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THE STATIONARY BEHAVIOUR OF FLUID LIMITS OF REVERSIBLE PROCESSES IS CONCENTRATED ON STATIONARY POINTS

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Abstract. Assume that a stochastic process can be approximated, when some scale parameter gets large, by a fluid limit (also called "mean field limit", or "hydrodynamic limit"). A common practice, often called the "fixed point approximation" consists in approximating the stationary behaviour of the stochastic process by the stationary points of the fluid limit. It is known that this may be incorrect in general, as the stationary behaviour of the fluid limit may not be described by its stationary points. We show however that, if the stochastic process is reversible, the fixed point approximation is indeed valid. More precisely, we assume that the stochastic process converges to the fluid limit in distribution (hence in probability) at every fixed point in time. This assumption is very weak and holds for a large family of processes, among which many mean field and other interaction models. We show that the reversibility of the stochastic process implies that any limit point of its stationary distribution is concentrated on stationary points of the fluid limit. If the fluid limit has a unique stationary point, it is an approximation of the stationary distribution of the stochastic process.

1. **Introduction.** This paper is motivated by the use of fluid limits in models of interacting objects or particles, in contexts such as communication and computer system modelling [6], biology [7] or game theory [3]. Typically, one has a stochastic process Y^N, indexed by a size parameter N; under fairly general assumptions, one can show that the stochastic process Y^N converges to a deterministic fluid limit φ [18]. We are interested in the stationary distribution of Y^N, assumed to exist and be unique, but which may be too complicated to be computed explicitly. The "fixed point assumption" is then sometimes invoked [16, 5, 23, 15]: it consists in approximating the stationary distribution of Y^N by a stationary point of the deterministic fluid limit φ. In the frequent case where the fluid limit φ is described by an Ordinary Differential Equation (ODE), say of the form $\dot{y} = F(y)$, the stationary points are obtained by solving $F(y) = 0$. If Y^N is an empirical measure, convergence to a deterministic limit implies propagation of chaos, i.e. the states of different objects are asymptotically independent, and the distribution of any particular object at

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any time is obtained from the fluid limit. Under the fixed point assumption, the stationary distribution of one object is approximated by a stationary point of the fluid limit.

A critique of the fixed point approximation method is formulated in [2], which observes that one may only say, in general, that the stationary distribution of Y^N converges to a stationary distribution of the fluid limit. For a deterministic fluid limit, a stationary distribution is supported by the Birkhoff center of the fluid limit, which may be larger than the set of stationary points. An example is given in [2] where the fluid limit has a unique stationary point, but the stationary distribution of Y^N does not converge to the Dirac mass at this stationary point; in contrast, it converges to a distribution supported by a limit cycle of the ODE. If the fluid limit has a unique limit point, say y^*, to which all trajectories converge, then this unique limit point is also the unique stationary point and the stationary distribution of Y^N does converge to the Dirac mass at y^* (i.e. the fixed point approximation is then valid). However, as illustrated in [2], this assumption may be difficult to verify, as it often does not hold, and when it does, it may be difficult to establish. For example, in [10] it is shown that the fixed point assumption does not hold for some parameter settings of a wireless system analyzed in [5], due to limit cycles in the fluid limit.

In this paper we show that there is a class of systems for which such complications may not arise, namely the class of reversible stochastic processes. Reversibility is classically defined as a property of time reversibility in stationary regime [14]. There is a large class of processes that are known to be reversible, for example product-form queuing networks with reversible routing, or stochastic processes in [15], which describes the occupancy of inter-city telecommunication links; in Section 5 we give an example motivated by crowd dynamics. In such cases, we show that the fluid limit must have stationary points, and any limit point of the stationary distribution of Y^N must be supported by the set of stationary points. Thus, for reversible processes that have a fluid limit, the fixed point approximation is justified.

2. Assumptions and notation.

2.1. A collection of reversible random processes.

Let E be a Polish space and let d be a measure that metrizes E. Let $\mathcal{P}(E)$ be the set of probability measures on E, endowed with the topology of weak convergence. Let $\mathcal{C}_b(E)$ be the set of bounded continuous functions from E to $\mathbb{R}$, and similarly $\mathcal{C}_b(E \times E)$ is the set of bounded continuous functions from $E \times E$ to $\mathbb{R}$.

