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COUPLING OF MICROSCOPIC AND PHASE TRANSITION MODELS AT BOUNDARY

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Abstract. This paper deals with coupling conditions between the classical microscopic Follow The Leader model and a phase transition (PT) model. We propose a solution at the interface between the two models. We describe the solution to the Riemann problem.

1. Introduction.

The present paper provides modeling techniques for the coupling of a microscopic and a macroscopic model of vehicular traffic at interfaces.

We underline that a significant effort was devoted by applied mathematician to the modeling of vehicular traffic via partial differential equations. Such effort enriched the already wide spectrum of available mathematical models, therefore researchers and practitioners face now an unprecedented opportunity in model choice. In turn this poses the problem of identifying the “best” model, depending on the specific application features. The solution may be found in combining different models using coupling conditions at interfaces. The latter approach may be chosen either to take advantage of different characteristics of models or to differentiate geographically scale and grain level description.

Let us start revising briefly some of the most well known models of vehicular traffic. At microscopic level, one of the most studied approach is the so called Follow the Leader, where each driver adjusts its velocity to the one of the vehicle in front. More precisely, the velocity change is proportional to difference in velocities and inversely proportional to some positive power of the vehicles distance. The fluid dynamic approach at macroscopic level was, in turn, initiated by the seminal work of Lighthill-Whitham and Richards [16, 20]. This model well describes the evolution of free traffic, but it is not accurate when the traffic is congested. Hence second order models, i.e. system with two equations, were introduced. Among these we mention the Payne-Whitham [18, 21] model, which was invalidated by Daganzo [8] in 1995.

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Aw and Rascle in [3] then proposed the conservation of a "modified" momentum. The same model was independently suggested by Zhang in [23], thus we refer to it as the Aw–Rascle–Zhang model. In 2002, Colombo proposed a second order model with phase transitions [6], in order to describe the different behaviors of traffic in free and congested regimes. It is based on two phases: the free and the congested one. In free flow the model is a classical LWR model, whereas in the congested flow it is a system of two equations: the first one is the conservation of the number of vehicles, while the second one describes the evolution of a linearized momentum. Various modifications of this model were considered, for instance combining LWR and ARZ model [12] and the more recent refinement [5]; see also [7].

The relations among microscopic and (fluid dynamic) macroscopic were studied in few papers. In [2], the authors showed how solutions to the Follow the Leader model, with appropriate rescaling, in the limit tend to solutions of the ARZ model. In [17] an hybrid model is built based on results from [2]. Coupling at boundaries has been investigated first only under a numerical point of view (see, for instance, [13, 14]). Then in [15] a coupling at boundaries of the Follow the Leader and ARZ model was proposed.

The simulation of large networks and fitting of data from mobile sensors is mostly performed using the classical LWR model or modifications of it; see [22]. The combination of different scales thus requires the definition of coupling conditions at boundaries between the Follow the Leader model and the LWR ones. The main obstacle along this path is the fact that the Follow the Leader model is naturally coupled with second order ones (that means two equations) as explained above. In this paper we define coupling conditions for the Follow the Leader and the phase transition (briefly PT) model of [5].

We assume that there is a fixed interface, located at $x = 0$, between the Follow the Leader and the PT model. We give coupling conditions between these two models, based on similar considerations as in [15]. We describe in details how to solve the Riemann problem for such coupling. The solution of such Riemann problem is based on the solution for an initial-boundary value problem for the PT model. Therefore, for the latter problem, we also prove existence of solutions by means of the wave-front tracking technique.

The paper is organized as follows. In Section 2, we recall the definition of the Follow the Leader model, whereas in Section 3 the phase transition one, introduced in [5], is presented. Moreover in Section 3.1, we prove existence of solutions for an initial-boundary value problem for the PT model. Sections 4 and 5 deal with coupling conditions and with the analysis for the Riemann problems respectively for FTL-PT model and for PT-FTL model.

2. Description of the Follow the Leader model.

The *Follow The Leader* model [11, 19] is described by the system of ordinary differential equations

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = C_\gamma \frac{v_{i+1} - v_i}{(x_{i+1} - x_i)^{\gamma+1}}, \end{cases} \tag{1}$$

where x_i and v_i denote respectively the position and the speed of the i-th vehicle ($i \in \{1, \ldots, I\}$). Here we suppose that $x_1 < x_2 < \cdots < x_I$, $C_\gamma = \frac{(\Delta X)^{\gamma+1}}{\tau}$, $\gamma \geq 0$ and ΔX is the length of a single car and $\tau > 0$ is a relaxation time. The dynamic of the I-th car is assumed to be known. By recursion, one can find the position and the velocity of each car, provided the initial position $\bar{x}_1, \ldots \bar{x}_{I-1}$ and velocity

$\bar{v}_1, \ldots, \bar{v}_{I-1}$ are known. By (1), each driver adjusts its velocity accordingly to the relative velocity with respect that of the next driver and to the distance from the next car. More precisely, the velocity change is proportional to the difference in velocities and inversely proportional to vehicles' distance.

