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ASYMPTOTICS OF AN OPTIMAL COMPLIANCE-NETWORK PROBLEM

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Abstract. We consider the problem of the optimal location of a Dirichlet region in a d-dimensional domain Ω subjected to a given force f in order to minimize the p-compliance of the configuration. We look for the optimal region among the class of all closed connected sets of assigned length l. Then we let the length l tend to infinity and we look at the Γ-limit of a suitable rescaled functional, from which we get information of the asymptotic distribution of the optimal region. We also study the case where the Dirichlet region is a discrete set of finite cardinality.

1. **Introduction.** We consider the problem of finding the best location of the Dirichlet region Σ in a d-dimensional domain Ω associated to an elliptic equation in divergence form, namely

$$\begin{cases} -\Delta_p u = f \text{ in } \Omega \setminus \Sigma \\ u = 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases} \tag{1}$$

where f is a nonnegative function belonging to $L^q(\Omega)$, q being the conjugate exponent of p and $\Delta_p u$ stands for $\mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})$. We are interested in the minimization of the p-compliance functional defined by

$$C(\Sigma) = \int_\Omega f u_{f,\Sigma,\Omega} dx,$$

where $u_{f,\Sigma,\Omega}$ is the unique distributional solution of the equation (1). The admissible class of control variables Σ considered here is the class of all closed connected sets with given one dimensional Hausdorff measure. It is easy to obtain the optimal configuration Σ_l which minimize the compliance functional (see Theorem 2.1 in section 2) as a consequence of Šverák's result (see e.g. [4], [11]). We are interested in the asymptotic behavior of Σ_l as $l \to +\infty$; more precisely, we want to obtain the limit distribution of Σ_l as a limit probability measure that minimizes the Γ-limit functional of the suitable rescaled p-compliance functional. In the literature, there are similar results among which we may cite location problems studied in [2], irrigation problems in [10] and compliance in [6]. The present paper is an extension of the result [5], where a two-dimensional case was studied. The proofs follow the guidelines of [5] and difficulties are mainly of technical nature. In [5], the two-dimensional

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setting has been used in the proofs of Γ-lim inf and Γ-lim sup inequalities. More precisely, Lemma 1 in [5] (analogous of Lemma 3.1 in this paper), which is crucial for the Γ-lim inf inequality, follows from the classical Poincaré inequality while in this paper we consider the Poincaré inequality using the notion of p-capacity of a set. For the construction of the recovering sequence of the Γ-lim sup inequality, the analogous of Lemma 3.4 in this paper is enough in the case where the dimension is two. The case of higher dimension requires more effort since the boundary of the unit cube is not a one-dimensional set. To overcome this difficulty, we prove another result (Lemma 3.5) which studies the difference between two solutions of the p-Laplacian equation with two different Dirichlet boundary conditions. This result together with Lemma 3.4 are sufficient for the construction of the recovering sequence. In the last section, we deal with the case where the Dirichlet region is a discrete set with a finite numbers of elements under the assumption that $p > d$. This problem is in connection with the location problem studied in [2].

2. **The p-compliance under length constraint.** Let $p > d-1$ be fixed and $q = p/(p-1)$ be the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}^d$ and a positive real number l, we define the class $\mathcal{A}_l(\Omega)$ by

$$\mathcal{A}_l(\Omega) := \{\Sigma \subset \overline{\Omega}, \quad \text{closed and connected}: \quad 0 < \mathcal{H}^1(\Sigma) \leq l\}.$$

For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity, we denote by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \setminus \Sigma \\ u &= 0 \text{ in } \Sigma \cup \partial\Omega, \end{cases}$$

that is $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega \setminus \Sigma). \tag{2}$$

By the maximum principle, the nonnegativity of the function f implies that of u. For $f \geq 0$, we define the p-compliance functional as follows:

$$\begin{aligned} C(\Sigma) = F_p(\Sigma, f, \Omega) &= \int_\Omega f u_{f,\Sigma,\Omega} dx = \int_\Omega |\nabla u_{f,\Sigma,\Omega}|^p dx \\ &= q \max \left\{ \int_\Omega (v - \frac{1}{p} |\nabla v|^p) dx : v \in W_0^{1,p}(\Omega \setminus \Sigma) \right\}, \end{aligned}$$

where q stands for the conjugate exponent of p. The existence of the minimal p-compliance configuration is just a consequence of a generalized Šverák compactness-continuity result (see [4]). For the convenience of the reader, we briefly describe the existence of the optimal sets.

Let $\{\Sigma_n\}_n \subset \mathcal{A}_l(\Omega)$ be a minimizing sequence of the compliance functional, then there exists a constant $A > 0$ such that $\sup_n\{C(\Sigma_n)\} \leq A$. Since $\{\Sigma_n\}_n$ is a sequence of closed connected subsets of Ω such that $\sup_n \mathcal{H}^1(\Sigma_n) \leq l$, by Blaschke theorem (compactness of the sequence $\{\Sigma_n\}_n$ in the Hausdorff topology) and by Gołab theorem (lower semicontinuity of the $\mathcal{H}^1$ with respect to the Hausdorff topology), up to extracting a subsequence, $\{\Sigma_n\}_n$ converges in Hausdorff distance to some $\Sigma \in \mathcal{A}_l(\Omega)$ and $\mathcal{H}^1(\Sigma) \leq \liminf_{n\to\infty} \mathcal{H}^1(\Sigma_n)$. For the lower semicontinuity of the compliance functional we need the Šverák continuity-compactness result which is stated as follow: Let $\{D_n\}_n$ be a sequence of open and bounded sets contained in a fix bounded set B. If we assume that the number of the connected components of the complements of D_n in B is uniformly bounded by some number s, then $\{D_n\}_n$ converges in the Hausdorff topology to some open and bounded set $D \subset B$ and the number of the connected components of the complement of D is less or equal to s. Moreover, if we denote by $u_n \in W_0^{1,p}(D_n)$ the distributional solution of the p-Laplacian equation $-\Delta_p u_n = f$ in D_n for some f in $W^{-1,q}(B)$, then up to subsequence, $\{u_n\}_n$ converges strongly in $W^{1,p}(B)$ (u_n are extended by zero outside D_n) to a function u which is the distributional solution of the equation $-\Delta_p u = f$ in D. This result is interesting only in the case where p satisfies $d-1 < p \leq d$ because the case where $p > d$ is trivial due to the fact that functions in $W^{1,p}(B)$ are continuous and the convergence of solutions follows easily. To apply this result to our problem, we choose $D_n = \Omega \setminus \Sigma_n$ and notice that $\{\Omega_n\}_n$ converges to $D = \Omega \setminus \Sigma$ in the Hausdorff topology where Σ is the limit of Σ_n (note that to avoid terminology mix up, we did not differentiate between the Hausdorff convergence of compact sets and of open sets). From the continuity with respect to the domains variation of solutions, the lower semicontinuity follows easily and also the existence of an optimal shape as given in the result below.

Theorem 2.1. *For any real number $l > 0$, Ω bounded open subset of $\mathbb{R}^d$, $d \geq 2$ and f a nonnegative function belonging to $L^q(\Omega)$, the problem*

$$\min\{C_p(\Sigma) : \Sigma \in \mathcal{A}_l(\Omega)\} \tag{3}$$

admits at least one solution.