We are given a collection of probability spaces $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ indexed by $N = 1, 2, 3, ...$ and for every N we have a process Y^N defined on $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$. Time is continuous. Let $D_E[0, \infty)$ be the set of cádlág functions $[0, \infty) \to E$; Y^N is then a stochastic process with sample paths in $D_E[0, \infty)$.

We denote by $Y^N(t)$ the random value of Y^N at time $t \geq 0$. Let $E^N \subset E$ a set on which $Y^N(0)$ is concentrated, so that $\mathbb{P}^N(Y^N(0) \in E^N) = 1$.

We assume that, for every N, the process Y^N is Feller, in the sense that for every $t \geq 0$ and $h \in \mathcal{C}_b(E)$, $\mathbb{E}^N \left[h(Y^N(t)) \middle| Y^N(0) = y_0 \right]$ is a continuous function of $y_0 \in E$. Examples of such processes are continuous time Markov chains as in [17], or linear interpolations of discrete time Markov chains as in [4], or the projections of a Markov process as in [13].

Definition 2.1. A probability $\Pi^N \in \mathcal{P}(E)$ is *invariant* for Y^N if $\Pi^N(E^N) = 1$ and for every $h \in \mathcal{C}_b(E)$ and every $t \geq 0$:

$$\int_E \mathbb{E}^N \left[h\left(Y^N(t)\right) \middle| Y^N(0) = y\right] \Pi^N(dy) = \int_E h(y)\Pi^N(dy)$$

We are interested in reversible processes, i.e. processes that keep the same stationary law under time reversal. A weak form of such a property is defined as follows

Definition 2.2. Assume Π^N is a probability on E such that $\Pi^N(E^N) = 1$, for some N. We say that Y^N is *reversible* under Π^N if for every time $t \geq 0$ and any $h \in \mathcal{C}_b(E \times E)$:

$$\begin{aligned}&\textstyle\int_E \mathbb{E}^N \left[h\left(y, Y^N(t)\right) \middle| Y^N(0) = y\right] \Pi^N(dy)\\ &\quad = \textstyle\int_E \mathbb{E}^N \left[h\left(Y^N(t), y\right) \middle| Y^N(0) = y\right] \Pi^N(dy)\end{aligned}$$

Note that, necessarily, Π^N is an invariant probability for Y^N. If Y^N is an ergodic Markov process with enumerable state space, then Definition 2.2 coincides with the classical definition of reversibility by Kelly in [14]. Similarly, if Y^N is a projection of a reversible Markov process X^N, as in [11], then Y^N is reversible under the projection of the stationary probability of X^N; note that in such a case, Y^N is not Markov.

2.2. **A limiting, continuous semi-flow.** Further, let φ be a deterministic process, i.e. a measurable mapping

$$\begin{array}{rccc}\varphi: & [0,\infty) \times E & \to & E\\ & t, y_0 & \mapsto & \varphi_t(y_0)\end{array}$$

We assume that φ_t is a semi-flow, i.e.

1. $\varphi_0(y) = y$,
2. $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s \geq 0$ and $t \geq 0$,

and we say that φ is "space continuous" if for every $t \geq 0$, $\varphi_t(y)$ is continuous in y.

Definition 2.3. We say that $y \in E$ is a *stationary point* of φ if $\varphi_t(y) = y$ for all $t \geq 0$

In cases where E is a subset of $\mathbb{R}^d$ for some integer d, the semi-flow φ may be an autonomous ODE, of the form $\dot{y} = F(y)$; here the stationary points are the solutions of $F(y) = 0$.

Definition 2.4. We say that the semi-flow φ is *reversible* under the probability $\Pi \in \mathcal{P}(E)$ if for every time $t \geq 0$ and any $h \in \mathcal{C}_b(E \times E)$:

$$\int_E h(y, \varphi_t(y))\Pi(dy) = \int_E h(\varphi_t(y), y)\Pi(dy) \tag{1}$$

As we show in the next section, reversible semi-flows must concentrate on stationary points.