3. **Description of the phase transition model.** We describe the phase transition model, introduced in [5], with the Newell-Daganzo velocity function for congestion. The model consists in two phases: the free and the congested. These phases are described by the sets

$$\begin{cases} \Omega_f = \left\{(\rho, q) \in [0, R] \times [-1, +\infty[:\ q = \frac{R(\rho-\sigma)}{\sigma(R-\rho)},\ 0 \le \rho \le \sigma_+\right\} \\ \Omega_c = \left\{(\rho, q) \in [0, R] \times [-1, +\infty[:\ v_c(\rho, q) \le V,\ \frac{q^-}{R} \le \frac{q}{\rho} \le \frac{q^+}{R}\right\} \end{cases} \tag{2}$$

where $V > 0$ is the velocity in the free phase, $R > 0$ is the maximal density, $-1 < q^- < 0 < q^+ < 1$, $\frac{R}{5} < \sigma < R$ and $v_c : [\sigma_-, R] \times [q^-, q^+] \to [0, +\infty[$, defined by

$$v_c(\rho, q) = v_c^{eq}(\rho)(1+q) = \frac{V\sigma}{R-\sigma}\left(\frac{R}{\rho} - 1\right)(1+q)\,, \tag{3}$$

is the velocity in the congested phase; see Figure 1. Here the constants $\sigma_- > 0$ and $\sigma_+ > 0$ are defined respectively by the ρ-component of the solutions in Ω_c to the systems

$$\begin{cases} v_c(\rho, q) = V, \\ \frac{q}{\rho} = \frac{q^-}{R}, \end{cases} \qquad \text{and} \qquad \begin{cases} v_c(\rho, q) = V, \\ \frac{q}{\rho} = \frac{q^+}{R}. \end{cases}$$

We denote with $v(\rho, q)$ the extension of v_c to $]0, R] \times [-1, +\infty)$. Note that $v(\rho, q) = V$ if $(\rho, q) \in \Omega_f$. Introduce the function

$$\Phi(\rho, v) = (\Phi_1(\rho, v), \Phi_2(\rho, v)) = \left(\rho, \frac{v\,(R-\sigma)}{V\sigma}\frac{\rho}{R-\rho} - 1\right) \tag{4}$$

which maps admissible (ρ, v) to element in $\Omega_f \cup \Omega_c$. More precisely, the function Φ transforms the coordinates (ρ, v) into (ρ, q). For a later use, let us consider the function $d : (\Omega_f \cup \Omega_c)^2 \to \mathbb{R}$ defined, for every $(\rho_1, q_1), (\rho_2, q_2) \in \Omega_f \cup \Omega_c$, by

$$d\left((\rho_1, q_1), (\rho_2, q_2)\right) = \left|v(\rho_1, q_1) - v(\rho_2, q_2)\right| + \left|\omega(\rho_1, q_1) - \omega(\rho_2, q_2)\right|, \tag{5}$$

where $\omega : \Omega_f \cup \Omega_c \to \mathbb{R}$ is the continuous function defined by

$$\omega(\rho, q) = \begin{cases} \frac{q}{\rho} & \text{if } \ \frac{q}{\rho} \ge \frac{q^-}{R}, \\ V\rho - \sigma_- V + \frac{q^-}{R} & \text{otherwise.} \end{cases} \tag{6}$$

It is easy to see that d is a metric on $\Omega_f \cup \Omega_c$ and that it measures the distance of two points along the Riemann invariants.

The model in the free phase Ω_f reads

$$\partial_t \rho + \partial_x(\rho V) = \partial_t \rho + V \partial_x(\rho) = 0, \tag{7}$$

while in the congested phase Ω_c reads

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0. \end{cases} \tag{8}$$

The eigenvalues for (8) are

$$\lambda_1(\rho, q) = (1+2q)v_c^{eq}(\rho) + \rho(1+q)(v_c^{eq})'(\rho), \quad \lambda_2(\rho, q) = (1+q)v_c^{eq}(\rho), \tag{9}$$

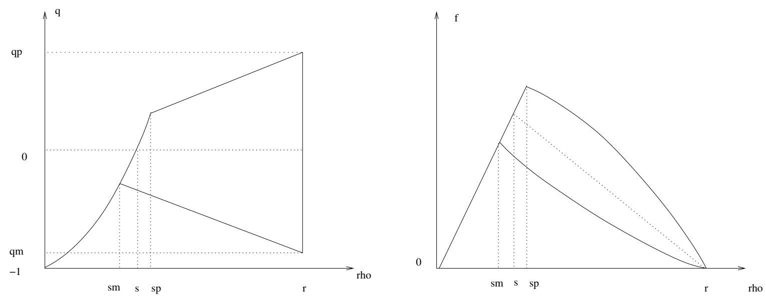

FIGURE 1. The fundamental diagram for the phase transition model in (ρ, q) and $(\rho, \rho v_c)$ coordinates.

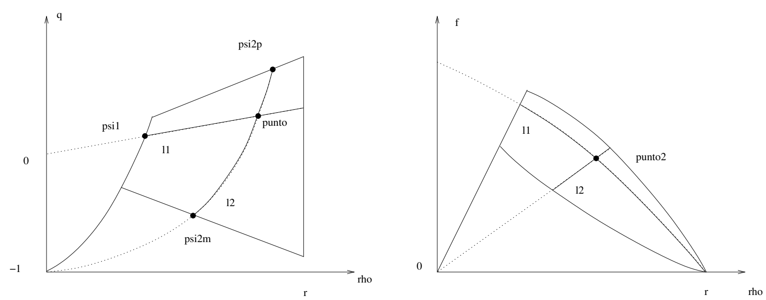

FIGURE 2. Shape of Lax curves in (ρ, q) and $(\rho, \rho v_c(\rho, q))$ planes.

while the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad r_2 = \begin{pmatrix} v_c^{eq}(\rho) \\ -(1+q)(v_c^{eq})'(\rho) \end{pmatrix}. \tag{10}$$

