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EXPLICIT CONSTRUCTION OF SOLUTIONS TO THE BURGERS EQUATION WITH DISCONTINUOUS INITIAL-BOUNDARY CONDITIONS

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Abstract. A solution of the initial-boundary value problem on the strip $(0,\infty)\times[0,1]$ for scalar conservation laws with strictly convex flux can be obtained by considering gradients of the unique solution V to an associated Hamilton-Jacobi equation (with appropriately defined initial and boundary conditions). It was shown in Frankowska (2010) that V can be expressed as the minimum of three value functions arising in calculus of variations problems that, in turn, can be obtained from the Lax formulae. Moreover the traces of the gradients V_x satisfy generalized boundary conditions (as in LeFloch (1988)). In this work we illustrate this approach in the case of the Burgers equation and provide numerical approximation of its solutions.

1. **Introduction.** Some models of traffic flow involve the following scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \quad \text{on } \in (0,\infty)\times[0,1] \\ u(\cdot,0) = u_\ell(\cdot), \;\; u(\cdot,1) = u_r(\cdot) \;\text{ on } (0,\infty) \;\text{ (boundary values)} \\ u(0,\cdot) = u_0(\cdot) \;\text{ on } [0,1] \;\text{ (initial value)} \end{cases} \tag{1.1}$$

with concave flux function f, see for instance [11, 12, 13] and the references contained therein. Following the classical literature on this topic, in [9, 6, 10] the above conservation laws was studied for convex f. When f is concave, then $-f$ is convex and therefore it is enough to find a solution $\bar{u}$ to $\bar{u}_t + [(-f)(-\bar{u})]_x = 0$ satisfying $\bar{u}(\cdot,0) = -u_\ell(\cdot)$, $\bar{u}(\cdot,1) = -u_r(\cdot)$, $\bar{u}(0,\cdot) = -u_0(\cdot)$ and to define $u = -\bar{u}$. For this reason it is enough to study the case of a convex flux f.

For initial value problems existence and uniqueness of entropy solutions was investigated by many authors in a much more general framework (see for instance [9, 5, 3]). In the presence of boundary conditions, weak solutions satisfying boundary

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conditions pointwise, in general, do not exist. In [2] the authors proposed a way to interpret smooth boundary conditions and obtained existence and uniqueness results for solutions having bounded total variation (BV solutions). It was observed in [10] that for scalar conservation laws on $(0,\infty)\times[0,\infty)$ with strictly convex smooth f the "left" boundary conditions from [2] are equivalent to : for a.e. $t>0$

$$\begin{cases} u(t,0)=u_\ell(t) \ \ \& \ \ f'(u_\ell(t))\geq 0 \\ \\ \text{or } f'(u(t,0))\leq 0 \ \ \& \ \ f'(u_\ell(t))\leq 0 \\ \\ \text{or } f'(u(t,0))\leq 0, \ \ f'(u_\ell(t))\geq 0 \ \ \& \ \ f(u(t,0))\geq f(u_\ell(t)). \end{cases} \tag{1.2}$$

Furthermore, in [10] some existence and regularity results were derived for L^∞ initial-boundary values with (1.2) satisfied in a weak sense (not in the poitnwise manner).

Subsequently, in [1] the existence of weak solutions for nonlinear systems of conservation laws was investigated for the initial and the boundary data having small total variations. Up to now there are no uniqueness results for solutions of (1.1) with general discontinuous initial-boundary data, because they are not BV and so do not fit into the framework of [2], though in [10] uniqueness was proved for a very restrictive class of piecewise C^1 solutions.

In [6] it was observed that the inequality $f'(u(t,0))\leq 0$ in (1.2) can be replaced by an exact expression for $u(t,0)$. More precisely, when $u(t,0)\neq u_\ell(t)$, then it is equal to the trace of the solution of the Cauchy problem (i.e. the initial value problem on $[0,1]$ with no left boundary condition). An exact expression for the trace of the solution of the Cauchy problem was also provided.

Moreover, (1.2) has been linked to properties of weak traces of continuous viscosity solutions to the Hamilton-Jacobi equation

$$\begin{cases} V_t+f(V_x)=0 \quad \text{on } \in(0,\infty)\times[0,1] \\ \\ V(\cdot,0)=g_\ell(\cdot), \ \ V(\cdot,1)=g_r(\cdot) \ \ \text{(boundary values)} \\ \\ V(0,x)=h(0)+\int_0^x u_0(r)dr \ \ \text{for all } x\in[0,1] \ \ \text{(initial value)}, \end{cases} \tag{1.3}$$

where g_ℓ and g_r are appropriately defined continuous functions and $h(0)\in\mathbb{R}$. Let us underline that, under very general assumptions, for the initial value problems viscosity solutions to Hamilton-Jacobi equations are unique and for this reason the additional boundary conditions have to be interpreted in light of such uniqueness.

Observe that if $V\in C^2$ satisfies (1.3), then differentiating (1.3) with respect to x it follows that $u=V_x$ solves (1.1). When V is not differentiable, such a relation is less immediate and was demonstrated in [6].

The unique solution to (1.3) turns out to be the minimum of three value functions that can be obtained using the Lax formulae. In this paper we exploit this fact to get numerical approximations of solutions to (1.1).

In the difference with [10], where only the left boundary condition was present, because of the right boundary condition, additional shock waves may perturb, both, the left and the right boundary conditions. For this reason, results of [6] were proved only on $[0,T]\times[0,1]$ for some time $T>0$, in order to avoid such additional shocks. In their absence, results hold true on $\mathbb{R}_+\times[0,1]$.

Here we shall restrict our attention to the Burgers equation. More general study is postponed to a future work.

The outline of the paper is as follows. In Section 2 we provide some preliminary results. In Section 3 we recall main results from [6] that are used in Section 4 to get numerical approximation of solutions to (1.1).

2. **Preliminaries and notations.** For a locally Lipschitz mapping $\psi : \mathbb{R}^n \to \mathbb{R}$ its generalized gradient at $x \in \mathbb{R}^n$ is defined by

$$\partial\psi(x) = \overline{co}\,\{p \in \mathbb{R}^n \mid \exists\, y_i \to x \text{ such that } \lim_{i\to\infty} \psi'(y_i) = p\},$$

where $\overline{co}$ denotes the closed convex hull. Denote by $\partial_-\psi(x)$ the subdifferential of ψ at $x \in \mathbb{R}^n$ consisting of all $p \in \mathbb{R}^n$ such that $\liminf_{y\to x} \frac{\psi(y)-\psi(x)-\langle p,y-x\rangle}{|y-x|} \geq 0$.

For $\varphi,\ \psi : \mathbb{R}^n \to \mathbb{R}$, let $(\varphi \wedge \psi)(x) := \min\{\varphi(x), \psi(x)\}$.