2.3. **Convergence hypothesis.** We assume that, for every fixed t the processes Y^N converge in distribution to some space continuous deterministic process φ as $N \to \infty$ for every collection of converging initial conditions. More precisely:

Hypothesis 1. For every y_0 in E, every sequence $(y_0^N)_{N=1,2,...}$ such that $y_0^N \in E^N$ and $\lim_{N\to\infty} y_0^N = y_0$, and every $t \geq 0$, the conditional law of $Y^N(t)$ given $Y^N(0) = y_0^N$ converges in distribution to the Dirac mass at $\varphi_t(y_0)$. That is

$$\lim_{N\to\infty} \mathbb{E}^N \left[h(Y^N(t)) \middle| Y^N(0) = y_0^N \right] = h \circ \varphi_t(y_0) \tag{2}$$

for all $h \in \mathcal{C}_b(E)$ and any fixed $t \geq 0$. In the above, φ is a space continuous semi-flow.

Hypothesis 1 is commonly true in the context of fluid or mean field limits. The stronger convergence results results in [19, 17, 25, 6] imply that Hypothesis 1 is satisfied ; we give a detailed example illustrating this in Section 5. Similarly, [2] gives very general conditions (called H1 to H5) that ensure convergence of a stochastic process to its mean field limit; under these conditions, Hypothesis 1 is automatically satisfied (the deterministic process φ is then an ODE). Note that the results in these references are stronger than what we require in Hypothesis 1; for example in [17] there is almost sure, uniform convergence for all $t \in [0, T]$, for any $T \geq 0$; in [13] the convergence is on the set of trajectories.

Under Hypothesis 1, φ is called the *hydrodynamic limit* or simply *fluid limit* of Y^N.

3. Reversible semi-flows concentrate on stationary points.

Theorem 3.1. *Let φ be a space continuous semi-flow, reversible under Π. Let S be the set of stationary points of φ. Then Π is concentrated on S, i.e. $\Pi(S) = 1$.*

Proof. Denote with S^c the complement of the set of stationary points. Take some fixed but arbitrary $y_0 \in S^c$. By definition of S, there exists some $\tau > 0$ such that

$$\varphi_{2\tau}(y_0) \neq y_0 \tag{3}$$

Define $\varphi_\tau(y_0) = y_1$, $\varphi_\tau(y_1) = y_2$, so that $y_2 \neq y_0$.

For $y \in E$ and $\varepsilon > 0$ we denote with $B(y, \varepsilon)$ the open ball $= \{x \in E, d(x, y) < \varepsilon\}$. Let $\varepsilon = d(y_0, y_2) > 0$ and let $B_2 = B(y_2, \varepsilon/2)$. Since the semi-flow is space continuous, there is some $\alpha_1 > 0$ such that $B_1 = B(y_1, \alpha_1)$ and $\varphi_\tau(B_1) \subset B_2$. Also let $B_1' = B(y_1, \alpha_1/2)$. By the same argument, there exists some $\alpha_0 > 0$ such that $\alpha_0 < \varepsilon/2$, $B_0 = B(y_0, \alpha_0)$ and $\varphi_\tau(B_0) \subset B_1'$. We have thus:

$$\begin{aligned} \varphi_\tau(B_0) &\subset B_1' \subset B_1 \\ \varphi_\tau(B_1) &\subset B_2 \\ B_0 \cap B_2 &= \emptyset \end{aligned}$$

Let ξ be some continuous function $[0, +\infty) \to [0, 1]$ such that $\xi(u) = 1$ whenever $0 \leq u \leq 1/2$ and $\xi(u) = 0$ whenever $u \geq 1$ (for example take a linear interpolation). Now take

$$h(y, z) \stackrel{\text{def}}{=} \xi\left(\frac{d(y_0, y)}{\alpha_0}\right) \xi\left(\frac{d(y_1, y)}{\alpha_1}\right) \tag{4}$$

so that $h \in \mathcal{C}_b(E \times E)$ and

$$\begin{aligned} h(y, z) &= 0 \qquad \text{whenever } y \notin B_0 \text{ or } z \notin B_1 \\ h(y, z) &= 1 \qquad \text{whenever } d(y_0, y) < \alpha_0/2 \text{ and } z \in B_1' \end{aligned}$$