Moreover the first characteristic speed is genuinely nonlinear if $q \neq 0$ and it is linearly degenerate if $q = 0$. Instead the second characteristic speed is linearly degenerate. A deeper analysis of (8) is contained in [5]. We denote with $L_1(\rho; \rho_0, q_0)$ and $L_2(\rho; \rho_0, q_0)$ respectively the Lax curves of the first and second family for system (8); see Figure 2. We recall that $(\rho, L_1(\rho; \rho_0, q_0))$ are lines in the (ρ, q) plane passing through the origin and $(\rho, L_2(\rho; \rho_0, q_0))$ are lines in the $(\rho, \rho v)$ plane passing through the origin. Let us introduce the functions

$$\psi_1 : \Omega_c \to \Omega_c, \qquad \psi_2^- : \Omega_c \to \Omega_c, \qquad \psi_2^+ : \Omega_c \to \Omega_c, \tag{11}$$

such that, for every $(\rho_0, q_0) \in \Omega_c$, $\psi_1(\rho_0, q_0)$, $\psi_2^-(\rho_0, q_0)$ and $\psi_2^+(\rho_0, q_0)$ are defined respectively by the solutions in Ω_c to the systems

$$\begin{cases} \frac{q}{\rho} = \frac{q_0}{\rho_0}, \\ v_c(\rho, q) = V, \end{cases} \qquad \begin{cases} \frac{q}{\rho} = \frac{q^-}{R}, \\ v_c(\rho, q) = v_c(\rho_0, q_0), \end{cases} \qquad \begin{cases} \frac{q}{\rho} = \frac{q^+}{R}, \\ v_c(\rho, q) = v_c(\rho_0, q_0). \end{cases}$$

Moreover, define the flux function

$$\begin{array}{rcl} \varphi : \Omega_f \cup \Omega_c & \longrightarrow & \mathbb{R} \\ (\rho, q) & \longmapsto & \begin{cases} \rho V, & \text{if } (\rho, q) \in \Omega_f, \\ \rho v_c(\rho, q), & \text{if } (\rho, q) \in \Omega_c. \end{cases} \end{array}$$

For $(\rho, q) \in \Omega_f \cup \Omega_c$, define the following sets

$$\mathcal{T}_l\left(\rho, q\right) = \begin{cases} \left\{(\tilde\rho, \tilde q) \in \Omega_c : \tilde q = \frac{q}{\rho}\tilde\rho\right\}, & \text{if } (\rho, q) \in \Omega_c, \\ (\rho, q) \cup \left\{(\tilde\rho, \tilde q) \in \Omega_c : \begin{array}{c} \tilde q = \frac{q^-}{R}\tilde\rho, \\ \tilde\rho v_c(\tilde\rho, \tilde q) < V\rho \end{array}\right\}, & \text{if } (\rho, q) \in \Omega_f \setminus \Omega_c, \end{cases} \tag{12}$$

and

$$\mathcal{T}_r\left(\rho, q\right) = \begin{cases} \Omega_f, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{array}{l} \{(\tilde\rho, \tilde q) \in \Omega_c : v_c^{eq}(\tilde\rho)(1 + \tilde q) = v_c^{eq}(\rho)(1 + q)\} \\ \cup \left\{(\tilde\rho, \tilde q) \in \Omega_f : \tilde\rho V < \varphi(\psi_2^-(\rho, q))\right\}, \end{array} & \text{if } (\rho, q) \in \Omega_c, \end{cases} \tag{13}$$

These sets contain all the possible states, which can be connected to (ρ, q) with waves with negative and positive speed respectively, as the following proposition underlines; for a proof see [10, Proposition 3.2].

Proposition 1. *For $(\rho_l, q_l) \in \Omega_f \cup \Omega_c$ and $(\rho_r, q_r) \in \Omega_f \cup \Omega_c$, consider the Riemann problem*

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \textit{if } (\rho, q) \in \Omega_f,\ t > 0, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0, \end{cases} & \textit{if } (\rho, q) \in \Omega_c,\ t > 0, \\ (\rho, q)(0, x) = \begin{cases} (\rho_l, q_l), & \textit{if } x < 0, \\ (\rho_r, q_r), & \textit{if } x > 0. \end{cases} \end{cases} \tag{14}$$

1. *The Riemann problem (14) is solved with waves with non positive speed if and only if $(\rho_r, q_r) \in \mathcal{T}_l\left(\rho_l, q_l\right)$.*
2. *The Riemann problem (14) is solved with waves with non negative speed if and only if $(\rho_l, q_l) \in \mathcal{T}_r\left(\rho_r, q_r\right)$.*

Remark 1. Note that, if $(\rho, q) \in \Omega_c$, then the sets $\left\{(\tilde\rho, \tilde q) \in \Omega_c : \tilde q = \frac{q}{\rho}\tilde\rho\right\}$ and $\{(\tilde\rho, \tilde q) \in \Omega_c : v_c^{eq}(\tilde\rho)(1 + \tilde q) = v_c^{eq}(\rho)(1 + q)\}$, appearing in (12) and in (13), are composed by all the points in Ω_c which can be connected to (ρ, q) by a Lax curve respectively of the first family and by the second one.

Moreover the set $\left\{(\tilde\rho, \tilde q) \in \Omega_c : \tilde q = \frac{q^-}{R}\tilde\rho,\ \tilde\rho v_c(\tilde\rho, \tilde q) < V\rho\right\}$ in (12) contains all the points in Ω_c of the first Lax curve passing through (R, q^-) with sufficiently small flux. Finally, the set $\left\{(\tilde\rho, \tilde q) \in \Omega_f : \tilde\rho V < \varphi(\psi_2^-(\rho, q))\right\}$ consists in all the points of Ω_f with flux less than $\varphi(\psi_2^-(\rho, q))$.

The following results hold. The proofs are contained in [10].