As we have existence of at least one optimal set, we are interested in the asymptotic behavior of those optimal sets Σ_l of the problem (3) as $l \to +\infty$. To achieve this, let us associate to every $\Sigma \in \mathcal{A}_l(\Omega)$ a probability measure on $\overline{\Omega}$ that will be called associated measure, given by

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and define a functional $F_l : \mathcal{P}(\overline{\Omega}) \to [0, +\infty]$ by

$$F_l(\mu) = \begin{cases} l^{\frac{q}{d-1}} C_p(\Sigma) & \text{if} \quad \mu = \mu_\Sigma, \Sigma \in \mathcal{A}_l(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

The scaling factor $l^{\frac{q}{d-1}}$ is needed in order to avoid the functional degenerating to the trivial limit functional which vanishes everywhere. Our main result deals with the behavior as $l \to +\infty$ of the functional F_l, and we state it in terms of Γ-convergence.

Theorem 2.2. *The functional F_l defined in (4) Γ-converges, with respect to the weak* topology on the class $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$, to the functional F defined on $\mathcal{P}(\overline{\Omega})$ by*

$$F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx. \tag{5}$$

In this case μ_a is the density of the absolutely continuous part of μ with respect to the Lebesgue measure, and θ is a positive constant depending only on d and p and is defined by

$$\theta = \inf\{\liminf_{l \to +\infty} l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) : \Sigma_l \in \mathcal{A}_l(I^d)\} \tag{6}$$

$I^d = (0,1)^d$ *being the unit cube in* $\mathbb{R}^d$.

According to the general theory of Γ-convergence (see [7]), we deduce the following consequences of Theorem 2.2:

- if Σ_l is an optimal set of the minimization problem (3), then up to a subsequence the associated measure μ_{Σ_l} *weak** converges to μ as $l \to +\infty$, where μ is a minimizer of F (for the definition of F see (5))
- since F has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, the whole sequence μ_{Σ_l} converges to the unique minimizer μ of F given by $\mu = cf^{\frac{q(d-1)}{q+d-1}}\mathcal{L}^d$ where c is such that μ is a probability measure, that is $c = 1/\left(\int_\Omega f^{\frac{q(d-1)}{q+d-1}}dx\right)$
- the minimal value of F is equal to $\theta c^{\frac{q+d-1}{d-1}}$, and the sequence of the values $\inf\{F_p(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}_l(\Omega)\}$ is asymptotically equivalent to $l^{\frac{q}{d-1}}\theta c^{\frac{q+d-1}{d-1}}$.

3. Γ-convergence result.

We will split the proof of the Γ-convergence result into two steps corresponding to Γ-lim inf and Γ-lim sup inequalities.

3.1. Γ-lim inf inequality.

Before proving the $\Gamma - \liminf$ inequality, we need some results and constructions. We start with a construction of a set $G_{\varepsilon,l}$ which will be useful later. Let Ω be a domain contained in I_a^d where I_a^d is a cube of $\mathbb{R}^d$ of side $2a$ for some positive real number a. Let M be a union of d segments of length 1 joining at the center of the unit cube I^d and connecting two parallel faces of the unit cube in the given direction. The segments are made in such a way that their endpoints coincide with the middle points of the faces of I^d. We consider the set $G_{\varepsilon,l}$ to be the homogenization of the set M of order $\lfloor(\frac{\varepsilon l}{2ad})^{1/(d-1)}\rfloor$ into I_a^d. It is clear that, due to the particularity of the set M, the set $G_{\varepsilon,l}$ is connected and $\mathcal{H}^1(G_{\varepsilon,l}) \approx \varepsilon l$.

The following Lemma justifies the scaling factor in (4)

Lemma 3.1. *Let* $Q_R \subset \mathbb{R}^d$ *be a cube of side* R *and* $A \subset \overline{Q}_R$ *a closed subset of* $\overline{Q}_R$ *of positive p-capacity, then*

(1) *there exists a constant* $C = C(d,p)$ *such that, for all functions* $v \in C^\infty(\overline{Q}_R)$ *with nonnegative mean value and vanishing on* A, *we have*

$$\int_{Q_R} |v|^p dx \leq \frac{CR^d}{cap_p(A, Q_{2R})}\int_{Q_R} |\nabla v|^p dx,$$

where $cap_p(A, Q_{2R})$ *stands for the relative p-capacity of the set* A *inside* Q_{2R}.

(2) *For any* $\varepsilon > 0$, *any* $0 < l < +\infty$, *any domain* Ω *and any function with non zero mean value* $v \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l}) \subset W_0^{1,p}(\Omega)$ ($G_{\varepsilon,l}$ *is the network constructed above) it holds* $||v||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0)l^{\frac{1}{1-d}}||v||_{W_0^{1,p}(\Omega)}$, *where* $\varepsilon_0 = cap_p(M, 2I^d)$.

(3) *As a consequence, if we have a nonnegative function* $f \in L^q(\Omega)$, *then the function* $u_{f,G_{\varepsilon,l},\Omega}$ *satisfies* $||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d,\varepsilon,\varepsilon_0)l^{\frac{q}{1-d}}||f||_{L^q(\Omega)}^{1/(p-1)}$.

Proof. The first assertion is a variant of the well-known Poincaré inequality. See [9] for more details. For proving the second assertion, we first choose the function v to be a nonnegative smooth function not identically zero on a large cube I_a^d which vanish outside $\Omega \setminus G_{\varepsilon,l}$. We consider the subdivision of the cube I_a^d into subcubes which are coming from the homogenization of order $\lfloor(\frac{\varepsilon l}{2ad})^{1/(d-1)}\rfloor$ of the unit cube into I_a^d and consider the associated network $G_{\varepsilon,l}$. The side of subcubes is of order $l^{1/(1-d)}$. Let us denote the subcubes by Q_j. The set $I_a^d \setminus G_{\varepsilon,l}$ can be seen as the homogenization of order $k = \lfloor(\frac{\varepsilon l}{ad})^{1/(d-1)}\rfloor$ of $I^d \setminus M$ into I_a^d (M is the set constructed above). Let us set $\varepsilon_0 = cap_p(M, 2I^d)$ and notice that v vanishes on $G_{\varepsilon,l}$. By applying the first statement of this Lemma (v has positive mean value), it follows that

$$\int_{Q_j} |v|^p dx \leq \frac{Ck^{-d}}{cap_p(k^{-1}M, 2Q_j)} \int_{Q_j} |\nabla v|^p dx \leq \frac{Cl^{p/(1-d)}}{cap_p(M, 2I^d)} \int_{Q_j} |\nabla v|^p dx$$

and by summing up over j we get

$$\int_{I_a^d} |v|^p dx \leq \frac{C}{\varepsilon_0} l^{p/(1-d)} \int_{I_a^d} |\nabla v|^p dx.$$

Using the fact that v vanishes outside Ω, we may restrict the integrand to Ω, raise each term of the inequality to the power $1/p$ and thus getting the result by noticing that the L^p norm of the gradient $||\nabla v||_{L^p(\Omega)}$ stands for the norm $||v||_{W_0^{1,p}(\Omega)}$. The general case follows by density. For the last inequality, we use the weak version of the PDE and the Hölder inequality to obtain

$$\int_\Omega |\nabla u_{f,G_{\varepsilon,l},\Omega}|^p dx = \int_\Omega f u_{f,G_{\varepsilon,l},\Omega} dx \leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)}.$$

Since $u_{f,G_{\varepsilon,l},\Omega} \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,l})$, by the second part of this Lemma we get

$$\begin{aligned} ||u_{f,G_{\varepsilon,l},\Omega}||^p_{W_0^{1,p}(\Omega)} &\leq ||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} ||f||_{L^q(\Omega)} \\ &\leq C(d, \varepsilon_0, \varepsilon) l^{1/(1-d)} ||u_{f,G_{\varepsilon,l},\Omega}||_{W_0^{1,p}(\Omega)} ||f||_{L^q(\Omega)}, \end{aligned}$$

and the desired result follows. □

Before proving the Γ-lim inf inequality, we need the following estimate which will be helpful.