Consider a measurable mapping $\gamma : \mathbb{R}_+ \to \mathbb{R}$ and the scalar differential inclusion under state constraints

$$\left\{\begin{array}{lcl} m'(t) & \in & \gamma(t) + \chi(m(t))\mathbb{R}_+ \quad \text{a.e. in } \mathbb{R}_+ \\ m(t) & \geq & 0 \quad \text{for all } \ t \in \mathbb{R}_+ \\ m(0) & = & 0, \end{array}\right. \tag{2.1}$$

where

$$\chi(m) = \left\{\begin{array}{ll} 0 & m \neq 0 \\ 1 & m = 0. \end{array}\right.$$

Recall that $m(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is called a solution to differential inclusion (2.1) if it is locally absolutely continuous and satisfies (2.1). Set

$$\|\gamma\|_\infty := ess\ sup_{\,s\geq 0}\, |\gamma(s)| \in \mathbb{R} \cup \{+\infty\}.$$

Proposition 1 ([6])**.** *Assume that γ is essentially bounded on bounded sets. Then for all $T > 0$ the solution $m(\cdot)$ to the differential inclusion*

$$\left\{\begin{array}{lcl} m'(t) & \in & \gamma(t) + \chi(m(t))\mathbb{R}_+ \quad \text{a.e. in } [0,T] \\ m(t) & \geq & 0 \quad \text{for all } \ t \in [0,T] \\ m(0) & = & 0 \end{array}\right.$$

exists and is unique. Furthermore for $M_T := ess\ sup_{\,t\in[0,T]}\, |\gamma(t)|$, $m(\cdot)$ satisfies the inclusion

$$\left\{\begin{array}{lcl} m'(t) & \in & \gamma(t) + \chi(m(t))[0, M_T] \quad \text{a.e. in } [0,T] \\ m(t) & \geq & 0 \quad \text{for all } \ t \in [0,T] \\ m(0) & = & 0. \end{array}\right.$$

Consequently differential inclusion (2.1) has a unique solution $m(\cdot)$ defined on $\mathbb{R}_+$ and for a.e. $t \geq 0$, $|m'(t)| \leq 2ess\ sup_{\,s\in[0,t]}\, |\gamma(s)| \leq 2\|\gamma\|_\infty$.

We recall next some results from [6] on value functions of Calculus of Variations problems, where more general Lagrangians were considered.

Let $c > 0$ and $h : [0,1] \to \mathbb{R}$ be a $c-$Lipschitz function (i.e. Lipschitz with the Lipschitz constant c). For all $t \geq 0,\ x \in [0,1]$ consider the problem of Calculus of Variations under state constraint for the initial data $h(\cdot)$

$$V_1(t,x) = \min\left\{h(y(0)) + \frac{1}{2}\int_0^t y'(s)^2 ds \mid y(t) = x,\ y([0,t]) \subset [0,1]\right\}$$

and let $Y(t,x)$ be the set of all $y_{t,x} \in [0,1]$ such that

$$h(y_{t,x}) + \frac{(x-y_{t,x})^2}{2t} = \min_{y\in[0,1]}\left(h(y) + \frac{(x-y)^2}{2t}\right).$$

By [6], the set-valued map $\mathbb{R} \ni t \rightsquigarrow Y(t,0)$ is measurable, has closed nonempty images and is single-valued a.e. Consequently any selection $y(t) \in Y(t,0)$ is measurable. The same observation concerns the set-valued map $Y(t,1)$.

Propositions 3.2, 3.4 and the very proof of Proposition 3.1 from [6] imply the following result.

Proposition 2. *V_1 is Lipschitz continuous, $V_1(0,\cdot) = h(\cdot)$ and for any $t > 0$, $x \in [0,1]$ and for all $y_{t,x} \in Y(t,x)$ we have $|x - y_{t,x}| \le 4ct$, and*

$$V_1(t,x) = h(y_{t,x}) + \frac{(x - y_{t,x})^2}{2t}.$$

Notice that $\frac{(x-y)^2}{2t}$ may become large, when t is small. Hence the obtained estimates on $\frac{x-y_{t,x}}{2t}$ are very useful, since they allow to avoid large terms when getting $V_1(t,x)$ numerically as a minimum of an optimization problem.

The above function V_1 is the unique Lipschitz viscosity solution of the associated Hamilton-Jacobi equation with the initial condition $h(\cdot)$ (for the definition of viscosity solutions see for instance [4]). The following result concerns boundary conditions satisfied by V_1.

Theorem 2.1. *a) Consider any selections $y_\ell(s) \in Y(s,0)$ and $y_r(s) \in Y(s,1)$, $s > 0$. Then for all $t > 0$*

$$V_1(t,0) = h(0) - \frac{1}{2}\int_0^t \left(\frac{y_\ell(s)}{s}\right)^2 ds,$$

$$V_1(t,1) = h(1) - \frac{1}{2}\int_0^t \left(\frac{1 - y_r(s)}{s}\right)^2 ds.$$

b) For every $t > 0$ consider a selection $[0,1] \ni z \mapsto y_{t,z} \in Y(t,z)$. Then for all $x \in [0,1]$,

$$\lim_{t\to 0+} \int_0^x \frac{z - y_{t,z}}{t} dz = h(x) - h(0).$$

If in addition $h \in C^1$, then $\lim_{t\to 0+, z\to x} \frac{z-y_{t,z}}{t} = h'(x)$ for all $x \in (0,1)$.

Let $c > 0$ and $g_\ell : \mathbb{R}_+ \to \mathbb{R}$ be a $c-$Lipschitz function. For every $x \in [0,1]$, $t > 0$ consider the following problem of Calculus of Variations under state constraint for the left boundary condition $g_\ell(\cdot)$

$$V_2(t,x) = \min\left\{g_\ell(\tau) + \frac{1}{2}\int_\tau^t y'(s)^2 ds \,|\, \tau \le t,\ y(t) = x,\ y(\tau) = 0,\ y([\tau,t]) \subset [0,1]\right\}.$$

For $t > 0,\ x \in (0,1]$, let $\Upsilon_\ell(t,x)$ be the set of all $0 \le \tau_{t,x} < t$ such that

$$g_\ell(\tau_{t,x}) + \frac{x^2}{2(t - \tau_{t,x})} = \min_{\tau\in[0,t)} \left(g_\ell(\tau) + \frac{x^2}{2(t-\tau)}\right)$$

and $\Upsilon_\ell(t,0)$ be the set of all $0 \le \tau_{t,0} \le t$ such that $g_\ell(\tau_{t,0}) = \min_{\tau\in[0,t]} g_\ell(\tau)$.

The next Proposition can be deduced from Proposition 4.1 and the proof of Proposition 4.2 from [6].

Proposition 3. *For all $t > 0$, $x \in (0,1]$ and $0 < \tau_{t,x} \in \Upsilon_\ell(t,x)$, we have $x \le \sqrt{2c}\,(t - \tau_{t,x})$,*

$$V_2(t,x) = g_\ell(\tau_{t,x}) + \frac{x^2}{2(t - \tau_{t,x})}.$$

As before, the estimates on $\frac{x}{t-\tau_{t,x}}$ are very useful, since they allow to avoid large terms when getting $V_2(t,x)$ numerically as a minimum of an optimization problem.

For every $t > 0$ define

$$z_\ell(t) = \begin{cases} 0 & \text{if } 0 \in \Upsilon_\ell(t,x)\ \forall\, x \in [0,1] \\ \sup\{x \in (0,1] \mid \Upsilon_\ell(t,x) \cap (0,\infty) \neq \emptyset\} & \text{otherwise.} \end{cases}$$

The following result simplifies numerical approximations of V_2.