Lemma 3.1. *We have that*

$$\varphi\left(\mathcal{T}_l\left(\rho, q\right)\right) = \begin{cases} [0, \varphi(\psi_1(\rho, q))], & \textit{if } (\rho, q) \in \Omega_c, \\ [0, \rho V], & \textit{if } (\rho, q) \in \Omega_f \setminus \Omega_c, \end{cases} \tag{15}$$

and

$$\varphi\left(\mathcal{T}_r\left(\rho, q\right)\right) = \begin{cases} [0, \varphi(\psi_2^+(\rho, q))], & \textit{if } (\rho, q) \in \Omega_c, \\ [0, \sigma_+ V], & \textit{if } (\rho, q) \in \Omega_f. \end{cases} \tag{16}$$

Lemma 3.2. *The first eigenvalue λ_1 is strictly negative in Ω_c. The second eigenvalue λ_2 is positive in Ω_c.*

Proposition 2. *Since $-1 < q^- < 0 < q^+ < 1$, the Lax curves of the first family in the $(\rho, \rho v)$ plane, parameterized by ρ, are uniformly bi-Lipschitz in Ω_c. For every $0 < \bar{v} < V$, the Lax curves of the second family in the $(\rho, \rho v)$ plane, parameterized by ρ, are uniformly bi-Lipschitz in the set $\{(\rho, q) \in \Omega_c : v_c(\rho, q) \geq \bar{v}\}$.*

Definition 3.3. Let I be a real interval. We say that a function $(\rho, q) : I \to \Omega_f \cup \Omega_c$ satisfies the assumption **(H-1)** in I if there exists $0 < \bar{v} < V$ such that, for a.e. $x \in I$, $(\rho, q)(x) \in \Omega_f$ or $(\rho, q)(x) \in \Omega_c$ and $v_c((\rho, q)(x)) \geq \bar{v}$.

Definition 3.3 will be used together with Proposition 2 for controlling the variation of ρ and q by the metric d.

3.1. **A special boundary value problem.** In this part we focus the attention in a special boundary value problem for the phase transition model. More precisely, let us consider the following system

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \text{if } (\rho, q) \in \Omega_f,\ t > 0,\ x < 0, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0, \end{cases} & \text{if } (\rho, q) \in \Omega_c,\ t > 0,\ x < 0, \\ (\rho, q)(0, x) = (\bar{\rho}, \bar{q}), & \text{if } x < 0, \\ (\rho, q)(t, 0) = (\tilde{\rho}(t), \tilde{q}(t)), & \text{if } t > 0, \end{cases} \tag{17}$$

where $(\bar{\rho}, \bar{q}) \in \Omega_f \cup \Omega_c$ and $(\tilde{\rho}, \tilde{q}) : [0, +\infty) \to \Omega_f \cup \Omega_c$ is a bounded variation function.

Definition 3.4. A function $(\rho, q) \in C^0\left([0, T]; L^1(-\infty, 0)\right)$ is a solution to (17) if

1. (ρ, q) is a weak entropy solution to the phase transition model (7)-(8) for $(t, x) \in (0, T) \times (-\infty, 0)$;
2. $\lim_{t \to 0^+} (\rho(t, \cdot), q(t, \cdot)) = (\bar{\rho}, \bar{q})$ in the L^1-topology;
3. for a.e. $t \in (0, T)$, the function $(\rho, q)(t, \cdot)$ admits a version with bounded variation;
4. for a.e. $t \in (0, T)$, the Riemann problem for the phase transition model (7)-(8) with initial condition

$$\begin{cases} (\rho, q)(t, 0-), & \text{if } x < 0, \\ (\tilde{\rho}(t), \tilde{q}(t)), & \text{if } x > 0, \end{cases}$$

is solved with waves with positive speed. Here $(\rho, q)(t, 0-)$ denote the left trace of the version with bounded variation of (ρ, q) at $x = 0$.

Remark that the point 4 of the previous definition does not impose that the trace at $x = 0$ of the solution equals to the boundary condition. We require that the classical Riemann problem connecting the trace with the boundary condition is solved with waves with positive speed; see [1, 9]. In this situation, all the waves we consider should be admissible waves also for the phase transition model on the real line; in particular, for a phase transition wave, the left and right states belong respectively to $\Omega_f \setminus \Omega_c$ and to Ω_c.

The following result holds.

Theorem 3.5. *Fix $(\bar{\rho}, \bar{q}) \in \Omega_f \cup \Omega_c$, $T > 0$ and $(\tilde{\rho}, \tilde{q}) \in BV([0, T], \Omega_f \cup \Omega_c)$. Assume that $\bar{\rho} < R$ and $(\tilde{\rho}, \tilde{q})$ satisfies the assumption **(H-1)**, in the sense of*

Definition 3.3. There exists a solution $(\rho, q) \in C^0\left([0,T]; L^1(-\infty,0)\right)$ *to (17) in the sense of Definition 3.4.*

Proof. We construct a wave-front tracking approximate solution (ρ_ν, q_ν) to (17). Consider a sequence $(\tilde\rho_\nu, \tilde q_\nu)$ of piecewise constant functions, defined on $[0,T]$, having a finite number of discontinuities, satisfying assumption **(H-1)** and such that $\lim_{\nu\to+\infty}(\tilde\rho_\nu, \tilde q_\nu) = (\tilde\rho, \tilde q)$ in $L^1([0,T]; \Omega_f \cup \Omega_c)$. For every $\nu \in \mathbb{N}\setminus\{0\}$, we apply the following procedure. At time $t = 0$ we solve the Riemann problem for the phase transition model with initial condition

$$\begin{cases} (\bar\rho, \bar q), & \text{if } x < 0, \\ (\tilde\rho_\nu(0+), \tilde q_\nu(0+)), & \text{if } x > 0. \end{cases}$$

Since we are interested only in the region $x < 0$, we neglect the waves of the second family (they have strictly positive speed) and the phase transition waves with positive speed. Therefore we consider only the wave of the first family (it has strictly negative speed) and the phase transition wave with negative speed. The wave of the first family can be a rarefaction wave; in this case we split it with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$, traveling with the Rankine-Hugoniot speed. We repeat a similar procedure at every time $t \in (0,T)$ such that the function $(\tilde\rho_\nu, \tilde q_\nu)$ has a discontinuity at t. In this case we need to solve the Riemann problem for the phase transition model with initial condition

$$\begin{cases} (\rho_\nu, q_\nu)(t, 0-), & \text{if } x < 0, \\ (\tilde\rho_\nu(t+), \tilde q_\nu(t+)), & \text{if } x > 0, \end{cases}$$

where $(\rho_\nu, q_\nu)(t, 0-)$ denotes the left trace of the approximate solution (ρ_ν, q_ν) at time t. Moreover at every time at which two waves interact together we need to solve the corresponding Riemann problem. As usual, by slightly modifying the speed of waves, we may assume that at every positive time t at most one interaction happens and that the formation of rarefaction fans happens only at $x = 0$. There exists a positive constant $K > 0$ such that

$$d\left((\rho_1, q_1), (\rho_2, q_2)\right) \le K\left(|\rho_1 - \rho_2| + |q_1 - q_2|\right) \tag{18}$$

for every (ρ_i, q_i) $(i \in \{1,2\})$ produced by the wave-front technique.