It follows that $h(\varphi_\tau(z), z) = 0$ for every $z \in E$ and

$$\int_E h(y, \varphi_\tau(y))\Pi(dy) \geq \Pi(B(y_0, \alpha_0/2)) \tag{5}$$

Apply Definition 2.4, it comes $\Pi\left(B(y_0, \alpha_0/2)\right) = 0$; thus, for any non stationary point y_0 there is some $\alpha > 0$ such that

$$\Pi\left(B(y_0, \alpha)\right) = 0 \tag{6}$$

Thus $y_0 \in S^c$ is in an open set of mass 0 for Π. Let $\mathrm{supp}(\Pi)$ be the support of Π; its complement $(\mathrm{supp}(\Pi))^c$ is the union of all open sets of mass 0. It follows that $y_0 \in (\mathrm{supp}(\Pi))^c$.

This holds for any $y_0 \in S^c$, i.e. $S^c \subset (\mathrm{supp}(\Pi))^c$ or, in other words, $\mathrm{supp}(\Pi) \subset S$. Now, in a Polish space, $\Pi(\mathrm{supp}(\Pi)) = 1$ thus $\Pi(S) = 1$ as well. □

Note that it follows that a semi-flow that does not have any stationary point cannot be reversible under any probability.

4. Stationary behaviour of fluid limits of reversible processes.

Theorem 4.1. *Assume for every N the process Y^N is reversible under some probability Π^N. Assume the convergence Hypothesis 1 holds and that $\Pi \in \mathcal{P}(E)$ is an accumulation point for the weak topology [1] of the sequence Π^N. Then the fluid limit is reversible under Π. In particular, it follows from Theorem 3.1 that Π is concentrated on the set of stationary points S of the fluid limit φ.*

Proof. All we need to show is that Π verifies Definition 2.4. Let N_k be a subsequence such that $\lim_{k\to\infty} \Pi^{N_k} = \Pi$ for the weak topology on $\mathcal{P}(E)$. By Skorohod's representation theorem for Polish spaces [12, Thm 1.8], there exists a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which some random variables X^k for $k \in \mathbb{N}$ and X are defined such that

$$\begin{cases} \text{law of } X^k = \Pi^{N_k} \\ \text{law of } X = \Pi \\ X^k \to X \ \mathbb{P} - \text{a.s.} \end{cases}$$

Fix some $t \geq 0$ and $h \in \mathcal{C}_b(E \times E)$, and define, for $k \in \mathbb{N}$ and $y \in E$

$$\begin{aligned} a^k(y) &\stackrel{\text{def}}{=} \mathbb{E}\left(h\left(y, Y^{N_k}(t)\right) \middle| Y^{N_k}(0) = y\right) \\ b^k(y) &\stackrel{\text{def}}{=} \mathbb{E}\left(h\left(Y^{N_k}(t), y\right) \middle| Y^{N_k}(0) = y\right) \end{aligned}$$

Since Y^N is reversible under Π^{N_k}:

$$\int_E a^k(y)\Pi^{N_k}(dy) = \int_E b^k(y)\Pi^{N_k}(dy) \tag{7}$$

Hypothesis 1 implies that $\lim_{k\to\infty} a^k(x^k) = h(x, \varphi_t(x))$ for every sequence x^k such that $x^k \in E^{N_k}$ and $\lim_{k\to\infty} x^k = x \in E$. Now $X^k \in E^{N_k}$ $\mathbb{P}-$ almost surely, since the law of X^k is Π^{N_k} and Y^{N_k} is reversible under Π^{N_k} . Further, $X^k \to X$ $\mathbb{P}-$ almost surely; thus

$$\lim_{k\to\infty} a^k(X^k) = h(X, \varphi_t(X)) \quad \mathbb{P} - \text{ almost surely} \tag{8}$$

Now $a^k(X^k) \leq \|h\|_\infty$ and, thus, by dominated convergence:

$$\lim_{k\to\infty} \mathbb{E}\left(a^k(X^k)\right) = \mathbb{E}\left(h(X, \varphi_t(X))\right) \tag{9}$$

