (2.4) can describe the formation of shock waves in traffic flow.

It was shown in [21] by D. Li and the author that the finite time blow up must occur at the level of the first order derivative of the solution and L^p , $1 \leq p \leq \infty$ norms of the solution remain finite near the blowup time. This suggests that the finite time blow up is a shock wave. Despite the nonlocal nature of the problem, we identify scenarios of blowups for physical initial data. The list is certainly not exhaustive, nevertheless it is consistent with the blowups observed in numerical simulations in [18, 44]. Our results confirm the formation of shock waves in the nonlocal model (2.1) which corresponds to congestion formations in traffic flow.

Theorem 2.1. (*Local existence* [21]) *Let $\rho_0 \in H^m$ and $m \geq 2$ be an integer. Then there exists $T = T(\|\rho_0\|_{H^m}) > 0$ and a unique solution ρ to (2.1) such that $\rho \in C([0, T], H^m) \cap C^1([0, T], H^{m-1})$.*

In the theory of traffic flow, the function $\rho(t, x)$ represents the density which is normalized to the interval $[0, 1]$, i.e., typically, we have $0 \leq \rho \leq 1$. The following lemma justifies this fact.

Lemma 2.1 (*A priori L^∞ bound [21]*) Let $\rho_0 \in H^m$ and $m \geq 2$ be an integer. Let ρ be the corresponding maximal-lifespan solution obtained in Theorem 2.1 with lifespan $[0, T)$. Assume that $0 \leq \rho_0(x) \leq M$ and $0 < M \leq 1$. Then for any $0 \leq t < T$, we have

$$0 \leq \rho(t, x) \leq M, \quad \forall x \in \mathbb{R}.$$

Corollary 2.1 (*Blowup alternative, physical initial data [21]*) Let $\rho_0 \in H^m$ and $m \geq 2$ be an integer. Assume $0 \leq \rho_0(x) \leq 1$ for all $x \in \mathbb{R}$. Let ρ be the corresponding maximal-lifespan solution obtained in Theorem 2.1 with lifespan $[0, T)$. Then only one of the following occurs

- $T = +\infty$ and ρ is a global solution
- $0 < T < \infty$ and

$$\lim_{\tau \rightarrow T} \int_0^\tau \|\partial_x \rho(t, \cdot)\|_\infty dt = +\infty. \quad (2.5)$$

In particular, we have

$$\limsup_{t \rightarrow T} \|\partial_x \rho(t, \cdot)\|_\infty = +\infty. \quad (2.6)$$

Theorem 2.2. (*Existence of finite time blowups, scenario 1: collision with 0 or 1 [21]*)

Let $\rho_0 \in H^m(\mathbb{R})$ and $m \geq 2$ be an integer. Assume that $0 \leq \rho_0(x) \leq 1$ for all $x \in \mathbb{R}$. If there exist two points $-\infty < \alpha_1 < \alpha_2 < \infty$, such that $\rho_0(\alpha_1) = 0 < \rho_0(\alpha_2) = 1$, then ρ must blow up at some finite time $0 < T < \infty$, i.e.,

$$\limsup_{t \rightarrow T} \|\partial_x \rho(t, \cdot)\|_\infty = +\infty. \quad (2.7)$$

Moreover, for all $p \in [1, +\infty]$, the L^p norm of ρ remain finite as $t \rightarrow T$:

$$\limsup_{t \rightarrow T} \|\rho(t, \cdot)\|_p < \infty, \quad \forall 1 \leq p \leq \infty. \quad (2.8)$$

3. The second order traffic flow models. We established in [25] global solutions of ARZ model (1.4) with a nonconcave fundamental diagram. It was also proved that the zero relaxation limit of the solutions exists and is the unique entropy solution of the equilibrium equation. The nonconcave equilibrium flux is suggested from the experiment data, see Kerner and Rehborn [15]. It is interesting to note that a nonconcave fundamental diagram is a necessary condition to obtain complicated traffic flow patterns including clusters, see Kerner and Konhäuser [15], Jin and Zhang [14]. When the fundamental diagram changes its concavity, one of the characteristic fields of the system is neither linearly degenerate nor genuinely nonlinear. Furthermore, there is no dissipative mechanism in the relaxation system. Characteristics travel no faster than traffic, thus the model is anisotropic. Thus a different analysis is needed to establish the existence of an entropy solution. The analysis was based on an equivalent Lagrangian formulation. One of Oleinik's arguments in [40] was used to show our results.