Proposition 4. *Let $t > 0$.*

i) If $0 \in \Upsilon_\ell(t, z_\ell(t))$, then $\Upsilon_\ell(s,x) = \{0\}$ for all $x \in (z_\ell(t), 1]$ and $s \in (0,t]$.

ii) If for some $t > 0$ and $x \in (0,1]$ the set $\Upsilon_\ell(t,x)$ contains an element different from zero, then for all $s > t$, $0 \notin \Upsilon_\ell(s,x)$ and for all $y \in [0,x)$, $0 \notin \Upsilon_\ell(t,y)$.

iii) Let $t_0 := \inf\{t > 0 \mid z_\ell(t) = 1\}$. If $0 < t_0 < \infty$, then $z_\ell(t) = 1$ on $[t_0,\infty)$. If $0 \in \Upsilon_\ell(t_0, 1)$, then $z_\ell(\cdot)$ is strictly increasing on $(0,t_0)$.

Proposition 5. *Let $x \in (0,1]$ and $t_\ell(x) := \inf\{t > 0 \mid \Upsilon_\ell(t,x) \cap (0,\infty) \neq \emptyset\}$. Then V_2 is Lipschitz on $\{(t,x) \mid 0 < x \le z_\ell(t),\ t > t_\ell(x)\}$. Furthermore if for some $x \in (0,1]$, $t_\ell(x) < \infty$, then $V_2(\cdot,x)$ is $c-$Lipschitz on $[t_\ell(x),\infty)$.*

Let $u_0 : [0,1] \to \mathbb{R}$ and $u_\ell : \mathbb{R}_+ \to \mathbb{R}$ be measurable essentially bounded functions and $h(0) \in \mathbb{R}$. For all $x \in [0,1]$ define $h(x) = h(0) + \int_0^x u_0(r)dr$ and let V_1 be the corresponding value function. Consider a selection $\mathbb{R}_+ \ni s \mapsto y_\ell(s) \in Y(s,0)$ and set $\gamma(s) := \frac{1}{2}u_\ell(s)^2 - \left(\frac{y_\ell(s)}{s}\right)^2$ for all $s > 0$. Then, by Proposition 1, there exists a solution $m_\ell(\cdot) \ge 0$ to (2.1). For all $t \ge 0$ and $s > 0$ define

$$\Gamma_\ell(s) := m_\ell'(s) + \frac{y_\ell(s)^2}{2s^2}, \quad g_\ell(t) = h(0) - \int_0^t \Gamma_\ell(s)ds$$

and consider the value function V_2 corresponding to this $g_\ell(\cdot)$. The next Proposition shows that $g_\ell(\cdot)$ is the left boundary value of V_2 which is not larger than the left boundary value of V_1.

Proposition 6. *For all $t > 0$, $g_\ell(t) = V_2(t,0) \le V_1(t,0) - m_\ell(t)$.*

Consider any selections $y_{t,x} \in Y(t,x)$, $\tau_{t,x} \in \Upsilon_\ell(t,x)$. For all $(t,x) \in (0,\infty) \times (0,1]$ set

$$u_1(t,x) := \begin{cases} \frac{x}{t-\tau_{t,x}} & \text{if } V_2(t,x) < V_1(t,x) \\ \\ \frac{x-y_{t,x}}{t} & \text{otherwise.} \end{cases} \tag{2.2}$$

Our next result underlines the fact that for every $t > 0$ the state interval $[0,1]$ admits a subdivision in at most three subintervals : the one where $V_2(t,\cdot)$ is strictly smaller than $V_1(t,\cdot)$, the one where it is equal to it and the one where it is strictly larger.

Theorem 2.2. *For every $t > 0$ define*

$$\pi_\ell(t) = \begin{cases} 0 \ \text{ if } \{x \in [0, z_\ell(t)) \,|\, V_2(t,x) < V_1(t,x)\} = \emptyset \\ \sup\{x \in [0, z_\ell(t)) \,|\, V_2(t,x) < V_1(t,x)\} \ \text{ otherwise} \end{cases}$$

and

$$\Pi_\ell(t) = \begin{cases} 1 \quad \text{ if } \{x \in [0,1] \,|\, V_1(t,x) < V_2(t,x)\} = \emptyset \\ \inf\{x \in [0,1] \,|\, V_1(t,x) < V_2(t,x)\} \quad \text{ otherwise.} \end{cases}$$

Then $\pi_\ell(\cdot)$ *is continuous,* $\lim_{t\to 0+}\pi_\ell(t)=0$ *and*

$$\begin{array}{lll} i) & \forall\, x<\pi_\ell(t), & V_2(t,x)<V_1(t,x)\\ ii) & \forall\, x>\Pi_\ell(t), & V_2(t,x)>V_1(t,x)\\ iii) & \forall\, x\in[\pi_\ell(t),\Pi_\ell(t)], & V_2(t,x)=V_1(t,x)\\ iv) & \forall\, x\in(\pi_\ell(t),\Pi_\ell(t)), & y_{t,x}=0,\ \tau_{t,x}=0. \end{array}$$

Moreover if $\pi_\ell(t)<z_\ell(t)$*, then* $\pi_\ell(t)=\Pi_\ell(t)$*.*

The following theorem concerns the weak trace of u_1^2 when $x\to 0+$.

Theorem 2.3 (Boundary Value)**.** *If* $u_\ell(\cdot)\ge 0$ *a.e. in* $(0,\infty)$*, then* $\forall\, t>0$*,*

$$\lim_{x\to 0+}\frac{1}{2}\int_0^t u_1(s,x)^2ds=\int_0^t\Gamma_\ell(s)ds.$$

In an analogous way we investigate the right boundary conditions. Let $g_r:\mathbb{R}_+\to\mathbb{R}$ be a $c-$Lipschitz function. For every $x\in[0,1],\ t>0$ consider the following problem of Calculus of Variations under state constraint for the right boundary condition $g_r(\cdot)$.

$$V_3(t,x)=\min\{g_r(\tau)+\frac{1}{2}\int_\tau^t y'(s)^2ds\,|\,\tau\in[0,t],\,y(t)=x,\,y(\tau)=1,\,y([\tau,t])\subset[0,1]\}.$$

For all $t>0$ and $x\in[0,1)$, let $\Upsilon_r(t,x)$ be the set of all $0\le\tau_{t,x}<t$ such that

$$g_r(\tau_{t,x})+\frac{(1-x)^2}{2(t-\tau_{t,x})}=\min_{\tau\in[0,t)}\left(g_r(\tau)+\frac{(1-x)^2}{2(t-\tau)}\right).$$

and let $\Upsilon_r(t,1)$ be the set of all $0\le\tau_{t,1}<t$ such that $g_r(\tau_{t,1})=\min_{\tau\in[0,t]}g_r(\tau)$.

Proposition 7. *For any* $t>0$ *and* $x\in[0,1)$ *and for all* $0<\tau_{t,x}\in\Upsilon_r(t,x)$*, we have* $1-x\le\sqrt{2c}\,(t-\tau_{t,x})$ *and*

$$V_3(t,x)=g_r(\tau_{t,x})+\frac{(1-x)^2}{2(t-\tau_{t,x})}.$$

Let $u_r:\mathbb{R}_+\to\mathbb{R}$ be an essentially bounded measurable function. Consider a selection $y_r(s)\in Y(s,1)$ and define $\gamma(s):=\frac{1}{2}u_r(s)^2-\frac{1}{2}\left(\frac{1-y_r(s)}{s}\right)^2$ for all $s>0$. By Proposition 1 there exists a solution $m_r(\cdot)\ge 0$ to (2.1) defined on $\mathbb{R}_+$. For all $s>0,\ t\ge 0$ set

$$\Gamma_r(s):=m_r'(s)+\frac{1}{2}\left(\frac{1-y_r(s)}{s}\right)^2,\quad g_r(t)=h(1)-\int_0^t\Gamma_r(s)ds$$

and consider the value function V_3 corresponding to this $g_r(\cdot)$. Then, by Theorem 2.1, for all $t>0$,

$$V_3(t,1)\le h(1)-m_r(t)-\frac{1}{2}\int_0^t\left(\frac{1-y_r(s)}{s}\right)^2ds\ =\ V_1(t,1)-m_r(t).$$

That is on the right boundary of $[0,1]$, V_3 is not larger than V_1.