As in [10], we introduce the functional $W(t)$ measuring the strength of the waves at time t

$$W(t) = W_1(t) + W_2(t) + W_{PT}(t) \tag{19}$$

where

$$\begin{aligned} W_1(t) &= \sum_{x \in \mathcal{1}(t)} d\left((\rho_\nu(t,x-), q_\nu(t,x-)), (\rho_\nu(t,x+), q_\nu(t,x+))\right) \\ &= \sum_{x \in \mathcal{1}(t)} \left|v\left(\rho_\nu(t,x+), q_\nu(t,x+)\right) - v\left(\rho_\nu(t,x-), q_\nu(t,x-)\right)\right| \end{aligned} \tag{20}$$

$$\begin{aligned} W_2(t) &= \sum_{x \in \mathcal{2}(t)} d\left((\rho_\nu(t,x-), q_\nu(t,x-)), (\rho_\nu(t,x+), q_\nu(t,x+))\right) \\ &= \sum_{x \in \mathcal{2}(t)} \left|\omega\left(\rho_\nu(t,x+), q_\nu(t,x+)\right) - \omega\left(\rho_\nu(t,x-), q_\nu(t,x-)\right)\right| \end{aligned} \tag{21}$$

$$\begin{aligned} W_{PT}(t) &= \sum_{x\in\mathcal{PT}(t)} \left[\omega\left(\sigma_-, q^-\sigma_-/R\right) - \omega\left(\left(\rho_\nu, q_\nu\right)(t, x-)\right)\right] \\ &\quad + \sum_{x\in\mathcal{PT}(t)} \left[V - v_c\left(\left(\rho_\nu, q_\nu\right)(t, x+)\right)\right] \end{aligned} \tag{22}$$

measure respectively the strength of the waves of the first family, of the second family and of phase transition waves. The sets $\mathit{1}(t)$, $\mathit{2}(t)$ and $\mathcal{PT}(t)$ contain the points of discontinuity for the different types of waves. Note that the previous functionals may vary only at times $\bar{t}$ when two waves interact or when a wave reaches the boundary $x = 0$ or at the discontinuities for the functions $(\tilde{\rho}_\nu, \tilde{q}_\nu)$.

Before studying how W varies, we make the following observations.

A. If the initial condition $(\bar{\rho}, \bar{q})$ belongs to Ω_c, then every state, produced by the previous construction, belongs to Ω_c and to the Lax curve of the first family through $(\bar{\rho}, \bar{q})$. Since phase transition waves have left state belonging to $\Omega_f \setminus \Omega_c$, it is clear that from the boundary only waves of the first family can be generated. Interactions between these waves produce waves of the same family, since in Ω_c the system is Temple.

B. If the initial condition $(\bar{\rho}, \bar{q})$ belongs to $\Omega_f \setminus \Omega_c$, then the left most wave (if any) is a phase transition wave or a wave of the second family (until it comes back to $x = 0$). Moreover the state at the left of this wave is exactly $(\bar{\rho}, \bar{q})$, while all the other possible states between this wave and the boundary belong to Ω_c and to $q = \frac{q^-}{R}\rho$.

We need to estimate how the functional W varies at such times $\bar{t}$. There are several possibilities.

1. $\bar{t}$ is a point of discontinuity of $(\tilde{\rho}_\nu, \tilde{q}_\nu)$. Two different situations may happen.

   The state $(\rho_\nu(\bar{t}-, 0-), q_\nu(\bar{t}-, 0-))$ belongs to $\Omega_f \setminus \Omega_c$. In this case, this state is equal to $(\bar{\rho}, \bar{q})$. Moreover either no wave is produced or a phase transition wave (possibly followed by a wave of the first family) is generated at $x = 0$. Indicating with $(\hat{\rho}, \hat{q})$ the right state of the phase transition wave, we have $\Delta W_2(\bar{t}) = 0$,

   $$\begin{aligned} \Delta W_1(\bar{t}) &= v\left(\hat{\rho}, \hat{q}\right) - v\left(\psi_2^-\left(\tilde{\rho}_\nu(\bar{t}+), \tilde{q}_\nu(\bar{t}+)\right)\right) \geq 0 \\ \Delta W_{PT}(\bar{t}) &= \left[V - v\left(\hat{\rho}, \hat{q}\right)\right] + V\left[\sigma_- - \rho_\nu(\bar{t}-, 0-)\right] > 0 \\ \Delta W(\bar{t}) &= \left[V - v\left(\psi_2^-\left(\tilde{\rho}_\nu(\bar{t}+), \tilde{q}_\nu(\bar{t}+)\right)\right)\right] + V\left[\sigma_- - \bar{\rho}\right]. \end{aligned}$$

   Hence we deduce that $\Delta W(\bar{t}) \leq d\left(\left(\bar{\rho}, \bar{q}\right), \left(\tilde{\rho}_\nu(\bar{t}+), \tilde{q}_\nu(\bar{t}+)\right)\right)$.