Consider the second order model

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \frac{1}{\tau}(V_{op}(v) - u) \end{cases} \quad (3.1)$$

where v is specific volume, u is velocity, $\mu > 0$, $\tau > 0$, $V_{op}(v)$ is the equilibrium speed and specific volume relation and p is the pressure.

Let $(U(x - st), V(x - st))$ be the traveling wave solution, $\psi(z, t) = u(z, t) - U(z + x_0)$ and $\phi(z, t) = \int_{-\infty}^z (v(y, t) - V(y + x_0))dy$ which satisfies $\phi(\pm\infty, t) = 0$ for any $t \geq 0$ due to the conservation law in (3.1). Let L be the linear operator obtained by linearizing the system satisfied by (ϕ, ψ) around $(0, 0)$. Under the strict subcharacteristic condition (1.3), Li and Liu [31] and Li and Wu [35] established the nonlinear stability of traveling wave solutions with a nonconcave equilibrium flux.

Theorem 3.1. *Let $(V'_{op}(V))^2 < -p'(V) - sp''(V)V'$, then $(U(x - st), V(x - st))$ is linearly exponentially stable in some weighted spaces. To be precise, for each fixed small constant $\alpha > 0$, there exist constants $\delta_\alpha > 0$ and $M_\alpha > 0$ such that L generates a C_0 -semigroup denoted by $T_\alpha(t)$ on X_α satisfying*

$$\|T_\alpha(t)\|_{X_\alpha \rightarrow X_\alpha} \leq M_\alpha e^{-\delta_\alpha t}, \quad \text{for all } t \geq 0$$

where the weighted space X_α is defined by

$$X_\alpha = \{(\phi, \psi) \mid (\phi(z)w_\alpha(z), \psi(z)w_\alpha(z)) \in H^1(\mathbb{R}) \times L_2(\mathbb{R})\}, \quad w_\alpha(z) = e^{\alpha z} + e^{-\alpha z}$$

with norm $\|(\phi, \psi)\|_{X_\alpha} = \|(\phi w_\alpha, \psi w_\alpha)\|_{H^1(\mathbb{R}) \times L_2(\mathbb{R})}$.

Li and Liu proved in [32]-[34] global in time regularity and finite time singularity or shock formation of solutions simultaneously by showing the critical threshold phenomena systems arising from traffic flow. Assume the sub-characteristic condition. Let $r^\pm = (-p'(v))^{1/4}(u_x \mp \sqrt{-p'(v)}v_x)$.

Theorem 3.2. *Consider the relaxation system (3.1) subject to C^1 bounded initial data $(v_0, u_0)(x)$. Under the above assumptions, there exist $C_1 < C_2$, depending only on initial data (v_0, u_0) , such that $C_1 \leq v(t, x) \leq C_2, \forall x \in \mathbb{R}$, for $t \geq 0$ as long as the C^1 solution exists. Furthermore:*

i) *If at least one point $x \in \mathbb{R}$ either*

$$r^+(0, x) < -\frac{1}{2\tau} \int_{v^*}^{v_0(x)} \left(1 - \frac{u'_e(s)}{\sqrt{-p'(s)}}\right) (-p'(s))^{1/4} ds + \inf_{v \in [C_1, C_2]} G^+(v)$$

or

$$r^-(0, x) < -\frac{1}{2\tau} \int_{v^*}^{v_0(x)} \left(1 + \frac{u'_e(s)}{\sqrt{-p'(s)}}\right) (-p'(s))^{1/4} ds + \inf_{v \in [C_1, C_2]} G^-(v)$$

holds for some G^\pm , then the solution must develop a finite time singularity or shock where either r^+ or r^- goes to $-\infty$.

ii) *If the amplitude of initial data (u_0, v_0) is such that*

$$\inf_{v \in [C_1, C_2]} \left(\frac{\lambda_2^{3/2}(v)}{p''(v)} (\lambda_2(v) \pm u'_e(v)) \right) \geq \frac{1}{4} \int_{C_1}^{C_2} (\lambda_2(s) \mp u'_e(s)) \lambda_2^{-1/2}(s) ds,$$

then the solution remains smooth for all time, provided for all $x \in \mathbb{R}$ it holds

$$r^\pm(0, x) \geq -\frac{1}{2\tau} \int_{v^*}^{v_0(x)} \left(1 \mp \frac{u'_e(s)}{\sqrt{-p'(s)}}\right) (-p'(s))^{1/4} ds + \sup_{v \in [C_1, C_2]} G^\pm(v).$$

4. A discrete model. We derived [28] a class of dynamic traffic flow models from the PW model (1.2) that captured the essential features of traffic jams in *the unstable regions*. Consider the fundamental diagram which is not concave

$$q(\rho) = \rho v_e(\rho) = 5.0461\rho(1 + e^{\frac{\rho-0.25}{0.06}})^{-1} - 3.72 \cdot 10^{-6}. \quad (4.1)$$

We look for the traveling wave solutions of the PW model (1.2), namely, solutions of form $(P, V)(x - ct) = (P, V)(\xi)$ where $\xi = x - ct$ is the traveling wave variable. c is the traveling wave speed. We emphasize that we are looking for solutions on a ring road propagating with a negative speed

$$c < 0. \quad (4.2)$$

This reflects the fact that the vehicle clustering travels against the traffic flow.