Lemma 3.2. *Let $f, g \in L^q(\Omega)$ be given and u_f and u_g denote the solution of p-Laplacian equation with respective right hand side f, g and with Dirichlet boundary condition on $\Sigma_l' = \Sigma_l \cup G_{\varepsilon,l}$ (where Σ_l is an element of $\mathcal{A}_l(\Omega)$ and $G_{\varepsilon,l}$ the above constructed network), then*

$$l^{q/(d-1)} ||u_f - u_g||_{L^1(\Omega)} \leq C|\Omega|^{1/q} ||f - g||^{1/(p-1)}_{L^q(\Omega)},$$

where $C = C(d, p, \varepsilon_0, \varepsilon)$. In particular, if $\Omega = Q$ a cube centered at x_0, $g = f(x_0)$ and x_0 is a Lebesgue point for f, then

$$l^{q/(d-1)} ||u_f - u_g||_{L^1(Q)} \leq C|Q| \left(\frac{\int_Q |f(x) - f(x_0)|^q dx}{|Q|} \right)^{1/p} = |Q| r(Q).$$

Proof. For any $p \geq 2$, and any pair of vectors (z, w) we have the following monotonicity formula (see [8])

$$|z - w|^p \leq C(|z|^{p-2}z - |w|^{p-2}w) \cdot (z - w).$$

In our setting, we have $p > d - 1$ and $d \geq 3$ so we fulfill the requirement of the monotonicity formula. By choosing $z = \nabla u_f$ and $w = \nabla u_g$, we get

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \leq C ||u_f - u_g||_{L^p(\Omega)} ||f - g||_{L^q(\Omega)}.$$

From part 2 of Lemma 3.1, the inequality $||v||_{L^p(\Omega)} \leq Cl^{1/(1-d)} ||v||_{W_0^{1,p}(\Omega)}$ holds for every function v vanishing on Σ_l'. Since the function $u_f - u_g$ vanishes on Σ_l', we have

$$||u_f - u_g||^p_{W_0^{1,p}(\Omega)} \leq Cl^{1/(1-d)}||u_f - u_g||_{W_0^{1,p}(\Omega)}||f - g||_{L^q(\Omega)},$$

which gives

$$||u_f - u_g||_{W_0^{1,p}(\Omega)} \leq Cl^{1/(1-d)(p-1)}||f - g||^{1/(p-1)}_{L^q(\Omega)},$$

and using Hölder inequality, we get

$$\begin{aligned} ||u_f - u_g||_{L^1(\Omega)} &\leq |\Omega|^{1/q}||u_f - u_g||_{L^p(\Omega)} \\ &\leq C|\Omega|^{1/q}l^{1/(1-d)}||u_f - u_g||_{W_0^{1,p}(\Omega)} \\ &\leq C|\Omega|^{1/q}l^{q/(1-d)}||f - g||^{1/(p-1)}_{L^q(\Omega)}, \end{aligned}$$

then the first part of the statement follows. The second part is an obvious consequence of the first part. □

Remark 1. Since x_0 is a Lebesgue point for f, the number

$$r(Q) = \left(\frac{1}{|Q|}\int_Q |f(x) - f(x_0)|^q dx\right)^{1/p}$$

goes to zero whenever the cube Q shrinks around the point x_0.

In the following proposition, we prove that the Γ-lim inf functional is bounded below by the candidate limit functional F in (5).

Proposition 1. *Under the same hypotheses of Theorem 2.2, denoting by F^- the functional Γ-$\liminf_l F_l$, it holds $F^-(\mu) \geq F(\mu)$ for any $\mu \in \mathcal{P}(\overline{\Omega})$. This means that for any sequence $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ such that the associated sequence of measures μ_{Σ_l} weak* converges to μ, we have*

$$\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \geq F(\mu).$$

Proof. Let $\Sigma'_l = \Sigma_l \cup G_{\varepsilon,l}$ and set $u'_l = u_{f,\Sigma'_l,\Omega}$. Since $u_l \geq u'_l$, it is enough to estimate the integral $l^{\frac{q}{d-1}} \int_\Omega f u'_l dx$. It is obvious that $0 \leq u'_l \leq u_{f,G_{\varepsilon,l},\Omega}$ (by the maximum principle) and part 3 of Lemma 3.1 gives

$$||u_{f,G_{\varepsilon,l},\Omega}||_{L^p(\Omega)} \leq C(d, \varepsilon_0, \varepsilon, f)l^{\frac{q}{1-d}}.$$

It follows that $l^{\frac{q}{d-1}}u'_l$ is L^p bounded, so up to a subsequence $l^{\frac{q}{d-1}}u'_l \rightharpoonup w$ in $L^p(\Omega)$ for some function w. Thus

$$\lim_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega g u'_l dx = \int_\Omega g w dx, \quad \forall g \in L^q(\Omega).$$

To achieve our goal, it is enough to estimate w from below. We will show that, for almost any $x_0 \in \Omega$, it holds

$$w(x_0) \geq \theta \frac{f(x_0)^{1/(p-1)}}{(\mu_a + \varepsilon)^{\frac{q}{d-1}}}. \tag{7}$$

Now, we first estimate w on a cube Q centered at the point $x_0 \in \Omega$. We make the following assumption that we will refer to as assumption (A). x_0 is a Lebesgue point for f and

$$|Q|^{-1}\mu(Q) \to \mu_a(x_0) \quad \text{as} \quad \text{Q} \quad \text{shrinks around} \quad \text{x}_0.$$

We also assume that $f(x_0) > 0$ otherwise (7) would be trivial. We have

$$\lim_{l\to+\infty} l^{\frac{q}{d-1}} \int_Q u_l' dx = \int_Q w dx,$$

we use

$$u_l' \geq u_{f,\Sigma_l',Q} \geq u_{f(x_0),\Sigma_l',Q} - |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}| \quad \text{in} \quad Q,$$

where the first inequality comes from the fact that we add Dirichlet boundary condition on Q. The second part of Lemma 3.2 gives

$$\int_Q |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}| dx \leq l^{\frac{q}{1-d}} |Q| r(Q),$$

so passing to the limsup as $l \to +\infty$ yield

$$\limsup_{l\to+\infty} l^{\frac{q}{1-d}} \int_Q |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}| dx \leq |Q| r(Q). \tag{8}$$