   The state $(\rho_\nu(\bar{t}-, 0-), q_\nu(\bar{t}-, 0-))$ belongs to Ω_c. Either no wave is produced or a wave of the first family is generated at $x = 0$. We have $\Delta W_2(\bar{t}) = \Delta W_{PT}(\bar{t}) = 0$ and

   $$\Delta W(\bar{t}) = \Delta W_1(\bar{t}) = \left|v\left(\rho_\nu(\bar{t}-, 0-), q_\nu(\bar{t}-, 0-)\right) - v\left(\tilde{\rho}_\nu(\bar{t}+), \tilde{q}_\nu(\bar{t}+)\right)\right|.$$

   Hence $\Delta W(\bar{t}) \leq d\left(\left(\tilde{\rho}_\nu(\bar{t}-), \tilde{q}_\nu(\bar{t}-)\right), \left(\tilde{\rho}_\nu(\bar{t}+), \tilde{q}_\nu(\bar{t}+)\right)\right)$.

2. $\bar{t}$ is an interaction time between a phase transition wave and a wave of the first family. In this situation, the left wave, connecting the states $u_l \in \Omega_f \setminus \Omega_c$ and $u_m \in \Omega_c$, is the phase transition one, while the right wave, connecting the states $u_m \in \Omega_c$ and $u_r \in \Omega_c$, is of the first family. Note that $u_l = (\bar{\rho}, \bar{q})$, $\psi_2^-(u_m) = u_m$ and $\psi_2^-(u_r) = u_r$.

   This interaction can generate either a phase transition wave (possibly followed by a wave of the first family) or a wave of the second family: in this situation the number of waves does not increase. Moreover $W(\bar{t}+) \leq W(\bar{t}-)$.

3. $\bar{t}$ is an interaction time between a wave of the second family and a wave of the first family. The wave of the second family can only be generated by an interaction between a phase transition wave with a wave of the first family. Hence the left state is $(\bar{\rho}, \bar{q})$ and the right one is $\left(\sigma_-, \frac{q^-\sigma_-}{R}\right)$. This interaction can generate either a phase transition wave followed by a wave of the first family or a phase transition wave: in this situation the number of waves does not increase. Moreover $W(\bar{t}+) = W(\bar{t}-)$.
4. $\bar{t}$ is an interaction time between two waves of the first family. This interaction generates only a wave of the first family: in this situation the number of waves strictly decreases. Moreover $W(\bar{t}+) \leq W(\bar{t}-)$.
5. $\bar{t}$ is an interaction time between a phase transition wave with the boundary. In this situation no wave is produced. According to the previous observations, in a right neighborhood of $\bar{t}$, the solution is constantly equal to the initial condition $(\bar{\rho}, \bar{q})$. Therefore the number of waves is 0 and $W(\bar{t}+) = 0$.
6. $\bar{t}$ is an interaction time between a wave of the second family with the boundary. This case is entirely similar to the previous one.

This analysis proves that, for a.e. $t \in [0, T]$,

$$W(t) \leq K_1 \tag{23}$$

with

$$K_1 = d\left((\bar{\rho}, \bar{q}), (\tilde{\rho}(0+), \tilde{q}(0+))\right) + K\mathrm{Tot.Var.}\left(\tilde{\rho}, \tilde{q}\right).$$

Moreover the number of waves can increase only at times $\bar{t}$, points of discontinuity for $(\tilde{\rho}_\nu, \tilde{q}_\nu)$. For such times $\bar{t}$ the worst situation is the production of a phase transition wave followed by a rarefaction fan. In this case the number of waves produced at time $\bar{t}$ is bounded by $1 + \nu \left|(\tilde{\rho}_\nu, \tilde{q}_\nu)(\bar{t}+) - (\tilde{\rho}_\nu, \tilde{q}_\nu)(\bar{t}-)\right|$. Then the total maximum number of new waves is bounded by $N_\nu + \nu \cdot \mathrm{Tot.Var.}\,(\tilde{\rho}, \tilde{q})$, where N_ν is the number of discontinuities of $(\tilde{\rho}_\nu, \tilde{q}_\nu)$. This proves that the number of waves is finite.

Let us now consider the number of interactions. Interactions at point 1 can happen, by construction, at most a finite number of times. Since the number of waves is bounded, then also interactions at points 4, 5 and 6 can happen at most a finite number of times. Moreover interactions at point 2, producing a wave of the second family or a single phase transition, strictly decrease the number of waves; hence can happen at most a finite number of times. Consequently interactions at point 3 can happen at most a finite number of times. It remains to consider only interactions at point 2 producing both a phase transition and a wave of the first family. There exists a unique point $(\check{\rho}, \check{q}) \in \Omega_c$ such that the phase transition so generated connects $(\bar{\rho}, \bar{q})$ to $(\check{\rho}, \check{q})$. Obviously $(\check{\rho}, \check{q})$ depends only on $(\bar{\rho}, \bar{q})$ and it is determined in the $(\rho, \rho v)$ plane by the tangent point to the lower part of Ω_c of a line passing through $(\bar{\rho}, \bar{q})$. Moreover, since for every $t \in [0, T]$ there is at most one phase transition (the leftmost wave), in order to reproduce such type of interaction, both the phase transition and the wave of the first family must first interact with other waves. Hence we conclude that the number of interactions is finite.

Therefore for a.e. $t \in [0, T]$, by **(H-1)**, it is possible to obtain a uniform bound to $\mathrm{Tot.Var.}\,(\rho_\nu, q_\nu)(t, \cdot)$ and so to conclude as in [10]. □

Remark 2. In the proof of Theorem 3.5 an increment of the functional W, defined in (19), can happen only when the boundary data change. In this case the variation of W is proportional to the variation of the boundary data. Instead, in the case of an interaction of two waves, the functional W does not increase; see also [10, Lemma 4.29].

4. **The FTL-PT model.** This section deals with the coupling between the FTL model if $x < 0$ and the PT one if $x > 0$.