For every $t>0$ define

$$z_r(t)=\left\{\begin{array}{ll}1 & \text{if } 0\in\Upsilon_r(t,x)\quad\forall\,x\in[0,1)\\ \inf\{x\in[0,1)\mid\Upsilon_r(t,x)\cap(0,\infty)\neq\emptyset\} & \text{otherwise}\end{array}\right.$$

and consider any selections $y_{t,x} \in Y(t,x)$, $\tau_{t,x} \in \Upsilon_r(t,x)$. For $(t,x) \in (0,\infty)\times(0,1]$ set

$$u_2(t,x) := \begin{cases} \frac{x-1}{t-\tau_{t,x}} & \text{if } V_3(t,x) < V_1(t,x) \\ \\ \frac{x-y_{t,x}}{t} & \text{otherwise.} \end{cases} \tag{2.3}$$

Theorem 2.4. *For every $t > 0$ define*

$$\pi_r(t) = \begin{cases} 1 \text{ if } \{x \in (z_r(t), 1] \,|\, V_3(t,x) < V_1(t,x)\} = \emptyset \\ \inf\{x \in (z_r(t), 1] \,|\, V_3(t,x) < V_1(t,x)\} \text{ otherwise} \end{cases}$$

and

$$\Pi_r(t) = \begin{cases} 0 \quad \text{if } \{x \in [0,1] \,|\, V_1(t,x) < V_3(t,x)\} = \emptyset \\ \sup\{x \in [0,1] \,|\, V_1(t,x) < V_3(t,x)\} \quad \text{otherwise.} \end{cases}$$

Then $\pi_r(\cdot)$ is continuous, $\lim_{t\to 0+}\pi_r(t) = 1$ and

$$\begin{array}{lll} i) & \forall\, x > \pi_r(t), & V_3(t,x) < V_1(t,x) \\ ii) & \forall\, x < \Pi_r(t), & V_3(t,x) > V_1(t,x) \\ iii) & \forall\, x \in [\Pi_r(t), \pi_r(t)], & V_3(t,x) = V_1(t,x) \\ iv) & \forall\, x \in (\Pi_r(t), \pi_r(t)), & y_{t,x} = 1,\ \tau_{t,x} = 0. \end{array}$$

Moreover, if $\pi_r(t) > z_r(t)$, then $\pi_r(t) = \Pi_r(t)$.

The last result of this section concerns the weak trace of u_2^2 when $x \to 1-$.

Theorem 2.5 (Boundary Value)**.** *Assume that $u_r(\cdot) \le 0$ a.e. in $(0,\infty)$. Then for all $t > 0$,*

$$\lim_{x\to 1-} \frac{1}{2}\int_0^t u_2(s,x)^2 ds = \int_0^t \Gamma_r(s)ds.$$

3. **Solutions to Burgers equation.** We summarize here main results from [6] adapted to the case of Burgers equation. They involve the value functions discussed in the previous section. Let $u_0 : [0,1] \to \mathbb{R}$, $u_\ell : \mathbb{R}_+ \to \mathbb{R}$ and $u_r : \mathbb{R}_+ \to \mathbb{R}$ be measurable essentially bounded functions and $h(0) \in \mathbb{R}$. Consider h, g_ℓ, g_r, V_i, Γ_ℓ, Γ_r associated to these data as it was done in the previous section and define $W : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ by

$$W(t,x) = \min\{V_1(t,x), V_2(t,x), V_3(t,x)\}, \ \ W(0,x) = h(x).$$

Let $T > 0$ and $\pi_\ell(\cdot)$, $\pi_r(\cdot)$ be as in Theorems 2.2 and 2.4. Define $\alpha_T = T$ if $\pi_\ell(t) < 1$ and $\pi_r(t) > 0$ for all $t \in [0,T]$ and

$$\alpha_T = \min\{t \in [0,T] \,|\, \pi_r(t) = 0 \text{ or } \pi_\ell(t) = 1\}$$

otherwise.

Consider the Hamilton-Jacobi equation associated to the Burgers equation

$$V_t + \frac{1}{2}V_x^2 = 0 \tag{3.1}$$

Theorem 3.1. *$W(\cdot,\cdot) : [0,\alpha_T] \times [0,1] \to \mathbb{R}$ is the unique Lipschitz function satisfying the above Hamilton-Jacobi equation and the initial-boundary conditions in the following generalized sense :*

i) for all $(t,x) \in (0,\alpha_T) \times (0,1)$

$$p_t + \frac{1}{2}p_x^2 = 0, \quad \forall\, (p_t, p_x) \in \partial_- W(t,x)$$

ii) $W(0,\cdot) = h(\cdot)$, $W(\cdot,0) = g_\ell(\cdot)$, $W(\cdot,1) = g_r(\cdot)$.

Remark 1. (a) Note that our uniqueness result is stated on $[0, \alpha_T] \times [0, 1]$, where $\alpha_T \leq T$ is explicitly defined. If $\alpha_T = T$ for all $T > 0$, then the uniqueness holds true on $\mathbb{R}_+ \times [0, 1]$. If $\alpha_T < T$, then it may happen that either left or right boundary conditions are violated for some times in the interval (α_T, T).

(b) It can be shown that $W(\cdot, \cdot) : [0, \alpha_T] \times [0, 1] \to \mathbb{R}$ is also the unique Lipschitz continuous viscosity solution of (3.1), satisfying the initial-boundary conditions ii).

Proposition 8 (Regularity of W). *The gradient ∇W of W is continuous a.e. in $(0, \infty) \times (0, 1)$ in the following sense : if W is differentiable at some $(s, y) \in (0, \infty) \times (0, 1)$, then for any sequence $(s_i, y_i) \in (0, \infty) \times (0, 1)$ converging to (s, y) such that W is differentiable at (s_i, y_i) we have $\lim_{i \to \infty} \nabla W(s_i, y_i) = \nabla W(s, y)$.*

Let $u_1(\cdot, \cdot)$, $u_2(\cdot, \cdot)$ be as in (2.2) and (2.3). For all $(t, x) \in (0, \infty) \times (0, 1]$ define

$$u(t, x) := \begin{cases} u_1(t, x) & \text{if } (V_1 \wedge V_2)(t, x) < V_3(t, x) \\ u_2(t, x) & \text{otherwise.} \end{cases}$$

Our next result implies that solutions studied in [10] are equal to W_x.