It remains to estimate the second term. First of all let us define the number $L(l,Q) = \mathcal{H}^1(\Sigma_l^{'} \cap Q)$ and observe that

$$u_{f(x_0),\Sigma_l',Q} = f(x_0)^{1/(p-1)} u_{1,\Sigma_l',Q}. \tag{9}$$

For simplicity of the notation, we denote $u_{1,\Sigma_l',Q}$ by v_l. By a change of variables, if we assume the side of the cube Q to be λ and we define $v_{l,\lambda} := \lambda^{-q} v_l(\lambda x)$ (thinking, for instance that, both cubes are centered at the origin), we get $v_{l,\lambda} = u_{1,\lambda^{-1}\Sigma_l',I^d}$. It is easy to see that

$$\lambda^{-1}\Sigma_l^{'} \in \mathcal{A}_{L(l,Q)/\lambda}(I^d);$$

moreover, it holds $L(l,Q) \to +\infty$ as $l \to +\infty$, since

$$L(l,Q) \geq \mathcal{H}^1(G_{\varepsilon,l} \cap Q) \approx \varepsilon l |Q|. \tag{10}$$

Using (10) and the fact that $\mu_l = l^{-1}\mathcal{H}^1(\Sigma_l)$, we may estimate the ratio between $L(l,Q)$ and l. It follows from the weak* convergence of μ_l to μ that

$$\limsup_{l\to+\infty} \mu_l(Q) \leq \mu(\overline{Q}).$$

So we have

$$\limsup_{l\to+\infty} \frac{L(l,Q)}{l} \leq \mu(\overline{Q}) + \varepsilon|Q|. \tag{11}$$

Taking into account the definition of θ (for the definition of θ, we refer to (6)) and the change of variables $y = \lambda x$ we have,

$$\begin{aligned}
\liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \int_Q v_l(y) dy &= \liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \lambda^{d+q} \int_{I^d} v_{l,\lambda}(x) dx \\
&= \liminf_{l\to+\infty} \left(\lambda^{-1} L(l,Q)\right)^{\frac{q}{d-1}} \lambda^{d+q+\frac{q}{d-1}} \int_{I^d} v_{l,\lambda}(x) dx \\
&\geq \lambda^{d+q+\frac{q}{d-1}} \theta
\end{aligned}$$

hence the fact that $\lambda^d = |Q|$ allows to get

$$\begin{aligned}\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_Q v_l(y)dy &\geq \liminf_{l\to+\infty}\left(\frac{l}{L(l,Q)}\right)^{\frac{q}{d-1}} \liminf_{l\to+\infty} L(l,Q)^{\frac{q}{d-1}} \int_Q v_l(y)dy \\ &\geq \lambda^{d+q+\frac{q}{d-1}}\theta\left(\frac{1}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}} \\ &= \left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}}|Q|\theta.\end{aligned}$$

This estimate and the equations (8) and (9) give

$$\begin{aligned}\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_Q u_l' dx &\geq \liminf_{l\to+\infty} l^{\frac{q}{d-1}}\left(\int_Q v_l dx - \int_Q |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}|dx\right) \\ &= \liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_Q v_l dx - \limsup_{l\to+\infty} l^{\frac{q}{d-1}} \int_Q |u_{f,\Sigma_l',Q} - u_{f(x_0),\Sigma_l',Q}|dx \\ &\geq |Q|\left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}} \theta f(x_0)^{1/(p-1)} - |Q|r(Q).\end{aligned}$$

From this estimate, we deduce the lower bound of average of w on the cube Q which is

$$|Q|^{-1}\int_Q w dx \geq -r(Q) + \left(\frac{|Q|}{\mu(\overline{Q})+\varepsilon|Q|}\right)^{\frac{q}{d-1}} \theta f(x_0)^{1/(p-1)}.$$

By Remark 1 we know that $r(Q)$ tends to 0 when the cube Q shrinks to x_0, whenever x_0 is a Lebesgue point for f. Now we let the cube Q shrinks toward x_0 with x_0 satisfying assumption (A), then we get

$$w(x_0) \geq \frac{\theta f(x_0)^{1/(p-1)}}{(\mu_a(x_0)+\varepsilon)^{\frac{q}{d-1}}}.$$

It follows that

$$\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_l dx \geq \int_\Omega f w dx \geq \theta\int_\Omega \frac{f^q}{(\mu_a+\varepsilon)^{\frac{q}{d-1}}}dx.$$

Letting ε tend to 0 and using Fatou's Lemma, it holds

$$\liminf_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega f u_l dx \geq \theta\int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}}dx$$

as desired. □

3.2. Γ-lim sup **inequality.** For the proof of the Γ-lim sup inequality we introduce a definition and prove some preliminary results. We start with the definition of tiling set.

Definition 3.3. A set $\Sigma \in \mathcal{A}_l(I^d)$ is called tiling if $\Sigma \cap \partial I^d$ coincides with the 2^d vertices of I^d.

If $\Sigma \in \mathcal{A}_l(I^d)$ is tiling set and Σ_k is the homogenization of order k of Σ into I^d, then Σ_k remains connected and

$$\mathcal{H}^1(\Sigma_k) = k^{d-1}\mathcal{H}^1(\Sigma).$$

The following result study the asymptotic behavior of the rescaled state function associated to a homogenized set. It is a first step toward the construction of the recovering sequence.

Lemma 3.4. *Given $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ a tiling set, a domain $\Omega \subset \mathbb{R}^d$ and $f \in L^q(\Omega)$, we consider a sequence of sets*

$$\Sigma^k = \bigcup_{y \in k^{-1}\mathbb{Z}^d} (y + k^{-1}\Sigma_0 \cup \partial I^d) \cap \overline{\Omega}$$

and consider a sequence of functions $(u_k)_k$ given by

$$u_k = k^q u_{f,\Sigma^k,\Omega},$$

then $u_k \rightharpoonup c(\Sigma_0) f^{1/(p-1)}$ in $L^p(\Omega)$ as $k \to +\infty$, where $c(\Sigma_0)$ is a constant given by $\int_\Omega u_{1,\Sigma_0,I^d} dx$.

Proof. Let us set $\varepsilon_0 = cap_p(\Sigma_0) > 0$, then by part 3 of Lemma 3.1 the sequence $(u_k)_k$ is bounded in $L^p(\Omega)$. So up to a subsequence it converges weakly in $L^p(\Omega)$ to some function. Let us consider the subsequence (denoted by the same indices) $(u_k)_k$ and its weak limit $w_{f,\Sigma_0,\Omega}$. It is obvious that the pointwise value of this limit function depends only on the local behavior of f. In fact, we may produce small cubes around each point $x \in \Omega$ which do not affect each other and if $f = \sum_j f_j 1_{A_j}$ is piecewise constant (the pieces A_j being disjoint open sets, for instance), then for k large enough the value of u_k at $x \in A_j$ depends only of f_j (u_k vanishes on $k^{-1}\partial I^d$). From the rescaling property of the p-Laplacian operator Δ_p, if f is a piecewise constant function, it holds $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} w_{1,\Sigma_0,\Omega}$. It is clear that in the case $f = 1$, since we are simply homogenizing the function u_{1,Σ_0,I^d}, the limit of the whole sequence $(u_k)_k$ exists and does not depend on the global geometry of Ω, but it is a constant and it is the same constant if we have I^d instead of Ω. An easy computation shows that the constant is $c(\Sigma_0)$. It remains to extend the equality for non piecewise constant functions belonging to $L^q(\Omega)$. Let $f \in L^q(\Omega)$ be a generic function and $(f_n)_n$ a sequence of piecewise constant functions approaching f in $L^q(\Omega)$. Up to a subsequence it holds $k^q u_{f,\Sigma_k,\Omega} \rightharpoonup w_{f,\Sigma_0,\Omega}$ and $k^q u_{f_n,\Sigma^k,\Omega} \rightharpoonup f_n^{1/(p-1)} c(\Sigma_0)$ as $k \to +\infty$. By the first part of Lemma 3.2 it also holds

$$||k^q u_{f,\Sigma^k,\Omega} - k^q u_{f_n,\Sigma^k,\Omega}||_{L^1(\Omega)} \leq C||f - f_n||_{L^q(\Omega)}^{1/(p-1)}.$$