4.1. **The Riemann problem.** Here we describe how to solve the Riemann problem for the FTL-PT model. To this aim, we fix $I \in \mathbb{N}$ and the initial data

$$\begin{cases} \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_I \le 0 \\ \bar{v}_1, \dots, \bar{v}_I \in [0, V] \\ (\bar{\rho}, \bar{q}) \in \Omega_f \cup \Omega_c. \end{cases} \tag{24}$$

For defining a solution to this system, it is sufficient to prescribe the dynamics of the first vehicle at x_I (or of the second one at x_{I-1} in case $x_I = 0$) and to select a boundary data $\left(\rho^b, q^b\right) \in \mathcal{T}_r\left(\bar{\rho}, \bar{q}\right)$ for the PT model. Clearly this solution is valid until a car from the FTL model hits the boundary $x = 0$. However, the procedure can be restarted at every time a car from the FTL model reaches the boundary $x = 0$, providing in such a way a solution for all times.There are two different situations: $x_I < 0$ or $x_I = 0$.

First case. $\bar{x}_I < 0$. The solution we propose here holds for all times $t \in [0, \bar{t}]$, where $\bar{t}$ is the first time such that $x_I(\bar{t}) = 0$. In this situation, the PT model is not influenced by the FTL one. Hence, the solution for the PT model is given by

$$(\rho(t, x), q(t, x)) = (\bar{\rho}, \bar{q})$$

for every $x > 0$ and $t \in [0, \bar{t}]$; thus $\left(\rho^b, q^b\right) = (\bar{\rho}, \bar{q})$. Moreover the car at x_I moves according to the ODE system

$$\begin{cases} \dot{x}_I = v_I \\ \dot{v}_I = C_\gamma \bar{\rho}^\gamma \frac{v(\bar{\rho}, \bar{q}) - v_I}{(\Delta X - \bar{\rho} x_I)^\gamma} \\ x_I(0) = \bar{x}_I \\ v_I(0) = \bar{v}_I \end{cases}$$

for $t \in [0, \bar{t}]$; i.e. it moves in the same way as if it sees a car in position $\frac{\Delta X}{\bar{\rho}}$ with speed $v(\bar{\rho}, \bar{q})$.

Second case. $\bar{x}_I = 0$. Here the first car at $x_I = 0$ exits from the FTL model and the leader becomes the vehicle at x_{I-1}. The solution proposed here is valid until the new leading vehicle reaches the boundary $x = 0$. According to [15, Section 5.1], we define $\tilde{\rho} = -\frac{\Delta X}{\bar{x}_{I-1}}$ as the presumed density at $x = 0$. The following possibilities, about the choice of $\left(\rho^b, q^b\right)$, occur.

1. $\Phi\left(\tilde{\rho}, \bar{v}_I\right) \in \Omega_f \setminus \Omega_c$ and $\Phi\left(\tilde{\rho}, \bar{v}_I\right) \in \mathcal{T}_r\left(\bar{\rho}, \bar{q}\right)$. Define $\left(\rho^b, q^b\right) = \Phi\left(\tilde{\rho}, \bar{v}_I\right)$.
2. $\Phi\left(\tilde{\rho}, \bar{v}_I\right) \in \Omega_f \setminus \Omega_c$ and $\Phi\left(\tilde{\rho}, \bar{v}_I\right) \notin \mathcal{T}_r\left(\bar{\rho}, \bar{q}\right)$. This means that $(\bar{\rho}, \bar{q}) \in \Omega_c \setminus \Omega_f$, otherwise $\mathcal{T}_r\left(\bar{\rho}, \bar{q}\right) = \Omega_f$. Define $\left(\rho^b, q^b\right) = \psi_2^-\left(\bar{\rho}, \bar{q}\right)$.
3. $\Phi\left(\tilde{\rho}, \bar{v}_I\right) \in \Omega_c$. Define $\left(\rho^b, q^b\right) \in \Omega_c \cap \mathcal{T}_r\left(\bar{\rho}, \bar{q}\right)$ as the unique state such that
$$L_2\left(\bar{\rho}; L_1\left(\rho^b; \Phi\left(\tilde{\rho}, \bar{v}_I\right)\right)\right) = \left(\bar{\rho}, \bar{q}\right).$$
4. $\Phi\left(\tilde{\rho}, \bar{v}_I\right) \notin \Omega_f \cup \Omega_c$ and $\Phi_2\left(\tilde{\rho}, \bar{v}_I\right) > \frac{q^+}{R} \Phi_1\left(\tilde{\rho}, \bar{v}_I\right)$. Define $\left(\rho^b, q^b\right) = \psi_2^+\left(\bar{\rho}, \bar{q}\right)$ if $(\bar{\rho}, \bar{q}) \in \Omega_c$, and define $\left(\rho^b, q^b\right) = \Phi(\sigma_+, V\sigma_+)$ if $(\bar{\rho}, \bar{q}) \in \Omega_f$.

5. $\Phi(\tilde{\rho},\bar{v}_I) \notin \Omega_f \cup \Omega_c$, $\Phi_2(\tilde{\rho},\bar{v}_I) < \frac{q^-}{R}\Phi_1(\tilde{\rho},\bar{v}_I)$ and $(\bar{\rho},\bar{q}) \in \Omega_f$. Define (ρ^b,q^b) as the unique solution to the system

$$\begin{cases} \frac{q^b}{\rho^b} = \frac{\Phi_2(\tilde{\rho},\bar{v}_I)}{\Phi_1(\tilde{\rho},\bar{v}_I)} \\ (\rho^b,q^b) \in \Omega_f. \end{cases}$$

6. $\Phi(\tilde{\rho},\bar{v}_I) \notin \Omega_f \cup \Omega_c$, $\Phi_2(\tilde{\rho},\bar{v}_I) < \frac{q^-}{R}\Phi_1(\tilde{\rho},\bar{v}_I)$ and $(\bar{\rho},\bar{q}) \in \Omega_c \setminus \Omega_f$. Define $(\rho^b,q^b) = \psi_2^-(\bar{\rho},\bar{q})$.