Theorem 3.2. *For every $T > 0$, let α_T be defined as above. Then for almost all $(t, x) \in (0, \alpha_T) \times (0, 1)$, $u(t, x) = W_x(t, x)$. Furthermore,*

i) $u(\cdot, \cdot)$ is continuous at a.e. $(t, x) \in (0, \infty) \times (0, 1)$

ii) For all $t > 0$, $u(t, \cdot)$ is continuous at a.e. $x \in (0, 1)$.

iii) For every $t \in (0, \alpha_T]$, $u(t, \cdot)$ has left and right limits at each point $x \in (0, 1)$ and the Lax shock inequality $u(t, x-) \geq u(t, x+)$ holds true.

iv) $u(\cdot, \cdot)$ is essentially bounded on $[0, \alpha_T] \times [0, 1]$ by a constant depending only on ess $\sup_{r \in [0,1]} |u_0(r)|$, ess $\sup_{s \in [0, \alpha_T]} |u_\ell(s)|$ and ess $\sup_{s \in [0, \alpha_T]} |u_r(s)|$.

The next two theorems imply that u satisfies initial-boundary conditions pointwise or weakly, as in [10].

Below we denote by $sign(a)$ the sign function equal to 1 when $a > 0$, to -1 when $a < 0$ and to 0 when $a = 0$.

Theorem 3.3. *Assume that $u_0(\cdot)$, $u_\ell(\cdot) \geq 0$, $u_r(\cdot) \leq 0$ are continuous and let $T > 0$. Then for a.e. $t \in [0, \alpha_T]$ and all $\bar{x} \in (0, 1)$, the following limits do exist*

$$u(t, 0) := \lim_{x \to 0+} u(t, x), \; u(t, 1) := \lim_{x \to 1-} u(t, x), \; u(0, \bar{x}) := \lim_{t \to 0+, \; x \to \bar{x}} u(t, x)$$

and $\frac{1}{2} u(t, 0)^2 = \Gamma_\ell(t)$, $\frac{1}{2} u(t, 1)^2 = \Gamma_r(t)$, $u(0, \bar{x}) = u_0(\bar{x})$.

Furthermore at $x = 0$ one of the following two relations holds true :

$$\begin{array}{ll} i) & u(t, 0) = u_\ell(t) \\ ii) & u(t, 0) = -\frac{y_\ell(t)}{t} \leq -u_\ell(t) \end{array}$$

and at $x = 1$ one of the following two relations holds true :

$$\begin{array}{ll} i)' & u(t, 1) = u_r(t) \\ ii)'' & u(t, 1) = \frac{1 - y_r(t)}{t} \geq |u_r(t)|. \end{array}$$

Moreover, if $m_\ell(t) > 0$, then $\lim_{x \to 0+} sign(u(t, x)) = 1$; if $m_\ell(t) = 0$ and $u_\ell(t) < \frac{y_\ell(t)}{t}$, then $\lim_{x \to 0+} sign(u(t, x)) = -1$.

Furthermore if $m_r(t) > 0$, then $\lim_{x \to 1-} sign(u(t, x)) = -1$; if $m_r(t) = 0$ and $|u_r(t)| < \frac{1 - y_r(t)}{t}$, then $\lim_{x \to 1-} sign(u(t, x)) = 1$.

When the initial/boundary conditions are discontinuous, then the following weaker result holds true.

Theorem 3.4. *Assume that* $u_\ell(\cdot) \geq 0$ *and* $u_r(\cdot) \leq 0$ *a.e. in* $(0,\infty)$ *and let* $T > 0$. *Then for all* $t \in [0, \alpha_T]$,

$$\lim_{x\to 0+} \frac{1}{2}\int_0^t u(s,x)^2 ds = \int_0^t \Gamma_\ell(s)ds, \quad \lim_{x\to 1-} \frac{1}{2}\int_0^t u(s,x)^2 ds = \int_0^t \Gamma_r(s)ds$$

and for all $x \in (0,1)$

$$\lim_{t\to 0+} \int_0^x u(t,r)dr = \int_0^x u_0(r)dr.$$

Theorem 3.5 (Trace of Sign)**.** *Assume that* $u_\ell(\cdot) \geq 0$ *and* $u_r(\cdot) \leq 0$ *a.e. in* $(0,\infty)$. *Let* $\pi_\ell(t)$, $\pi_r(t)$, $\Pi_\ell(t)$, $\Pi_r(t)$ *be defined as in Theorems 2.2, 2.4 and* $y_\ell(t) \in Y(t,0)$, $y_r(t) \in Y(t,1)$ *be such that*

$$y_\ell(t) = \max\{y \,|\, y \in Y(t,0)\}, \quad y_r(t) = \min\{y \,|\, y \in Y(t,1)\}.$$

Then for all $T > 0$ *and every* $t \in (0, \alpha_T)$

$$\begin{aligned} \Pi_\ell(t) > 0 &\implies \lim_{x\to 0+} sign(u(t,x)) = 1, \\ \Pi_\ell(t) = 0 \;\&\; y_\ell(t) > 0 &\implies \lim_{x\to 0+} sign(u(t,x)) = -1, \\ \Pi_\ell(t) = y_\ell(t) = 0 &\implies \lim_{x\to 0+} u(t,x) = 0 \end{aligned}$$

and

$$\begin{aligned} \Pi_r(t) < 1 &\implies \lim_{x\to 0+} sign(u(t,x)) = -1, \\ \Pi_r(t) = 1 \;\&\; y_r(t) < 1 &\implies \lim_{x\to 1-} sign(u(t,x)) = 1, \\ \Pi_r(t) = y_r(t) = 1 &\implies \lim_{x\to 1-} u(t,x) = 0. \end{aligned}$$

Moreover if $\lim_{x\to 0+} sign(u(t,x)) = -1$, *then* $\lim_{x\to 0+} u(t,x) = \frac{-y_\ell(t)}{t}$ *and if* $\lim_{x\to 1-} sign(u(t,x)) = 1$, *then* $\lim_{x\to 1-} u(t,x) = \frac{1-y_r(t)}{t}$.

We next provide an existence result analogous to Theorem 3.3 when no sign conditions are imposed on $u_\ell(\cdot)$ and $u_r(\cdot)$.

Theorem 3.6. *Let* $\bar{u}_\ell(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, $\bar{u}_r(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, $u_0(\cdot) : [0,1] \to \mathbb{R}$ *be continuous. Define the continuous functions*

$$u_\ell(t) = \begin{cases} \bar{u}_\ell(t) & \text{if } \bar{u}_\ell(t) \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$u_r(t) = \begin{cases} \bar{u}_r(t) & \text{if } \bar{u}_r(t) \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

and let $T > 0$, α_T *and* $u(\cdot,\cdot)$ *be as in Theorem 3.3. Then for a.e.* $t \in [0,\alpha_T]$ *one of the following four conditions is verified for* $x = 0$

i)	$u(t,0) = \bar{u}_\ell(t)$	$\bar{u}_\ell(t) \geq 0$
ii)	$u(t,0) = 0$	$\bar{u}_\ell(t) < 0$
iii)	$u(t,0) = -\frac{y_\ell(t)}{t} \leq -\bar{u}_\ell(t)$	$\bar{u}_\ell(t) \geq 0$
iv)	$u(t,0) = -\frac{y_\ell(t)}{t}$	$\bar{u}_\ell(t) < 0.$

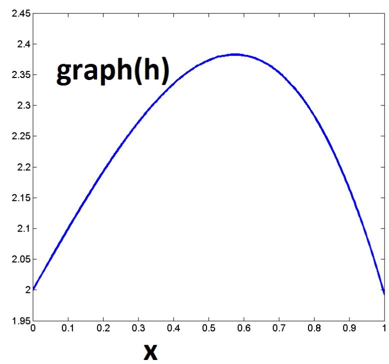
Similarly, for a.e. $t \in [0,\alpha_T]$ *one of the following four conditions is verified for* $x = 1$