Taking into account the lower semicontinuity of the $L^1(\Omega)$-norm with respect to the $L^p(\Omega)$-weak topology, we get, passing to the limit as $k \to +\infty$,

$$||w_{f,\Sigma_0,\Omega} - f_n^{1/(p-1)} c(\Sigma_0)||_{L^1(\Omega)} \leq C||f - f_n||_{L^q(\Omega)}^{1/(p-1)}.$$

We now pass to the limit as $n \to +\infty$ and using Fatou's Lemma (up to a subsequence f_n converges pointwise a.e. to f), we get $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)} c(\Sigma_0)$ and the proof is completed. □

This result remain true even if Σ_0 is not tiling. In fact we have never used the fact that Σ_0 is tiling in the proof. We keep it for the up coming construction. One problem in the previous Lemma is that, we have used the whole boundary of the unit cube which is not a one dimensional set (if $d \geq 3$) and consequently the set Σ^k is not a one dimensional set. In the following Lemma, we prove that $u_{f,\Sigma^k,\Omega}$ may be approximated by $u_{f,\Sigma_l^k,\Omega}$ where Σ_l^k is a one dimensional closed and connected set. This result is the second and key step for the construction of the recovering sequence.

Lemma 3.5. *Let $\Sigma \in \mathcal{A}_l(I^d)$ be a tiling set such that the corresponding rescaled state functions $l^{\frac{q}{1-d}} u_{f,\Sigma,I^d}$ are uniformly L^p bounded, then there exists $T_l \in \mathcal{A}_l(I^d)$*

such that $\mathcal{H}^1(T_l) \ll l$ *and if we denote by* $u_l = u_{f,\Sigma\cup T_l, I^d}$ *and* v_l *the solution of the equation*

$$\begin{cases} -\Delta_p u = f \;\; in \;\; I^d \setminus \Sigma \cup T_l \\ \quad u = 0 \;\; in \;\; \Sigma \cup T_l, \end{cases}$$

then $v_l \leq u_l + c_l l^{\frac{q}{1-d}}$ *on* I^d *where* c_l *is a constant depending on* l *and tends to zero as* l *goes to infinity.*

Proof. Let $\Sigma \in \mathcal{A}_l(I^d)$ be a tiling set such that the sequence $(\tilde{u}_l)_l = (l^{\frac{q}{d-1}} u_{f,\Sigma,I^d})_l$ is L^p bounded and denote by u_l the solution of the equation

$$\begin{cases} -\Delta_p u = f \;\; \text{in} \;\; I^d \setminus \Sigma \\ \quad u = 0 \;\; \text{on} \;\; \Sigma \cup \partial I^d, \end{cases}$$

and by v_l^k the solution of the equation

$$\begin{cases} -\Delta_p u = f \;\; \text{in} \;\; I^d \setminus \Sigma \cup \Sigma_k \\ \quad u = 0 \;\; \text{on} \;\; \Sigma \cup \Sigma_k, \end{cases}$$

where Σ_k is grid of length k contained in the boundary of I^d and converges to it in Hausdorff distance. Since Σ is tiling, we may choose Σ_k such that $\Sigma \cup \Sigma_k$ is connected for all k. For l fixed, $(\Sigma \cup \Sigma_k)_k$ is a sequence of connected sets which converges to the connected set $\Sigma \cup \partial I^d$ then by generalized Šverak continuity result (see [4]) the sequence $(v_l^k)_k$ converges strongly to u_l in $W^{1,p}(I^q)$ as $k \to +\infty$. As consequence $(l^{\frac{q}{d-1}} v_l^k)_l$ (as well as $l^{\frac{q}{d-1}}(v_l^k - u_l)$) is L^p bounded, more precisely there exists a constant c_k such that

$$\|l^{\frac{q}{d-1}}(v_l^k - u_l)\|_{L^p(I^d)} \leq c_k. \tag{12}$$

Moreover c_k may be as small as we want for k large enough. Now let k depend on l say $k = k(l)$ and consider the set $\Sigma_l = \Sigma \cup \Sigma_{k(l)}$. We may choose $k(l)$ such that $k(l) \ll l$ and $k(l) \to +\infty$ as $l \to +\infty$. This makes the length of Σ_l to be asymptotically equivalent to l. $(\Sigma_l)_l$ is a sequence of connected sets converging to the connected set $\overline{I}^d$ the closure of the unit cube then the associated sequence of solutions converges strongly to zero in $W^{1,p}(I^d)$ and $(l^{\frac{q}{d-1}} v_l^{k(l)})_l$ are L^p bounded. Moreover $l^{\frac{q}{d-1}} v_l^{k(l)}$ satisfies the inequality (12). From the maximum principle we get $v_l - u_l \geq 0$ (setting $v_l = v_l^{k(l)}$) and from the above boundedness and Hölder inequality it holds

$$0 \leq \int_{I^d} (v_l - u_l)dx \leq c_l l^{\frac{q}{1-d}}.$$

We obtain easily the existence of some constant c_l (it may be different from the above constant c_l but it goes to zero as $l \to +\infty$) such that the inequality

$$v_l - u_l \leq c_l l^{\frac{q}{1-d}}$$

holds in I^d and the proof is completed. □

Due to the terminology suggested in [5], the sets satisfying the hypothesis of Lemma 3.5 that is $\Sigma \cup T_l$ will be called almost boundary-covering sets. We have proved the Lemma for the unit cube but the result remains true for a cube of any side as well as an open domain with Lipschitz boundary. Now, we build an almost boundary-covering set that will be used for the construction of the recovering sequence for the Γ-lim sup inequality.