[1] i.e. there is a subsequence N_k such that $\lim_{k\to\infty} \Pi^{N_k} = \Pi$ for the weak topology on $\mathcal{P}(E)$

and similarly for b^k. Thus

$$\int_E h(y, \varphi_t(y))\Pi(dy) = \int_E h(\varphi_t(y), y)\Pi(dy) \tag{10}$$

□

In particular, if the semi-flow has a unique stationary point, we have:

Corollary 1. *Assume the processes Y^N are reversible under some probabilities Π^N. Assume Hypothesis 1 holds and:*

1. *the sequence $(\Pi^N)_{N=1,2,...}$ is tight;*
2. *the semi-flow φ has a unique stationary point y^*.*

It follows that the sequence Π^N converges weakly to the Dirac mass at y^*.

Proof. By [12, Thm 2.2], the set $\left\{\Pi^N, N = 1, 2, ...\right\}$ is relatively compact (for the weak topology) in $\mathcal{P}(E)$ therefore its closure, say $\mathcal{A}$, is compact. By [12, Thm 1.7], $\mathcal{P}(E)$ with the weak topology is metrizable (with Prokhorov's distance). By Theorem 3.1, any accumulation point in $\mathcal{P}(E)$ of the sequence $(\Pi^N)_{N=1,2,...}$ has support in the singleton $\{y^*\}$ and is thus equal to the Dirac mass at y^*, δ_{y^*}; thus, the sequence $(\Pi^N)_{N=1,2,...}$ has a unique accumulation point, equal to δ_{y^*}. In a metric compact space such as $\mathcal{A}$, a sequence such as Π^N that has a unique accumulation point converges to that point (e.g. [9, Prop. B.50]). Therefore, $\lim_{N\to\infty} d(\Pi^N, \delta_{y^*}) = 0$ where d is the Prokhorov's distance on $\mathcal{P}(E)$. Therefore also $\lim_{N\to\infty} \Pi^N = \delta_{y^*}$ for the weak topology in $\mathcal{P}(E)$. □

Recall that tightness means that for every $\varepsilon > 0$ there is some compact set $K \subset E$ such that $\Pi^N(K) \geq 1-\varepsilon$ for all N. If E is compact then $(\Pi^N)_{N=1,2,...}$ is necessarily tight, therefore condition 1 in the corollary is automatically satisfied. For mean field limits where E is the simplex in finite dimension, the corollary says that, if the pre-limit process is reversible, then the existence of a unique stationary point implies that the Dirac mass at this stationary point is the limit of the stationary probability of the pre-limit process.

Compare Corollary 1 to known results for the non reversible case [1]: there we need that the fluid limit φ has a unique limit point to which all trajectories converge. In contrast, here, we need a much weaker assumption, namely, the existence and uniqueness of a stationary point. It is possible for a semi-flow to have a unique stationary point, without this stationary point being a limit of all trajectories (for example because it is unstable, or because there are stable limit cycles as in [2]). In the reversible case, we do not need to show stability of the unique stationary point y^*.

5. **Example: Crowd dynamics.** In this section we give an example to illustrate the application of Theorem 4.1 – a detailed study of this example beyond the application of Theorem 4.1 is outside the scope of this paper. We consider the crowd dynamics model of [24]. The model captures the emergence of crowds in a city. A city is modelled as a fully connected bidirectional graph with I vertices, every vertex representing a square, where bars are located. There is a fixed total population N. People spend some time in a square and once in a while decide to leave a square and move to some other square. The original model is in discrete time, and at every time slot, the probability that a tagged person present in square i leaves square i is assume to be equal to $(1-c)^{N_i(t)-1}$ where $N_i(t)$ is the population of square i at time t. In this equation, c is the *chat* probability, and this model thus assumes that

a person leaves a square when it has no one to chat with. The model also assumes that departure events are independent. When a person leaves a square i, it moves to some other square j according to Markov routing, i.e. the next square is chosen independently of the past with probability $Q_{i,j}$ given by

$$Q_{i,j} = \frac{1}{d(i)} \tag{11}$$

where $d(i)$ is the degree of node i, i.e. the person picks a neighboring square j uniformly at random among all neighboring squares.