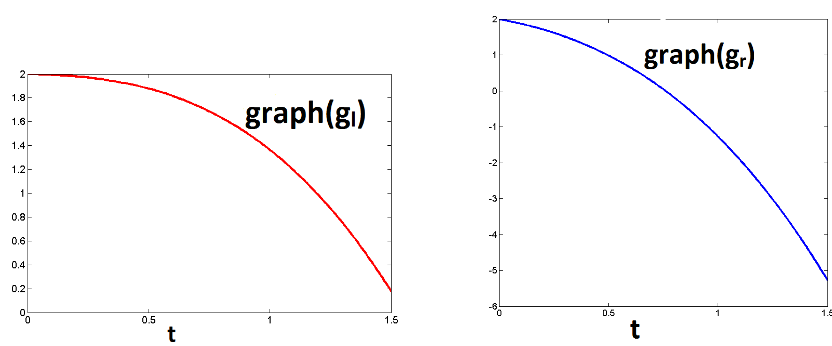
i)'	$u(t,1) = \bar{u}_r(t)$	$\bar{u}_r(t) \leq 0$
ii)'	$u(t,1) = 0$	$\bar{u}_r(t) > 0$
iii)'	$u(t,1) = \frac{1-y_r(t)}{t} \geq \lvert u_r(t)\rvert$	$\bar{u}_r(t) \leq 0$
iv)'	$u(t,1) = \frac{1-y_r(t)}{t}$	$\bar{u}_r(t) > 0.$

4. **Examples.** In this section we illustrate the above approach on three examples. The first one concerns continuous data for which, according to the previous section, generalized boundary conditions are satisfied pointwise, while the second and third examples involve piecewise continuous boundary conditions with only one discontinuity. Then the boundary conditions are satisfied in a weak sense described in Theorem 3.3.

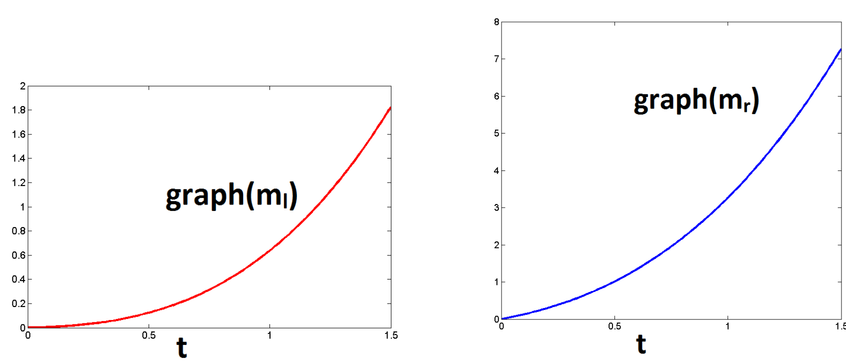
4.1. **Example 1.** Our first example shows that even when the initial and boundary data are continuous, it may happen that after some time a solution u to the Burgers equation depends only on the right boundary condition.

Let $u_0(x) = -3x^2 + 1, \quad u_\ell(t) = 0.3 + 1.5t, \quad u_r(t) = -1.5 - 2t.$

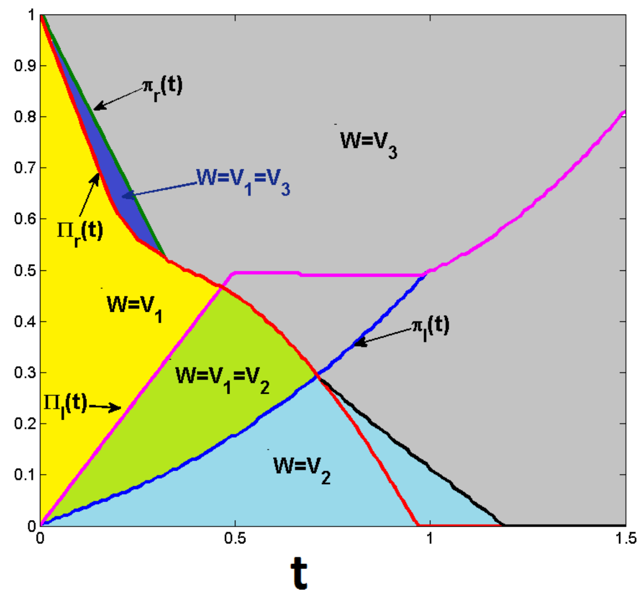




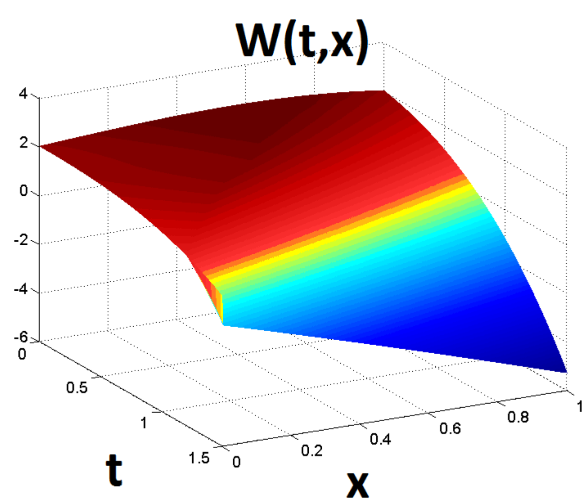
The initial value h and boundary values g_ℓ, g_r of the associated Hamilton-Jacobi equation are continuous.



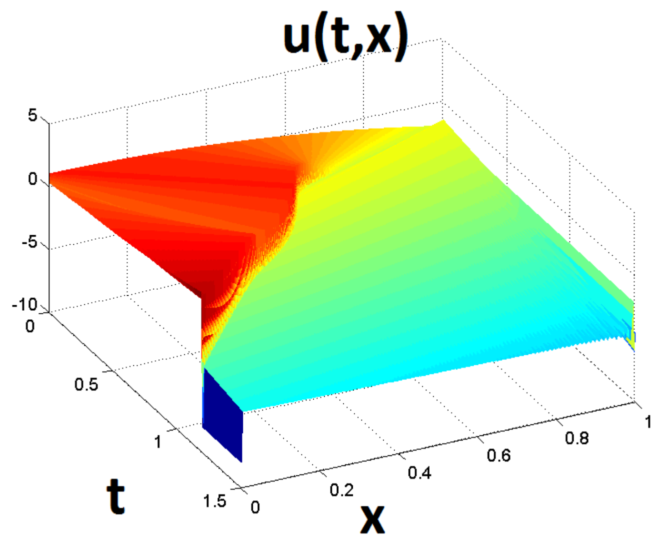
Functions m_ℓ and m_r are strictly increasing, $\Gamma_\ell(t) = \frac{1}{2}u_\ell(t)^2$ and $\Gamma_r(t) = \frac{1}{2}u_r(t)^2$.



The initial condition u_0 (that defines the value function V_1) has an influence on W up to time 0.5. For $t \in [1.2, 1.5]$, $W(t,x) = V_3(t,x)$ for all $x \in [0,1]$. That is the explicit expression for a solution to the Burgers equation depends only on the right boundary condition on the time interval $[1.2, 1.5]$.