Lemma 3.6. *For any $\varepsilon > 0$, there exists $l_0 > 0$ such that for all $l > l_0$ we find a set $\Sigma \in \mathcal{A}_l(I^d)$ which is almost boundary-covering, with*

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx < (1+\varepsilon)\theta$$

and consequently if we denote by $u_{1,\Sigma}$ the solution of the state equation which vanishes only on Σ and not on the whole boundary of I^d we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma} dx < (1+\varepsilon)\theta + c_l.$$

Proof. Given a small positive number δ $(0 < \delta \ll 1)$, by definition of θ, we may find a set $\Sigma_1 \in \mathcal{A}_{l_1}(I^d)$ such that

$$l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx < (1+\delta)\theta$$

and moreover the number l_1 may be chosen as large as we want. Now, we want enlarge the set Σ_1 to get a set Σ_2 which is almost boundary-covering. Let $\gamma = \cup_{j=1}^{2^d} S_j$ where S_j is the shortest segment joining Σ_1 to the j^{th} vertice of the unit cube I^d. We set $\Sigma_2 = \Sigma_1 \cup T_{l_1} \cup \gamma$ where T_{l_1} is the grid T_l in Lemma 3.5 with l replaced by l_1. Up to addition of one segment, we may assume Σ_2 connected. The length $l_2 = \mathcal{H}^1(\Sigma_2)$ does not exceed the number $l_1 + \mathcal{H}^1(T_{l_1}) + (2^d+1)\sqrt{d}$. It is possible to chose l_1 so that

$$\left(\frac{l_1 + \mathcal{H}^1(T_{l_1}) + (2^d+1)\sqrt{d}}{l_1}\right)^{\frac{q}{d-1}} \leq 1+\delta.$$

This implies

$$l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d} dx \leq \left(\frac{l_2}{l_1}\right)^{\frac{q}{d-1}} l_1^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_1,I^d} dx \leq (1+\delta)^2\theta.$$

Now if we are given a large number l, we homogenize the set Σ_2 of order $k = \lfloor \left(\frac{l}{l_2}\right)^{\frac{1}{d-1}} \rfloor$ into I^d and the homogenized set Σ belongs to $\mathcal{A}_{k^{d-1}l_2}(I^d)$ and is still almost boundary-covering. For this set Σ it holds (using the rescaling property of p-Laplacian operator)

$$(k^{d-1}l_2)^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx = l_2^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_2,I^d} dx.$$

Noticing that $l^{\frac{q}{d-1}} \leq \left(\frac{k+1}{k}\right)^q \left(k^{d-1}l_2\right)^{\frac{q}{d-1}}$, we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx \leq \left(\frac{k+1}{k}\right)^q (1+\delta)^2\theta.$$

If $l > l_2\delta^{-1}$, using the fact that $\delta \ll 1$, an easy computation shows that $1 + 1/k < 1+\delta$ so that we get

$$l^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma,I^d} dx \leq (1+\delta)^{2+q}\theta.$$

Finally, it is sufficient if δ is chosen so small that $(1+\delta)^{2+q} < 1+\varepsilon$ and $l_0 = l_2\delta^{-1}$, then the result follows. □

We have all the ingredients to prove the Γ-lim sup inequality. We will start from a particular class of measures. Let us call piecewise constant probability measures those probability measures $\mu \in \mathcal{P}(\overline{\Omega})$ which are of the form

$$\mu = \rho dx, \quad \text{with}, \quad \rho \in L^1(\Omega), \quad \int_\Omega \rho dx = 1, \quad \rho > 0,$$

for a piecewise constant function $\rho = \sum_{j=1}^m \rho_j I_{\Omega_j}$, the pieces Ω_j being disjoint Lipschitz open subsets with the possible exception of $\Omega_0 = \Omega \setminus \cup_{j=1}^m \Omega_j$.

Proposition 2. *Under the same hypotheses of Theorem 2.2, we have*

$$F^+(\mu) \leq F(\mu), \quad \textit{where} \quad F^+ = \Gamma - \limsup_{l \to +\infty} F_l,$$

for any piecewise constant measure $\mu \in \mathcal{P}(\overline{\Omega})$*. This means that for any such a measure* μ *and* $\varepsilon > 0$*, there exists a family of sets* $(\Sigma_l)_l \subset \mathcal{A}_l(\Omega)$ *such that the measure* μ_{Σ_l} *weak* converges to the measure* μ *and moreover*

$$\limsup_{l \to +\infty} l^{\frac{q}{d-1}} \int_\Omega f u_{f,\Sigma_l,\Omega} dx \leq (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^{\frac{q}{d-1}}} dx.$$

Proof. Apply Lemma 3.6 and take an almost boundary-covering set $\Sigma_0 \in \mathcal{A}_{l_0}(I^d)$ such that

$$l_0^{\frac{q}{d-1}} \int_{I^d} u_{1,\Sigma_0,I^d} dx < (1+\varepsilon)\theta.$$

We define the set Σ_l^j by homogenizing into Ω_j the set Σ_0 of order $k(l,j)$ that is

$$\Sigma_l^j = \overline{\Omega_j} \cap k(l,j)^{-1}(\mathbb{Z}^d + \Sigma_0).$$

Since Σ_0 is tiling , for $k(l,j)$ large enough Σ_l^j remains connected and

$$\mathcal{H}^1(\Sigma_l^j) = |\Omega_j| K(l,j)^{d-1} \mathcal{H}^1(\Sigma_0) \leq |\Omega_j| K(l,j)^{d-1} l_0.$$

Let $\Sigma_{l_1} \in \mathcal{A}_{l_1}(\Omega)$ be a set contained in the internal boundary of the union of Ω_j and converges to it in the Hausdorff topology as $l_1 \to +\infty$ (Σ_{l_1} may be obtained by homogenizing some kind of grid contained in ∂I^d of some order into $\cup_{j=0}^m \partial\Omega_j$). Due to the connectedness of Σ_{l_1}, the corresponding solution converges to the solution associated to the internal boundary of $\cup_{j=0}^m \Omega_j$ as well. Then we choose $\Sigma_l = \cup_{j=0}^m \Sigma_l^j \cup \Sigma_{l_1}$. We may assume Σ_l connected otherwise we add some segments to connect all the pieces. The family of sets Σ_l is admissible (i.e. $\Sigma_l \in \mathcal{A}_l(\Omega)$ and $\mu_{\Sigma_l} \rightharpoonup \mu$) if we have, as $l \to +\infty$,

$$\sum_{j=0}^m |\Omega_j| k(l,j)^{d-1} l_0 + l_1 \leq l \quad \text{and is asymptotic to} \quad l;$$

$$\frac{k(l,j)^{d-1} l_0}{l} \to \rho_j \quad \text{for} \quad j = 0, \cdots, m.$$

It is easy to see that all theses conditions are satisfied if we set

$$k(l,j) = \left\lfloor \left(\frac{l - l_1}{l_0} \rho_j \right)^{\frac{1}{d-1}} \right\rfloor.$$

Let us introduce the following sets

$$\Gamma_l^j = \overline{\Omega}_j \cap k(l,j)^{-1}(\mathbb{Z}^d + \partial I^d), \quad \Gamma_l = \bigcup_j \Gamma_l^j.$$

Thanks to Lemma 3.5 we have

$$\int_{\Omega_j} fk(l,j)^q u_{f,\Sigma_l^j,\Omega_j} dx \le \int_{\Omega_j} fk(l,j)^q u_{f,\Sigma_l\cup\Gamma_l^j,\Omega_j} dx + c_l l_0^{\frac{q}{1-d}}. \tag{13}$$

In fact, we consider subcubes $Q_{k(l,j)}$ which are obtained by the partition of Ω_j made by Γ_l^j, then in each subcube $Q_{k(l,j)}$, Lemma 3.5 gives

$$u_{f,\Sigma_l^j} \le u_{f,\Sigma_l^j,Q_{k(l,j)}} + c_l(k(l,j)l_0^{\frac{1}{d-1}})^{-q}.$$

By multiplying this inequality by f (notice that $f \ge 0$), Integrating over $Q_{k(l,j)}$ and summing up, we get

$$\int_{\Omega_j} fu_{f,\Sigma_l^j,\Omega_j} dx \le \int_{\Omega_j} fu_{f,\Sigma_l^j} dx \le \int_{\Omega_j} fu_{f,\Sigma_l^j\cup\Gamma_l^j,\Omega_j} dx + c_l(k(l,j)l_0^{\frac{1}{d-1}})^{-q},$$

where the first inequality comes from the maximum principle and the second is obtained by observing that on each cube $Q_{k(l,j)}$ it holds $u_{f,\Sigma_l^j\cup\Gamma_l^j,\Omega_j} = u_{f,\Sigma_l^j,Q_{k(l,j)}}$.