We showed in [28] that the dynamics of map are governed by the logistic map. The dynamics of the logistic map undergoes one stable steady state, a period-2 cycle, a period-4 cycle and further period-doublings to cycles of periods 8, 16, 32, ..., 2^n , ..., as the bifurcation parameter increases. The successive bifurcations come faster and faster. Ultimately the bifurcation parameter converges to a limiting value as $n \rightarrow +\infty$. What happens beyond the limiting parameter value? The answer is complicated: for many values of the parameter, the solution never settles down to any fixed point or a periodic orbit. Instead, the long-time behavior is aperiodic and exhibits sensitive dependence on initial data $P_1 \in [\delta, P_+]$ for some $0 < \delta < P_-$. This is a discrete time version of chaos. The results can explain the appearance of a phantom traffic jam, which is observed in real traffic flow.

5. A model with both relaxation and diffusion. It was found by asymptotic analysis and numerical simulations that the fine interplay between the relaxation and the diffusion may enhance physically interesting behavior such as soliton waves and oscillatory solutions [13, 14, 15, 19, 20, 28]. However, the rigorous stability theory for such systems has not been well studied. The delicate balance between the relaxation and the diffusion that leads to the nonlinear stability of the traveling wave fronts is identified to be that the diffusion coefficient is bounded by a constant multiple of the relaxation time. Such a result provides an important first step toward the understanding of the transition from stability to instability as the diffusion coefficient and the relaxation time vary in the physical problems.

We established in [30] the existence and stability of traveling wave solutions of a quasi-linear hyperbolic system with both relaxation and diffusion

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \frac{1}{\tau}(V_{op}(v) - u) + \mu u_{xx} \end{cases} \quad (5.1)$$

where v is specific volume, u is velocity, $\mu > 0$, $\tau > 0$, V_{op} is the equilibrium speed and p is the pressure. The last term on the right hand side of the second equation in (5.1) models viscosity with coefficient $\mu > 0$, a presumed tendency to adjust one's speed to that of the surrounding traffic.

Theorem 5.1. *Suppose the sub-characteristic condition, $-\sqrt{-p'(v)} < V'_{op}(v) < \sqrt{-p'(v)}$, holds and (v_-, v_+) is an admissible shock of equilibrium equation $v_t - V_{op}(v)_x = 0$. Suppose further that $0 < \mu \leq m\tau$, for some $m > 0$. Then there exists a traveling wave solution $(V, U)(x - st)$, which is unique up to a shift and the speed is sub-characteristic.*

Moreover, there exists a constant $\varepsilon_0 > 0$ such that if $|v_- - v_+| + \|v_0 - V\|_2 + \|u_0 - U\|_2 + \|(\phi_0, \psi_0)\|_0 \leq \varepsilon_0$ where $(\phi_0, \psi_0)(x) = (\int_{-\infty}^x (v_0 - V)(y) dy, (u_0 - U)(x))$

and $\int_{-\infty}^{+\infty} (v_0 - V)(x)dx = 0$, then the Cauchy problem has a unique global solution $(v, u)(x, t)$ satisfying $(v - V, u - U) \in C^0(0, \infty; H^2) \cap L^2(0, \infty; H^2)$ and

$$\sup_{x \in R} |(v, u)(x, t) - (V, U)(x - st)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

6. Concluding remarks. We surveyed some results on mathematical modeling of traffic flow. These models include the first order PDE models, a nonlocal PDE traffic flow model with Arrhenius look-ahead dynamics, and the second order PDE models, a discrete model describing chaotic behavior which can explain the appearance of a phantom traffic jam. We studied the well-posedness of such PDE problems, finite time blow-up or shock formation, front propagation, pattern formation and asymptotic behavior of solutions including the stability of the traveling waves. Traveling wave solutions are wave front solutions propagating with a constant speed and propagating against traffic. These problems are difficult to analyze because the PDE models are nonlinear and some of them are nonlocal PDEs, and the strengths of the waves are large. We will report more research progress along these directions in our future work.

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