In [24], the authors study by simulation the emergence of concentration in one square. They also show that for regular graphs (i.e. when all vertices have same degree) there is a critical value c^* such that for $c > c^*$ concentrations occur, whereas for $c < c^*$ the stationary distribution of people is uniform. The analysis is based on the study of stationary points for the empirical distribution. Note that, as mentioned in the introduction, the analysis with stationary points may, in general, miss the main part of the stationary distribution, and it is quite possible that the stationary distribution is not concentrated on stationary points (for example if there is a limit cycle [2]). A fluid flow approximation is proposed in [22], and similar results are found.

To understand whether the stationary point analysis is justified, we study the large N asymptotics for an appropriately rescaled version. To avoid unnecessary complications, we replace the original model of [24], which is in discrete time, by its continuous time counterpart. First consider the Markov process on state space $\{1, ...I\}^N$ which represents the state of every person in town, the state of a person being equal to a vertex. The intensity rate of the transition where person k at vertex i leaves for neighbouring vertex j is $\frac{\mu^N(N_i)}{d(i)}$ where N_i is the number of persons at vertex i and

$$\mu^N(N_i) \stackrel{\text{def}}{=} (1-c)^{N_i - 1} \tag{12}$$

The process of interest is then the counting process $X^N(t) = (N_1(t), ..., N_I(t))$; it is also a Markov process, on $\{1, ...I\}^N$. The process X^N is known in queuing theory as a queuing network of infinite server stations with state dependent service rate $\mu^N(N_i)$ and with Markov routing given by Eq.(11).

It follows from classical results on quasi-reversibility that X^N has product-form (see for example [20, Chapter 8]), i.e. it is ergodic (since the graph of squares is fully connected and the population is finite) and its stationary probability is given, for every $(n_1, ..., n_I) \in \mathbb{N}^I$ such that $n_1 + ... + n_I = N$ by

$$P^N(n_1, ..., n_I) \quad = \quad \eta^N \prod_{i=1}^{I} f_i^N(n_i) \tag{13}$$

In this formula, η^N is a normalizing constant, $f_i^N(n) \stackrel{\text{def}}{=} \frac{\theta_i^n}{n!} \prod_{m=1}^{n} \mu^N(m)$, and the vector $\theta = (\theta_1, ..., \theta_I)$ is the stationary distribution of the random walk on the graph of squares, i.e. a normalized solution of the equation $\theta Q = \theta$ where the matrix Q is given by Eq.(11). Note that it follows from Eq.(11) that

$$\theta_i = \frac{d(i)}{\sum_{j=1}^{I} d(j)} \tag{14}$$

To apply Theorem 4.1, we need to show that X^N is reversible. A product-form queuing network is, in general, not reversible. However, it is so if the Markov routing chain (i.e. the random walk on the graph) is reversible [21], which is the case here.

Theorem 5.1. *For every N, the process X^N is reversible.*

Proof. Take θ given by Eq.(14). Then $\theta_i Q_{i,j} = \theta_j Q_{j,i}$ for any pair (i,j), thus the Markov chain with transition matrix Q given by Eq.(11) is reversible. By [21], it follows that the product-form queuing network X^N is reversible. □

In [8], it is suggested to scale the chat probability as

$$c = \frac{s}{N} \tag{15}$$

in order to account for the fact that, for large populations, meetings tend to be limited by space or size of the friend's group. We use this scaling law and consider the re-scaled process Y^N of occupancy measures, i.e. given by

$$Y^N(t) = \left(\frac{N_1(t)}{N}, \ldots \frac{N_I(t)}{N} \right) = \frac{1}{N} X^N(t) \tag{16}$$

Obviously, for every N the process Y^N is Markov and is reversible. Further, it converges to an ODE, as we see next.