We choose l_1 to be a function of l (for example $l_1 = l^{\frac{d-1}{d}}$) in such a way that l_1 goes to $+\infty$ whenever l goes to $+\infty$. We are interested in the estimate of the value of $F_l(\Sigma_l)$. By Lemma 3.5 and inequality (13) we get

$$\begin{aligned}
&l^{\frac{q}{d-1}} \int_\Omega fu_{f,\Sigma_l,\Omega} dx \\
&= \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}} \int_{\Omega_j} fk(l,j)^q u_{f,\Sigma_l,\Omega} dx \\
&\le \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}} \left(\int_{\Omega_j} fk(l,j)^q u_{f,\Sigma_l,\Omega_j} dx + c_{l_1}\right) \\
&\le \sum_{j=0}^m \left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}} \left(\int_{\Omega_j} fk(l,j)^q u_{f,\Sigma_l^j\cup\Gamma_l^j,\Omega_j} dx + c_{l_1} + c_l l_0^{\frac{q}{1-d}}\right),
\end{aligned}$$

where c_{l_1} goes to zero as l_1 tends to infinity. By applying Lemma 3.4 to each Ω_j we get the following weak convergence in L^p.

$$k(l,j)^q u_{f,\Sigma_l^j\cup\Gamma_l^j,\Omega_j} \rightharpoonup c(\Sigma_0) f^{1/(p-1)} \quad \text{as} \quad l \to +\infty$$

and the term $\left(\frac{l}{k(l,j)^{d-1}}\right)^{\frac{q}{d-1}}$ converges to $\left(\frac{l_0}{\rho_j}\right)^{\frac{q}{d-1}}$ as $l \to +\infty$ for $j = 0, \cdots, m$. The choice of the set Σ_0 implies that $l_0^{\frac{q}{d-1}} c(\Sigma_0) < (1+\varepsilon)\theta$, so we have

$$\limsup_{l\to+\infty} l^{\frac{q}{d-1}} \int_{\Omega_j} fu_{f,\Sigma_l,\Omega} dx \le (1+\varepsilon)\theta\rho_j^{-\frac{q}{d-1}} \int_{\Omega_j} f^q dx, \quad \text{for } j = 0, \cdots, m$$

and summing up over j, it holds

$$\limsup_{l\to+\infty} l^{\frac{q}{d-1}} \int_\Omega fu_{f,\Sigma_l,\Omega} dx \le (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^{\frac{q}{d-1}}} dx.$$

□

We have to extend the result to non piecewise constant measures. By the general theory of Γ-convergence, we know that it is enough to prove the Γ-lim sup inequality on a class which is dense in energy. Hence, due to the lower semicontinuity of the functional F, it is sufficient to prove the following

Proposition 3. *For any measure $\mu \in \mathcal{P}(\overline{\Omega})$ there exists a sequence $(\mu_n)_n$ of piecewise constant measures such that μ_n weak* converges to μ and*

$$\limsup_n F(\mu_n) \leq F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d-1}}} dx.$$

Proof. First observe that the inequality is trivial whenever $F(\mu) = +\infty$. Assume now that $F(\mu) < +\infty$ and we shall start proving the inequality for measures which are absolutely continuous with respect to the Lebesgue measure and have positive densities bounded away from zero. Given a measure $\mu = \rho dx$, with $\rho \geq c > 0$, it is possible to find a sequence of measures $\mu_n = \rho_n dx$ such that $\rho_n \to \rho$ strongly in L^1 and ρ_n are piecewise constant with $\rho_n \geq c$. The pointwise *a.e* convergence of ρ_n to ρ may be assumed and the inequality $F(\mu) \geq \limsup_n F(\mu_n)$ follows easily (we have even an equality). So we have extended the result to any absolutely continuous measure with density bounded below away from zero. To get the result for any measure $\mu \in \mathcal{P}(\overline{\Omega})$, it is sufficient to prove that any measure μ may be approximated weakly* by absolutely continuous measure μ_n with densities bounded below away from zero and $\limsup_n F(\mu_n) \leq F(\mu)$. Let us take $\mu = \rho dx + \mu^s$, where μ^s is the singular part of the measure μ with respect to the Lebesgue measure and ρ the density of the absolutely continuous part. We construct a sequence of absolutely continuous measure μ_n by setting $\mu_n = ((1-1/n)\rho + a_n + \phi_n)dx$, where $a_n = n^{-1} \int_\Omega \rho dx$ and $\phi_n dx \rightharpoonup \mu^s$ with $\int_\Omega \phi_n dx = \int_{\overline{\Omega}} d\mu^s$. The fact that $F(\mu) < +\infty$ implies that ρ cannot vanish, hence $a_n > 0$ and $\rho_n = (1-1/n)\rho + a_n + \phi_n$ is bounded below by the positive constant a_n. We have as well that μ_n weak* converges to μ and

$$\begin{aligned} F(\mu_n) &= \theta \int_\Omega \frac{f^q}{((1-1/n)\rho + a_n + \phi_n)^{\frac{q}{d-1}}} \leq \theta \int_\Omega \frac{f^q}{((1-1/n)\rho)^{\frac{q}{d-1}}} dx \\ &= (1 - \frac{1}{n})^{-q/(d-1)} . F(\mu) \end{aligned}$$

Passing to the lim sup on the inequality above, we get the desired result. □

4. **Some estimate of θ.** In this section, we will prove some estimate on the constant θ and in particular we will show that θ is neither 0 nor $+\infty$ so that our limit functional is not trivial. We have

Proposition 4.

$$\theta < +\infty.$$

Proof. Let $\Sigma_l \in \mathcal{A}_l(I^d)$ be a tiling set. For any positive integer n, let us denote by Σ_l^n the homogenization of the set Σ_l of order n into I^d. Clearly, Σ_l^n is connected and $\mathcal{H}^1(\Sigma_l^n) \leq n^{d-1} l$. Using the rescaling property of the p-Laplacian operator, it follows that

$$\theta \leq \liminf_n (n^{d-1} l)^{\frac{q}{d-1}} F_p(\Sigma_l^n, 1, I^d) = l^{\frac{q}{d-1}} F_p(\Sigma_l, 1, I^d) < +\infty$$

which concludes the proof. □

Proposition 5.

$$\theta \geq \frac{(d-1)q^{-q}}{(q+d-1) w_{d-1}^{\frac{q}{d-1}}},$$

where w_r is the volume of unit ball in $\mathbb{R}^r$.