The solution W to the Hamilton-Jacobi equation is continuous, but its graph exhibits zones of strong decrease corresponding to switches between the value functions involved into the definition of W.

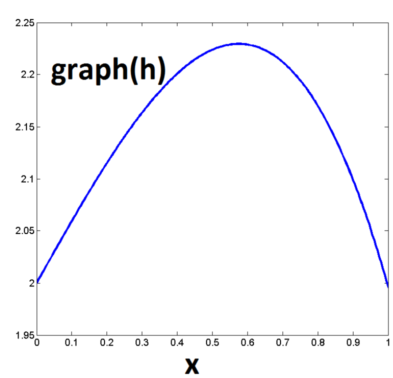


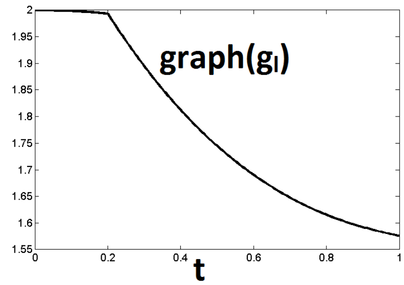
A solution $u(\cdot,\cdot)$ of the Burgers equation, expressed using derivatives of W with respect to x, is discontinuous. Its discontinuity is however diminish for time $t > 1.2$, when $W = V_3$, that is when only the right boundary condition plays a role.

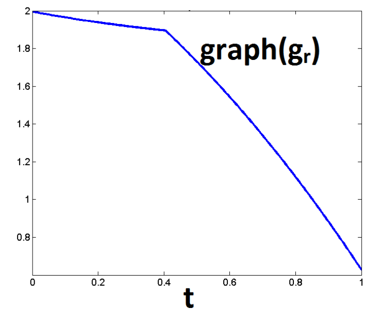
4.2. **Example 2.** Our second example illustrates that for some initial-boundary conditions it may happen that for sufficiently large time both left and right boundary conditions determine a solution u to the Burgers equation, while the influence of the initial condition does disappear.

Let $u_0(x) = -1.8x^2 + 0.6$ and

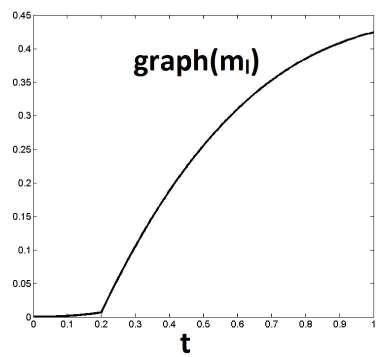
$$u_\ell(t) = \begin{cases} 0.1 + 1.5t & \text{if } t < 0.2 \\ 1.7 - 1.2t & \text{otherwise} \end{cases}, \quad u_r(t) = \begin{cases} -1.5 - 0.8t & \text{if } t > 0.4 \\ -0.8 + 0.49t & \text{otherwise.} \end{cases}$$

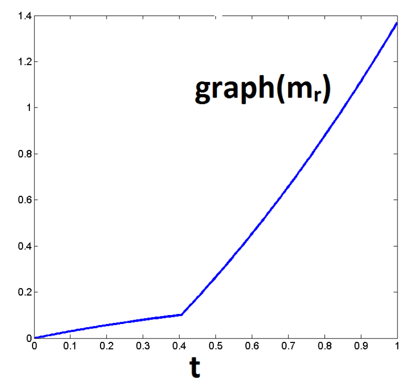




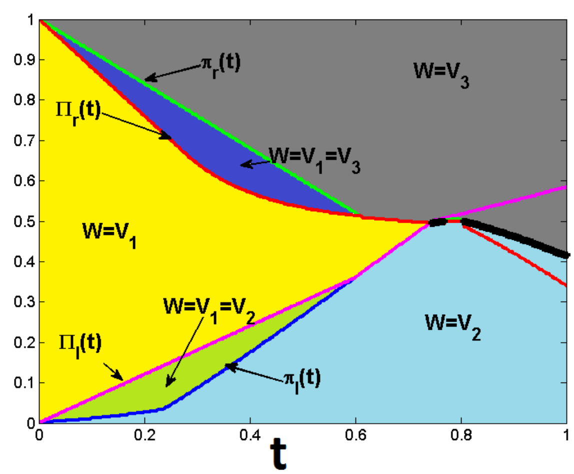


The initial value h and boundary values g_l, g_r of the associated Hamilton-Jacobi equation are continuous.

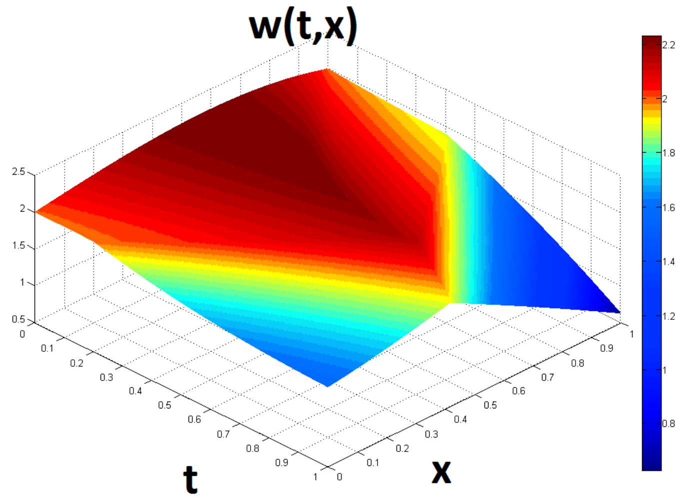




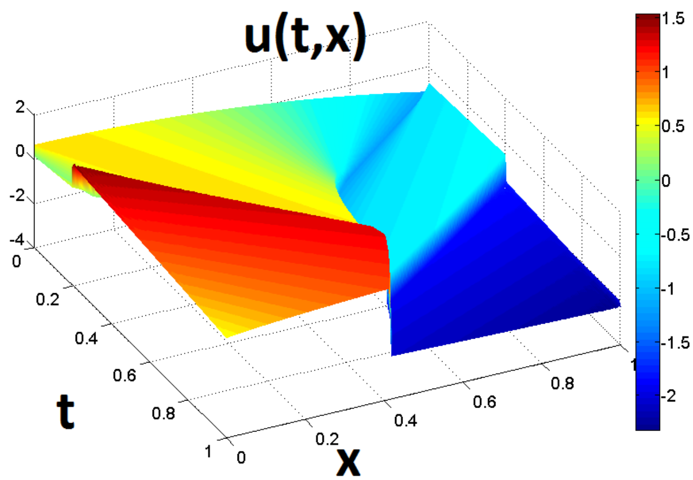
Notice that the functions $m_l(\cdot) = 0$ and $m_r(\cdot)$ are strictly increasing.



The value function V_1 is involved in definition of W up to time 0.72. For $t \in [0.72, 1]$, $W(t,x) = (V_2 \wedge V_3)(t,x)$ for all $x \in [0,1]$. In the above, the bold black line corresponds to states (t,x) where $V_2(t,x) = V_3(t,x)$. That is the initial condition is not involved into the explicit definition of solution of the Burgers equation on the time interval $[0.72, 1]$.



The function W solving the Hamilton-Jacobi equation is continuous, but its graph exhibits zones of strong decrease.