To establish the convergence of Y^N, we compute its drift

$$V^N(y) \stackrel{\text{def}}{=} \lim_{dt \to 0} \frac{\mathbb{E}\left(Y^N(t+dt) - y \middle| Y^N(t) = y \right)}{dt} \tag{17}$$

defined for every possible value y of $Y^N(t)$. When the occupancy measure is y, there are $N_i = N y_i$ people in square i, and the rate of departure from square i is $N_i \mu(N_i) = N y_i \mu(N y_i)$; the delta to the occupancy measure due to one person moving from square i to square j is $\frac{-e_i + e_j}{N}$, where e_i is the row vector with a 1 in position i and 0 elsewhere. Therefore

$$V^N(y) = \sum_{i,j} N y_i \mu(N y_i) Q_{i,j} \frac{-e_i + e_j}{N}$$

Taking into account Eq.(15), it comes

$$V^N(y) = \sum_{i,j} y_i \left(1 - \frac{s}{N} \right)^{N y_i - 1} Q_{i,j} \left(-e_i + e_j \right) \tag{18}$$

Let Δ_I denote the simplex, i.e.

$$\Delta_I = \left\{ y \in \mathbb{R}^I, y_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{I} y_i = 1 \right\}$$

When $N \to \infty$, $V^N(y)$ converges for every $y \in \Delta_I$ to

$$V(y) \stackrel{\text{def}}{=} \sum_{i,j} y_i e^{-s y_i} Q_{i,j} \left(-e_i + e_j \right) \tag{19}$$

This suggests that the fluid limit of Y^N, if it exists, would be the deterministic process $y(t)$, with sample paths in $\mathbb{R}^d$, obtained as solution of the ODE $\frac{dy}{dt} = V(y)$.

As we show next, this is indeed the case and follows from "Kurtz's theorem" [25, Theorem 10.2.1]. Before that, we rewrite the ODE more explicitly as

$$\frac{dy_i}{dt} = -y_i e^{-sy_i} + \sum_j y_j e^{-sy_j} Q_{j,i} \tag{20}$$

Theorem 5.2. *Assume that the initial condition y_0^N of the process Y^N is deterministic and converges to some $y_0 \in \Delta_I$. Let φ be the semi-flow defined by the ODE (20), i.e. $\varphi_t(y_0)$ is the solution of the ODE (20) with initial condition $y(0) = y_0$ (this solution exists and is unique by the Cauchy Lipschitz theorem). Then for each $T > 0$ and $\varepsilon > 0$:*

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{0\le t\le T} \left\| Y^N(t) - \varphi_t(y_0) \right\| > \varepsilon \right) = 0 \tag{21}$$

(the notation $\|\|$ stands for any norm on $\mathbb{R}^I$).

It follows that Hypothesis 1 is verified.

Proof. We apply Theorem 10.2.1 in [25]. We need to find a sequence of numbers $\delta_N \to 0$ such that the following three conditions hold:

$$\lim_{N\to\infty} \sup_{y\in\Delta_I^N} \left\| V^N(y) - V(y) \right\| = 0 \tag{22}$$

$$\sup_N \sup_{y\in\Delta_I^N} A^N(y) < \infty \tag{23}$$

$$\lim_{N\to\infty} \sup_{y\in\Delta_I^N} A^N_{\delta_N}(y) = 0 \tag{24}$$

In the above, Δ_I^N is the set of feasible states of Y^N, i.e. the set of $y \in \Delta_I$ such that Ny is integer, $A^N(y)$ is the expected norm of jump per time unit, and $A^N_{\delta_N}(y)$ is the absolute expected norm of jump per time unit due to jumps travelling further than δ_N.

We now show Eq.(22). First consider the case $y \in \Delta_I^N$ such that $y_i > 0$ (thus we have $1/N \le y_i \le 1$). We apply the inequality $\left|e^{-x} - e^{-x'}\right| \le |x - x'|$, valid for $x \ge 0$ and $x' \ge 0$, to $x = -(Ny_i - 1)\log\left(1 - \frac{s}{N}\right)$, $x' = sy_i$ and obtain:

$$\left| \left(1 - \frac{s}{N}\right)^{Ny_i - 1} - e^{-sy_i} \right| \le \left| (Ny_i - 1)\log\left(1 - \frac{s}{N}\right) + sy_i \right|$$