Proof. First, we prove that

$$F_p(\Sigma_l, 1, I^d) \geq q^{-q} D_q(\Sigma_l \cup \partial I^d),$$

where $D_r(\Sigma) = \int_{I^d} h_\Sigma(x)^r dx$ and $h_\Sigma(x) = d(x, \Sigma)$ is the distance from x to the set Σ. For every real number A and for every real number $r > 1$, we have

$$\begin{aligned} F_p(\Sigma_l, 1, I^d) &= q \max \left\{ \int_{I^d} (v - \frac{1}{p} |\nabla v|^p) dx : v \in W_0^{1,p}(I^d \setminus \Sigma_l) \right\} \\ &\geq q \int_{I^d} (A h_{\Sigma_l \cup \partial I^d}(x)^r - \frac{1}{p} |\nabla (A h_{\Sigma_l \cup \partial I^d}(x)^r)|^p) dx. \end{aligned}$$

It is well known that the distance function is 1-Lipschitz and satisfies $|\nabla h_{\Sigma_l \cup \partial I^d}| = 1$ (and consequently $|\nabla (h_{\Sigma_l \cup \partial I^d})^r| = r(h_{\Sigma_l \cup \partial I^d})^{r-1}$). Choosing $r = q$ the conjugate exponent of p, we get

$$F_p(\Sigma_l, 1, I^d) \geq q(A - A^q \left(\frac{q^p}{p}\right)) \int_{I^d} h_{\Sigma_l \cup \partial I^d}(x)^q dx.$$

The result follows by optimizing on A (the optimal choice is $A = q^{-q}$). In [10] it has been proved that for any set $\Sigma_l \in \mathcal{A}_l(I^d)$ it holds

$$\liminf_l l^{\frac{q}{d-1}} \int_{I^d} h_{\Sigma_l}(x)^q dx \geq \frac{d-1}{(q+d-1) w_{d-1}^{\frac{q}{d-1}}}.$$

Here the same proof may be adapted by doing some modification and getting the same result even if $\Sigma_l \cup \partial I^d$ is not a one dimensional set i.e.

$$\liminf_l l^{\frac{q}{d-1}} \int_{I^d} h_{\Sigma_l \cup \partial I^d}(x)^q dx \geq \frac{d-1}{(q+d-1) w_{d-1}^{\frac{q}{d-1}}},$$

and the desired result holds. □

5. **Asymptotics of p-compliance-location problem.** In this section, we consider the case where the control variable is a discrete set of finite elements. Let $p > d$ be fixed and $q = p/(p-1)$ the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}^d$ and n a given positive integer, we define

$$\mathcal{A}_n(\Omega) := \{\Sigma \subset \overline{\Omega} : 0 < \mathcal{H}^0(\Sigma) \leq n\}.$$

For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity(since $p > d$, every point has positive p-capacity), we denote as before by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{in } \Sigma \cup \partial\Omega, \end{cases}$$

that is $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega \setminus \Sigma). \tag{14}$$

For $f \geq 0$, we define the p-compliance functional as before and the existence of the minimal p-compliance configuration is a consequence of the continuity of Sobolev functions when $p > d$.

Theorem 5.1. *For any integer $n > 0$, Ω bounded open subset of $\mathbb{R}^d$, $d \geq 2$ and f a nonnegative function belonging to $L^q(\Omega)$, the problem*

$$\min\{C_p(\Sigma) : \Sigma \in \mathcal{A}_n(\Omega)\} \tag{15}$$

admits at least one solution.

As before, we are interested in the asymptotic behavior of the optimal set Σ_n of the problem (15) as $n \to +\infty$. Let us associate to every $\Sigma \in \mathcal{A}_n(\Omega)$ a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = n^{-1}\delta_\Sigma$$

and define a functional $G_n : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$G_n(\mu) = \begin{cases} n^{\frac{q}{d}} C_p(\Sigma) & \text{if} \quad \mu = \mu_\Sigma, \Sigma \in \mathcal{A}_n(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{16}$$

The scaling factor $n^{\frac{q}{d}}$ is needed in order to avoid the sequence of functionals G_n to converges to the trivial limit functional which vanishes everywhere. Again the main result deals with the behavior as $n \to +\infty$ of the functional G_n, and is stated in terms of Γ-convergence.

Theorem 5.2. *The functional G_n defined in (16) Γ-converges, with respect to the weak* topology on the class $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$, to a functional G defined on $\mathcal{P}(\overline{\Omega})$ by*

$$G(\mu) = \theta_1 \int_\Omega \frac{f^q}{\mu_a^{\frac{q}{d}}} dx, \tag{17}$$

where μ_a stands for the density of the absolutely continuous part of μ with respect to the Lebesgue measure, and θ_1 is a positive constant depending only on d and p and is defined by

$$\theta_1 = \inf\{\liminf_{n\to+\infty} n^{\frac{q}{d}} F_p(\Sigma_n, 1, I^d) : \Sigma_n \in \mathcal{A}_n(I^d)\} \tag{18}$$

$I^d = (0,1)^d$ being the unit cube in $\mathbb{R}^d$.

We deduce the following consequences of Theorem 5.2:

- if Σ_n is an optimal set of the minimization problem (15), then up to a subsequence the associated sequence of measures μ_{Σ_n} converges weak* to the measure μ as $n \to +\infty$, where μ is a minimizer of G;
- since G has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, the whole sequence μ_{Σ_n} converges to the unique minimizer μ of G given by $\mu = cf^{\frac{qd}{q+d}}\mathcal{L}^d$ where c is such that μ is a probability measure, that is $c = 1/\left(\int_\Omega f^{\frac{qd}{q+d}} dx\right)$;
- the minimal value of G is equal to $\theta_1 c^{\frac{q+d}{d}}$, and the sequence of values $\inf\{F_p(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}_n(\Omega)\}$ is asymptotically equivalent to $n^{\frac{q}{d}} \inf\{G(\mu) : \mu \in \mathcal{P}(\overline{\Omega})\}$.

This problem is in connection with the location problem, that is, the minimization of the average distance functional $\int_\Omega f(x)d_\Sigma(x)dx$ where $d_\Sigma(x)$ is the distance from x to the set Σ and $\Sigma \in \mathcal{A}_n(\Omega)$ (see [2] for more details) . We will not prove Theorem 5.2 since its proof follows the line of the proof of Theorem 2.2 but we will point out some necessary modifications. Lemma 3.1 is crucial for the proof of the Γ-$\liminf$ inequality. This Lemma remains valid in the case of discrete set provided that the power $d-1$ is replaced by d. In that case it suffices that v vanishes on one point since point has positive p-capacity (remember that $p > d$). Another important element in the proof of the Γ-lim inf inequality is the set $G_{\varepsilon,l}$. Here, we will call it $G_{\varepsilon,n}$ and its construction is obtained by the homogenization of order $\lfloor \left(\frac{\varepsilon n}{2ad}\right)^{1/d} \rfloor$ of the center of the unit cube into the cube $I_a^d = (-a,a)^d$ which contains Ω. For the Γ-lim sup inequality, proofs are essentially the same except the fact that we do not need tiling set and replace l by n. We conclude this section with the estimate of the constant θ_1. To prove the finiteness it suffice to use the set Σ_n which is the homogenization of order n of the center of the unit cube into the unit cube. For the lower bound, the proof follows that of Proposition 5 and gives

$$\theta_1 \geq \frac{d}{(q+d)w_d^{\frac{q}{d}}}.$$

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