Moreover, given the boundary term (ρ^b,q^b) for the PT model, the car at x_{I-1}, according to the previous case, moves according to the ODE system

$$\begin{cases} \dot{x}_{I-1} = v_{I-1} \\ \dot{v}_{I-1} = C_\gamma \left(\rho^b\right)^\gamma \frac{v(\rho^b,q^b)-v_{I-1}}{(\Delta X - \rho^b x_{I-1})^\gamma} \\ x_{I-1}(0) = \bar{x}_{I-1} \\ v_{I-1}(0) = \bar{v}_{I-1} \end{cases}$$

for $t \in [0,\bar{t}]$; i.e. it moves in the same way as if it sees a car in position $\frac{\Delta X}{\rho^b}$ with speed $v(\rho^b,q^b)$.

Remark 3. The case $\bar{x}_I = 0$ presents various possibilities in the choice of the boundary datum $\left(\rho^b,q^b\right)$. The key point is that $\left(\rho^b,q^b\right)$ should belong to the set $\mathcal{T}_r(\bar{\rho},\bar{q})$, in order to produce waves with positive speed. The choice of $\left(\rho^b,q^b\right)$ in the first three possibilities is the natural one, since $\Phi(\tilde{\rho},\tilde{v}_I) \in \Omega_f \cup \Omega_c$. In the last three cases, instead, we need to project the point $\Phi(\tilde{\rho},\tilde{v}_I)$ in the set $\Omega_f \cup \Omega_c$ and some arbitrariness appears. A similar projection problem was considered in [4]. Our projection has some differences with respect to that in [4] and this is due to the constraint $\left(\rho^b,q^b\right) \in \mathcal{T}_r(\bar{\rho},\bar{q})$. For example, if $(\bar{\rho},\bar{q}) \in \Omega_f$, then the boundary data belongs to Ω_f and this explains the point 5 and the second part of 4. In point 6, we decide to project to Ω_c, but also some projections on Ω_f could be reasonable.

5. The PT-FTL model.

This section deals with the coupling between the PT model if $x < 0$ and the FTL one if $x > 0$.

5.1. The Riemann problem.

Here we describe how to solve the Riemann problem for the PT-FTL model. To this aim, we fix $I \in \mathbb{N}$ and the initial data

$$\begin{cases} (\bar{\rho},\bar{q}) \in \Omega_f \cup \Omega_c \\ 0 \le \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_I \\ \bar{v}_1,\ldots,\bar{v}_I \in [0,V]. \end{cases} \tag{25}$$

Let $\Upsilon : [0,V] \to [0,R]$ be the unique continuous function such that

$$\begin{cases} \Phi(\Upsilon(v),v) \in \Omega_c \\ \Phi_2(\Upsilon(v),v) = \frac{q^+}{R}\Upsilon(v), \end{cases}$$

where Φ and Φ_2 are defined in (4). First, we need to define the point $(\rho^b,q^b) \in \Omega_f \cup \Omega_c$. Consider the solution to the classical Riemann problem for the phase transition model with initial condition $(\bar{\rho},\bar{q})$ if $x < 0$ and $\Phi(\Upsilon(\bar{v}_1),\bar{v}_1)$ if $x > 0$. Define (ρ^b,q^b) as the left trace at $x = 0$ and $t = 1$ of this solution.

We have two different situations.

1. $\rho^b \geq \frac{\Delta X}{\bar{x}_1}$. In this case we have an emission procedure at time $t = 0$; see [15]. For simplicity, we rearrange the index of cars for $x > 0$ with the map $i \mapsto i+1$, in such a way the leading car is at position $\bar{x}_{I+1}$. The new car in the FTL model is now at position $\bar{x}_1 = 0$ with a initial speed $\bar{v}_1 = \bar{v}_2$. The dynamics for the FTL is now completely determined, provided the dynamics of the first car is known.

   Define $\tilde{\rho}(t) = \Upsilon(v_1(t))$, $\tilde{q}(t) = \Phi_2\left(\Upsilon(v_1(t)), v_1(t)\right)$. The solution for the phase transition model is given by the solution to (17), which exists by Theorem 3.5. This solution is valid for $t \in [0, \bar{t}]$, where $\bar{t} > 0$ (next emission time) is the first time such that

$$\frac{1}{\bar{t}} \int_0^{\bar{t}} \rho(t, 0-) dt = \frac{\Delta X}{x_1(\bar{t})}. \tag{26}$$

2. $\rho^b < \frac{\Delta X}{x_{I+1}}$. The dynamics for the FTL is now completely determined, provided the dynamics of the first car is known.

   Define $\tilde{\rho}(t) = \Upsilon(v_1(t))$, $\tilde{q}(t) = \Phi_2\left(\Upsilon(v_1(t)), v_1(t)\right)$. The solution for the phase transition model is given by the solution to (17), which exists by Theorem 3.5. This solution is valid for $t \in [0, \bar{t}]$, where $\bar{t} > 0$ (next emission time) is the first time such that (26) holds.

At times $\bar{t}$ such that (26) holds, we perform an emission procedure (see [15]) in this way. We rearrange the index of cars for $x > 0$ with the map $i \mapsto i+1$, in such a way the leading car is at position $\bar{x}_{I+1}$. The new car in the FTL model is now at position $\bar{x}_1 = 0$ with a initial speed

$$\bar{v}_1 = \frac{\int_0^{\bar{t}} \rho(s, 0-) v_c\left(\rho(s, 0-), q(s, 0-)\right) ds}{\int_0^{\bar{t}} \rho(s, 0-) ds}$$

The dynamics for the FTL is now completely determined, provided the dynamics of the first car is known. The procedure restarts as in the point 2.

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