The right handside is convex in y_i thus its maximum for $y_i \in [1/N, 1]$ is obtained at one end of the interval. Thus

$$\left| \left(1 - \frac{s}{N}\right)^{Ny_i - 1} - e^{-sy_i} \right| \le a_N(s) \stackrel{\text{def}}{=} \max\left[\frac{s}{N}, \left| (N-1)\log\left(1 - \frac{s}{N}\right) + s \right| \right]$$

Second, multiply by y_i and note that $y_i \le 1$, it follows that, whenever $y \in \Delta_I^N$ and $y_i > 0$:

$$\left| y_i \left(1 - \frac{s}{N}\right)^{Ny_i - 1} - y_i e^{-sy_i} \right| \le a_N(s)$$

and this is also obviously true if $y_i = 0$. It follows that

$$\sup_{y\in\Delta_I^N} \left\| V^N(y) - V(y) \right\| \le a_N(s) \sum_{i,j} \left\| -e_i + e_j \right\| Q_{i,j}$$

from where Eq.(22) follows since $\lim_{N\to\infty} a_N(s) = 0$.

We now show Eq.(23). We take the sup norm on $\mathbb{R}^d$ so that $\|-e_i+e_j\|=1$ for $i \neq j$; thus

$$\begin{aligned} A^N(y) &\stackrel{\text{def}}{=} \sum_{i,j} y_i \left(1-\frac{s}{N}\right)^{Ny_i-1} Q_{i,j} \|-e_i+e_j\| \\ &= \sum_{i,j} y_i \left(1-\frac{s}{N}\right)^{Ny_i-1} Q_{i,j} \leq \sum_{i,j} Q_{i,j} = I \end{aligned}$$

which shows Eq.(23).

We now show Eq.(24). We take $\delta_N = \frac{1}{N}$. The jumps of Y^N are of the form $\frac{-e_i+e_j}{N}$ and thus $\left\|\frac{-e_i+e_j}{N}\right\| \leq \delta_N$. Thus

$$A^N(y)_{\delta_N} \stackrel{\text{def}}{=} \sum_{i,j} y_i \left(1-\frac{s}{N}\right)^{Ny_i-1} Q_{i,j} \|-e_i+e_j\| \mathbf{1}_{\{\left\|\frac{-e_i+e_j}{N}\right\|>\delta_N\}} = 0$$

which trivially shows Eq.(24). By Theorem 10.2.1 in [25], this establishes Eq.(21).

It follows (as a much weaker convergence) that for any fixed T, $Y^N(T)$ converges in probability to the deterministic $\varphi_T(y_0)$. Thus there is also convergence in law, since convergence in probability to a deterministic variable implies convergence in distribution, i.e. Eq.(2) in Hypothesis 1 is verified. It remains to see that φ_t is space continuous: this follows from the fact that the right-handside of the ODE is Lipschitz continuous and from the Cauchy Lipschitz theorem. □

It follows from Theorem 4.1 that any limit point of the stationary probability Eq.(13) is concentrated on the stationary points of the ODE (20). This justifies a posteriori the method in [24], which looked only at stationary points.

For the case of a regular graph (this is the case studied analytically in [24]), the stationary points can be obtained explicitly (Theorem 6.1 in [8]). In particular, there is a critical value s^* below which there is only one stationary point, equal to the uniform distribution $y^* = (\frac{1}{I}, ..., \frac{1}{I})$ and above which there are other stationary points. The critical value is given in [8] and is equal to

$$s^* = \min_{K=1,\ldots,I-1} \min_{\alpha>1} ((I-K)\alpha + K\phi(\alpha)) \tag{25}$$

with $\phi(x) \stackrel{\text{def}}{=} -W_0(-xe^{-x})$, W_0 being the Lambert-W function of index 0. For example for $I=3$, the critical value is $s^* \approx 2.7456$.

We can apply Corollary 1: since the state space $E = \Delta_I$ is compact, it follows that for $s < s^*$, the stationary distribution given by Eq.(13), re-scaled by $1/N$, converges as $N \to \infty$ to the uniform distribution. This illustrates the interest of the reversibility results in this paper; we do not need to show that all trajectories converge to the single stationary point – its uniqueness and the reversibility argument are sufficient.

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