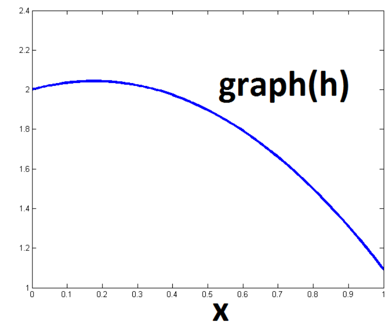


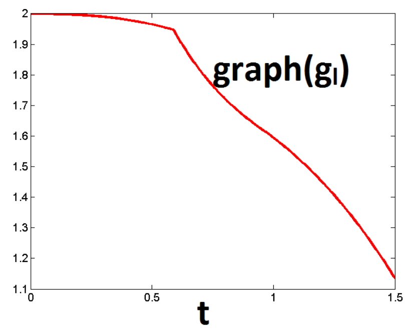
The solution $u(\cdot,\cdot)$ of the Burgers equation, expressed using derivatives of W with respect to x, is discontinuous. The largest discontinuity regions correspond switches between value functions determining W.

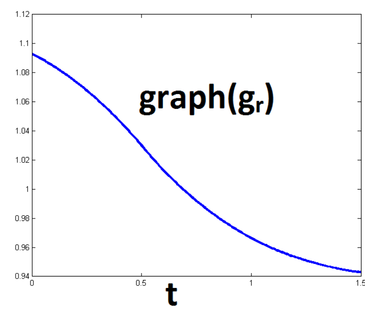
4.3. **Example 3.** Our third example illustrates that it may happen that for a period of time only the initial and the right boundary conditions determine a solution u to the Burgers equation, but then again both boundary conditions and the initial condition play a role.

Let $u_0(x) = -2.8x + 0.5$,

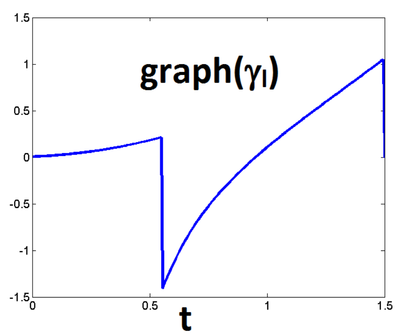
$$u_l(t) = 0.1 + t, \qquad u_r(t) = \begin{cases} -0.4 - 0.4t & \text{if } t < 0.5 \\ -0.8 + 0.4t & \text{otherwise} \end{cases}$$



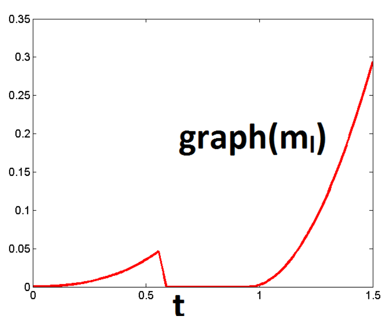


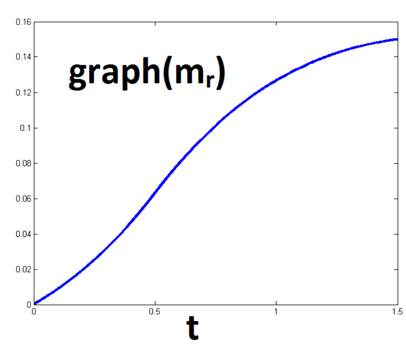


The initial value h and boundary values g_l, g_r of the associated Hamilton-Jacobi equation are continuous.

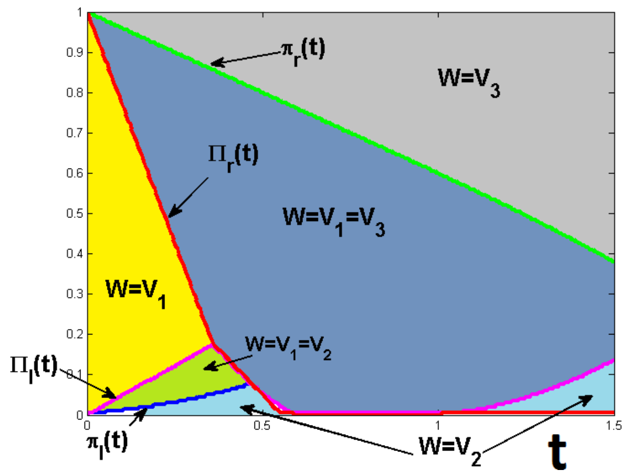


The function γ_ℓ is discontinuous.

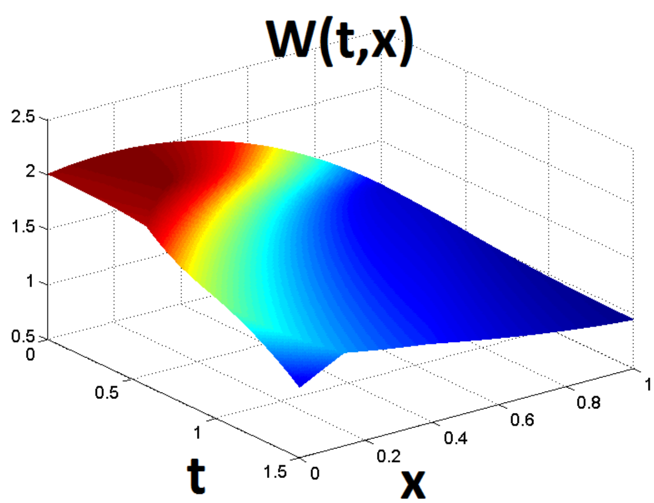




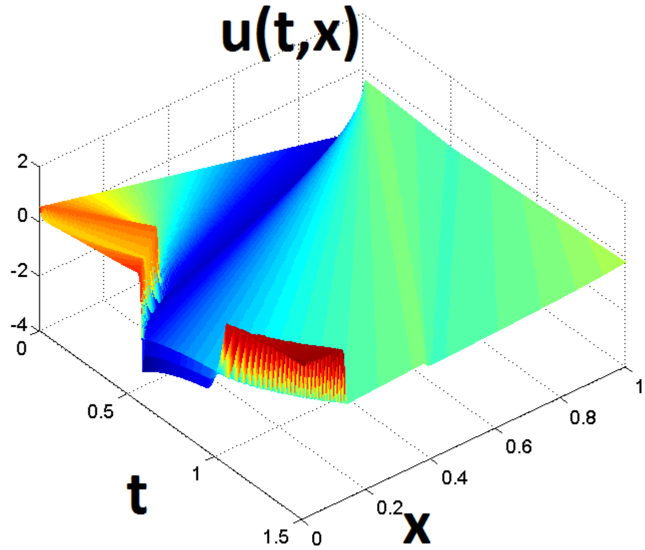
Notice that the function $m_l(\cdot) = 0$ on the time interval $[0.55, 1]$. Hence it is not monotone in this example, while $m_r(\cdot)$ is strictly increasing.



The value function V_2 is involved in definition of W up to time 0.55 and then again for time larger than 1.1. For $0.55 < t < 1.1$, $W(t,x) = (V_1 \wedge V_3)(t,x)$ for all $x \in [0,1]$. That is the left boundary condition is not involved into the explicit definition of solution of the Burgers equation on the time interval $[0.55, 1.1]$.



The function W solving the Hamilton-Jacobi equation is continuous, but its graph exhibits zones of strong decrease.



The solution $u(\cdot,\cdot)$ of the Burgers equation, expressed using derivatives of W with respect to x, is discontinuous. The largest discontinuity regions concerns switches between value functions.

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